Inferences for two parameter Teissier distribution in case of fuzzy progressively censored data

Sudhanshu Vikram Singh[1], Vikas Kumar Sharma[2], Sanjay Kumar Singh[3]

•

[1]Department of Mathematics, Institute of Infrastructure
Techonology Research And Management, Ahmedabad, India
[2,3]Department of Statistics, Institute of Science, Banaras Hindu University, Varanasi, India
sudhanshu.vikram22061991@gmail.com[1], vikasstats@rediffmail.com[2],
singhsk64@gmail.com[3]

Abstract

In process of observing data, it is sometimes not possible to obtain data precisely and fuzzy methods are useful for analyzing such data sets. In this article, we propose location-scale family of the Teissier distribution for fitting fuzzy censored data sets. The maximum likelihood, least squares and Bayes estimators of the parameters of the Teissier distribution are constructed in the presence of the progressively fuzzy censored samples. In addition to that statistical properties of the distribution are also derived. Fitting of the tensile strengths of the carbon fibers is done using the proposed distribution with comparison to the location-scale families of the exponential, Maxwell and Lindley distributions. We found that the Teissier distribution can be effectively used for fitting complete and fuzzy censored data as well.

Keywords: Location-scale Teissier distribution, Fuzzy lifetime data, Type-II progressive censoring scheme, Mean residual life, Moments, Maximum likelihood estimator, Least squares estimator, Bayes estimator

1. Introduction

Uncertainty is associated with our daily life activities. We have to take some actions/decisions to minimize the risk or maximize the gains in business and other activities. We make some probabilistic statements to assess the nature of the random phenomenon under consideration, so that our decision makes our life better. In this article, we are concerned about the life of man-made systems/products and propose a method of estimating the parameter associated with the random phenomenon. Lifetimes are not deterministic and they are random in nature. This type of uncertainty arises due the random nature of the phenomenon and are efficiently dealt by the statistical methods where the lifetimes are considered as random variables having certain probability distributions that are characterized by some constants (called parameters).

In conventional statistical inferences, random observations are drawn from the population and inferences are made about the parameters provided that the data are observed precisely. In many cases, it may not be possible to obtain the data in precise form because of many unavoidable reasons such as the lack of information, human errors, measurement errors and other practical difficulties. In fact, the real measurements of the continuous variables are never precisely obtained and have some errors. For instance, locations of the objects in space, live positions of ships on radar screen and space time data are a few examples among many. There is uncertainty always attached with the individual observation. Such type of uncertainty cannot be dealt by the conventional statistical procedures. In such cases, the fuzzy set theory is useful for explaining the random behavior of the observations. Until 1960, the probability theories and statistics are used to model the uncertainty. In 1968, [7] introduced the probability measure for fuzzy events, see also [6]. [22] discussed various methods for fuzzy data estimation and hypothesis testing. From the application point of view, [5] proposed the statistical analysis of fuzzy data. [8, 21] proposed to use Bayesian approach to estimate the parameters and reliability function for the distributions using the fuzzy lifetime data. Recently, [11] presented classical and Bayesian procedure for estimating the parameter of Rayleigh distribution based on Type-II progressively hybrid censored data under fuzzy setup.

In the field of surveys and life testing experiments, it may be possible to come across the situation of incomplete/lost data due to lack of time, cost and some other constraints. Some of the experimental units have some complete information and the rest reports non-occurrence. The units for which exact failure information is available are called complete samples and the remaining units are called censored observations. For analyzing such censored data, it is essential to use censoring schemes. Conventional Type-I and Type-II censoring schemes have been studied by many authors, see [13], [15] and [14] for modeling censored data sets.

Under Type-I censoring scheme, n units are put under observation and the experiment is carried-out up to a predetermined time. In the case of the Type-II censoring scheme, the experiment is terminated when pre-fixed numbers of units are observed to have failed. It is worth to mention here that these schemes do not enough flexible to incorporate removals of the experimental units at the stages other than the final terminal stage. In this context, [9] proposed a censoring scheme, called the progressive censoring scheme, which has an advanced feature that allows the removal of the units at the intermediate stages. Since then, an extensive list of literature is seen devoted to the progressive censoring. We follow [17] and [35] for the detailed theoretical aspects, estimation methods and applications of this scheme. [2] provided two simulation algorithms for generating the progressive Type-II censored samples based on exponential and uniform random variable transformations. [18] proposed classical and Bayes estimation for the flexible Weibull parameters under the progressive Type-II censoring scheme. [19] discussed the different estimation methods for estimating the parameters of Rayleigh distribution on the basis of progressive Type-II censoring scheme.

Suppose n experimental units are put under observation and experiment is terminated after a prefixed number (m) of observations are observed. At each stage, we progressively drop some prefixed number of units from the remaining units. Suppose that as the first failure occurs, some units, say R_1 are removed from the remaining $(n-1)$ units. At the second stage, R_2 units are removed from the remaining $(n-R_1-2)$ units. Similarly, at th mth stage x_m is observed, R_m units are removed from the remaining $(n-R_1-\cdots-R_{m-1}-m)$ units such that $(m+\sum_{i=1}^{m} R_i = n)$. We terminate the experiment at time point x_m. This is how we obtain the progressively censored data $(x_1, x_2, \ldots, x_m)$ with prefixed removals $(R_1, R_2, \ldots, R_m)$.

The objective of this paper is to develop the various estimation methods to estimate the parameters of two-parameter Teissier distribution with fuzzy censored data. The Teissier distribution was first introduced in [23] to model the frequency of the mortality due to ageing only i.e. deaths are protected from the accidents and disease. Later on, [24] used this distribution for the reliability analysis. After that it has been overlooked in the literature. [25] rediscovered the Teissier distribution and derived its statistical properties. For adding more flexibility to the model, [26] introduced a two parameter extension of Teissier distribution called Power Muth distribution. [34] proposed another two parameter extension of the Teissier distribution, called exponentiated Teissier distribution. Recently, [36] proposed the Teissier generalised family of the distributions and produced various flexible probability distributions.

The rest of the paper is organized as follows. In section 2, the progressive censoring and different methods of estimation are discussed. Section 3 deals with the brief introduction of fuzzy random variable and related concepts. We introduce location-scale family of the Teissier distributions in section 4 and discussed it's properties here. Section 5 discusses the estimation of parameters of the Teissier distribution for the Type-II progressive censoring under under fuzzy setup. Simulation study and real data application are discussed in section 6 and section 7, respectively. At last, overall conclusions are given in section 8.

2. Progressive Censoring and Estimation Methods

Statistical inference mainly focuses on the estimation of the unknown model parameters using the available observed data. As discussed in the above section, we may encounter the situation where the observed data is censored. The estimation procedures can be easily implemented for censored data, [35]. In this section, we describe the maximum likelihood estimation (MLE), least squares estimation (LSE) and Bayes estimation procedures for the Type-II progressively censored data.

Under the MLE, we maximize the joint density function (called likelihood function) of observed random sample over the parameter space. If we are given realizations of the n i.i.d. random variables, the likelihood function is nothing but the product of the respective densities. Suppose that X is a random variable governed by the probability distribution function (pdf) $f(x,\theta)$ and $x_1, x_2, ..., x_n$ are the n i.i.d. random samples drawn from it. The likelihood function is $L(\theta; x) = \prod_{i=1}^{n} f(x_i,\theta)$, where θ is the parameter of interest.

Since, in this paper, we consider progressively Type-II censored data, we define the likelihood function (following [35]) as given by

$$L(\theta) = C\prod_{i=1}^{m} f\left(x_i\right)\left(1 - F\left(x_i\right)\right)^{R_i}, \tag{1}$$

where $C = \prod_{i=1}^{m-1}\left(n - \sum_{j=1}^{i}(r_i+1)\right)$ and $f(x)$ and $F(x)$ are the pdf and cdf of the assumed probability distribution. The MLE $\hat{\theta}$ of θ maximize the likelihood function given in the equation (1). For some distributions, the closed-form MLEs are not possible to derive and iterative methods are used for numerical computation of the estimates.

In the process of estimating the parameters, [1] proposed the method of least squares for estimating the beta parameters. That are obtained by minimizing the sum of squares of the discrepancies between the observed and expected distributions. For the progressively Type-II censored data, [12] have constructed the LSEs of the parameters of the generalised inverted exponential distribution. For the Type-II progressively censored sample, we have

$$E\left[F(x_i)\right] = 1 - \prod_{j=m-i+1}^{m} \alpha_j, \tag{2}$$

where $\alpha_j = \frac{a_i}{1+a_i}$ and $a_i = i + \sum_{j=m-i+1}^{m} R_i$. The LSEs are obtained by minimising the following function

$$S(\theta) = \sum_{i=1}^{m}\left[F(x_i) - E(F(x_i))\right]^2. \tag{3}$$

The equation (3) are usually optimized numerically as the closed form solution is not possible in most of the cases.

In recent decades, Bayesian perspective received a great attention by researchers for statistical inferences. In Bayesian framework, we use the Bayes theorem to update the probability for a hypothesis after observing the data as evidence. It facilitates to use the prior information for formulating a better method of estimation. Bayes procedure is discussed in section 5.

3. Fuzzy random variable and membership function

Before defining the fuzzy random variable, some basic definitions are needed to explore. Let $S = (\Omega, A, P_\theta)$ be the probability space. Here (Ω, A) is a measurable space and P_θ is the probability measure defined on measurable space (Ω, A).

Definition 1. Membership function for any set Ω is the function from Ω to real interval $[0,1]$. Here the value of $\mu_{\tilde{A}}(x)$ at x define the "Grade of membership" or "degree of truthfulness ".

$$\mu_{\tilde{A}}(x) : \Omega \rightarrow [0,1]$$

Here membership function is not limited to 0 and 1 only, but takes any value between $[0,1]$.

Definition 2. Here Ω is a universal set. Fuzzy set $\tilde{A}$ in Ω is denoted by an ordered set of pairs $(x, \mu_{\tilde{A}}(x))$. The first component of which denotes elements of the set $\tilde{A}$ and second denotes the degree of membership of that elements in set $\tilde{A}$.

If $sup\mu_{\tilde{A}}(x) = 1$, then $\tilde{A}$ is called normal fuzzy set.

Definition 3. Let X be the universal set. Then the support of a fuzzy set $\tilde{A}$, i.e. $S(\tilde{A})$ is the set of all points $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

Definition 4. Let X be the universal set. A fuzzy set $\tilde{A}$ is said to be convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)), \quad x_1, x_2 \in X, 0 < \lambda < 1.$

Definition 5. A fuzzy set $\tilde{A}$ is a fuzzy number if it is normal, convex and its support is bounded.

In general, there are two fuzzy numbers that are mostly used Triangular fuzzy number and Trapezoidal fuzzy number. Triangular fuzzy number is denoted by $\tilde{x} = (u, v, w)$ with corresponding membership function as

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{x-u}{v-u}, & u \leq x \leq v \\ \frac{w-x}{w-v} & v \leq x \leq w \\ 0, & otherwise \end{cases} \tag{4}$$

Fuzzy random variable can be defined as random variable, the value of which is not real but fuzzy number which is nothing but a particular kind of fuzzy set, see [7], [6] and [22].

4. Location-Scale Teissier distribution

In any probability distribution, location parameter determines the shift of the distribution or origin on the horizontal axis. Sometimes, in reliability and life testing experiments, it is possible that failure will absolutely not happen before a given time. In this regard, location parameter have some profound effect in reliability and life-testing experiments. Here, we introduce the Teissier distribution with location parameter denoted by μ.

Definition 6. Consider random variable Y that follows the Teissier distribution defined in [23] with scale parameter θ. Then the random variable $Y = X + \mu$ is said to follow the 2-parameter Teissier (2-T) distribution indexed by location parameter μ and scale parameter θ. The pdf and cdf of the 2-T distribution are given as follows

$$f(x;\mu,\theta) = \theta e(e^{\theta(x-\mu)}) \left(e^{\theta(x-\mu)} - 1\right) \exp\left(-e^{\theta(x-\mu)}\right), x > \mu > 0, \theta > 0, \tag{5}$$

$$F(x;\mu,\theta) = 1 - \exp\left[\theta(x-\mu) - e^{\theta(x-\mu)} + 1\right], x > \mu > 0, \theta > 0. \tag{6}$$

The corresponding hazard function of the 2-T distribution is given by

$$h(x;\mu,\theta) = \theta\left(e^{\theta(x-\mu)} - 1\right), x > \mu > 0, \theta > 0.$$

Note that $h^{'}(x;\mu,\theta) = \theta^2 e^{\theta(x-\mu)} + \left(e^{\theta(x-\mu)} - 1\right) > 0, \forall \;\; x > \mu > 0, \theta > 0$. It can be concluded that the hazard function of the 2-T distribution is monotonic increasing.

Fig. 1 displays the various shapes of the pdf of the 2-T distribution. We can see that the distribution is unimodal and right skewed. So, it is flexible enough to fit a wide varieties of right skewed data sets. Fig. 2 displays the hazard rate function which has increasing shapes for different parameter values. Since, the 2-T has increasing hazard rate, so it may be a good choice for modelling data set in reliability and survival analysis where the things are more likely to fail with the increasing time.

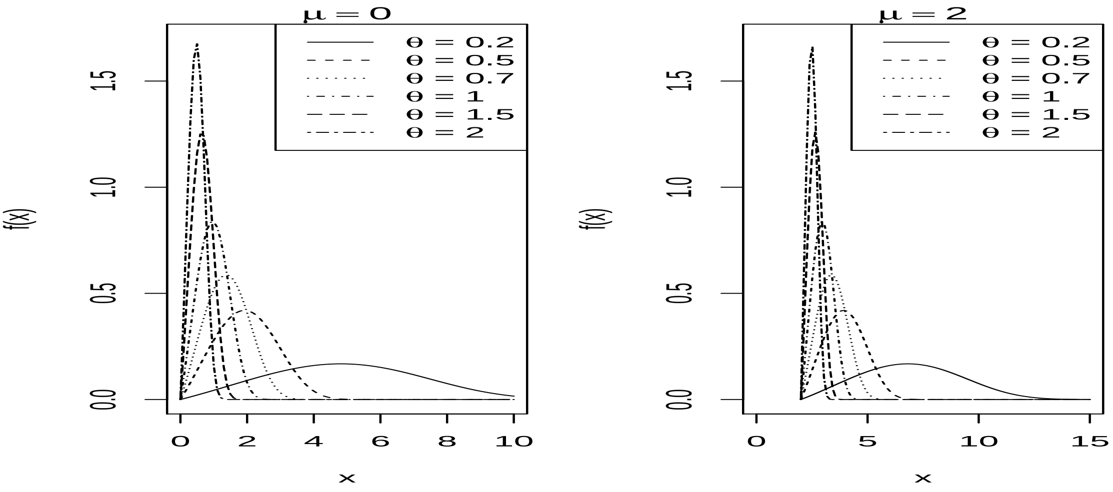

Figure 1: *Density function for different values of θ for given $\mu = 0$ and $\mu = 2$*

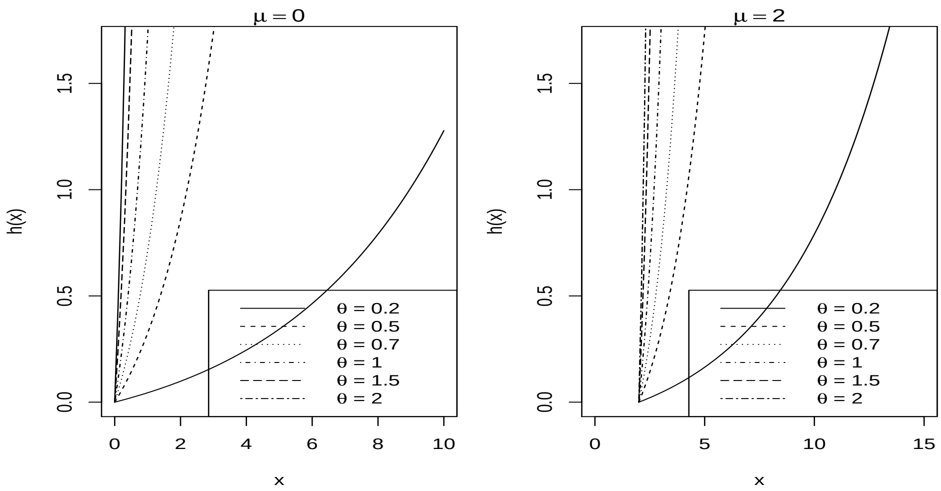

Figure 2: *Hazard function for different values of θ for given $\mu = 0$ and $\mu = 2$*

4.1. Properties

In this section, the basic properties of the 2-T distribution are derived. We derive moments, quantile function, moments generating function and mean residual life function.

4.1.1 Quantile function

The quantile function or inverse cumulative distribution function is more precisely used to generate the computer pseudo random data from associated probability distribution and explore its basic statistical properties such as measures of central value, dispersion, skewness and kurtosis. The qth quantile of any random variable X is defined as the solution of the equation $F_X(\eta_q) = q, 0 < q < 1$ i.e. $\eta_q = F_X^{-1}(q)$, where $F(.)$ is the cdf. The closed-form expression of the 2-T distribution is obtained in terms of the Lambert-W function. The Lambert-W function is used to obtain the solution of the equation $W(z)e^{W(z)} = z, z \in C$. For more details about the Lambert-W function see [31]. For the 2-T distribution η_q is the solution of the equation

$$1 - \exp\left[\theta(\eta_q - \mu) - e^{\theta(\eta_q - \mu)} + 1\right] = q, \quad 0 < q < 1. \tag{7}$$

On simplifying the equation (7), it turns out to be

$$e^{\theta(\eta_q-\mu)} = \theta(\eta_q-\mu) + (1-\log(1-q)).$$

Here we note that $e^\theta > 0, \forall\, \theta > 0$ and $e^\theta \neq 1, \forall\, \theta > 0$ also $(\eta_q-\mu) \neq 0\ \forall\ \eta_q > \mu$. So, this equation can be solved by using the Lambert-W function. η_q is given by

$$\eta_q = \mu - \frac{1}{\theta} + \frac{1}{\theta}\log(1-q) - \frac{1}{\theta}W_{-1}\left(\frac{u-1}{e}\right) \tag{8}$$

where W_{-1} represents the negative branch fo Lambert-W function.

4.1.2 Mean Residual Life

The mean residual life (MRL) is an important criterion of reliability measure of non-negative random variables. Sometimes it is more relevant than the hazard rate function, mainly in repair and replacement problems because it relates only to the risk of immediate failure than entire residual life function. The MRL or mean remaining life of the random variable X beyond the value x, denoted by $r(x)$ is given as

$$r(x) = E\left((X-x)/X > x\right) = \int_x^\infty \frac{\bar{F}(y)dy}{\bar{F}(x)}. \tag{9}$$

The analytical expression of the MRL function for the 2-T distribution is given as follows

$$r(x) = \frac{1}{\bar{F}(x)}\int_x^\infty \exp\left(\theta(y-\mu) - e^{\theta(y-\mu)} + 1\right)dx. \tag{10}$$

In equation (10), we have made change of variable $(x-\mu) = z$ and again make substitution as $e^{\theta z} = t$ in the equation. Then we obtain the expression of MRL of the 2-T distribution as

$$r(x) = \frac{\exp\left(1-e^{\theta(x-\mu)}\right)}{\theta\bar{F}(x)}, \tag{11}$$

where $\bar{F}(x)$ is the survival function of the random variable X.

4.1.3 Moment Generating Function

The analytical expression for moment generating function of random variable X following the 2-T distribution, having the density function as $f_X(x)$ is defined as

$$M_X(t) = E(e^{tx}) = \int_\mu^\infty e^{tx} f_X(x)dx. \tag{12}$$

Proposition 1. The moment generating function of the 2-T Teissier distribution is given by

$$M_X(t) = e^{\mu t+1}\left[\Gamma\left(2+\frac{t}{\theta},1\right) - \Gamma\left(1+\frac{t}{\theta},1\right)\right], -\infty < t < \infty,$$

where Γ denotes the upper incomplete gamma function.

Proof. The proof is straight forward. ■

4.1.4 Moments

For calculating the moments of the 2-T distribution, we make use of Generalized integro-exponential function, the integral representation of which is given as

$$E_a^b(t) = \frac{1}{\Gamma(b+1)}\int_1^\infty (\log v)^b v^{-a} e^{-tv} dv, m = 0,1,2,..... \tag{13}$$

where $t, a \in C$ and $\Gamma(p) = \int_0^\infty e^{-x}x^{p-1}$ is the ordinary gamma function.

Proposition 2. For random variable X following the distribution (5), the kth moment is given by

$$E[X^k] = e\mu^k \sum_{i=0}^{k} \frac{\binom{k}{i}}{(\mu\theta)^i} \left[E_{-1}^i(1) - E_0^i(1)\right]. \tag{14}$$

Proof. Let X be any random variable and following any particular distribution $f_X(x)$. Then k^{th} moment of the random variable X is given as

$$E[X^k] = \int x^k f_X(x)dx. \tag{15}$$

Now for the moments of the 2-T distribution, $f_X(x)$ here is as defined in (5) also make substitution as $(x-\mu) = y$, the above equation turns out to be as

$$E[X^k] = \theta e \int_0^\infty (\mu+y)^k e^{\theta y} \left(e^{\theta y} - 1\right) \exp\left(-e^{\theta y}\right) dy.$$

Now using Binomial expansion in the expression $(\mu+y)^k$ and using the substitution $e^{\theta y} = t$ in the above equation. Then, it turns out to be as

$$E[X^k] = \mu^k e \sum_{i=0}^{k} \frac{\binom{k}{i}}{(\mu\theta)^i} \left[\int_1^\infty (\log t)^i t e^{-t} dt - \int_1^\infty (\log t)^i e^{-t} dt\right].$$

Here, by using the generalized integro-exponential function in the above equation it will be same as (14). ■

In this section, the analytical expression of the particular case i.e. for $k = 1$ (Mean) is also derived.

Corollary 1. Let X be the random variable following the 2-T distribution given in (5), then for given μ and θ, the mean of the distribution will be given by

$$E[X] = \left(\mu + \frac{1}{\theta}\right)$$

where E stands for the expectation.

Proof. For given μ and θ, mean value of the 2-T distribution is defined as $E[X] = \int_\mu^\infty x f_X(x)dx$. Here, $f_X(x)$ is same as defined in (5). Now, making substitution as $(x-\mu) = y$ we get

$$E[X] = \theta e \int_0^\infty (\mu+y)\, e^{\theta y} \left(e^{\theta y} - 1\right) \exp\left(-e^{\theta y}\right) dy$$

By making substitution $e^{\theta y} = t$ in the above equation and using analytical expression $\int_1^\infty \log z(z-1)e^{-z}dz = \frac{1}{e}$, we get required result. ■

5. Estimation for fuzzy progressively censored data

In this section, for the given fuzzy progressively censored data, we obtain the MLEs, LSEs and Bayes estimators of the parameters μ and θ.

5.1. Maximum likelihood estimation

Following the [7], the likelihood for fuzzy data can expressed as

$$\ell(\theta, \mu; x) = A \prod_{i=1}^{m} \int f(x_i) \left[1 - F(x_i)\right]^{R_i} \mu_{\tilde{x}_i}(x)dx. \tag{16}$$

Let (x_i, r_i) be the progressive sample. For 2-T distribution, the log $-likelihood$ function can be obtained by putting the value (6) and (5) in (16). Differentiating the above equation with respect to μ and θ and equating to zero. We obtain

$$\frac{d}{d\mu}\log l = \sum_{i=1}^{m} \frac{\int \left(f_\mu(x_i)\left[1 - F\left(x_i\right)\right]^{R_i} - R_i\left[1 - F\left(x_i\right)\right]^{R_i - 1} F_\mu(x_i) \right) \mu_{\tilde{x}_i}(x)dx}{\int f\left(x_i\right)\left[1 - F\left(x_i\right)\right]^{R_i} \mu_{\tilde{x}_i}(x)dx} = 0,$$

$$\frac{d}{d\theta}\log l = \sum_{i=1}^{m} \frac{\int \left(f_\theta(x_i)\left[1 - F\left(x_i\right)\right]^{R_i} - R_i\left[1 - F\left(x_i\right)\right]^{R_i - 1} F_\theta(x_i) \right) \mu_{\tilde{x}_i}(x)dx}{\int f\left(x_i\right)\left[1 - F\left(x_i\right)\right]^{R_i} \mu_{\tilde{x}_i}(x)dx} = 0,$$

where $f_\mu(x) = \frac{d}{d\mu}f(x)$ and $F_\mu(x) = \frac{d}{d\mu}F(x)$. Above equations can not be solved analytically. That's why some numerical techniques such as Newton-Raphson are required to solve these equations.

5.2. Least squares estimation

The equations (2) and (6) provide the method of obtaining the LSEs for fuzzy data. The LSEs can be obtained by minimising S. So differentiating S with respect to μ and θ and equating to 0, we obtain

$$\frac{dS}{d\mu} = \sum_{i=1}^{m} \int \left[F(x_i) - E(F(x_i))\right] F_\mu(x_i)\mu_{\tilde{x}_i}(x)dx = 0,$$

$$\frac{dS}{d\theta} = \sum_{i=1}^{m} \int \left[F(x_i) - E(F(x_i))\right] F_\theta(x_i)\mu_{\tilde{x}_i}(x)dx = 0,$$

where $E\left(F(x_i)\right) = 1 - \prod_{j=m-i+1}^{m} \alpha_j$, $\alpha_j = \frac{a_i}{1+a_i}$, $a_i = i + \sum_{j=m-i+1}^{m} R_i$, and $\frac{dS}{d\mu}$, $\frac{dS}{d\theta}$ are the first order derivatives of the $F(x)$ with respect to μ and θ. Equations $\frac{dS}{d\mu} = 0$ and $\frac{dS}{d\theta} = 0$ provide the LSEs for μ and θ. But there is no closed form solution of the above equations. A suitable iterative search method such as Newton-Raphson is is needed to obtain the LSEs of the parameters.

5.3. Bayes Estimation

In statistical inferences, Bayesian estimation turns out as a valid and powerful alternative of classical or traditional perspectives of the parameter estimation. In this section, Bayes estimates of the parameters of the 2-T distribution are derived using the Type-II progressive censoring scheme where data is given in form of the fuzzy numbers. Here, we assume that the parameters (μ, θ) follow the independent gamma priors denoted by $\pi_1(\mu)$ and $\pi_2(\theta)$ respectively. Then, the probability density function of μ and θ are given as follows

$$\pi_1(\mu) = \frac{p^n}{\Gamma(n)} \exp(-p\mu)\mu^{n-1}, \quad \mu, p, n > 0,$$

$$\pi_2(\theta) = \frac{q^a}{\Gamma(n)} \exp(-q\theta)\theta^{b-1}, \quad \theta, b, q > 0.$$

The joint posterior density of μ and θ for given observed data is defined by

$$\pi(\mu, \theta|\tilde{x}) = \frac{\pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)}{\int_0^\infty \int_0^\infty \pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)d\mu d\theta},$$

where $l(\theta, \mu; x)$ is the log-likelihood defined in the equation (16). Bayes estimates of any function of μ and θ i.e. $g(\mu, \theta)$ under the squared error loss function is defined as

$$E(g(\mu, \theta)) = \frac{\int_0^\infty \int_0^\infty g(\mu, \theta)\pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)d\mu d\theta}{\int_0^\infty \int_0^\infty \pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)d\mu d\theta}. \tag{17}$$

The above integral (17) can not be solved analytically. So we have to use some approximations to calculate the Bayes estimates. Here, we proposed Markov Chain Monte Carlo technique to obtain the Bayes estimates of the parameters of the 2-T distribution. In this paper, we use Metropolis-Hastings (MH) algorithm to simulate the posterior samples.

In order to obtain the reasonable results by simulation in a limited amount of time, the choice of an effective proposal distribution is crucial. Since the target density is unknown, the choice of the proposal distribution is very difficult. To overcome this difficulty [33] provided a possible adaptive algorithm as a remedy which adapts continuously to the target distribution. The basic idea is to update the proposal distribution by using the knowledge we acquired so far about the target population. For the simulation algorithm researchers may see [37].

6. Simulation Study

To estimate the unknown constants of the proposed distribution, various estimation methods are proposed such as MLE, LSE, and Bayes estimation in the previous sections. To access the long-run performance and to choose the best possible estimators for the 2-T parameters, simulation experiments are carried-out. During simulations, we generate the Type-II progressively fuzzy censored pseudo-random data from the 2-T distribution with various choices of the 2-T parameters. Without loss of generality, we here present the simulation results for a parameter combination $(\mu = 1, \theta = 1.5)$. To compare the performance of these estimators, we calculate the bias and mean squared error (MSE) for each estimators. For generating the progressive Type-II censored data, we use an algorithm given by ([2]). After getting the pseudo data, each realization of x is then fuzzified by using the triangular fuzzy number defined in (4). In generating the random sample form the 2-T distribution, we have to use Lambert-W function which is easily available in R software.

Further, for the fixed value of μ and θ, we take the different combinations of (n, m) along with four removal schemes for variation purposes. The removal schemes are as follows:
Scheme 1: $R_1 = (n - m)$ and $R_2 = R_2 = ...R_m = 0$,
Scheme 2: $R_1 = R_2 = ...R_{m-1} = 0$ and $R_m = n - m$,
Scheme 3: $R_1 = R_2 = ...R_{n-m} = 1$ and $R_{n-m+1} = ...R_{2m-n} = 0$,
Scheme 4: $R_1 = R_2 = ...R_{(2m-n)} = 0$ and $R_{2m-n+1} = ...R_m = 0$.

On the basis of simulation results summarized in the Tables (1 and 2), we observe that even for the small sample size, the performance of all the estimators is quite satisfactory. The average estimates are nearer to the true values of the parameters and MSEs of all the estimators decrease as the sample size increases (i.e. m increases for the given n) under the different removal schemes.

Table 2: *Average estimate, Bias and MSE of the estimators of μ for given $\mu = 1, \theta = 1.5$*

(n,m,s)	MLE	BIAS	MSE	LSE	BIAS	MSE	BAYES	BIAS	MSE
(20,12,1)	1.05117	0.08489	0.01194	0.96279	0.147329	0.03393	0.95877	0.10030	0.01661
(20,12,2)	1.07121	0.09659	0.01546	0.96337	0.128139	0.02589	0.93566	0.12577	0.02813
(20,12,3)	1.05428	0.08559	0.01219	0.96157	0.13481	0.02856	0.95331	0.10465	0.01845
(20,12,4)	1.05842	0.08893	0.01322	0.95713	0.13284	0.02776	0.94327	0.11840	0.02488
(20,16,1)	1.04958	0.08302	0.01146	0.9722	0.12815	0.02577	0.96858	0.09272	0.01336
(20,16,2)	1.05754	0.08701	0.01252	0.96564	0.12048	0.02289	0.96034	0.10296	0.01680
(20,16,3)	1.04896	0.08189	0.01119	0.97146	0.12415	0.0242	0.96601	0.09257	0.01345
(20,16,4)	1.05377	0.08582	0.01221	0.96663	0.12211	0.02320	0.96038	0.10094	0.01654
(30,18,1)	1.03790	0.06643	0.00736	0.97335	0.11831	0.02198	0.97923	0.07149	0.00784
(30,18,2)	1.05625	0.07593	0.00966	0.97553	0.10164	0.01621	0.98319	0.06926	0.00736
(30,18,3)	1.03848	0.06589	0.00734	0.97729	0.10477	0.01742	0.98215	0.08784	0.01240
(30,18,4)	1.04099	0.06827	0.00786	0.96974	0.10467	0.01727	0.97611	0.07657	0.00913
(30,24,1)	1.03706	0.06560	0.00719	0.97992	0.10319	0.01662	0.97835	0.07379	0.00745
(30,24,2)	1.04694	0.07004	0.00821	0.97807	0.09683	0.01470	0.09792	0.06645	0.00721
(30,24,3)	1.03493	0.06352	0.00674	0.97905	0.09887	0.01541	0.98565	0.06580	0.00653
(30,24,4)	1.03961	0.06692	0.00756	0.97831	0.09554	0.01432	0.98573	0.06517	0.00643
(40,24,1)	1.02919	0.05493	0.00509	0.98192	0.10178	0.01636	0.98354	0.06807	0.00702
(40,24,2)	1.05041	0.06566	0.00721	0.98207	0.08668	0.01182	0.98933	0.05802	0.00510
(40,24,3)	1.02968	0.05493	0.00503	0.98149	0.08933	0.01259	0.98579	0.06193	0.00582
(40,24,4)	1.03283	0.05694	0.00540	0.97724	0.08872	0.01242	0.98870	0.05761	0.00505
(40,32,1)	1.02840	0.05495	0.00503	0.98604	0.08961	0.01249	0.99025	0.05438	0.00451
(40,32,2)	1.04047	0.05944	0.00593	0.98374	0.08232	0.01073	0.98886	0.05601	0.00478
(40,32,3)	1.02827	0.05382	0.00483	0.98503	0.08418	0.01117	0.99035	0.05371	0.00441
(40,32,4)	1.03192	0.05598	0.00526	0.98272	0.083117	0.01073	0.98901	0.05568	0.00472
(50,32,1)	1.02491	0.04841	0.00389	0.98714	0.08689	0.01173	0.99131	0.04960	0.00373
(50,32,2)	1.04350	0.05681	0.00542	0.98590	0.07658	0.00919	0.99125	0.04857	0.00361
(50,32,3)	1.02472	0.04763	0.00381	0.98443	0.07822	0.00967	0.98994	0.05062	0.00391
(50,32,4)	1.02843	0.04942	0.00416	0.98371	0.07742	0.00948	0.98932	0.05171	0.00407

Table 1: *Average estimate, Bias and MSE of the estimators of θ for given $\mu = 1, \theta = 1.5$*

(n,m,s)	MLE	BIAS	MSE	LSE	BIAS	MSE	BAYES	BIAS	MSE
(20,12,1)	1.68417	0.26014	0.13438	1.51204	0.32128	0.19246	1.51421	0.46034	0.09226
(20,12,2)	1.87181	0.43052	0.39091	1.50953	0.32969	0.19795	1.51804	0.43837	0.15056
(20,12,3)	1.70755	0.28655	0.15911	1.51094	0.33063	0.21401	1.51649	0.13176	0.10492
(20,12,4)	1.74806	0.32708	0.21514	1.49362	0.32598	0.19692	1.52243	0.00117	0.13382
(20,16,1)	1.65856	0.23182	0.10179	1.50694	0.27225	0.13749	1.51162	0.42873	0.06524
(20,16,2)	1.72482	0.28867	0.16376	1.48766	0.26682	0.12266	1.51452	0.16994	0.08835
(20,16,3)	1.65894	0.23057	0.10174	1.50268	0.26759	0.12866	1.50970	0.07517	0.06535
(20,16,4)	1.69341	0.26412	0.13576	1.49197	0.26952	0.12975	1.51735	0.03427	0.08529
(30,18,1)	1.62509	0.19206	0.06812	1.49969	0.24981	0.10879	1.51535	0.56535	0.04742
(30,18,2)	1.79499	0.33826	0.11789	1.50243	0.26144	0.11812	1.52215	0.78643	0.07443
(30,18,3)	1.63589	0.20816	0.08106	1.50654	0.25692	0.11976	1.51554	0.38108	0.05667
(30,18,4)	1.65963	0.23143	0.10024	1.48741	0.25252	0.11179	1.52062	0.64902	0.06721
(30,24,1)	1.61181	0.17401	0.05427	1.49836	0.21281	0.07755	1.51166	0.13018	0.03632
(30,24,2)	1.67945	0.22550	0.09407	1.49155	0.21491	0.07673	1.51314	0.12987	0.03987
(30,24,3)	1.60598	0.17006	0.05202	1.49456	0.21026	0.07416	1.51199	0.13882	0.03696
(30,24,4)	1.63389	0.19729	0.07220	1.49172	0.2089	0.07286	1.51423	0.12571	0.04457
(40,24,1)	1.59452	0.15782	0.04424	1.50070	0.21230	0.07591	1.51460	0.03188	0.03336
(40,24,2)	1.76290	0.29066	0.15152	1.50191	0.22435	0.07381	1.52099	0.06421	0.05131
(40,24,3)	1.60391	0.17089	0.05283	1.49815	0.21323	0.07699	1.51675	0.03746	0.03862
(40,24,4)	1.62594	0.19079	0.06559	1.48809	0.21711	0.07946	1.51199	0.21665	0.03781
(40,32,1)	1.58081	0.14022	0.03471	1.49854	0.18253	0.05580	1.50969	0.19968	0.02466
(40,32,2)	1.66169	0.19758	0.06913	1.49313	0.18304	0.05512	1.51179	0.21566	0.03059
(40,32,3)	1.58256	0.140137	0.03532	1.4959	0.17741	0.05193	1.50996	0.19452	0.02508
(40,32,4)	1.60563	0.16096	0.04668	1.49024	0.17888	0.05241	1.51156	0.21259	0.03001
(50,32,1)	1.57342	0.13123	0.02990	1.50051	0.17974	0.05394	1.50665	0.09304	0.02300
(50,32,2)	1.72174	0.24448	0.10128	1.49803	0.19029	0.05928	1.51015	0.11409	0.03324
(50,32,3)	1.58079	0.13939	0.03447	1.49515	0.18043	0.05417	1.50771	0.11663	0.02528
(50,32,4)	1.60111	0.15821	0.04490	1.49576	0.18461	0.05669	1.50946	0.12165	0.03084

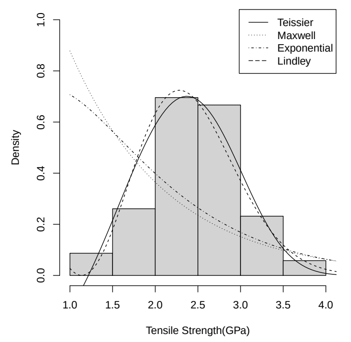

Figure 3: *Fitted density of various distributions*

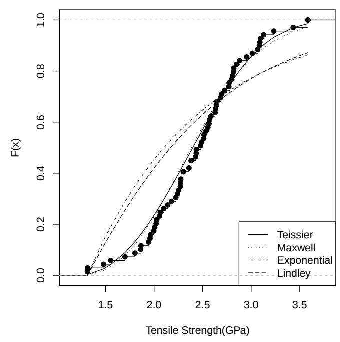

Figure 4: *Fitted CDF of various distributions*

7. Real Data Application

To demonstrate the application of the proposed distribution and estimation methods, a real data set has been considered, that represent the tensile strength, measured in Giga-Pascal (GPa), of 69 carbon fibers tested under tension at gauge lengths of 20mm, see [27]. They conducted single-filament tensile tests on carbon fibers of differing gauge lengths. These experimental results were further reported and analyzed by [28] and [29]. [30] also used this data to show the usefulness of the three-parameter Birnbaum-Saunders distribution and the inverse Gaussian distribution. Set of data points is given as follows :

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585

As we have stated in the first section that this paper aims to propose the use of the 2-T distribution for fuzzy data sets, we assume that the tensile strength of carbon fibers are observed with some degrees of imprecision and fit the 2-T distribution over the other competing distributions. Here, the triangular fuzzy number is used to model the unknown value of $\tilde{x} = (u, v, w)$. The corresponding membership function of each value of observed data point, say x, is given by

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{x-(x_i-h)}{h}, & x_i - h \leq x \leq x_i, \\ \frac{(x_i+h)-x}{h}, & x_i \leq x \leq x_i + h, \\ 0, & otherwise, \end{cases} \tag{18}$$

where $h = 0.05x_i$.

Table 3: *MLEs, AIC, BIC and KS-statistics along with p-value for complete data set without fuzzy*

Model	MLEs $\hat{\alpha}$	MLEs $\hat{\lambda}$	NLL	AIC	BIC	KS	p-value
Teissier	1.217	0.834	49.982	103.965	108.433	0.089	0.641
Maxwell	1.136	0.761	51.291	106.582	111.051	0.095	0.557
Exponential	1.312	0.878	78.001	158.001	164.469	0.310	0.000
Lindley	1.312	1.265	72.943	147.887	154.355	0.276	0.000

Table 4: *MLEs, AIC, BIC and KS-statistics along with p-value for complete fuzzy data*

Model	MLEs $\hat{\alpha}$	MLEs $\hat{\lambda}$	NLL	AIC	BIC
Teissier	1.216	0.834	196.3064	396.613	401.081
Maxwell	1.136	0.761	197.6153	399.230	403.699
Exponential	1.312	0.877	224.3245	450.649	457.117
Lindley	1.312	1.264	219.2673	440.535	447.002

In order to proceed with the fuzzy set-up, we would like to first assess the goodness-of-fit of the proposed distribution to model the given data set. For this purpose, we use some goodness-of-fit criteria such as Kolmogorov-Smirnov (KS) statistic, Akaike information criterion (AIC) and Bayesian information criterion (BIC) to compare the fitting of the competing distributions. For the comparison purposes, we take two parameter families of the exponential, Maxwell and Lindley distributions, which are very popular distributions in statistical literature. Table 3 consists of the MLEs and negative log-Likelihood (NLL) values for all four distributions for the carbon fibers data. Table (4) also shows the different model selection criteria such as AIC, BIC, and KS-statistic along with the p-value. From the table, it is observed that the 2-T distribution has the lowest AIC, BIC and KS values for given data-set. As we can note that the 2-T distribution has the smallest statistic values among others, it may be used to model the tensile strength data set over the considered distributions. Since, we aimed to propose the estimation under fuzzy set-up, the fitting of all four distributions under fuzzy environment is also presented. The fitting results are presented in table (4). We can also note here that the 2-T distribution also has the smallest AIC and BIC values under fuzzy set up. It can be concluded that the 2-T is the best fitting model among others for the fuzzy and without fuzzy data problems as well.

From the Tables (3) and (4) it is found that 2-T distribution is quite enough flexible to model the uncertainty arises due to the randomness and fuzziness. Table (5) includes the estimates of the parameters under the MLE, LSE and Bayesian estimation methods for the 2-T distribution under different progressive Type-II censoring schemes. These parameters are calculated for different schemes as well as for the different sample sizes. The 95 percent confidence intervals are also provided for the 2-T parameters. For the Bayesian estimation, the non-informative priors are considered to estimate μ and θ. Fig. (5) represents the corresponding density and trace plots for the simulated posterior samples based on the real data-set. Here, the trace plots display the random scatter of the sample around the mean values and do not have trend which emphasis the fact that the model has converged. Also, the density plots show the unique modality of the parameters.

Table 5: *MLEs, LSEs and Bayes estimates under progressive fuzzy sample*

m	Schemes	MLEs		LSEs		Bayes		CI		HPD	
		$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$
	1	1.242	1.208	1.513	1.510	1.213	1.170	(1.15,1.33)	(1.03,1.39)	(1.11,1.30)	(0.99,1.33)
40	2	1.211	0.828	1.343	0.876	1.140	0.752	(1.08,1.35)	(0.68,0.97)	(0.95,1.29)	(0.60,0.88)
	3	1.250	1.038	1.641	1.589	1.216	0.999	(1.16,1.34)	(0.88,1.19)	(1.10,1.30)	(0.83,1.16)
	4	1.233	0.915	1.587	1.302	1.190	0.875	(1.13,1.34)	(0.77,1.06)	(1.05 ,1.30)	(0.72,1.02)
	1	1.243	1.142	1.512	1.401	1.215	1.111	(1.15,1.33)	(0.98,1.30)	(1.10,1.30)	(0.94,1.26)
45	2	1.217	0.841	1.366	0.900	1.152	0.771	(1.09,1.34)	(0.70,0.98)	(0.98,1.29)	(0.64,0.90)
	3	1.255	1.057	1.623	1.510	1.225	1.026	(1.17,1.33)	(0.91,1.20)	(1.13,1.30)	(0.88,1.18)
	4	1.230	0.888	1.552	1.176	1.193	0.858	(1.12,1.34)	(0.75,1.02)	(1.06,1.30)	(0.72,0.99)
	1	1.244	1.085	1.513	1.311	1.215	1.058	(1.15,1.34)	(0.94,1.23)	(1.10,1.30)	(0.91,1.20)
50	2	1.220	0.847	1.389	0.926	1.156	0.786	(1.10,1.34)	(0.72,0.97)	(1,1.29)	(0.66,0.91)
	3	1.255	1.049	1.590	1.391	1.226	1.020	(1.17,1.33)	(0.91,1.19)	(1.13,1.30)	(0.88,1.16)
	4	1.228	0.874	1.508	1.082	1.189	0.846	(1.11,1.34)	(0.75,1.00)	(1.04,1.30)	(0.71,0.97)
	1	1.241	0.974	1.490	1.134	1.204	0.945	(1.15,1.33)	(0.85,1.09)	(1.01,1.30)	(0.83,1.07)
60	2	1.220	0.843	1.410	0.949	1.169	0.800	(1.10,1.34)	(0.73,0.95)	(1.01,1.29)	(0.69,0.91)
	3	1.246	0.971	1.507	1.148	1.215	0.945	(1.15,1.33)	(0.85,1.09)	(1.11,1.30)	(0.83,1.07)
	4	1.226	0.859	1.435	0.975	1.187	0.831	(1.11,1.34)	(0.75,0.97)	(1.05,1.30)	(0.73,0.95)

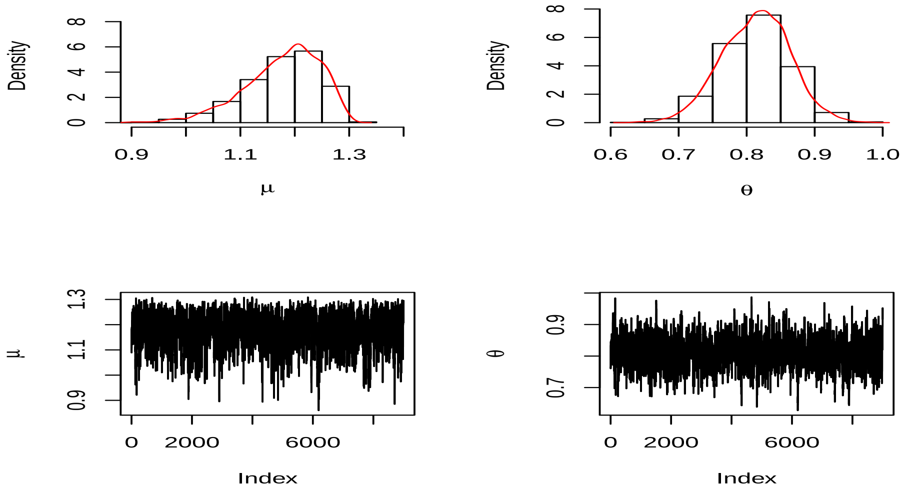

Figure 5: *The density and trace plots of simulated μ and θ for complete data*

8. Conclusion

In real world situations, it is always possible that the observed lifetime data might be observed imprecisely and may be represented in the form of fuzzy numbers. Therefore, a suitable statistical methodology is required to handle these type of data sets. In this article, we introduced a two parameter Teissier distribution to model the fuzzy censored lifetime data sets. We derived the some useful properties of the distribution such as mean residual life, moments, moment generating function and quantile function. We also discussed different methods to estimate the parameters of Teissier distribution by using the maximum likelihood, least square and Bayesian techniques. In order to assess the validity and applications of the estimation procedures, we presented an extensive simulation study. Lastly, a real data set is considered to discuss the applicability of the distribution. From the data, it is found that Two parameter Teissier distribution may be a better choice over other competing distributions such as Maxwell, Exponential and Lindley. The objective of the article was to introduce a new two parameter distribution to model the fuzzy censored data set having increasing failure rate which is generally found in the reliability and survival analysis. Future objectives may be threefold. The first one may be In future, with the objective of modelling the fuzzy data set, the more flexibility can be added by introducing more parameters in the distribution. Also, we can work in future with different censoring schemes available in the literature.

References

[1] Swain, James J. and Venkatraman, Sekhar and Wilson, James R. (1988). Least-squares estimation of distribution functions in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29:271–297.

[2] Balakrishnan, N. and Sandhu,R. A. (1995). A simple simulational algorithm for generating progressive Type-II censored samples. *The American Statistician*, 49:229–230.

[3] Ng, H.K. T. and Luo, L. and Hu, Y. and Duan, F. (2012). Parameter estimation of three-parameter Weibull distribution based on progressively type-II censored samples. *Journal of Statistical Computation and Simulation*, 82:1661–1678.

[4] Balakrishnan, N. and Cramer, Erhard (2014). A simple simulational algorithm for generating progressive Type-II censored samples. *Stat. Indust. Tech.*

[5] Viertl, Reinhard (2011). Statistical methods for fuzzy data. *John Wiley & Sons*.

[6] Zimmermann, Hans-Jürgen (2011). Fuzzy set theory—and its applications. *Springer Science & Business Media*.

[7] Zadeh, Lotfi Asker (1968). Probability measures of fuzzy events. *Journal of mathematical analysis and applications*, 23:421–427.

[8] Pak, Abbas and Parham, Gholam Ali and Saraj, Mansour (2014). Inferences on the competing risk reliability problem for exponential distribution based on fuzzy data. *IEEE Transactions on reliability*, 63:2–12.

[9] Cohen, A. Clifford (1963). Progressively censored samples in life testing. *Technometrics*, 5:327–339.

[10] Cheng, R.C. H. and Amin, N.A. K.(1983). Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society: Series B (Methodological)*, 45:394–403.

[11] Chaturvedi, Ankita and Singh, Sanjay Kumar and Singh, Umesh (2018). Statistical Inferences of Type-II Progressively Hybrid Censored Fuzzy Data with Rayleigh Distribution. *Austrian Journal of Statistics*, 3:40–62.

[12] Krishna, Hare and Kumar, Kapil (2013). Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample. *Journal of Statistical Computation and Simulation*, 83:1007–1019.

[13] Nelson, W. (1982). Applied Life Data Analysis. New York: A John Wiley & Sons. *Inc*.

[14] Lawless, Jerald F. (2011). Statistical models and methods for lifetime data. *John Wiley & Sons*.

[15] Cohen, A. Clifford (1991). Order statistics and inference: Estimation methods. *Academic Press*.

[16] Balakrishnan, Narayanaswamy and Balakrishnan, Narayanaswamy and Aggarwala, Rita (2000). Progressive censoring: theory, methods, and applications. *Springer Science & Business Media*.

[17] Balakrishnan, N. and Cramer, Erhard (2014). The art of progressive censoring. *Stat. Indust. Tech*.

[18] Singh, Sanjay Kumar and Singh, Umesh and Sharma, Vikas Kumar and Kumar, Manoj (2015). Estimation for flexible Weibull extension under progressive Type-II censoring. *Journal of Data Science*, 13:21–41.

[19] Pak, Abbas and Parham, Gholam Ali and Saraj, Mansour (2014). Inference for the Rayleigh distribution based on progressive Type-II fuzzy censored data. *Journal of Modern Applied Statistical Methods*, 13:19.

[20] Pak, Abbas and others (2009). Statistical inference for the parameter of Lindley distribution based on fuzzy data. *Brazilian Journal of Probability and Statistics*, 31:502–515.

[21] Pak, Abbas and Parham, Gholam Ali and Saraj, Mansour (2014). Reliability estimation in Rayleigh distribution based on fuzzy lifetime data. *International Journal of System Assurance Engineering and Management*, 5:487–494.

[22] Buckley, James J. (2005). Fuzzy statistics: hypothesis testing. *Soft Computing*, 9:512–518.

[23] Teissier, G. (1934). Recherches sur le vieillissement et sur les lois de la mortalité. *Annales de physiologie et de physicochimie biologique*, 10:237–284.

[24] Muth, Eginhard J. (1977). Reliability models with positive memory derived from the mean residual life function. *The Theory and Applications of Reliability*, 2:401–435.

[25] Jodra, Pedro and Jimenez-Gamero, Maria Dolores and Alba-Fernandez, Maria Virtudes (2015). On the Muth distribution. *Mathematical Modelling and Analysis*, 20:291–310.

[26] Jodra, Pedro and Gomez, Hector Wladimir and Jimenez-Gamero, Maria Dolores and Alba-Fernandez, Maria Virtudes (2017). The Power Muth Distribution. *Mathematical Modelling and Analysis*, 22:186–201.

[27] Bader, M. G. and Priest, A. M. (1982). Statistical aspects of fibre and bundle strength in hybrid composites. *Progress in science and engineering of composites*, 1129–1136.

[28] Smith, Richard L. (1991). Weibull regression models for reliability data. *Reliability Engineering & System Safety*, 34:55–76.

[29] Watson,A. Smith and Smith, R. L. (1985). An examination of statistical theories for fibrous materials in the light of experimental data. *Journal of Materials Science*, 20:3260–3270.

[30] Durham, S. D. and Padgett, W. J. (1997). Cumulative damage models for system failure with application to carbon fibers and composites. *Technometrics*, 39:34–44.

[31] Corless, Robert M. and Gonnet, Gaston H. and Hare, David E. G. and Jeffrey, David J. and Knuth, Donald E. (1996). On the LambertW function. *Advances in Computational mathematics*, 5:329–359.

[32] Milgram, M. S. (1985). The generalized integro-exponential function. *Mathematics of computation*, 44:443–458.

[33] Haario, Heikki and Saksman, Eero and Tamminen, Johanna and others (2001). An adaptive Metropolis algorithm. *Bernoulli*, 7:223–242.

[34] Sharma, Vikas Kumar and Singh, Sudhanshu V. and Shekhawat, Komal (1997). Exponentiated Teissier distribution with increasing, decreasing and bathtub hazard functions. *Journal of Applied Statistics*, 1–23.

[35] Balakrishnan, Narayanaswamy and Balakrishnan, Narayanaswamy and Aggarwala, Rita (2000). Progressive censoring: theory, methods, and applications. *Springer Science & Business Media*.

[36] Singh, Sudhanshu V. and Elgarhy, Mohammed and Ahmad, Zubair and Sharma, Vikas Kumar and Hamedani, Gholamhossein G. (1997). New Class of Probability Distributions Arising from Teissier Distribution. *Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy: Proceedings of the First International Conference, MMCITRE 2020*, 41.

[37] Griffin, J. E. and Walker, S. G. (2013). On adaptive metropolis–hastings methods. *Statistics and Computing*.