Record-based Transmuted Power Lomax Distribution: Properties and its Applications in Reliability

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Abstract

In this paper, we consider a record-based transmuted version of Power Lomax distribution and it is named as Record-based Transmuted Power Lomax (RTPL) distribution. Further, we present several statistical properties of the proposed distribution such as moments, quantiles, stochastic ordering, order statistics, and its explicit expressions. Some of its reliability measures such as survival function, hazard function, cumulative hazard function, mean residual time, and mean inactivity time is also discussed. The maximum likelihood method is used to estimate the parameters of the RTPL distribution and this new extended model is applied to a real datasets to access the suitability and applicability of the model based on well-known information criteria and test for goodness of fit. The simulation study is performed to verify the efficiency and asymptotic behavior of the maximum likelihood estimators.

Keywords: Record-based Transmuted map, Power Lomax distribution, Lambert *W* function, Maximum Likelihood Estimation.

1. INTRODUCTION

Record values and record statistics are routine and central points for monitoring many aspects of human life in date to date activities and it has a lot of real-life applications. In particular, the industry has many products which fail at times due to stress. For example, an electronic component ceases to function in an environment of high temperature, and a battery dies under the stress due to over use. But the precise breaking stress or failure point varies even among identical items. Hence in such experiments, measurements may be made sequentially and only the record values are observed. Thus, the number of measurements made is considerably smaller than the complete sample size. This "measurement saving" method can be important when the measurement of these experiments is costly if the entire sample was destroyed. There are situations in which an observation is sorted only if it is a record value. This includes studies in meteorology, hydrology, economics, athletic events, and life testing studies.

In 1952, Chandler introduced the study of record values and discussed lots of the most important and basic properties of records. Let $X_1, X_2,...$ be the sequence of the random variables, there are two types of the record values such as upper and lower records. We say that X_n be the upper record value if $X_n > \max\{X_1, X_2,..., X_{n-1}\}, n = 2,3,....$ this means that X_n which is more than all previous $X's$, and X_n be the lower record value if $X_n < \min\{X_1, X_2,..., X_{n-1}\}, n = 2,3,....$. In two situations X_1 is considered the first upper or lower record value. The upper records can be used in many real-life phenomena when compares to the lower records. Now if together with some sequence $X_1, X_2, \ldots$ one considers $Y_1 = -X_1,\ Y_1 = -X_2, \ldots,\ Y_n = -X_n, \ldots$, then it becomes evident that the lower record times for $Y's$ is coinciding with the corresponding upper record times of $X's$.

Balakrishnan et.al [9] proposed a record-based transmuted map to generate new probability models. Let $X_1, X_2, \ldots, X_n$ be a sequence of an independent and identically distributed random variable with a distribution function $G(x)$. Let $X_{U(1)}$ and $X_{U(2)}$ be the two upper records from the above sequence of independent and identically distributed random variables. Define a random variable Y as follows:

$$Y = \begin{cases} X_{U(1)}, \text{ with probability } 1-p \\ X_{U(2)}, \text{ with probability } p \end{cases}$$

Where, $p \in [0,1]$, then

$$F_Y(x) = (1-p)P(X_{U(1)} \le x) + p\,P(X_{U(2)} \le x)$$

The record-based transmuted cumulative distribution function is obtained as

$$F_Y(x) = G(x) + p.\overline{G}(x)\log \overline{G}(x); \text{ for } x \in R,\ 0 \le p \le 1 \tag{1}$$

The corresponding probability density function is given by

$$f_Y(x) = g(x)\left[1 - p - p.\log \overline{G}(x)\right]; \text{ for } x \in R,\ 0 \le p \le 1 \tag{2}$$

Balakrishnan et al. [9] also introduced a few new record-based transmuted (RT) probability distributions like RT-exponential (RTE) distribution, RT-Linear exponential (RTLE) distribution, RT-Weibull (RTW) distribution, etc. Vijay Kumar et al. [8] studied the Record-Based Transmuted Generalized Linear Exponential Distribution with increasing, decreasing, and bathtub-shaped failure rates.

The Lomax distribution, is also known as Pareto Type II distribution and it is proposed by K.S. Lomax (1954). It is also classified as heavy-tailed distribution and referred as a shifted Pareto distribution, which is widely used in survival analysis. It is popularly used as an alternative to power-law, exponential, gamma, and Weibull distribution for modeling heavy-tailed data in the domain of business, Economics, and Actuarial science. A random variable X follows the Lomax distribution with the shape parameters $\beta > 0$ and the scale parameter $\lambda > 0$ and its cumulative distribution function is given by

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\beta};\ x > 0 \tag{3}$$

The corresponding probability density function is given below

$$f(x) = \frac{\beta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-\beta-1}; x > 0 \tag{4}$$

El-Houssainy et al. [14] mentioned that the Power Lomax (PL) distribution is obtained by using the power transformation that is $Y = X^{1/\beta}$. The random variable X is said to follow the three-parameter PL distribution with the shape parameters $\alpha, \beta > 0$ and scale parameter $\lambda > 0$ if the cumulative distribution function of $x > 0$ is given by

$$F(x) = 1 - \lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha} \tag{5}$$

The probability density function of the power Lomax distribution is given by

$$f(x) = \alpha\beta\lambda^{\alpha} x^{\beta-1}\left(\lambda + x^{\beta}\right)^{-\alpha-1} \tag{6}$$

In the literature, some extensions of the Lomax distribution were developed and further showed that these resultant distributions are better than the baseline distribution, and the following will be the list of a few such extensions of the Lomax distribution. Abdul-Moniem et al. [1] introduced Exponentiated Lomax (EL) distribution, Muhammad Rajab et al.[19] proposed Beta-Lomax (BL) distribution, Cordeiro G. M et al. [11] developed gamma-Lomax (GL) distribution, El-Bassiouny et al. [13] studied Exponential Lomax distribution, Singh Yadav et al.[20] investigated on Inverse Lomax (IL) distribution, Masood Anwar et al. [6] presented the Half-logistic Lomax (HLL) distribution, and Sanaa Al-Marzouki et al. [4] developed the Exponentiated power Lomax distribution.

The remaining part of this paper is organized as follows: In Section 2, we introduce Record based Transmuted power Lomax (RTPL) distribution and present some of its special cases. In Section 3, we derive some structural properties including quantile function, moments, Lorenz curve, Bonferroni curve, entropy, and order statistics. In Section 4, we present the simulation study to measure the precision and asymptotic nature of parameter estimates of the proposed distribution. In Section 5, we discuss the maximum likelihood estimates (MLEs) of the model parameters. In Section 6, we considered two data set for illustrating the suitability and goodness of fit of the RTPL distribution. In Section 7, we conclude the study with a summary of results.

2. RECORD-BASED TRANSMUTED POWER LOMAX DISTRIBUTION

A non-negative integer-valued random variable X is said to follow Record based transmuted Power Lomax distribution with scale parameter $\lambda > 0$, shape parameters $\alpha, \beta > 0$ and $p \in [0,1]$ if its cumulative distribution function is of the following form

$$F(x) = 1 - \frac{\lambda^{\alpha}}{\left(x^{\beta} + \lambda\right)^{\alpha}}\left[1 - p\log\left(\frac{\lambda}{x^{\beta} + \lambda}\right)^{\alpha}\right] \tag{7}$$

And the corresponding probability density function of the RTPL distribution is given by

$$f(x) = \frac{\alpha\beta\lambda^{\alpha} x^{\beta-1}}{\left(x^{\beta} + \lambda\right)^{\alpha+1}}\left[1 - p.\left(1 + \log\left(\frac{\lambda}{x^{\beta} + \lambda}\right)^{\alpha}\right)\right] \tag{8}$$

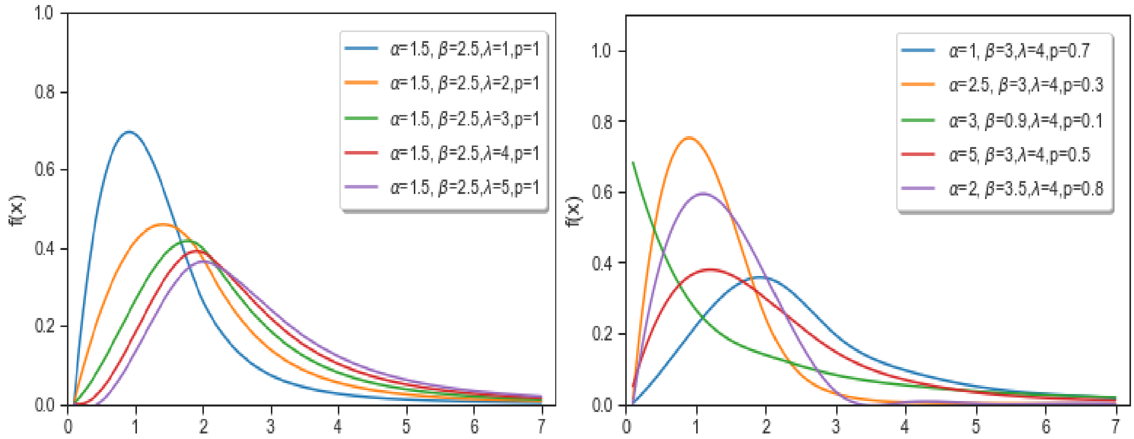

Figure: 1a Figure: 1b

Figure 1. The probability density plot of the RTPL distribution

The shapes of the probability density function of RTPL distributions for different values of the parameters can be described in Figure 1. From Figure 1a, it is observed that the curves are unimodal and positively skewed which represents the density plot with fixed $\alpha = 0.5$, $\beta = 2.5$, $p = 1$ and λ those with different values. From Figure 1b, it can be observed that the curve is left-skewed and reversed J shaped and the curves represent fixed values such that $\lambda = 4$and assign different values to the other three parameters of RTPL distribution.

2.1.Reliability Analysis

In this section, we define the survival function, hazard rate function, reversed hazard rate function, and cumulative hazard rate function of the RTPL distribution.

The survival function of RTPL distribution is obtained as follows

$$S(x) = \frac{\lambda^{\alpha}}{\left(x^{\beta}+\lambda\right)^{\alpha}}\left[1 - p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right] \tag{9}$$

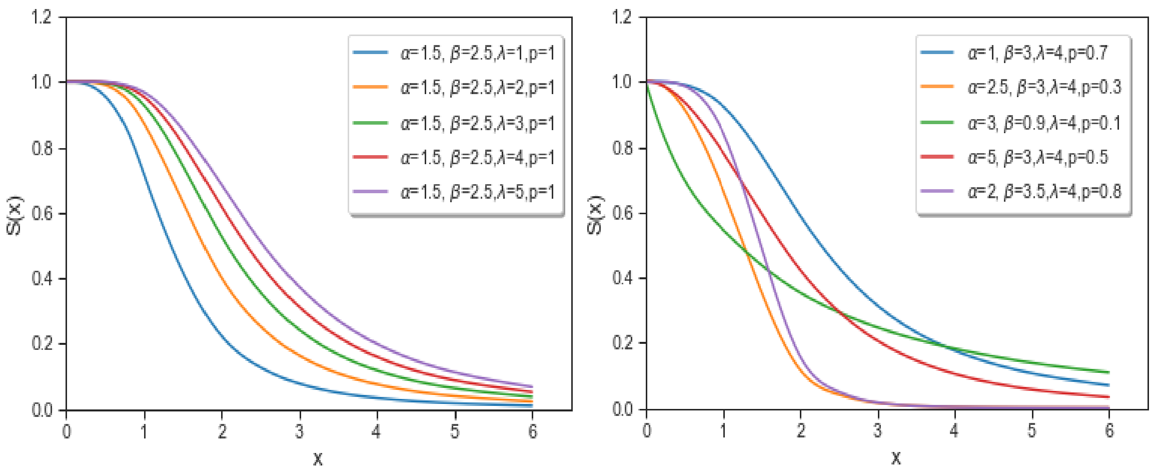

Figure: 2a Figure: 2b

Figure 2. The plots of the Survival function of RTPL distribution

The hazard rate function of RTPL distribution defined as

$$h(x)=\frac{\alpha\beta x^{\beta-1}\left[1-p\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right]}{\left(x^{\beta}+\lambda\right)\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]} \tag{10}$$

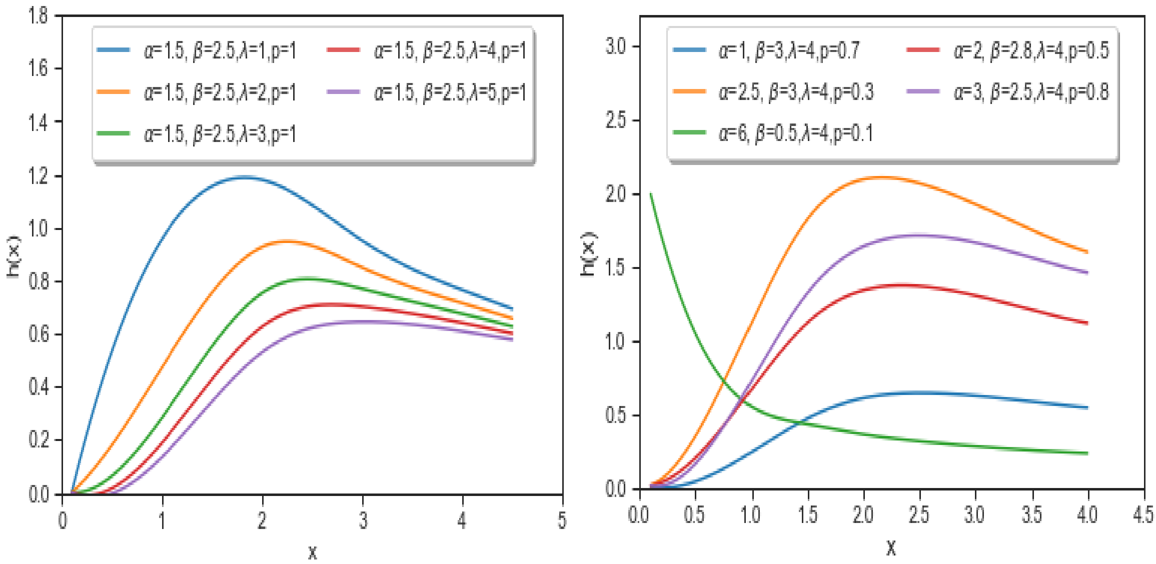

Figure: 3a

Figure: 3b

Figure 3. The hazard rate plot for RTPL distribution

The cumulative hazard function of RTPL distribution is as follows

$$H(x)=-\log\left[\frac{\lambda^{\alpha}}{\left(x^{\beta}+\lambda\right)^{\alpha}}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right]$$

(11)

The reversed hazard function of the RTPL distribution is given as

$$\tau(x)=\frac{\alpha\beta\lambda^{\alpha}x^{\beta-1}\left(x^{\beta}+\lambda\right)^{-(\alpha+1)}\left(1-p.\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right)}{1-\lambda^{\alpha}\left(x^{\beta}+\lambda\right)^{-\alpha}\left(1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)} \tag{12}$$

The survival function plot with parameters $\alpha=1.5, \beta=2.5$ and $p=1$ is represented in Figure 2a and different values of the parameter λ and Figure 2b shows the survival function plot of RTPL for $\lambda=4$ and assigning different values for the parameters α, β, and p. The survival curves are decreasing as time increases. Figure 3a displays the hazard rate plot for $\alpha=1.5, \beta=2.5, p=1$ and λ with changing value, which describes the curves are increasing, decreasing, and Figure 3b shows that hazard rate plot for $\lambda=4$, and varying parameter values for α, β and p the curves are increasing and reversed *J* shaped.

Special cases: For different values of the parameters, the following distributions are obtained as the special case of the RTPL distribution.

Case 1: If the value of $p = 0$, then the distribution function given in (7) reduced to the power Lomax distribution.
Case 2: If the value of $p = 0$ and $\beta = 1$, then the distribution function given in (7) reduced to the Lomax distribution.

3. STATISTICAL AND MATHEMATICAL PROPERTIES

This section deals with some important properties of the proposed RTPL distribution such as quantile function, moments, inverted moments, entropy, stochastic ordering, and order statistics.

3.1. Quantile Function

The quantile function plays an important role when simulating random variables from a probability distribution. The quantile function of the RTPL distribution function is defined as follows

$$1-\frac{\lambda^{\alpha}}{\left(x^{\beta}+\lambda\right)^{\alpha}}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]=u \tag{13}$$

The closed-form expression of the quantile function has been obtained by using the Lambert W-function such as:

$$W(\nu)e^{W(\nu)}=\nu$$

Where v is the complex number. For real numbers, $\nu \geq -\frac{1}{e}$, the Lambert W function has only two branches W_0 which takes the value in $\left[-1,\infty\right)$ and W_{-1} which takes the value in $\left[-\infty,-1\right)$ and for $\nu \in \left[-\frac{1}{e},0\right)$. It can be verified that $\frac{u-1}{pe^{1/p}} \in \left[-\frac{1}{e},0\right)$, and $-\frac{1}{p}+\log\left(\frac{\lambda^{\alpha}}{x^{\beta}+\lambda^{\alpha}}\right) \in \left[-\infty,-1\right)$

Now using the negative branch of the Lambert W function in the above equation we get,

$$W_{-1}\left(\frac{u-1}{pe^{1/p}}\right)=-\frac{1}{p}+\log\left(\frac{\lambda^{\alpha}}{x^{\beta}+\lambda^{\alpha}}\right) \tag{14}$$

Thus, by solving the equation (13), we get the quantile function as given below

$$x_p=\left(\lambda e^{-\frac{1}{\alpha}\left(W_{-1}\left(\frac{u-1}{pe^{1/p}}\right)+\frac{1}{p}\right)}-\lambda\right)^{1/\beta} \tag{15}$$

The median of the probability distribution can be obtained by taking the u as 0.5 in the above quantile function.

3.2. Method of Moments

The r^{th} raw moment of the random variable X having RTPL distribution is obtained by substituting the equation (8) as follows:

$$\mu_r' = E(X^r) = \int_0^\infty x^r \frac{\alpha\beta\lambda^\alpha x^{\beta-1}}{(x^\beta+\lambda)^{\alpha+1}}\left[1-p.\left(1+\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right)\right]dx$$

$$= \frac{\alpha\beta}{\lambda}\int_0^\infty x^{r-\beta-1}\left(1+\frac{x^\beta}{\lambda}\right)^{\alpha+1}\left[1-p.\left(1+\log\left(1+\frac{x^\beta}{\lambda}\right)^\alpha\right)\right]dx$$

By taking $y=\dfrac{x^\beta}{\lambda}$ and applying the transformation method and $y=\dfrac{w}{1-w}$ in the above equation, we get

$$\mu_r' = E(X^r) = \alpha\lambda^{r/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}\frac{\Gamma\left(\frac{r}{\beta}+k+1\right)\Gamma\left(\alpha-\frac{r}{\beta}-k\right)}{\Gamma(\alpha+1)}+(1-p)\frac{\Gamma\left(\frac{r}{\beta}+k\right)\Gamma\left(\alpha-\frac{r}{\beta}\right)}{\Gamma(\alpha+1)}\right] \tag{16}$$

The first two moments of the distribution can be derived from equation (16) and it is given as

$$\mu_1' = \alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}\frac{\Gamma\left(\frac{1}{\beta}+k+1\right)\Gamma\left(\alpha-\frac{1}{\beta}-k\right)}{\Gamma(\alpha+1)}+(1-p)\frac{\Gamma\left(\frac{1}{\beta}+k\right)\Gamma\left(\alpha-\frac{1}{\beta}\right)}{\Gamma(\alpha+1)}\right] \tag{17}$$

$$\mu_2' = \alpha\lambda^{2/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}\frac{\Gamma\left(\frac{2}{\beta}+k+1\right)\Gamma\left(\alpha-\frac{2}{\beta}-k\right)}{\Gamma(\alpha+1)}+(1-p)\frac{\Gamma\left(\frac{2}{\beta}+k\right)\Gamma\left(\alpha-\frac{2}{\beta}\right)}{\Gamma(\alpha+1)}\right] \tag{18}$$

The r^{th} incomplete moment of the RTPL distribution can be obtained by using the equation (8) as follows:

$$\phi_r(t) = \int_x^\infty x^r \frac{\alpha\beta\lambda^\alpha x^{\beta-1}}{(x^\beta+\lambda)^{\alpha+1}}\left[1-p.\left(1+\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right)\right]dx$$

$$= \alpha\lambda^{r/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{r}{\beta}+k+1\right),\left(\alpha-\frac{r}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{r}{\beta}+k\right),\left(\alpha-\frac{r}{\beta}\right)\right)\right] \tag{19}$$

By taking $r=1$ in the equation (19) to get the1^{st} incomplete moment of RTPL distribution and it is given as

$$\phi_1(t) = \alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right] \tag{20}$$

The r^{th} central moment of RTPL distribution is defined as follows

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu_1')^n \frac{\alpha\beta\lambda^\alpha x^{\beta-1}}{(x^\beta+\lambda)^{\alpha+1}}\left[1-p.\left(1+\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right)\right]dx$$

$$=\sum_{k=0}^{r}\alpha\lambda^{r-k/\beta}(-\mu_1')^k\binom{r}{k}\left\{(1-p)B\left(\left(\frac{r-k}{\beta}+1\right),\left(\alpha-\frac{r-k}{\beta}\right)\right)+p\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!}B\left(\left(\frac{r-k}{\beta}+1\right),\left(\alpha-\frac{r-k}{\beta}-m\right)\right)\right\} \quad (21)$$

The r^{th} inverted moment of RTPL distribution is defined and obtained as follows

$$\mu_r^* = \int_{-\infty}^{\infty} x^{-r} \frac{\alpha\beta\lambda^\alpha x^{\beta-1}}{(x^\beta+\lambda)^{\alpha+1}}\left[1-p.\left(1+\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right)\right]dx$$

$$=\alpha\lambda^{-r/\beta}\left\{(1-p)B\left(\left(1-\frac{r}{\beta}\right),\left(\alpha+\frac{r}{\beta}\right)\right)+p\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(k-\frac{r}{\beta}+1\right),\left(\alpha+\frac{r}{\beta}-k\right)\right)\right\} \quad (22)$$

Mean residual life (MRL) or life expectancy at time *t* is the expected additional life length for a unit, which is still alive at time *t*. The mean residual lifetime of the RTPL distribution is defined as follows

$$m_x(t) = E(X-t/X>t), \quad t>0$$

$$=\frac{\mu-\phi_1(t)}{S(t)}-t, \quad \text{where} \quad \mu=\mu_1'$$

$$=\frac{\mu-\left[p\alpha^2\lambda^{1/\beta}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+\alpha(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\lambda^\alpha(x^\beta+\lambda)^{-\alpha}\left[1-p\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right]}-t \quad (23)$$

Mean inactivity time (MIT) is the waiting time to elapsed since the failure of an item is on the condition that the failure can be occurred in *(0,t)*. The mean inactivity time of the proposed RTPL distribution is obtained as

$$\psi_x(t) = E(t-X/X<t)$$

$$=t-\frac{\phi_1(t)}{F(t)}$$

$$=t-\frac{p\alpha^2\lambda^{1/\beta}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+\alpha(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)}{1-\left(\lambda^\alpha(x^\beta+\lambda)^{-\alpha}\left[1-p\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right]\right)} \quad (24)$$

3.3. Measures of Inequality and Uncertainty

In this section, the measures of uncertainty and three inequality measures of the RTPL distribution have been derived. The Lorenz curve of the RTPL distribution can be derived by using the first incomplete moment in (20) and the moment in (17) is obtained as follows

$$LO(x)=\frac{\phi_1(x)}{E(x)}$$

$$=\frac{\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]} \quad (25)$$

The Bonferroni curve of the RTPL distribution is obtained by using (7) and the Lorenz curve in (25) is given below

$$BO(x)=\frac{LO(x)}{F(x)}$$

$$=\frac{\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left(1-\frac{\lambda^{\alpha}}{(x^{\beta}+\lambda)^{\alpha}}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]} \quad (26)$$

The Zenga Index of the RTPL distribution is obtained as

$$Z=1-\frac{\mu^{-}_{(x)}}{\mu^{+}_{(x)}} \quad (27)$$

Where, $\mu^{-}_{(x)}=\frac{1}{F(x)}\int_0^x xf(x)dx$ and $\mu^{+}_{(x)}=\frac{1}{1-F(x)}\int_0^\infty xf(x)dx$

$$\mu^{-}_{(x)}=\frac{\alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left(1-\frac{\lambda^{\alpha}}{(x^{\beta}+\lambda)^{\alpha}}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)} \quad (28)$$

$$\mu^{+}_{(x)}=\frac{\alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left(\frac{\lambda^{\alpha}}{(x^{\beta}+\lambda)^{\alpha}}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)} \quad (29)$$

By substituting the equations (28) and (29) in (27), we get the Zenga Index of the RTPL distribution.

Entropy is one of the important tools for measuring the uncertainty of the random variables and the information provided by such variables. In some cases, the random variables in the probability distribution are associated with some sort of uncertainty, and entropy can be used to quantify them.

The Rényi entropy can be derived by using the equation (8) is defined as follows

$$RE_x(\delta)=\frac{\delta}{1-\delta}\log\left(\int_{-\infty}^{\infty}f(x)^{\delta}dx\right)$$

$$=\frac{\delta}{1-\delta}\left[\log\left(\frac{\alpha\beta}{\lambda}\right)+\log\left(\sum_{k=0}^{\infty}\sum_{i,j=0}^{\infty}\binom{\delta}{k}\binom{k}{j}\frac{p^k\alpha^j j(-1)^{i+j+k+1}}{n\lambda^n\lambda^{-\frac{\delta\beta+i\beta-\beta+1}{\beta}}}\left(\frac{\Gamma\left(\delta+i-2+\frac{1}{\beta}\right)\Gamma\left(\frac{1}{\beta}-\delta-i-\alpha+2\right)}{\Gamma\left(\frac{1}{\beta}+\alpha\right)}\right)\right)\right] \tag{30}$$

3.4. Order Statistics

The order statistics play a vital role in predicting the failure time of certain items by using previously observed failures. Let $X_1, X_2, \dots, X_n$ be a random sample of size n, and let $X_{r:n}$ denotes that i^{th} order statistic, then the pdf of $X_{r:n}$ is given by

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}F(x)^r f(x)(1-F(x))^{n-r}$$

We can rewrite the above equation as follows:

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}F(x)^{r-1}f(x)(1-F(x))^{n-r}$$

Substituting the equations (7) and (8) in the above equation, we can write

$$f_{k:n}(x)=\left(\frac{\alpha\beta}{\lambda B(r,n-r+1)}\right)\sum_{i=0}^{r-1}\sum_{k=0}^{\infty}(-1)^i\binom{r-1}{i}\left(\frac{\Gamma(n+i-r+1)}{k!\Gamma(n+i-r)}\left(p.\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\right)^k\right)\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right) \tag{31}$$

3.5. Record Statistics

Let $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$ be the upper record values from a sequence of identically and independently distributed random variables from the RTPL distribution. The pdf of n^{th}upper record value $X_{U(n)}$ of the RTPL distribution is defined by

$$f_{U(n)}(x)=\frac{1}{\Gamma n}[-\log(1-F(x))]^{n-1}f(x)$$

$$=\left(\frac{\alpha\lambda^{\alpha}\beta x^{\beta-1}}{\Gamma n(\lambda+x^{\beta})^{\alpha+1}}\right)\left[-\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\left(1-p.\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\right)^{n-1}\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)\right] \tag{32}$$

The pdf of n^{th}lower record value $X_{L(n)}$ of the RTPL distribution is defined by

$$f_{L(n)}(x)=\frac{1}{\Gamma n}\left[-\log(F(x))\right]^{n-1} f(x)$$

$$f_{L(n)}(x)=\left(\frac{\alpha\lambda^{\alpha}\beta x^{\beta-1}}{\Gamma n\left(\lambda+x^{\beta}\right)^{\alpha+1}}\right)\left[-\log\left(1-\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\left(1-p.\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\right)\right)^{n-1}\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)\right]$$

(33)

3.6.Stochastic Ordering

The ordering of probability distributions particularly among lifetime distributions plays an important role in the statistical literature. We consider stochastic orders, namely, the hazard rate, the mean residual life, and the likelihood ratio order for two independent RTPL random variables under a restricted parameter space. It can be recalled that if a family has a likelihood ratio ordering, it has the monotone likelihood ratio property. If X and Y are independent random variables with a cumulative distribution function F_X and F_Y respectively, then X is said to be smaller than Y in the

- stochastic order $X \le_{st} Y$ if $F_X(x) \ge F_Y(x)$ for all x
- hazard rate order $X \le_{hr} Y$ if $h_X(x) \ge h_Y(x)$ for all x
- mean residual life order $X \le_{mrl} Y$ if $m_X(x) \ge m_Y(x)$ for all x
- likelihood ratio order $X \le_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results are well known for establishing stochastic ordering of probability distributions. The likelihood ratio is given as follows

$$\frac{f_X(x)}{f_Y(x)}=\frac{\alpha_1\beta_1\lambda_1^{\alpha_1}x^{\beta_1-1}\left(x^{\beta_2}+\lambda_2\right)^{\alpha_2+1}\left[1-p_1.\left(1+\log\left(\frac{\lambda_1}{x^{\beta_1}+\lambda_1}\right)^{\alpha_1}\right)\right]}{\alpha_2\beta_2\lambda_2^{\alpha_2}x^{\beta_2-1}\left(x^{\beta_1}+\lambda_1\right)^{\alpha_1+1}\left[1-p_2.\left(1+\log\left(\frac{\lambda_2}{x^{\beta_2}+\lambda_2}\right)^{\alpha_2}\right)\right]} \quad (34)$$

By taking the logarithm on both sides of the likelihood ratio which is given in the equation (34) then we get,

$$\log\left[\frac{f_X(x)}{f_Y(x)}\right]=\log\left[\frac{\alpha_1\beta_1\lambda_1^{\alpha_1}x^{\beta_1-1}\left(x^{\beta_2}+\lambda_2\right)^{\alpha_2+1}\left[1-p_1.\left(1+\log\left(\frac{\lambda_1}{x^{\beta_1}+\lambda_1}\right)^{\alpha_1}\right)\right]}{\alpha_2\beta_2\lambda_2^{\alpha_2}x^{\beta_2-1}\left(x^{\beta_1}+\lambda_1\right)^{\alpha_1+1}\left[1-p_2.\left(1+\log\left(\frac{\lambda_2}{x^{\beta_2}+\lambda_2}\right)^{\alpha_2}\right)\right]}\right] \quad (35)$$

$$\frac{d}{dx}\log\left[\frac{f_X(x)}{f_Y(x)}\right]=\frac{\beta_1-1}{x}-\frac{\beta_2-1}{x}-\left(\frac{(\alpha_1+1)\beta_1 x^{\beta_1-1}}{x^{\beta_1}+\lambda_1}\right)+\left(\frac{(\alpha_2+1)\beta_2 x^{\beta_2-1}}{x^{\beta_2}+\lambda_2}\right)+ \frac{\alpha_1\beta_1 p_1 x^{\beta_1-1}\left(x^{\beta_1}+\lambda_1\right)^{-1}}{\left[1-p_1.\left(1+\log\left(\frac{\lambda_1}{x^{\beta_1}+\lambda_1}\right)^{\alpha_1}\right)\right]}-\frac{\alpha_2\beta_2 p_2 x^{\beta_2-1}\left(x^{\beta_2}+\lambda_2\right)^{-1}}{\left[1-p_2.\left(1+\log\left(\frac{\lambda_2}{x^{\beta_2}+\lambda_2}\right)^{\alpha_2}\right)\right]} \tag{36}$$

Now if $\alpha_1=\alpha_2=\alpha$, $\beta_1=\beta_2=\beta$, $p_1=p_2=p$ and $\lambda_1>\lambda_2$ then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}\le 0$ implies that $X\le_{lr} Y$ and hence $X\le_{lr} Y$, $X\le_{hr} Y$ $X\le_{mlr} Y$ and $X\le_{st} Y$.

4. MAXIMUM LIKELIHOOD ESTIMATION METHOD

In this section, the maximum likelihood method is used to estimate the unknown parameters of the RTPL distribution and the information matrix is obtained to observe the asymptotic behavior of the parameters of RTPL distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the RTPL distribution with unknown parameters $\alpha, \beta, \lambda,$ and p then the likelihood function is given by

$$L=\prod_{i=1}^{n}\frac{\alpha\beta\lambda^{\alpha}x^{\beta-1}}{\left(x^{\beta}+\lambda\right)^{\alpha+1}}\left[1-p\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right]$$

The log-likelihood function of the RTPL distribution is given below

$$l=n\log\alpha+n\log\beta+n\alpha\log\lambda-(\alpha+1)\sum_{i=1}^{n}\log\left(\lambda+x_i^{\beta}\right)+(\beta-1)\sum_{i=1}^{n}\log x_i +\sum_{i=1}^{n}\log\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right) \tag{37}$$

Taking first-order partial derivatives of the equation (37) to find the unknown parameters,

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha}+n\log\lambda-\sum_{i=1}^{n}\log\left(x_i^{\beta}+\lambda\right)+\sum_{i=1}^{n}\frac{p\log\lambda-p\log\left(x_i^{\beta}+\lambda\right)}{\lambda\left(1-p\left[1+\alpha\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)}=0 \tag{38}$$

$$\frac{\partial\log L}{\partial\beta}=\frac{n}{\beta}-(\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\beta}\log x_i}{\left(x_i^{\beta}+\lambda\right)}+\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\frac{\alpha p x_i^{\beta}\log x_i\left(x_i^{\beta}+\lambda\right)^{-1}}{\left(1-p\left[1+\alpha\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)}=0 \tag{39}$$

$$\frac{\partial\log L}{\partial\lambda}=\frac{n\alpha}{\lambda}-\sum_{i=1}^{n}\frac{(\alpha+1)}{\left(x_i^{\beta}+\lambda\right)}-\sum_{i=1}^{n}\frac{\alpha p x_i^{\beta}\left(x_i^{\beta}+\lambda\right)^{-1}}{\lambda\left(1-p\left[1+\alpha\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)}=0 \tag{40}$$

$$\frac{\partial \log L}{\partial p} = -\sum_{i=1}^{n} \frac{\left[1+\alpha \log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]}{\lambda\left(1-p\left[1+\alpha \log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)} = 0 \tag{41}$$

Then the maximum likelihood estimates of the parameters $\alpha, \beta, \lambda,$ and p can be obtained by solving the partial differential equations in (38) to (41). The Fisher information I_{ij} matrix for RTPL distribution is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}$$

$$I_{11} = E\left[-\frac{\partial^2 \log L}{\partial \alpha^2}\right], \; I_{22} = E\left[-\frac{\partial^2 \log L}{\partial \beta^2}\right], \; I_{33} = E\left[-\frac{\partial^2 \log L}{\partial \lambda^2}\right], I_{44} = E\left[-\frac{\partial^2 \log L}{\partial p^2}\right]$$

$$I_{12} = I_{21} = E\left[-\frac{\partial^2 \log L}{\partial \alpha\, \partial \beta}\right], \; I_{13} = I_{31} = E\left[-\frac{\partial^2 \log L}{\partial \alpha\, \partial \lambda}\right], \; I_{14} = I_{41} = E\left[-\frac{\partial^2 \log L}{\partial \alpha\, \partial p}\right],$$

$$I_{23} = I_{32} = E\left[-\frac{\partial^2 \log L}{\partial \beta \partial \lambda}\right], I_{24} = I_{42} = E\left[-\frac{\partial^2 \log L}{\partial \beta\, \partial p}\right], I_{34} = I_{43} = E\left[-\frac{\partial^2 \log L}{\partial \lambda\, \partial p}\right]$$

The Fisher information matrix can be obtained by deriving the second-order partial of the log-likelihood function in equation (37) for unknown parameters. So we obtain the asymptotic $100(1-\alpha)\%$ confidence intervals for the unknown parameters of RTPL α, β, λ, and p can be easily obtained by using the equation given below

$$\alpha \in \left[\hat{\alpha} - z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}}, \hat{\alpha} + z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}}\right], \beta \in \left[\hat{\beta} - z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}, \hat{\beta} + z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}\right] \text{ and}$$

$$\lambda \in \left[\hat{\lambda} - z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}, \hat{\lambda} + z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}\right], p \in \left[\hat{p} - z_{\frac{\alpha}{2}}\sqrt{I_{44}^{-1}}, \hat{p} + z_{\frac{\alpha}{2}}\sqrt{I_{44}^{-1}}\right]$$

Where $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ quantile of the standard normal distribution.

5. MONTE CARLO SIMULATION

This section deals with the simulation study by generating the samples from the proposed distribution. The idea behind the Monte Carlo simulation is to generate a series of experimental samples using the random number sequence and it creates a fluctuating convergence process. The inverse transformation method is the most commonly used technique to generate random variates of the distribution. If a random variates R follows a uniform distribution with [0,1], the random variates $X = F^{-1}(R)$ have a continuous cumulative probability distribution $F(X)$. In this case, the inverse function is defined as

$$X = F^{-1}(R) = \min\{x : F(x) \geq R\}; \quad \text{for } 0 \leq R \leq 1$$

The procedure for generating random variates using the inverse transformation method is

Step 1: Generate a uniformly distributed random number sequence R between the interval [0,1].

Step 2: Calculate the random variates X of the RTPL distribution by using the equation given below,

$$x_p = \left(\lambda e^{-\frac{1}{\alpha}\left(W_{-1}\left(\frac{u-1}{pe^{1/p}} \right) + \frac{1}{p} \right)} - \lambda \right)^{1/\beta}$$

We study the performance of MLE of the RTPL distribution by conducting various simulations for different sample sizes and different parameter values. After generating random samples, it can be used to obtain the mean estimate, average bias, and root mean square error of the maximum likelihood estimators of the distribution.

a) Mean estimate of the MLE $\hat{\upsilon}$ of the parameter $\upsilon = \alpha, \beta, \lambda, \text{ and } p$:

$$\frac{1}{N}\sum_{i=1}^{N} \hat{\upsilon}$$

b) The average bias of the MLE $\hat{\upsilon}$ of the parameter $\upsilon = \alpha, \beta, \lambda, \text{ and } p$:

$$\frac{1}{N}\sum_{i=1}^{N} (\hat{\upsilon} - \upsilon)$$

c) Root mean squared error of the MLE $\hat{\upsilon}$ of the parameter $\upsilon = \alpha, \beta, \lambda, \text{ and } p$:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N} (\hat{\upsilon} - \upsilon)^2}$$

Table 1. Average Bias, Root mean square error of the estimates based on MLE by Monte Carlo Simulation of RTPL distribution for different sample sizes.

N	**Parameter**	$Case\ I: \alpha = 3, \beta = 2.5, \lambda = 4, p = 0.5$			$Case\ II: \alpha = 3, \beta = 2.5, \lambda = 4, p = 1$		
		Mean	AB	RMSE	Mean	AB	RMSE
25	α	4.63043	1.63043	6.54560	4.13564	1.13564	4.42860
	β	2.87287	0.37287	0.91905	3.27358	0.77358	1.02376
	λ	7.87230	3.87230	12.0427	12.8557	8.85574	18.3768
	p	0.36628	-0.13371	0.32787	0.56351	-0.43648	0.63158
50	α	4.79498	1.79498	6.37452	3.21649	0.21649	2.91639
	β	2.65200	0.15200	0.56537	3.35949	0.85949	1.07604
	λ	8.58040	4.58040	13.3039	9.45730	5.45730	13.0895
	p	0.35866	-0.14134	0.31642	0.63158	-0.36841	0.44482
75	α	4.70200	1.70200	5.25124	2.85943	-0.14056	2.78340
	β	2.57257	0.07257	0.47150	3.39907	0.89907	1.08594
	λ	8.06813	4.08045	11.2990	8.56134	4.56134	12.5419
	p	0.39320	-0.10679	0.31475	0.63080	-0.36919	0.45047
100	α	4.88606	1.88606	5.17449	2.58303	-0.41696	2.06095
	β	2.52120	0.02120	0.42337	3.39977	0.89977	1.03959
	λ	8.08045	4.06813	10.4641	7.72486	3.72485	10.9990
	p	0.39184	-0.10816	0.31473	0.63409	-0.36590	0.44544

Table 2. Average Bias, Root mean square error can be obtained by Monte Carlo Simulation of RTPL distribution for different sample sizes.

n	Parameter	$Case\ I: \alpha=0.5, \beta=0.6, \lambda=0.4, p=0.1$			$Case\ II: \alpha=0.5, \beta=0.6, \lambda=0.4, p=1$		
		Mean	AB	RMSE	Mean	AB	RMSE
50	α	0.63005	0.13004	0.46860	0.46542	-0.03457	0.27323
	β	0.66297	0.06297	0.24380	0.39145	0.29992	0.29433
	λ	0.50398	0.10398	0.78359	0.34823	1.39085	2.27508
	p	0.29443	0.19443	0.35607	0.34648	-0.51504	0.69344
100	α	0.55927	0.05927	0.22920	0.69993	-0.10854	0.16292
	β	0.63287	0.03287	0.15135	0.75728	0.25728	0.25213
	λ	0.37730	-0.02269	0.30049	0.81821	0.86313	1.28071
	p	0.29320	0.19320	0.36410	0.83007	-0.59811	0.62799
500	α	0.52201	0.02201	0.10083	1.79085	-0.15176	0.16203
	β	0.60772	0.00772	0.05899	1.26313	0.21821	0.24087
	λ	0.34190	-0.05809	0.15553	0.79885	0.39885	0.66036
	p	0.24686	0.14686	0.32035	0.65827	-0.39769	0.48515
1000	α	0.51573	0.01573	0.07803	0.40188	-0.15352	0.15928
	β	0.60490	0.00490	0.04176	0.48495	0.20107	0.24376
	λ	0.34769	-0.05230	0.12756	0.60230	0.25827	0.46895
	p	0.21904	0.11904	0.28422	0.66413	-0.33586	0.40010

Table 1 shows the simulation study is repeated for $N = 1000$ times each which has its sample size is given by $n = 25, 50, 75, 100$ and for two different cases such parameter values are shown as $Case\ I: \alpha = 3, \beta = 2.5, \lambda = 4, p = 0.5$, and $Case\ II: \alpha = 3, \beta = 2.5, \lambda = 4, p = 1$. Table 2 describes the simulation study is repeated for $N = 10000$ times each with sample size n = 50, 100, 500, 1000 and by taking parameter values $Case\ I: \alpha = 0.5, \beta = 0.6, \lambda = 0.4, p = 0.1$ and $Case\ II: \alpha = 0.5, \beta = 0.6, \lambda = 0.4, p = 1$. In the simulation study, we present the mean, average bias, and RMSE values of the parameters $\alpha, \beta, \lambda,$ and p for different sample sizes. From the results, we can verify that as the sample size n increases, the RMSEs decay toward zero. The average bias for the parameters is slightly larger for small to moderate sample sizes but tends to get smaller as the sample size n increases. We also observe that for all the parametric values, the bias decrease as the sample size n increases. Hence the ML estimates of RTPL distribution are consistent and efficient.

6. APPLICATIONS

In this section, we consider two real data sets for illustrating the suitability of the RTPL distribution in real-time applications, the first data set consists of the breaking stress of carbon fibers with the length of 50mm, and the second data involves the exact failure time of Kevlar 373/epoxy that is subject to constant pressure can be discussed by using the maximum likelihood method of estimation and goodness of fit test. The model selection is carried out by using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), and the CAIC (consistent Akaike information criteria).

$$AIC = -2L(\hat{\theta}) + 2q$$

$$BIC = -2L(\hat{\theta}) + q\log(n)$$

$$CAIC = -2L(\hat{\theta}) + \frac{2qn}{n-q-1}$$

Where $L(\hat{\theta})$ denotes the log-likelihood function evaluated at the MLEs, p is the number of parameters, and n is the sample size. Here, θ denotes the parameters $\theta = \alpha, \beta, \lambda, p$. An iterative procedure is applied to solve the equations (38), (39), (40), and (41). The model with minimum AIC (or BIC, CAIC) values is chosen as the best model to fit the given data sets.

Data Set 1: The data set contains exact times of failure. More precisely, it consists of the life of fatigue fracture of Kevlar 373/epoxy that is subject to constant pressure (at the 90% stress level) until all had failed. Analysis of this data set can also be found in [16]. These data are listed as: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773,1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 3. Summary of statistics for Data set 1.

n	**Minimum**	Q_1	**Median**	**Mean**	Q_3	**Maximum**
69	0.0251	0.8645	1.5728	1.5675	2.0903	3.7455

Table 4. The Parameter estimates of the RTPL distributions for Data set 1.

Probability Models	**Parameter Estimates**			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{p}$
RTPL	142.646	1.25041	134.721	0.84430
Expo-Lomax	14.3067	2.973127e+02	3.299822e-03	-
H-L	4.669135e+02	1.965831e-03	-	-
Lomax	4569.5	6914.7	-	-

Table 5. The log-likelihood, information criteria, and Goodness of fit test for Data set 1.

Accuracy Measures	**Probability Models**			
	RTPL	**Expo-Lomax**	**H-L**	**Lomax**
-log L	87.698	91.753	93.440	100.062
AIC	183.39	189.50	190.91	204.12
BIC	192.33	196.20	195.37	208.59
CAIC	184.02	189.87	191.09	204.31
$\boldsymbol{W_n^2}$	0.1678	0.2094	0.5941	1.1240
$\boldsymbol{A_n^2}$	1.1275	1.2655	3.0836	5.5955
$\boldsymbol{D_n}$	0.1175	0.1183	0.1639	0.2221

Table 5, provides the estimated values of the parameters and likelihood values for all the fitted distributions. From this, minimum values of the information criterion represent the fitness of the new model and we conclude that the RTPL distribution is best when compared to Lomax distribution, Half-logistic Lomax (HL) [6], and Exponentiated Lomax (Expo-Lomax) [1] distributions. The test statisticsD_n, W_n^2, A_n^2 have the smallest values for the Kevlar 373/epoxy data set under the RTPL distribution when compare to other suitable models. The RTPL distribution is approximately a better model for this real dataset.

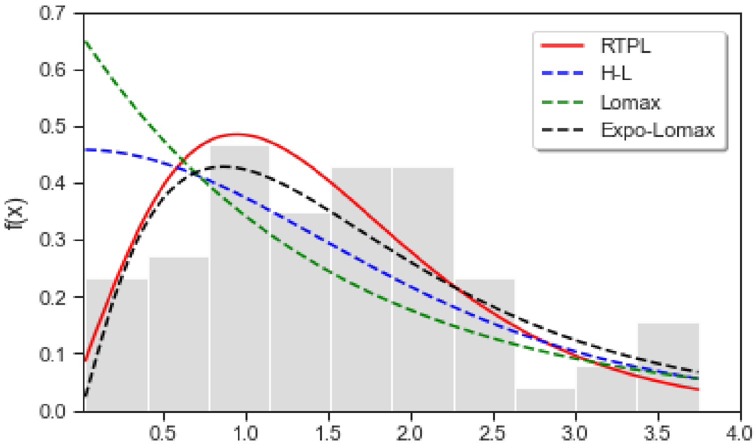

Figure 4: Estimated pdf plot for the data set 1

Figure 4 shows the fitted pdf plot, in which the histogram represents data points, and the curves show the fitness of the four comparable distributions. This plot shows that the RTPL model provides an adequate fit to the lifetime of fatigue fracture of the Kevlar 373/epoxy datasetwhen compared to the other advisory models.

Data Set 2: This dataset describes the breaking stress of carbon fibers with a length (GPA) of 50mm. The data has been taken from [17]. The data is given as follows: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.5, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88 , 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 6. Summary of statistics for Data set 2.

n	**Minimum**	Q_1	**Median**	**Mean**	Q_3	**Maximum**
66	0.390	2.178	2.835	2.759	3.277	4.900

Table 7. The Parameter estimates of the RTPL distributions for Data set 2.

Probability Models	**Parameter Estimates**				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{p}$	$\hat{\gamma}$
RTPL	98.6077	2.77527	1161.283	0.74846	-
Expo-PL	5690.105	1.11941	7.95276	-	6600.825
PL	105.423	2.08411	784.590	-	-
H-L	7.176263e+02	7.599327e-04	-	-	-
Lomax	2686.160	7663.882	-	-	-

Table 8: The log-likelihood, information criteria, and Goodness of fit test for Data set 2.

Accuracy Measures	**Probability Models**				
	RTPL	**Expo-PL**	**PLomax**	**H-L**	**Lomax**
-log L	85.5269	94.2648	98.4301	122.4363	133.0312
AIC	179.053	196.529	202.860	248.872	270.0624
BIC	187.812	205.288	209.429	253.252	274.4417
CAIC	179.709	197.185	203.247	249.063	270.2529

W_n^2	0.06810	0.38229	1.8567	12.206	13.422
A_n^2	0.40351	1.9864	8.6186	2.4891	2.7094
D_n	0.07319	0.16162	0.29427	0.3269	0.3474

Table 8 presents the estimated values of the parameters for all the fitted distributions. From this we conclude that the RTPL distribution provides the best fit to the given data set when compared to Lomax distribution, Half-logistic Lomax (HL) [6], Exponentiated Power Lomax (Expo-PL) [4], and Power Lomax (PLomax) [14] distributions. The values of tests statistics such as the Kolmogorov-Smirnov D_n, Cramér-von Mises W_n^2, Anderson and Darling A_n^2can be used to measure the goodness of fit of the RTPL distribution while concerning the other models through the breaking stress of carbon fibers data. Hence, the RTPL distribution approximately provides an adequate fit for the dataset.

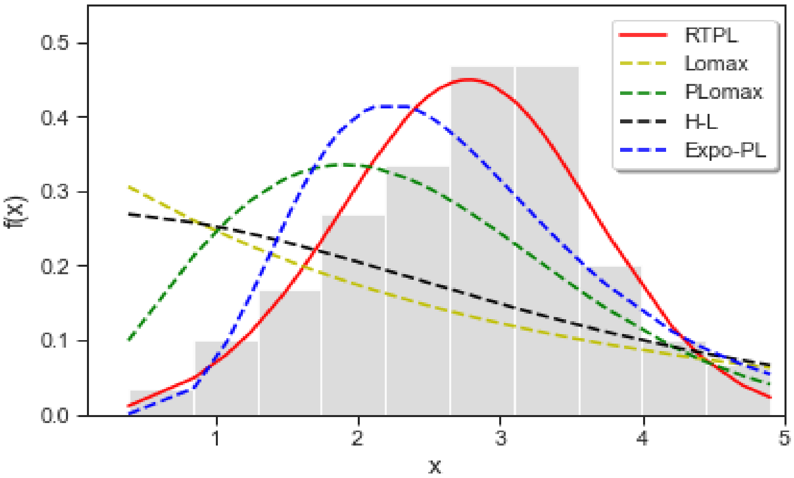

Figure 5: Estimated pdf plot for the data set 2

The fitted pdf plot is displayed in Figure 5, this plot shows that the histogram represents the data points and the curve shows the fitness of the five different distribution which is chosen for this comparative study. From this, we conclude that the RTPL model provides an adequate fit to the breaking stress of the carbon fibers data set, when compared to the other suitable models.

7. CONCLUSION

In this paper, a new extension of the four-parameter Lomax distribution is proposed and it is named as Record-based Transmuted Power Lomax distribution based on Record based transmuted map. The usefulness of this newly proposed model is illustrated by using two real data sets. This results illustrate that the proposed model provides a consistently better fit than the other existing suitable models. The graphical representation of the hazard rate of RTPL model has been explored and the obtained shapes are increasing, decreasing, and reversed J shaped. The maximum likelihood estimation method is used to estimate the unknown parameters of the RTPL distribution. The performance of the maximum likelihood estimates is investigated through the Monte Carlo simulation study to generate a random sample by using the quantile function and we observed that the proposed distribution shows a better fit when the sample size increases. The results of the Kolmogorov Smirnov test, Cramer Von Mises test, Anderson Darlings test, and important information criterions conclude that the RTPL model is provided goodness fit and emerge as better model compared to the other models. It is evident that, it has a lot of scope and real time applications in many field of science.

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