# On the Minimum of Exponential and Teissier Distributions

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**Abstract**

*In reliability theory minimum of two random variables has a significant meaning, and models with increasing failure rates play a vital role. Motivated by these facts, in this article, a two-parameter lifetime distribution with an increasing failure rate is constructed by considering the method of a minimum of two independent random variables following the exponential and Teissier distributions and studied in detail. Several exciting features, such as moments, quantiles, Bonferroni and Lorenz curves, entropies, stress–strength reliability, moments of a residual lifetime, and order statistics, are derived for the proposed distribution. For the estimation purpose, eight different techniques have been used, including maximum likelihood, ordinary least square, weighted least square, Cramer-von Mises, maximum product spacing, Anderson-Darling, right-tailed Anderson-Darling, and bootstrapping (parametric and nonparametric). The performance of these estimators is compared using three real datasets. The exact Fisher information matrix elements are derived, and confidence intervals based on the information matrix and bootstrapping techniques are constructed. A simulation study is carried out to see the efficiency of the maximum likelihood in terms of mean square error and bias. Negative log-likelihood, Akaike information criteria, Bayesian information criteria, Consistent Akaike information criteria, and Hannan-Quinn information criteria are the goodness-of-fit statistics employed. Furthermore, other nonparametric test statistics such as Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises are used for model selection. Moreover, three real datasets related to epidemiology, seismology, and reliability are modeled and compared with exponential, exponentiated exponential, Lindley, exponentiated Lindley, Rayleigh, exponentiated Rayleigh, Gompertz, exponentiated Gompertz, Weibull, and exponentiated Weibull distributions to demonstrate how the suggested model performs in practice. And it is observed that the proposed distribution provides a better fit among all considered models, according to most of the test statistics. The proposed lifetime distribution is unimodal and capable of modeling positive datasets with an increasing failure rate which contains Gompertz one-parameter model as a particular case. It is a simple model with only two parameters resulting from expressions for different characteristics that are analytically tractable. So, it is expected that it will be helpful in various disciplines where such types of data exist, such as reliability, lifetime modeling, and survival analysis.*

**Keywords:** Probability distribution, Moments, Information Matrix, Maximum likelihood estimator, Bootstrap, Simulations.

## 1. Introduction

To model, the frequency of mortality associated with aging alone, Teissier [1] developed an increasing failure rate distribution, known as the Teissier distribution (TD). Muth [2] pointed out that the TD has a heavier tail than some classical distributions like the gamma, Weibull, and log-normal distributions. The TD was used by Rinne [3] to model the lifetime of a real dataset related to used motor cars. Leemis and McQueston [4] established a univariate distributional

relationship, in which this model was reconsidered and renamed the "Muth distribution." Some statistical features of Muth distribution were thoroughly investigated by Jodra et al. [5]. Irshad et al. [6, 7] studied inference and some other extensions of the Muth distribution. Saroj et al. [12] introduced inverse muth distribution.
Recently, exponentiated Teissier distribution (ETD) has been given by Sharma et al. [11]. Eghwerido [12], and Poonia and Aazad [13] applied the alpha power transform (APT) technique of Mahdavi and Kundu [14] on TD and ETD, respectively. The exponential distribution (ED) is a well-known classical distribution with some distinguishing features such as a constant hazard rate and memorylessness. Related references on exponential distributions can be found in the literature, for example, see Gupta and Kundu [15], Nadarajah and Haghighi [16] and Mahdavi and Kundu [14]. Gompertz proposed that human mortality increases exponentially with age. Makeham extended Gompertz's suggestion of competing risks by adding one and two parameters to the standard Gompertz distribution known as Gompertz Makeham-I (GMD-I) and Gompertz Makeham-II (GMD-II) distributions, respectively. Chapter 10 of Marshall and Olkin [17] provides a comprehensive review of the Gompertz and all extensions made by Makeham. According to chapter 10 of Marshall and Olkin [17], GMD-I has three cases. The cdf of the second case of GMD-I is given as

$$F(x;\theta,\beta,\xi) = 1 - e^{-\xi(e^{\theta x}+\beta\theta x-1)} \; \theta > 0, -1 \leq \beta < 0, \; \xi > 0. \tag{1}$$

However, TD is a particular case of the second case of GMD-I when $\xi = 1$ and $\beta = -1$. The case of $\beta = -1$ is under communication. Recently, many lifetime distributions has been developed in reliability theory, Deepthy and Sebastian [9] developed Burr III Modified Weibull Distribution, Manoharan and Kavya [13] extended Lomax distribution to construct a reliability model. Some probable scenarios that arise in real-life applications because of the distribution of the minimum of two random variables are fascinating, see chapters 5 and 17 of Marshall and Olkin [17].

Suppose $X_1$ and $X_2$ are two independent random variables follow Teissier and exponential distribution with parameters $\theta$ and $\theta\lambda$ respectively. The cumulative density functions (cdfs) of $X_1$ and $X_2$ are given by (for $x > 0,\ \theta > 0,\ \lambda \geq 0$)

$$F_{X_1}(x;\theta) = 1 - e^{\theta x - e^{\theta x}+1}, \;\; F_{X_2}(x;\theta,\lambda) = 1 - e^{-\theta\lambda x}, \tag{2}$$

respectively. Suppose $X$=Minimum$\{X_1, X_2\}$. The cdf, pdf, survival function, hazard rate function (hrf), cumulative hazard rate and reversed hazard rate of METD are given by (for $x > 0,\ \theta > 0,\ \lambda > 0$)

$$F(x;\theta,\lambda) = 1 - e^{1+\theta x - e^{\theta x} - \theta\lambda x}, \tag{3}$$

$$f(x;\theta,\lambda) = \theta\left(\lambda - 1 + e^{\theta x}\right) e^{1+\theta x - e^{\theta x} - \theta\lambda x}, \tag{4}$$

$$S(x;\theta,\lambda) = e^{1+\theta x - e^{\theta x} - \theta\lambda x}, \; h(x;\theta,\lambda) = \theta\left(\lambda - 1 + e^{\theta x}\right), \tag{5}$$

$$H(x;\theta,\lambda) = (e^{\theta x} + \theta\lambda x - \theta x - 1), \; r(x;\theta,\lambda) = \frac{\theta\left(\lambda - 1 + e^{\theta x}\right) e^{\theta x+1}}{e^{\theta\lambda x + e^{\theta x}} - e^{\theta x+1}}. \tag{6}$$

Interestingly, the pdf of METD can be obtained from second case of GMD-I as a special case by substituting $\xi = 1$ and $\beta = \lambda - 1$ in Eqn.(1), in this case $0 \leq \lambda < 1$. Unfortunately, this case has not received much attention in the literature. However, the METD model work for $\lambda \geq 1$ also. It should be noted that Gompertz's one-parameter distribution is a particular case of METD when $\lambda = 1$ in METD.
The rest of the article is organized as follows. In Section 2, some statistical properties of the METD have been derived. Section 3 deals with the estimation of the parameters of METD. In Section 4, simulation is carried out. In section 5 three applications are presented to show that the proposed distribution can be used quite effectively in analyzing the real-life datasets. Finally, section 6 provides some conclusions.

## 2. Statistical Properties

In this section some basic features of the proposed distribustions such as shape analysis of pdf and hrf of METD, moments, quantiles, Bonferroni and Lorenz Curve, Renyi entropy, stress-strength reliability (ssr), moments of residual life function and order statistics are studied.

### 2.1. Shape of pdf and hrf

The pdf of METD is log-concave as

$$(\log f(x))'' = \frac{-\theta^2 e^{\theta x}\left[(e^{\theta x}-1)^2 + 2\lambda(e^{\theta x}-1) + (\lambda-1/2)^2 + 3/4\right]}{(e^{\theta x}+\lambda-1)^2}, \tag{7}$$

is negative for all $x > 0,\ \theta > 0,\ \lambda > 0$. Moreover, $\lim_{x\to 0^+} f(x) = \theta\lambda$ and $\lim_{x\to\infty} f(x) = 0$. The pdf is decreasing for $\lambda \geq 1$ and having a unique mode at $\frac{1}{\theta}\log\left(\frac{1}{2}\left(3-2\lambda+\sqrt{5-4\lambda}\right)\right)$ for $0 \leq \lambda < 1$, see Fig.1(a). Also $(h(x))' = \theta^2 e^{\theta x} > 0 \implies$ hrf is exponentially increasing.

### 2.2. Moment generating function and moments

By using $u = e^{\theta x}$, the moment generating function (mgf) of METD can be written as

$$M_X(t) = e\left[E^0_{\lambda-\frac{t}{\theta}-1}(1) - (1-\lambda)E^0_{\lambda-\frac{t}{\theta}}(1)\right], \tag{8}$$

where

$$E^l_s(z) = \frac{1}{\Gamma(l+1)}\int_1^\infty (\log u)^l e^{-zu} u^{-s}\, ds, \tag{9}$$

$l > -1$, $s \in \mathbb{R}$ and $\Gamma(.)$ is the gamma function. Moreover, the $r$th derivative of $M_X(t)$ at $t = 0$ also the $r$th moment about origin, can be given as

$$E(X^r) = M_X^{(r)}(0) = e\theta^{-r}\Gamma(r+1)\left(E^r_{\lambda-1}(1) - (1-\lambda)E^r_\lambda(1)\right). \tag{10}$$

Using the moments, mean, variance, skewness and excess kurtosis of the METD can be calculated and shown in Fig.1.

### 2.3. Quantile function

By inverting the cdf of METD, the quantile function of METD($\theta,\lambda$) can be expressed as

$$\xi_p = \begin{cases} \frac{1}{\theta}\log\left[(\lambda-1)W_{-1}\left(\frac{e^{\frac{1}{\lambda-1}}(1-p)^{\frac{1}{1-\lambda}}}{\lambda-1}\right)\right] & \text{if } \lambda < 1 \\ \frac{1}{\theta}\log\left[(\lambda-1)W\left(\frac{e^{\frac{1}{\lambda-1}}(1-p)^{\frac{1}{1-\lambda}}}{\lambda-1}\right)\right] & \text{if } \lambda > 1 \end{cases},\quad \frac{1}{\theta}\log\left[1-\log(1-p)\right] \text{ if } \lambda = 1 \tag{11}$$

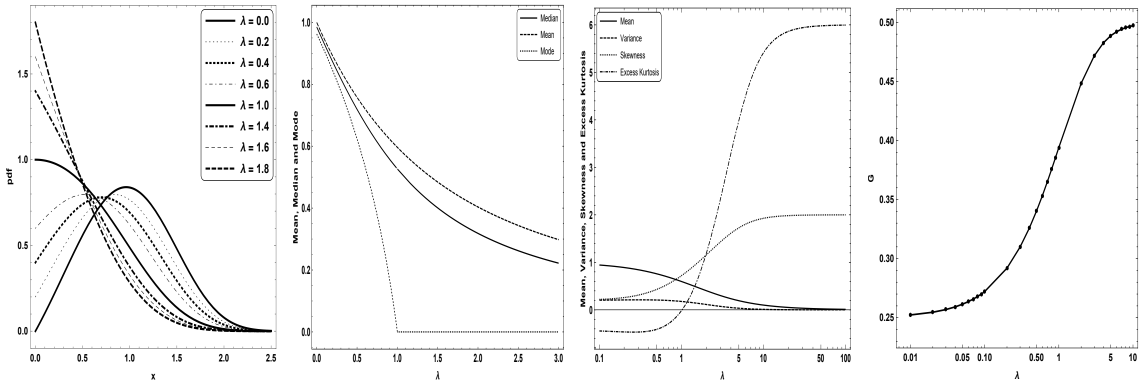
where $p \in (0,1)$, $W(.)$ represent the Lambert-$W$ function( see Jodra [18]) and $W_{-1}(.)$ is the negative branch of the Lambert-$W$ function, $p = 0.25, 0.50, 0.75$ corresponds to the first, second(median) and third quartiles.

### 2.4. Bonferroni and Lorenz Curve

The Bonferroni curve, Lorenz curve and Gini coefficient are defined as $B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(t)\,dt$, $L(p) = \frac{1}{\mu}\int_0^p F^{-1}(t)\,dt$, and $G = 1 - 2\int_0^1 L(p)\,dp$, respectively, where $0 < p < 1$, $\mu = \int_0^1 F^{-1}(t)\,dt$ and $F^{-1}(.)$ is the quantile function. Lorenz curve for METD is given as

$$L(p) = 1 - \frac{J\left(\theta,\xi_p,1,\lambda-1,1\right) - (1-\lambda)J\left(\theta,\xi_p,1,\lambda,1\right)}{J(\theta,0,1,\lambda-1,1) - (1-\lambda)J(\theta,0,1,\lambda,1)}, \tag{12}$$

where $J(\theta,t,r,s,z) = \frac{1}{\Gamma(r+1)}\int_{e^{\theta t}}^\infty (\log u)^r e^{-zu} u^{-s}\, du$.




(a) (b) (c) (d)

**Figure 1:** *Plots of (a) Probability density function, (b) Mean, median, and mode, and (c) Mean, variance, skewness, and excess kurtosis, (d) Gini coefficient, for the METD when $\theta = 1$.*

## 2.5. Renyi entropy

The Renyi entropy of a non-negative continuous random variable $X$ with pdf $f(x)$ is a measure of variation and is defined as $I_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty f(x)^\gamma dx$, where $\gamma > 0$ and $\gamma \neq 1$. If $X \sim$METD$(\theta, \lambda)$ then the Renyi entropy of $X$ is given by (using $u = e^x$)

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left[e^\gamma\theta^{\gamma-1}H(\gamma, \gamma(\lambda-1)+1, \lambda-1, \gamma)\right], \tag{13}$$

where $H(z, s, c, p) = \int_1^\infty u^{-s}e^{-zu}(c+u)^p\,du,\ c > -1$.

## 2.6. Stress strength reliability

Suppose $X_1$ and $X_2$ are two random variables from the METD family such that $X_1 \sim$ METD$(\theta_1, \lambda_1)$ and $X_2 \sim$ METD$(\theta_2, \lambda_2)$. The ssr of METD is specified as

$$R = P(X_1 > X_2) = \int_0^\infty f_{X_1}(x)F_{X_2}(x)\,dx. \tag{14}$$

using $u = e^{\theta x}$, the expression of $R$ can be written as

$$R = 1 - e^2\left(Q\left(\theta, \theta\lambda_2 - \theta + \lambda_1 - 1\right) + (\lambda_1 - 1)\,Q\left(\theta, \theta\left(\lambda_2 - 1\right) + \lambda_1\right)\right), \tag{15}$$

where $Q(\theta, s) = \int_1^\infty u^{-s}e^{-(u+u^\theta)}\,du$ and $\theta = \theta_2/\theta_1$.

## 2.7. Moments of residual life function

The $r$th moment of residual life of a random variable $X$ with pdf $f(x)$ and cdf $F(x)$ is given by

$$m_r(t) = E[(X-t)^r|X > t] = \frac{1}{1-F(t)}\int_t^\infty (x-t)^r f(x)\,dx, \tag{16}$$

where $r = 1, 2, 3, ....$

The mean and variance of residual life may be expressed as $m(t) = m_1'(t) - t$, and $V(t) = m_2'(t) - (m_1'(t))^2$ where $m_r'(t) = \frac{1}{1-F(t)}\int_t^\infty x^r f(x)\,dx$. By using $u = e^{\theta x}$ and defining $J(\theta, t, r, s, z) = \frac{1}{\Gamma(r+1)}\int_{e^{\theta t}}^\infty (\log u)^r e^{-zu}u^{-s}\,du$ , the numerator of $m_r'(t)$ can be written as

$$\int_t^\infty x^r f(x)\,dx = e\Gamma(r+1)\theta^{-r}\left[J(\theta, t, r, \lambda-1, 1) + (\lambda-1)J(\theta, t, r, \lambda, 1)\right]. \tag{17}$$

## 2.8. Order statistics

Assume that $X_1, X_2, ..., X_n$ is a random sample of size n drawn from a population with the pdf $f(x)$ and the related order statistics $X_{(1)}, X_{(2)}, ..., X_{(n)}$. The pdf of $r$th order statistics of METD is given as

$$f_{X_{(r)}}(x) = \frac{\theta}{B(r, n-r+1)} \left(\lambda - 1 + e^{\theta x}\right) (1 - \phi(x))^{r-1} (\phi(x))^{n-r+1}, \tag{18}$$

where $\phi(x) = e^{1-\theta\lambda x+\theta x - e^{\theta x}}$, $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $x > 0$, $\theta > 0, \beta > 0, r \leq n$. The $m$th moment of $r$th order statistics $X_{(r)}$ is derived as

$$E[X_{(r)}^m] = \frac{\theta^{-m}\Gamma(m+1)}{B(r, n-r+1)} \sum_{i=0}^{n-r} \sum_{j=0}^{i+r-1} (-1)^{i+j} e^{j+1} \binom{i+r-1}{j} \binom{n-r}{i} \left[E_{q_j}^m(j+1) + (\lambda-1) E_{q_j+1}^m (j+1)\right], \tag{19}$$

where $E_s^l(z)$ given in Eqn. (9) and $q_j = (j+1)(\lambda-1)$.

# 3. Estimation of Parameters

For the estimation purpose, eight different techniques have been used, including maximum likelihood (MLE), ordinary least square (OLS), weighted least square (WLS), Cramer-von Mises (CVM), maximum product spacing (MPS), Anderson-Darling (AD), right-tailed Anderson-Darling (RTAD), and bootstrapping (parametric and nonparametric). For the OLS and MPS techniques, see Swain et al. [19], and Cheng and Amin [20], respectively. A general theory of various estimation techniques can be found in Sharma et al. [11] and Dey et al. [?]. For parametric and nonparametric bootstrap estimation, 1000 samples were generated according to the algorithm given by Kharazmi et al. [21]. The discussed estimation procedures are applied to two real datasets, and the results are shown in section 5.

## 3.1. Method of the Maximum likelihood

The maximum likelihood (MLE) approach is the most extensively used approach for parameter estimation. This approach has numerous flexible properties, including consistency, asymptotic efficiency, and invariance. Suppose $x_1, x_2, ..., x_n$ be a sample of size n from the METD. For the vector of parameters $\Theta = (\theta, \lambda)$, the log-likelihood function of METD is given as

$$l(\Theta) = n \log\theta + \sum_{i=1}^{n} \left[1 - e^{\theta x_i} + \theta(1-\lambda)x_i\right] + \sum_{i=1}^{n} \log\left[\lambda - 1 + e^{\theta x_i}\right]. \tag{20}$$

The MLEs of the parameters have been calculated numerically using Mathematica 12.3. After differentiation of the log-likelihood function with respect to the parameters $\theta$ and $\lambda$, the components of the score vector $U(\Theta)$ can be expressed as

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{x_i e^{\theta x_i}}{\lambda - 1 + e^{\theta x_i}} + \sum_{i=1}^{n} x_i(1 - \lambda - e^{\theta x_i}),\ U_\lambda = \sum_{i=1}^{n} \frac{1}{\lambda - 1 + e^{\theta x_i}} - \theta \sum_{i=1}^{n} x_i. \tag{21}$$

The MLEs of the parameters can be obtained by setting these equations to zero and solving them. Another advantage of the MLE approach is that it is useful to construct approximated confidence interval (ACI) of the parameters, see Lawless [?]. The exact $2 \times 2$ information matrix $I(\Theta)$ required for interval estimate of METD parameters defined as $I(\Theta) = \begin{pmatrix} I_{\theta,\theta} & I_{\theta,\lambda} \\ I_{\lambda,\theta} & I_{\lambda,\lambda} \end{pmatrix}$. The members of the $I(\Theta|x)$ for the METD are given as

$$I_{\theta,\theta} = -E\Big(\frac{\partial^2 l(\Theta|x)}{\partial\theta^2}\Big) = \theta^{-2}\big[2e\big\{(\lambda-1)(E_{\lambda-1}^2(1) - J_{\lambda-1}^2(\lambda-1)) + E_{\lambda-2}^2(1)\big\} + 1\big], \tag{22}$$

$$I_{\theta,\lambda} = I_{\lambda,\theta} = -E\Big(\frac{\partial^2 l(\Theta|x)}{\partial\theta\partial\lambda}\Big) = \frac{1}{\theta}\Big[e(E^1_{\lambda-1}(1) + (\lambda-1)E^1_{\lambda}(1) + J^1_{\lambda-1}(\lambda-1))\Big], \qquad (23)$$

and

$$I_{\lambda,\lambda} = -E\Big(\frac{\partial^2 l(\Theta|x)}{\partial\lambda^2}\Big) = eJ^0_{\lambda}(\lambda-1), \qquad (24)$$

where the integral $E^l_s(z)$ is defined in Eqn.(9) and the integral $J^r_s(k) = \frac{1}{\Gamma(r+1)}\int_1^\infty \frac{e^{-u}u^{-s}\log^r(u)}{k+u}\,du$. The multivariate normal distribution $N_2(0, I(\hat{\Theta})^{-1})$ may be used to generate confidence intervals for model parameters under usual regularity conditions.

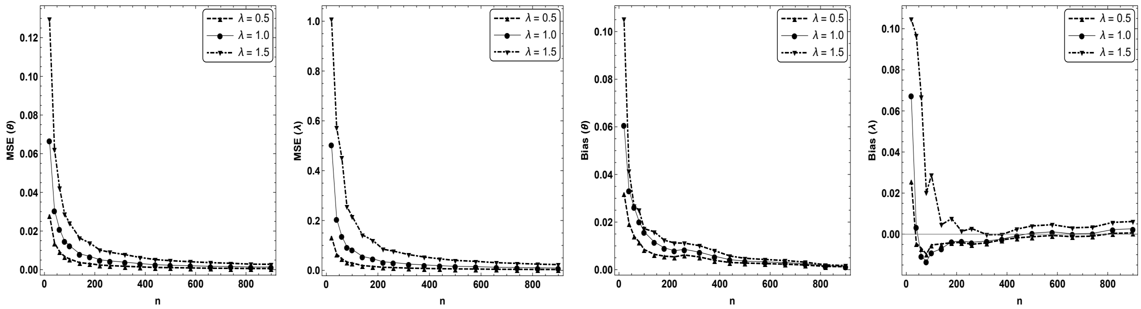



**Figure 2:** *The MSE and bias of the parameters for the simulated samples.*

## 4. Simulation

To investigate the performance of the MLEs for the METD parameters, several simulations are explored for different sample sizes. The samples are generated from Eqn. (11), using the inverse cdf technique. The parameters values are taken as $\theta = 1$ and $\lambda = 0.5, 1.0$ and 1.5 and the sample sizes are selected as $n = 20, 40, 60, 80, 100, 140, 180, 220, 260, 320, 380, 440, 500, 580, 660, 740, 820$ and 900. Each sample size is repeated 1000 times, biases and mean squared errors are calculated. Fig.2 displays the results of the simulation. The trends in Fig.2 reveal that as the sample size increases, the MSEs and bias of the MLEs decay toward zero, as expected by first-order asymptotic theory.

## 5. Data Analysis

Three real datasets are being used to demonstrate the practical significance of the METD model. The first dataset was downloaded from the webportal of the World Health Organization (https://covid19.who.int/) on October 12, 2021, which denotes the daily number of deaths in South Africa due to the novel coronavirus from May 11, 2020, to June 28, 2020. Currently, the first dataset is slightly modified by the WHO. The second dataset is related to seismology and is taken from the Wolfram data repository, which indicates the earthquake waiting times in days. The third dataset is taken from Murthy et al. [25], which is about aircraft windshield data and demonstrates the service times of windshields that had not failed at the time of observation.

- Dataset I
  8, 12, 1, 13, 19, 9, 14, 3, 22, 26, 27, 30, 28, 10, 22, 52, 43, 28, 25, 34, 32, 40, 22, 50, 37, 56, 60, 44, 46, 82, 82, 48, 74, 70, 69, 57, 88, 57, 49, 63, 94, 46, 53, 61, 111, 103, 87, 48, 73.
- Dataset II
  840, 1901, 40, 139, 246, 157, 695, 1336, 780, 1617, 145, 294, 335, 203, 638, 44, 562, 1354, 436, 937, 33, 721, 454, 30, 735, 121, 76, 36, 384, 38, 150, 710, 667, 129, 365, 280, 46, 40, 9, 92, 434, 402, 556, 209, 82, 736, 194, 99, 599, 220, 584, 759, 304, 83, 887, 319, 375, 832, 263, 460, 567, 328.

- Dataset III
  0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

Table 1 give a brief summary of the datasets and fitted METD. The datasets are starting from non-zero and right-skewed. Therefore, METD may be the right choice to model these datasets. Parametric and non-parametric bootstrap techniques also have been applied for estimation purpose by adopting the methodology of Efron and LePage [22]. According to the described methodology in the estimation section, the exact information matrix for the datasets I, II and III are given as: $\begin{pmatrix} 14517.2 & 77.4383 \\ 77.4383 & 0.7927 \end{pmatrix}$, $\begin{pmatrix} 3.0789 \times 10^6 & 484.359 \\ 484.359 & 0.0799 \end{pmatrix}$ and $\begin{pmatrix} 30.5202 & 3.54375 \\ 3.54375 & 0.787313 \end{pmatrix}$, respectively. Inverse of the exact information matrix for the datasets I, II and III are given as: $\begin{pmatrix} 0.0001 & -0.0140 \\ -0.0140 & 2.6341 \end{pmatrix}$, $\begin{pmatrix} 6.8421 \times 10^{-6} & -0.0414 \\ -0.0414 & 263.358 \end{pmatrix}$ and $\begin{pmatrix} 0.0686361 & -0.308936 \\ -0.308936 & 2.66068 \end{pmatrix}$, respectively. The interval estimates of the parameters based on the expected information matrix are given as: (i) For the dataset I, $\hat{\theta} \in (0.0134, 0.0201)$, $\hat{\lambda} \in (0.0253, 0.9342)$. (ii) For the dataset II, $\hat{\theta} \in (0, 0.0012)$, $\hat{\lambda} \in (0, 7.2586)$ and (iii) For the dataset III, $\hat{\theta} \in (0.3018, 0.4312)$, $\hat{\lambda} \in (0.0812, 0.8867)$. The point and interval estimates of the parameters based on the parametric bootstrap are given as: (i) For the dataset I, $\hat{\theta} = 0.0170, \hat{\lambda} = 0.4903, \hat{\theta} \in (0.0137, 0.0206), \hat{\lambda} \in (0.1577, 1.0143)$. (ii) For the dataset II, $\hat{\theta} = 0.0007, \hat{\lambda} = 3.1266, \hat{\theta} \in (0.0003, 0.0012), \hat{\lambda} \in (1.4779, 6.6621)$ and (iii) For the dataset III, $\hat{\theta} = 0.3709, \hat{\lambda} = 0.4771, \hat{\theta} \in (0.3096, 0.4411), \hat{\lambda} \in (0.1274, 0.9156)$ . The point and interval estimates of the parameters based on the non-parametric bootstrap are given as: (i) For the dataset I, $\hat{\theta} = 0.0169, \hat{\lambda} = 0.4885, \hat{\theta} \in (0.0141, 0.0201), \hat{\lambda} \in (0.1315, 0.9469)$. (ii) For the dataset II, $\hat{\theta} = 0.0007, \hat{\lambda} = 3.1244, \hat{\theta} \in (0.0003, 0.0012), \hat{\lambda} \in (1.4249, 6.0172)$ and (iii) For the dataset III, $\hat{\theta} = 0.3696, \hat{\lambda} = 0.4913, \hat{\theta} \in (0.3115, 0.4367), \hat{\lambda} \in (0.1812, 0.9456)$ All the seven different estimation

**Table 1:** *Summary of the datasets and fitted METD.*

| | Size | Min | $Q_{.25}$ | Median | $Q_{.75}$ | Max | Mean | Skew | Ex-Ku |
|---|---|---|---|---|---|---|---|---|---|
| Data I | 49 | 1 | 25 | 46 | 61 | 111 | 45.46 | 0.42 | -0.52 |
| METD | - | 0 | 23.99 | 43.26 | 64.22 | $\infty$ | 45.47 | 0.41 | -0.41 |
| Data II | 62 | 9 | 129 | 328 | 667 | 1901 | 437.21 | 1.49 | 2.52 |
| METD | - | 0 | 136.87 | 323.22 | 624.12 | $\infty$ | 437.20 | 1.48 | 2.57 |
| Data III | 63 | 0.04 | 1.09 | 2.06 | 2.81 | 5.14 | 2.08 | 0.43 | -0.26 |
| METD | - | 0 | 1.09 | 1.98 | 2.94 | $\infty$ | 2.08 | 0.41 | -0.41 |

approaches, as given in the estimation section, have been applied to estimate the parameters of METD for the both datasets, and results are shown in Table 2 with different test statistics and ranking based on Kolmogorov-Smirnov (KS) test. From Table 2, it may be concluded that according to the KS test, the maximum likelihood (ML) and CVM are the best estimator among considered estimation procedures for dataset I and II respectively, whereas MPS is the worst techniques among all considered methods for the both datasets. For third dataset CVM is most effective estimator and AD is not much efficient among all estimators.

The METD is compared with the following distributions: exponential distribution (ED), exponentiated exponential distribution (EED) of Kundu and Gupta [15], Teissier distribution (TD) of Teissier [1], exponentiated Teissier distribution (ETD) of Sharma et al. [11], Rayleigh distribution (RD), exponentiated Rayleigh distribution (ERD), Lindley distribution, exponentiated Lindley distribution, Gompertz Distribution (GOD), exponentiated Gompertz Distribution (EGOD) of El-Gohary et al. [24], Weibull distribution (WD), and exponentiated Weibull distribution (EWD) of Mudholkar and Srivastava [23]. Several goodness-of-fit (gof) statistics are used for model selection, including negative log-likelihood (NLL), Akaike (AIC), Bayesian (BIC), and Consistent

**Table 2:** *Estimation of parameters by different techniques and various test statistics with ranking(r).*

| Method | $\hat{\theta}$ | $\hat{\lambda}$ | NLL | OLSS | WLSS | CVMS | MPSS | ADS | RTADS | KS | $p$-Value(KS) | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Dataset I | | | | | | | | | | | | |
| ML | 0.0168 | 0.4797 | 227.402 | 0.0175 | 6.0462 | 0.0178 | -4.0207 | 0.1138 | 0.0556 | 0.0475 | 0.9996 | 1 |
| OLS | 0.0162 | 0.5409 | 227.453 | 0.0159 | 5.0371 | 0.0190 | -4.0177 | 0.1182 | 0.0548 | 0.0507 | 0.9989 | 4 |
| WLS | 0.0160 | 0.5599 | 227.496 | 0.0161 | 4.9512 | 0.0202 | -4.0172 | 0.1262 | 0.0593 | 0.0518 | 0.9984 | 6 |
| CVM | 0.0168 | 0.4873 | 227.403 | 0.0172 | 5.9974 | 0.0177 | -4.0206 | 0.1136 | 0.0556 | 0.0498 | 0.9991 | 3 |
| MPS | 0.0157 | 0.5926 | 227.587 | 0.0171 | 5.1453 | 0.0228 | -4.0168 | 0.1450 | 0.0706 | 0.0551 | 0.9964 | 7 |
| AD | 0.0166 | 0.5003 | 227.408 | 0.0166 | 5.5836 | 0.0178 | -4.0195 | 0.1122 | 0.0536 | 0.0489 | 0.9993 | 2 |
| RTAD | 0.0165 | 0.5208 | 227.421 | 0.0161 | 5.3265 | 0.0182 | -4.0187 | 0.1137 | 0.0530 | 0.0513 | 0.9987 | 5 |
| Dataset II | | | | | | | | | | | | |
| ML | 0.0006 | 3.2178 | 438.650 | 0.0343 | 17.6435 | 0.0335 | -4.5439 | 0.2896 | 0.1375 | 0.0610 | 0.9642 | 6 |
| OLS | 0.0005 | 3.8710 | 438.714 | 0.0331 | 17.7131 | 0.0338 | -4.5406 | 0.2960 | 0.1444 | 0.0582 | 0.9764 | 2 |
| WLS | 0.0005 | 3.8398 | 438.690 | 0.0338 | 17.4445 | 0.0341 | -4.5409 | 0.2905 | 0.1404 | 0.0596 | 0.9708 | 4 |
| CVM | 0.0006 | 2.9352 | 438.662 | 0.0340 | 17.9474 | 0.0329 | -4.5459 | 0.2940 | 0.1378 | **0.0577** | **0.9784** | **1** |
| MPS | 0.0002 | 8.7240 | 438.942 | 0.0364 | 18.2424 | 0.0386 | -4.5389 | 0.3158 | 0.1631 | 0.0612 | 0.9631 | 7 |
| AD | 0.0006 | 3.4156 | 438.657 | 0.0339 | 17.5170 | 0.0336 | -4.5426 | 0.2889 | 0.1376 | 0.0600 | 0.9687 | 5 |
| RTAD | 0.0006 | 3.1760 | 438.651 | 0.0338 | 17.6555 | 0.0330 | -4.5439 | 0.2900 | 0.1370 | 0.0588 | 0.9742 | 3 |
| Dataset III | | | | | | | | | | | | |
| ML | 0.3665 | 0.4840 | 98.1613 | 0.0347 | 16.0498 | 0.0379 | -4.6152 | 0.2623 | 0.1334 | 0.0681 | 0.9127 | 4 |
| OLS | 0.3799 | 0.4184 | 98.2516 | 0.0322 | 16.8590 | 0.0325 | -4.6204 | 0.2722 | 0.1394 | 0.0637 | 0.9458 | 2 |
| WLS | 0.3682 | 0.4751 | 98.1627 | 0.0341 | 16.0314 | 0.0370 | -4.6156 | 0.2612 | 0.1329 | 0.0673 | 0.9188 | 3 |
| CVM | 0.3888 | 0.3811 | 98.4149 | 0.0332 | 18.4910 | 0.0315 | -4.6256 | 0.3026 | 0.1577 | **0.0625** | **0.9534** | **1** |
| MPS | 0.3480 | 0.5702 | 98.3114 | 0.0442 | 18.2744 | 0.0514 | -4.6127 | 0.3125 | 0.1647 | 0.0709 | 0.8870 | 6 |
| AD | 0.3660 | 0.3162 | 99.2066 | 0.1814 | 67.5575 | 0.1848 | -4.6290 | 1.0657 | 0.5165 | 0.0959 | 0.5740 | 7 |
| RTAD | 0.3695 | 0.4734 | 98.1657 | 0.0340 | 16.0688 | 0.0367 | -4.6160 | 0.2611 | 0.1325 | 0.0687 | 0.9074 | 5 |

Akaike (CAIC), and Hannan-Quinn (HQIC) information criteria. Other robust test statistics, including Kolmogorov-Smirnov (KS), Anderson-Darling (AD), and Cramer-von Mises (CVM), as well as the $p$-value of KS, are examined for model selection in addition to these information metrics. The model with the lowest test statistics value and the highest $p$-value was chosen as the best model among all competitors. Table 3 shows all the relevant gof statistics of the fitted distributions for both datasets. For dataset I, according to AD, CVM, and KS tests, METD achieved the first rank, whereas according to AIC, BIC, CAIC, and HQIC, METD achieved the second rank, and ETD achieved the first rank among all competitor models. According to NLL, EGOD, ETD, and EWD have better ranks than METD. Therefore, as METD is a simple model in comparison with exponentiated models and has a significant $p$-value, it is concluded that METD may be a good choice to model the first dataset.
METD ranked first in NLL, AD, CVM, and KS tests for dataset II, while METD ranked second in AIC, BIC, CAIC, and HQIC tests, and ED ranked first among all competitors models. ED has no significant $p$-value. Therefore, it is concluded that METD provides a reasonable fit for the second dataset in comparison with all competitor distributions. For third dataset, METD is consistently achieved best rank according to all gof test statistics. Only it is second under NLL and KS test. According to the AIC and KS test, the top four models are selected for both datasets. Histograms of datasets with pdfs of distributions, are displayed in Fig. 3. Once again, these plots confirm the conclusion that the METD is an appropriate model for these datasets.

## 6. Conclusion

The METD is a two-parameter distribution proposed in this study. The hazard rate function of the METD is exponentially increasing and the probability density function is unimodal ($0 \leq \lambda < 1$) and decreasing ($\lambda \geq 1$). Two different datasets of different characteristics (unimodal and decreasing) are provided to show the practical significance of the present distribution. Furthermore, the proposed distribution can be used as an alternative to some well-known distributions such as exponential, Lindley, Rayleigh, Gompertz, Weibull, and their exponentiated models, and it is expected that it will provide a better fit for similar datasets than the models discussed in this paper. The METD demonstrated in this study shows its ability to model

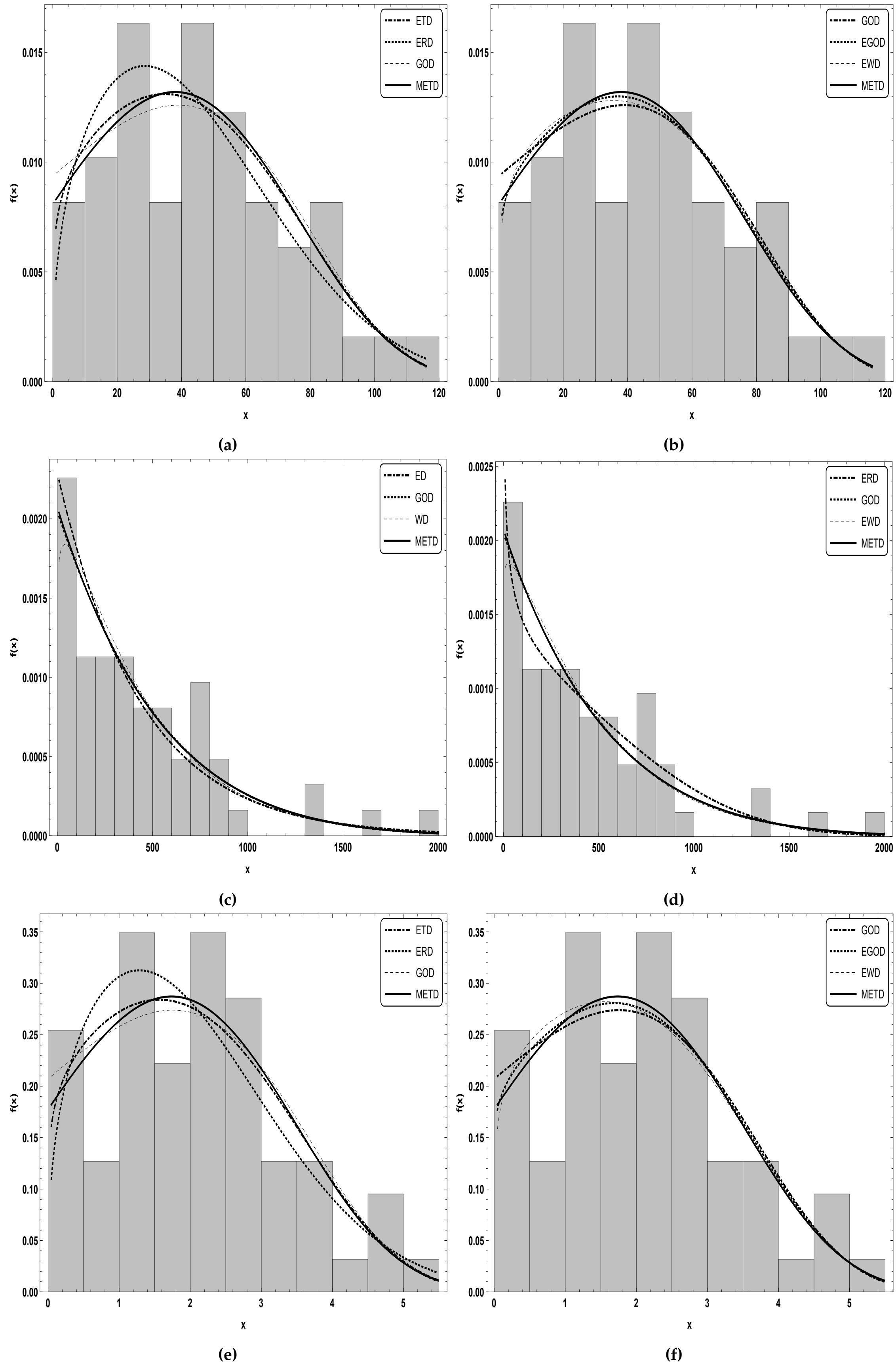



**Figure 3:** *Histogram with some better fitted pdfs for dataset I (a),(b), dataset II (c),(d) and dataset III (e),(f) where better models according to AIC in first column and better models according to KS test in second column.*

**Table 3:** *Various goodness-of-fit statistics, the number of parameters (NP) with respective ranking r of all fitted distributions for the first and second datasets.*

| Distribution | NP | NLL | *r* | AIC | *r* | BIC | *r* | CAIC | *r* | HQIC | *r* | AD | *r* | CVM | *r* | KS | PV(KS) | *r* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Dataset I | | | | | | | | | | | | | | | | | | |
| ED | 1 | 236.035 | 13 | 474.070 | 13 | 475.962 | 13 | 476.962 | 13 | 474.788 | 13 | 2.8483 | 12 | 0.5423 | 13 | 0.1999 | 0.0344 | 13 |
| EED | 2 | 231.116 | 11 | 466.232 | 11 | 470.015 | 11 | 472.015 | 11 | 467.667 | 11 | 0.6772 | 10 | 0.1161 | 11 | 0.1207 | 0.4377 | 11 |
| TD | 1 | 233.764 | 12 | 469.528 | 12 | 471.42 | 12 | 472.42 | 12 | 470.246 | 12 | 3.0822 | 13 | 0.3962 | 12 | 0.1780 | 0.0787 | 12 |
| ETD | 2 | 227.391 | 2 | 458.782 | 1 | 462.566 | 1 | 464.566 | 1 | 460.218 | 1 | 0.1155 | 2 | 0.0180 | 2 | 0.0532 | 0.9977 | 5 |
| RD | 1 | 230.394 | 10 | 462.787 | 9 | 464.679 | 6 | 465.679 | 5 | 463.505 | 9 | 0.8837 | 11 | 0.0876 | 8 | 0.1026 | 0.6419 | 8 |
| ERD | 2 | 228.079 | 6 | 460.157 | 4 | 463.941 | 4 | 465.941 | 6 | 461.593 | 4 | 0.2162 | 6 | 0.0378 | 6 | 0.0838 | 0.8529 | 6 |
| LD | 1 | 230.188 | 9 | 462.376 | 8 | 464.268 | 5 | 465.268 | 4 | 463.094 | 8 | 0.5572 | 9 | 0.0998 | 10 | 0.1155 | 0.4933 | 10 |
| ELD | 2 | 230.165 | 8 | 464.330 | 10 | 468.114 | 10 | 470.114 | 10 | 465.765 | 10 | 0.5232 | 8 | 0.0911 | 9 | 0.1132 | 0.5191 | 9 |
| GOD | 2 | 227.437 | 5 | 458.875 | 3 | 462.659 | 3 | 464.659 | 3 | 460.310 | 3 | 0.1311 | 5 | 0.0204 | 5 | 0.0506 | 0.9989 | 4 |
| EGOD | 3 | 227.383 | 1 | 460.766 | 5 | 466.442 | 8 | 469.442 | 8 | 462.919 | 5 | 0.1163 | 3 | 0.0181 | 3 | 0.0479 | 0.9995 | 2 |
| WD | 2 | 228.807 | 7 | 461.614 | 7 | 465.397 | 7 | 467.397 | 7 | 463.049 | 7 | 0.2932 | 7 | 0.0442 | 7 | 0.0874 | 0.8158 | 7 |
| EWD | 3 | 227.401 | 3 | 460.803 | 6 | 466.478 | 9 | 469.478 | 9 | 462.956 | 6 | 0.1204 | 4 | 0.0187 | 4 | 0.0484 | 0.9994 | 3 |
| **METD** | **2** | **227.402** | **4** | **458.805** | **2** | **462.588** | **2** | **464.588** | **2** | **460.240** | **2** | **0.1138** | **1** | **0.0178** | **1** | **0.0475** | **0.9996** | **1** |
| Dataset II | | | | | | | | | | | | | | | | | | |
| ED | 1 | 438.986 | 7 | 879.971 | 1 | 882.098 | 1 | 883.098 | 1 | 880.806 | 1 | 0.3666 | 7 | 0.0534 | 7 | 0.0744 | 0.8562 | 8 |
| EED | 2 | 438.748 | 5 | 881.496 | 5 | 885.750 | 5 | 887.750 | 5 | 883.166 | 5 | 0.3574 | 6 | 0.0486 | 6 | 0.0738 | 0.8627 | 7 |
| TD | 1 | 487.937 | 13 | 977.874 | 13 | 980.001 | 13 | 981.001 | 13 | 978.709 | 13 | 32.2274 | 13 | 3.8607 | 13 | 0.3786 | $2.67 \times 10^{-8}$ | 13 |
| ETD | 2 | 442.409 | 10 | 888.818 | 9 | 893.073 | 9 | 895.073 | 9 | 890.489 | 9 | 1.2232 | 10 | 0.1945 | 10 | 0.1106 | 0.4035 | 10 |
| RD | 1 | 464.709 | 12 | 931.417 | 12 | 933.544 | 12 | 934.544 | 12 | 932.252 | 12 | 13.7411 | 12 | 1.4724 | 12 | 0.2574 | 0.0004 | 12 |
| ERD | 2 | 439.801 | 8 | 883.602 | 8 | 887.857 | 7 | 889.857 | 7 | 885.273 | 7 | 0.4831 | 8 | 0.0589 | 8 | 0.0658 | 0.9347 | 3 |
| LD | 1 | 446.268 | 11 | 894.536 | 11 | 896.663 | 11 | 897.663 | 10 | 895.371 | 11 | 3.6521 | 11 | 0.3507 | 11 | 0.1594 | 0.0765 | 11 |
| ELD | 2 | 438.755 | 6 | 881.510 | 6 | 885.764 | 6 | 887.764 | 6 | 883.180 | 6 | 0.3317 | 4 | 0.0421 | 4 | 0.0684 | 0.9138 | 5 |
| GOD | 2 | 438.656 | 2 | 881.312 | 3 | 885.566 | 3 | 887.566 | 3 | 882.982 | 3 | 0.2905 | 2 | 0.0337 | 2 | 0.0614 | 0.9622 | 2 |
| EGOD | 3 | 441.424 | 9 | 888.847 | 10 | 895.229 | 10 | 898.229 | 11 | 891.353 | 10 | 0.8585 | 9 | 0.1241 | 9 | 0.0848 | 0.7306 | 9 |
| WD | 2 | 438.703 | 4 | 881.406 | 4 | 885.660 | 4 | 887.660 | 4 | 883.076 | 4 | 0.3406 | 5 | 0.0440 | 5 | 0.0700 | 0.9006 | 6 |
| EWD | 3 | 438.692 | 3 | 883.384 | 7 | 889.766 | 8 | 892.766 | 8 | 885.890 | 8 | 0.3230 | 3 | 0.0403 | 3 | 0.0669 | 0.9261 | 4 |
| **METD** | **2** | **438.650** | **1** | **881.301** | **2** | **885.555** | **2** | **887.555** | **2** | **882.971** | **2** | **0.2896** | **1** | **0.0335** | **1** | **0.0610** | **0.9641** | **1** |
| Dataset III | | | | | | | | | | | | | | | | | | |
| ED | 1 | 109.299 | 13 | 220.597 | 13 | 222.740 | 13 | 223.740 | 13 | 221.440 | 13 | 3.8816 | 12 | 0.7789 | 13 | 0.2077 | 0.0074 | 13 |
| EED | 2 | 103.547 | 10 | 211.093 | 10 | 215.380 | 11 | 217.380 | 11 | 212.779 | 11 | 1.3151 | 10 | 0.2329 | 10 | 0.1437 | 0.1339 | 10 |
| TD | 1 | 106.974 | 12 | 215.947 | 12 | 218.090 | 12 | 219.090 | 12 | 216.790 | 12 | 3.9012 | 13 | 0.4427 | 12 | 0.1705 | 0.0453 | 12 |
| ETD | 2 | 98.288 | 4 | 200.577 | 3 | 204.863 | 3 | 206.863 | 3 | 202.263 | 3 | 0.3049 | 4 | 0.0464 | 3 | 0.0761 | 0.8311 | 5 |
| RD | 1 | 102.492 | 9 | 206.984 | 8 | 209.127 | 8 | 210.127 | 5 | 207.827 | 8 | 1.2469 | 9 | 0.0841 | 6 | 0.0958 | 0.5754 | 6 |
| ERD | 2 | 99.198 | 6 | 202.397 | 4 | 206.683 | 4 | 208.683 | 4 | 204.083 | 4 | 0.5116 | 6 | 0.0903 | 7 | 0.1067 | 0.4383 | 7 |
| LD | 1 | 104.578 | 11 | 211.156 | 11 | 213.299 | 10 | 214.299 | 10 | 211.999 | 10 | 2.1351 | 11 | 0.4159 | 11 | 0.1564 | 0.0821 | 11 |
| ELD | 2 | 101.888 | 8 | 207.776 | 9 | 212.063 | 9 | 214.063 | 9 | 209.462 | 9 | 0.9654 | 8 | 0.1682 | 9 | 0.1300 | 0.2170 | 9 |
| GOD | 2 | 98.276 | 3 | 200.553 | 2 | 204.840 | 2 | 206.840 | 2 | 202.239 | 2 | 0.3033 | 3 | 0.0466 | 4 | 0.0679 | 0.9143 | 1 |
| EGOD | 3 | 98.231 | 2 | 202.463 | 5 | 208.893 | 5 | 211.893 | 7 | 204.992 | 5 | 0.2890 | 2 | 0.0428 | 2 | 0.0694 | 0.9009 | 3 |
| WD | 2 | 100.318 | 7 | 204.635 | 7 | 208.922 | 6 | 210.922 | 6 | 206.321 | 7 | 0.6425 | 7 | 0.0929 | 8 | 0.1086 | 0.4164 | 8 |
| EWD | 3 | 98.327 | 5 | 202.654 | 6 | 209.084 | 7 | 212.084 | 8 | 205.183 | 6 | 0.3105 | 5 | 0.0473 | 5 | 0.0760 | 0.8325 | 4 |
| **METD** | **2** | **98.161** | **1** | **200.323** | **1** | **204.609** | **1** | **206.609** | **1** | **202.008** | **1** | **0.2623** | **1** | **0.0379** | **1** | **0.0681** | **0.9127** | **2** |

Covid-19, earthquake waiting times and service times data appropriately.

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