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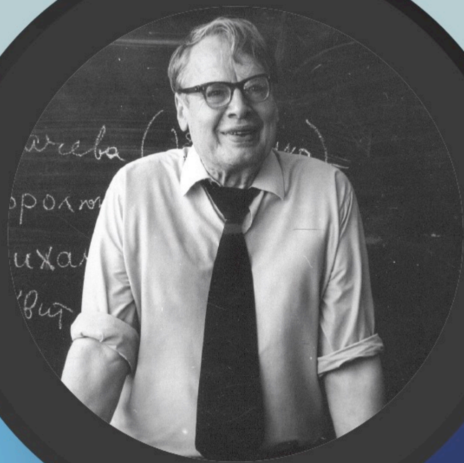

RELIABILITY

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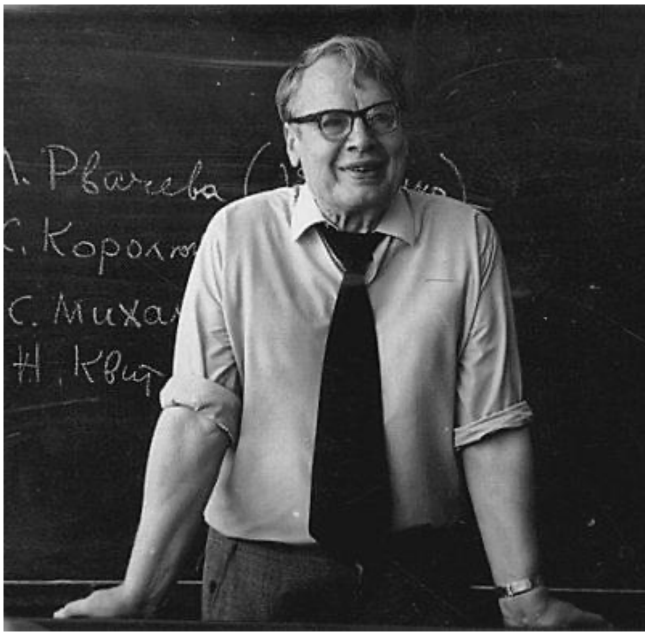

RELIABILITY:

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e-Journal ***Reliability: Theory & Applications*** publishes papers, reviews, memoirs, and bibliographical materials on Reliability, Quality Control, Safety, Survivability and Maintenance.

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Table of Contents

The system in industries is greatly impacted by failure. Eliminating these defects is therefore essential for enhancing system performance. This study aims to assess the range of repair/replacement facilities in the GDC (Gravity Die Casting) system at the Piston Foundry Plant. Two sub-units are connected to one main unit, which makes up the GDC system. Any component failure results in system failure. In this situation, the system will first attempt to be repaired, and if that is unsuccessful, it will be replaced. To operate effectively, the primary unit needs to be built of aluminium alloy (Al). Lack of raw materials is what leads to a system failing. Using semi-Markov processes and the regenerating point method, the aforementioned measurements were computed numerically and graphically. The results of this study are unusual since no prior research has concentrated on the GDC system repair/replacement facilities at piston foundries. The conclusions, according to the discussion, are very helpful for businesses who manufacture pistons and utilise the GDC system.

This paper aims to compare some bridge systems with multiple types of components in stochastic, hazard rate, and likelihood ratio order. Such systems are generally used in the designing and production industries. These systems are supported by a buffer store that balances the fluctuation in two production lines during the production process. The survival signature tool and distortion function technique are employed to compare the performance of four different bridge systems. Survival signature and henceforth survival function is computed for each considered system. The findings of comparisons are facilitated with the help of tables and figures. The comparison of large size coherent systems based on the structure-function approach is quite challenging. As this study is based on survival signature, so it is not so complex and has future scope.

In this present piece of work, we have considered a lifetime distribution based on trigonometric function called SSE(e)-distribution and discuss its various properties which have not been added previously by host as well as any other authors. This distribution is useful and a good contribution in research under trigonometric function. We are deriving some more useful properties such as moments, conditional moments, mean deviation about mean, mean deviation about median, order statistics etc. Estimation of parameter has been done for both classical and Bayesian paradigms under Type-II censored sample. Simulation study has also been carried out to know the progress of the estimators in the sense of having smallest risk (over the sample space) at the long-run use.

This paper deals with the assessment of various reliability factors of a real-life manufacturing system having inspection facility. This multistate manufacturing system have five workstations those are connected in series configuration as: W1, W2, W3, W4, W5. Workstations W2 and W4 has the configuration 2-out-of -3: G and 1-out-of-3: F. Due to failure of the any of the workstation, whole manufacturing system can completely fail. Apart from this machine failure can also make system down. To avoid sudden failure in the system pre-emptive maintenance strategy has been adopted. This is a corrective maintenance action before a failure occurs and scheduled during off days. Risk analysis is done because of fault of W5 workstation in material quality inspection. Probability distributions like exponential time distribution is followed by all failures and general time distribution by all repairs. To study the probabilistic behavior of the system in different possible transition states, Markov process have been used. Supplementary variable technique and copula method of finding joint probability distribution have been used to obtained various reliability features such as steady state behavior of the system, reliability function, availability, Mean time to failure, sensitivity analysis and profit analysis.

In reliability theory minimum of two random variables has a significant meaning, and models with increasing failure rates play a vital role. Motivated by these facts, in this article, a two-parameter lifetime distribution with an increasing failure rate is constructed by considering the method of a minimum of two independent random variables following the exponential and Teissier distributions and studied in detail. Several exciting features, such as moments, quantiles, Bonferroni and Lorenz curves, entropies, stress–strength reliability, moments of a residual lifetime, and order statistics, are derived for the proposed distribution. For the estimation purpose, eight different techniques have been used, including maximum likelihood, ordinary least square, weighted least square, Cramer-von Mises, maximum product spacing, Anderson-Darling, right-tailed Anderson-Darling, and bootstrapping (parametric and nonparametric). The performance of these estimators is compared using three real datasets. The exact Fisher information matrix elements are derived, and confidence intervals based on the information matrix and bootstrapping techniques are constructed. A simulation study is carried out to see the efficiency of the maximum likelihood in terms of mean square error and bias. Negative log-likelihood, Akaike information criteria, Bayesian information criteria, Consistent Akaike information criteria, and Hannan-Quinn information criteria are the goodness-of-fit statistics employed. Furthermore, other nonparametric test statistics such as Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises are used for model selection. Moreover, three real datasets related to epidemiology, seismology, and reliability are modeled and compared with exponential, exponentiated exponential, Lindley, exponentiated Lindley, Rayleigh, exponentiated Rayleigh, Gompertz, exponentiated Gompertz, Weibull, and exponentiated Weibull distributions to demonstrate how the suggested model performs in practice. And it is observed that the proposed distribution provides a better fit among all considered models, according to most of the test statistics. The proposed lifetime distribution is unimodal and capable of modeling positive datasets with an increasing failure rate which contains Gompertz one-parameter model as a particular case. It is a simple model with only two parameters resulting from expressions for different characteristics that are analytically tractable. So, it is expected that it will be helpful in various disciplines where such types of data exist, such as reliability, lifetime modeling, and survival analysis.

About the Teacher. Alexander D. Soloviev[1]

(September 6, 1927 - April 6, 2001)

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•
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Abstract

The article talks about a remarkable man and an outstanding scientist – Alexander Dmitrievich Soloviev. He was Doctor of Physics and Mathematics, Professor, Laureate of the State Prize of the USSR, Professor of the Probability Theory Department of the Faculty of Mechanics and Mathematics at Lomonosov Moscow State University. Alexander Dmitrievich lived an amazing creative life that can serve as an example for modern researchers. Victor Kashtanov, his student, shares his recollections and reflections on some episodes in the life of his teacher and friend.

Key words: memories, Soloviev, mathematicians, reliability assessment, redundancy
.

Such events happen in life that you only realize their significance many years later. One of such events in my life I consider my meeting with Alexander Dmitrievich Soloviev (Doctor of Physical and Mathematical Sciences, Professor, Laureate of the State Prize of the USSR, Professor of the Department of Probability Theory of the Faculty of Mechanics and Mathematics, Moscow State University named after M.V. Lomonosov, Professor, Department of Probability Theory, Faculty of Mechanics and Mathematics, Moscow State University, M.V. Lomonosov). Already, being a young specialist, working for a year after graduating from the Faculty of Mechanics and Mathematics of Moscow State University, in 1958 I went to work at Scientific Research Institute № 17 (NII-17) in the Mathematical Laboratory. At that time, A. D. Soloviev, a thirty-year-old associate professor at the Department of Mathematical Analysis of the Faculty of Mechanics and Mathematics, Moscow State University, was "moonlighting" in this laboratory (this was the former name of my part-time job). It was there that our first meeting took place. Later it turned out that we could have met earlier. I was told that he led classes in mathematical analysis in some groups of our course, but in our student group he was not. Later, my classmates talked about him as a good teacher - knowledgeable, fair, calm.

In the mathematical laboratory, Alexander Dmitrievich solved a wide variety of problems. Naturally, practical problems were solved. Consequently, solutions had to be brought to numbers, complex cumbersome formulas had to be simplified, and the accuracy of the obtained approximated results had to be evaluated. This is where his highest mathematical qualification as an analyst became apparent. In 1955 Alexander Dmitriyevich defended his PhD thesis "The problem of moments for integer analytic functions", the thesis supervisor was the corresponding member of the USSR Academy of Sciences, Professor A. O. Gelfond. It is also necessary to note fruitful cooperation of Alexander Dmitrievich with Prof. M.A. Evgrafov (see joint works published in the Doklady of the

[1] Reprinted from: Kashtanov, V.A. About the teacher. Alexander Soloviev. Dependability. - 2010. - №2. - Moscow. Publishing House "TECHNOLOGIES". - C. 71-79.

USSR). (see joint papers published in Reports of the Academy of Sciences of the USSR: "On one general criterion of basis", vol. 113, no. 3; "Determination of convergence class of interpolation series for some problems", no. 113, no. 5; "On one class of reversible operators in the ring of analytic functions", no. 114, no. 6) - an important expert on asymptotic methods (M.A. Evgrafov "Asymptotic estimates and integer functions", Moscow, 1962). Alexander Dmitrievich had a perfect command of subtle analytical methods for constructing asymptotic expansions and asymptotic evaluations, which he successfully used in the study of practical problems.

Mathematicians were also faced with the problem of studying stochastic models. At creation of radio equipment, the tasks of random processes processing, construction of estimations of correlation functions and spectral densities were solved. Tasks of evaluating reliability of developed equipment also arose. I have in front of me report "Mathematical Problems of Reliability of Radio-Electronic Equipment", signed by Aleksandr Dmitrievich (original signature) and approved by the management on March 19, 1958. The content of the report is surprising. Firstly, it is felt that there is still no unified terminology accepted in this science. Therefore, the reliability of an element is understood as the probability of failure-free operation, there is no concept of failure rate, this function is called a reliability characteristic. On the other hand, the concept of an aging element is used, estimates are constructed from below of the probability of no-failure operation of an aging element, which depend on the numerical characteristics determined by statistical tests. It is characteristic for all works of Alexander Dmitrievich – to bring mathematical formulas to practical use because the numerical characteristics can be obtained by the results of statistical tests, the estimates from below give the guaranteed value of the indicator. Reliability of systems with arbitrary structure is investigated, reliability of elements of which depends on a condition of others.

The content of a simple ordinary technical report shows how far Alexander Dmitrievich has advanced in formulating, solving, and understanding reliability problems.

In the late 50's - early 60's of the last centuries there were quantitative accumulations of results in the mathematical reliability theory, separate mathematical models under different, sometimes very significant, limitations were investigated, in a certain sense the terminology was formed, and specialists began to speak the same language and to understand each other better. Alexander Dmitrievich took an active part in forming the principles of constructing the mathematical theory of reliability. Suffice it to point out the work "Mathematical justification of the reliability theory", published in 1958 (Radioelectronic Industry, No.4).

Much later, a joint work "Mathematics and Reliability Theory" by B.V. Gnedenko and A.D. Soloviev was written (Izdatel'nye Znanie. New in Life, Science and Technology. Series "Mathematics, Cybernetics". №10. 1982), in which the authors outlined the history and their participation in this process.

In 1960, the leadership of the laboratory (Yuri Alexandrovich Arkhangelsky, later Doctor of Physical and Mathematical Sciences, Professor of the Department of Theoretical Mechanics at the Faculty of Mechanics and Mathematics, Moscow State University) offered us young specialists, who had worked in the laboratory for three years, to go to graduate school. To my indescribable joy, Alexander Dmitrievich agreed to be my supervisor. I hope I did not let my teacher down in the future, since I was his first student, who defended both his master's and doctoral dissertations. So, our scientific cooperation and collaboration began.

Since the early 60's Alexander Dmitrievich has been working with postgraduate students. But from the beginning of the 70's after his return from Cuba (Alexander Dmitriyevich spent several years there, being engaged in teaching and scientific work) the work with graduate students acquired a mass character. It should be noted that more than 30 postgraduate students under the guidance of Alexander Dmitrievich defended their doctoral theses. Some of them later became Doctor of Science. Let me mention the names of Doctor of Physical and Mathematical Sciences, Professor O. P. Vinogradov, Doctor of Physical and Mathematical Sciences, Professor A. M. Zubkov, Doctor of Physical and Mathematical Sciences, Professor O. Sakhobov. Alexander Dmitrievich never left his graduate students to the mercy of fate. His help was concrete and substantial. He spent a lot of time talking with a graduate student, showing ways to solve the problem, correcting mistakes.

Criticism was always benevolent. He was generous in imparting knowledge and new results. Suffice it to say that I had no joint work with my teacher until 1983.

If we adhere to chronology, then the beginning and the middle of the 1960s contain the formation of the remarkable scientific team of B.V. Gnedenko, Yu.K. Belyaev, A.D. Soloviev, organization of the seminar on mathematical reliability theory at the Mechanical and Mathematical Faculty of MSU, organization of the Reliability Study and cycles of lectures at the Polytechnic Museum on reliability and progressive methods of quality control of products. Of special note is the writing of the monograph "Mathematical Methods in the Reliability Theory" in 1965 (Moscow. Nauka.1965).

Writing a monograph summarizing the results of the development of the mathematical theory of reliability was an overdue necessity. Now, evaluating the appearance of this book, we can say that it has shaped domestic mathematical reliability theory and determined further ways of its development. It is a milestone in development of the domestic mathematical theory of reliability.

Boris Vladimirovich Gnedenko defined in this monograph the subject of the mathematical theory of reliability, highlighting the life cycles of complex technical systems, defining the theoretical basis of the theory and the ultimate practical tasks facing it.

"A general scientific discipline that studies general methods and techniques to be followed in designing, manufacturing, accepting, transporting and operating products to ensure maximum efficiency in the process of their use, as well as developing general methods of calculating the quality of devices according to the known qualities of their components" - this is the definition by B. V. Gnedenko.

The book, published in 1965, on the one hand, summed up the development of the domestic mathematical theory of reliability, on the other hand, determined the further directions of development of this theory, defined the relationship of the mathematical theory of reliability with the classical probability theory, the theory of random processes with the theory of mass service, mathematical statistics.

On this basis, the following areas were intensively developed at the time of writing:

- Problems of predicting reliability and durability, studies of distributions of positive random variables and their properties (aging and aging distributions),
- reliability characteristics of various structures (systems) under given distributions of no-failure times of their individual parts (study of distributions of functions from a set of random quantities) for restorable and non-restorable systems,
- evaluation of reliability characteristics based on test results, construction of various test plans.

At the stage of theory formation these very directions were considered by the authors to be the main ones. Therefore, basing on the conception of formation of mathematical reliability theory proposed by B. V. Gnedenko, the authors set forth in the monograph modern (at that time) results on estimation of reliability characteristics by test results (Y. K. Belyaev) and on research of reliability characteristics of different structures with elements of operation - restoration of failed elements (A. D. Soloviev).

In the sections of the monograph written by Alexander Dmitrievich the strengths of his mathematical and analytical qualifications were evident. Yu. K. Belyaev wrote about it in "Reliability" magazine (№4, 2006). He meant the time when their joint work had not yet begun. Let us cite this quote: "At that time, I learned from V.A. Kashtanov that very similar problems interested Alexander Dmitrievich Soloviev, whom Victor Alexeevich considered (and it was in fact) an unsurpassed virtuoso of asymptotic methods of mathematical analysis.

To the period of the early 1960s we start negotiations of Alexander Dmitrievich with Andrey Nikolayevich Kolmogorov about his transfer to the "Probability Theory" department. When this transfer took place, a group of authors was formed, which created a classical work called "Mathematical Methods in Reliability Theory".

Much later, reviewing his scientific work, Alexander Dmitrievich wrote in a letter to Igor Nikolayevich Kovalenko about the stages of his scientific activity (we will cite this excerpt in full):

"My scientific work went through several stages:

1. 1960s-70s. The construction of a mathematical theory of reliability;
2. 1970-80s. Creation of asymptotic theory, which allows to estimate the reliability of restored systems under small load. The main thing here is the proof of *the limit theorems of the uniform type*, in which in the limit transition all parameters and functions defining the system change, and the limit transition itself is defined by some small functional;
3. 1980s-90s. The transition from limit theorems to asymptotically exact bilateral inequalities from which, in particular, the limit theorems themselves follow;
4. I've had several themes in recent years:
   - The study of restorable systems with arbitrary service disciplines. Finding asymptotically optimal disciplines;
   - Reliability assessment of restorable systems with high redundancy and finite load."

As can be seen from the above quote, these periods have no clear boundaries, they overlap. For us it will be important to highlight the main ideas and achievements of Alexander Dmitrievich in these years. In the same letter Alexander Dmitrievich indicates his monograph "Mathematical Methods in the Reliability Theory" as the main work of the first period, which was republished many times in different countries. There are 7 editions of this monograph in Moscow, Berlin, Bucharest, New York, Budapest, and Japan during 1965-1972.

In addition, more than 40 articles were written during this period. If we evaluate in general the places of publication and the nature of publications, we can trace a deep interest of practitioners (industry representatives) in the theoretical research of mathematicians. The main publications that published Alexander Dmitrievich's works are departmental journals: "Radioelectronic Industry," "Problems of Radioelectronics," and "Automation and Computer Engineering. Three articles were published in the collected articles "Cybernetics to the service of Communism" (1964), which collected papers delivered at three scientific seminars: on reliability theory (Dependability Section of the Scientific Council on Cybernetics under the Presidium of the USSR Academy of Sciences); on mass service theory (the Mechanical and Mathematical Department of the Lomonosov Moscow State University, the Moscow State University, the Moscow Engineering Physics Institute, the Central Research Institute of the USSR Academy of Sciences); and on mass service theory (the Department of Physics and Technology of the Moscow State University). A seminar on the theory of reliability was held jointly by the Department of Mechanics and Mathematics, Lomonosov Moscow State University, and the Popov Radio Engineering and Telecommunications Scientific Research Institute. A.S. Popov). The academic journals "Proceedings of the USSR Academy of Sciences, Technical Cybernetics" and "Proceedings of the USSR Academy of Sciences, OTN, Power Engineering and Automation" also published Alexander Dmitrievich's works of that period.

For the second and third periods of AD's creative activity we refer to the monograph "Questions of the Mathematical Theory of Reliability", Moscow, Radio and Communications, 1983. In the preface to this monograph B.V. Gnedenko, describing the sections belonging to Alexander Dmitrievich, wrote: "The author managed to find an elegant manner of exposition, which allowed him to put an extensive material in a comparatively small volume. It is also noteworthy that the author does not make assumptions about the exponential distribution of the duration of no-failure operation or recovery time, and he managed to obtain general results under very broad assumptions. It is also important that in the limit theorems he obtained very accurate bilateral inequalities, which can be successfully used in practical situations.

Note that in addition to the "traditional" sections and research directions written by A. D. Soloviev and Yu. K. Belyaev, other sections appeared in the 1983 book. I. N. Kovalenko complemented the material with the subsection "Methods of statistical modeling" (the chapter "Multidimensional Markov Processes that describe complex systems and their statistical modeling" and the chapter "Analytical-statistical method of calculation of characteristics of high-reliability systems"). There also appeared the section "Problems of Optimization of Reliability and Efficiency of Functioning" written by E.Yu. Barzilovich, V.A. Kashtanov and I.A. Ushakov. If in the initial works the quality (efficiency) was estimated by distribution of the time of no-failure operation or by mathematical expectation of this time, then in the later works other indicators were investigated, which were defined as functionals built on trajectories of random processes describing evolution of a technical system. The solution of the problem was completed by the optimization of these functionalities (indicators).

From the mid-1970s to the mid-80s, Alexander Dmitrievich published about 30 works, many of them written jointly with his graduate students.

Note that Alexander Dmitrievich publishes his results during this period in the "Znanie" publishing house. This is since at the end of the 1960s the reliability cabinet begins to work on the premises of the Polytechnical Museum in Moscow, where the seminar on reliability and progressive methods of quality control of products, where mathematicians consult for industry representatives on practical problems of reliability arising during development of various apparatuses, is held. In the large auditorium of the Polytechnic Museum cycles of lectures on reliability for engineers are organized. The materials of the lectures are published in separate brochures. These were reviews of mathematical methods in reliability theory, the volume of materials was 40-50 pages. Let us point out three issues under the title "Fundamentals of the mathematical theory of reliability" (Moscow, Znanie, 1975). Of course, the mathematical results are adapted to the audience, but this gives the theory a practical orientation. Communication with the engineering audience allowed us to feel the applied problems, to describe new models, to formulate new problems. (Note that Alexander Dmitrievich had a great experience of delivering lectures at the Faculty of Mechanics and Mathematics of Moscow State University for specialists with higher engineering education and wishing to improve their mathematical qualification, for the so called "engineering stream"). Let's give the name of one of Alexander Dmitrievich's works, published after his lecture course for engineers: "Heuristic derivation of reliability characteristics of standby systems with fast restoration" (Moscow, Znanie, 1968). Such approach of Alexander Dmitrievich to the presentation of the material testifies to his desire to give practitioners a tool that could be easily used when solving practical problems.

The practical orientation of Alexander Dmitrievich's mathematical works is constantly traced. For him, the main task was not only to get some dependence (formula, equation, limit theorem), but also to show how this result can be used, to bring the research to number and to develop practical recommendations. He sought to formulate simple sufficient conditions, the verification of which allowed this mathematical result to be used in practice.

As noted above, Alexander Dmitrievich's research in the 80-90s dealt with subtle issues of asymptotic analysis of mass service and reliability models. He solves more complicated problems of constructing asymptotically accurate bilateral estimators, which allow not only to obtain the limiting values of the characteristics under study, but also to determine the convergence rates. The solution of these problems is related to the problem of summing up a random number of random terms. In his book "Boris Gnedenko in Memoirs of Students and Associates" (URSS. Moscow, 2006) I. N. Kovalenko wrote: "Boris Gnedenko's great merit was introducing into the mathematical theory of reliability the methods of summation theory of independent random variables. This stimulated the creation of a new direction - the limit theorems of the theory of redundant systems in the "triangular" scheme. The greatest contribution to the development of this direction was made by A.D. Solov'ev and his students...".

The works devoted to the construction of asymptotically exact bilateral estimates of the characteristics under study were mentioned above. In a certain sense, the chapters written by Alexander Dmitrievich in his 1983 monograph are a milestone work. These materials summarized the results of the studies that began with the publications of 1976-1977.

There are numerous examples in this book in which these two-way estimates can be used:

- Loaded duplication with recovery;
- Lightweight duplication with prophylactics;

- Temporary Reservation;
- Loaded duplication with recovery.

Let's mention works written together with O. Sahobov "Two-sided estimations of reliability of restored systems" (Izvestiya AS UzSSR, series Physics-Mat, ¹5, 1977), "Two-sided estimations of reliability in general redundancy model with one repair unit" (Izvestiya AS USSR. Technical Cybernetics, ¹4, 1977), "Two-sided estimations for system failure probability on one period of regeneration" (Izvestiya AS UzSSR, series Physics-Mat, ¹2, 1977).

In the 90's Alexander Dmitrievich published several works on reliability estimation of different systems. In collaboration with D.G. Konstantinidis he wrote the paper "Uniform reliability estimation of a complex restorable system with unlimited number of repair units" (Vestnik (Herald) of MSU, Series Mathematics, Mechanics. No.3, 1991), "Reliability Assessment of a Complex Reconstruction System with Unlimited Number of Repair Units" (Probability Theory and its Applications, vol. 37, issue 1, 1992), "Reliability Assessment of a Cold Reserving with Restoration Model in Case of Unlimited Number of Repair Units" (jointly with A. P. Polyakov, Vestnik (Herald) of MSU, Math. No.5, 1992), "An Estimation of the Average Lifetime of Reconstructed Systems" (jointly with N.G. Karaseva, Vestnik (Herald) of MSU, Math. №5, 1998).

All of Alexander Dmitrievich's co-authors mentioned in the latter papers were his graduate students, so it is obvious that all the mathematical ideas presented in these papers belong to his supervisor.

Researchers have long seen the connection between reliability models and mass maintenance models. Alexander Dmitrievich, investigating reliability models of restorable systems, devoted several his works to the analysis of various maintenance and restoration disciplines.

These works solve the problem of finding optimal recovery (maintenance) disciplines, which is fundamentally important from a practical point of view. One of the first works in this direction was the article "Optimal maintenance of restoring systems" (together with V.V. Kozlov. Izvestia of the Academy of Sciences of the USSR. Technical Cybernetics, Nos.3,4, 1977). This was followed by the paper "On a System with Maintenance Discipline of the First Demand with Minimum Remaining Length" (jointly with A. V. Pechinkin and S. F. Yashkov. Izvestia of the Academy of Sciences of the USSR. Technical Cybernetics, No.5, 1979). Let us also note the paper "Analysis of M/G/1/∞ system for different service disciplines" (The Theory of Mass Service. Proceedings of the All-Union School-Seminar. M. VNIISI, 1881), in which a wide range of service disciplines is investigated, a review of the results is given, and a list of characteristics obtained in closed form for various disciplines is given.

Besides his works devoted to the analysis of mass-service and reliability models, Alexander Dmitrievich wrote some historical and mathematical works. Let us mention the works related to the history of asymptotic methods of analysis, devoted to the problems very close to his scientific interests. These include the work "On the History of the Creation of the Passage Method" (published jointly with S. S. Petrova. SPb. These include: "On the History of the Creation of the Passage Method" (together with S. S. Petrova, SPb), "Historical and Mathematical Studies, Vol. 35, 1994", "P. A. Nekrasov and the Central Limit theorem of Probability Theory" (M. Historical and Mathematical Studies, Second Series, Issue 2(37), 1997), and "Asymptotic Methods of Laplace" (M. Historical and Mathematical Studies, Second Series, Issue 4(39), 1999).

Next, let us proceed to characterize the content of works on the mathematical theory of reliability and the ideas embedded in them.

The first works concerned the construction of probabilistic characteristics of reliability of systems by the characteristics of its individual parts. In other words, it was about the study of functions from random variables. However, when studying the process of functioning, when the model includes restoration of failed subsystems, it becomes necessary to consider the evolution of the system in time. Therefore, random processes are used to describe the model. In his first works, Alexander Dmitrievich used Markov processes. Regarding one of them (1964) it was written: "A. D. Soloviev's large work "On Reserving without Recovery" is a serious study, where the theory of reserving is systematically outlined, and many questions are far advanced based on random processes of death and reproduction" (see material from the editors in the article "Cybernetics - to the Service of Communism").

Alexander Dmitrievich sought to set and solve problems as much as possible under general assumptions with respect to the initial assumptions. Where it was possible to abandon some special assumptions (for example, exponentiality of some initial distributions), the problem was solved in general assumptions. With this approach, it is very rare to get a closed-form result. Then Alexander Dmitrievich's high erudition as a specialist in asymptotic methods worked. In his works, the terms asymptotic distribution, rare event, fast recovery, and the like appear. And all this in the study of reliability models. Here are the titles of some of these works: "Asymptotic Distribution of the Lifetime of a Duplicated Item" (Proceedings of the Academy of Sciences of the USSR, Technical Cybernetics, No. 5, 1964), "One Combinatorial Identity and its Application to the Problem of the First Occurrence of a Rare Event" (Probability Theory and its Applications, vol. XI, vol. 2, 1966), "Reserving with Fast Recovery" (Izvestia AS USSR, Technical Cybernetics, No.1, 1970), "Asymptotic Behavior of the Moment of the First Occurrence of a Rare Event in a Regenerating Process" (Izvestia AS USSR, Technical Cybernetics, No.6, 1971), "Asymptotic Analysis of Post-Failure Reliability Characteristics" (Proceedings of III All-Union School Meeting on Mass Service Theory, vol. 1, MSU, 1976).

Several works on asymptotic analysis dealt with mass service models. The relationship between reliability models and mass service models was mentioned above.

In 1972 Alexander Dmitrievich successfully defended his doctoral thesis "Systems of mass service with fast service" in the council of the Faculty of Mechanics and Mathematics of Moscow State University. Here is a citation from the abstract of this thesis, which fully demonstrates the characteristic features of Alexander Dmitrievich as a mathematician. Here is what he wrote in the thesis abstract.

Let us note two characteristic features of the work:

1. Almost everywhere the limit theorems have a uniform form, in other words, all the initial distributions and parameters change in the limit transition, and the topology of the limit transition is given by some small functional on distributions and parameters;
2. Each limit theorem looked for the most effective conditions, that is, conditions expressed explicitly and quite simply through the initial characteristics.

The study of numerous specific models of mass service and reliability eventually made it possible to develop a general basic mathematical model of a random process describing the evolution of the system under study, which can be used to judge the efficiency of its functioning.

It turns out to be a *regenerating random process* for which some event may occur at some point during the regeneration period. In specific models this event can be treated as the first loss of demand, system failure, etc. adverse events. Already in the 1983 monograph we find paragraphs and sections "Limit theorems for regenerating processes, exact distribution of the moment of the first event occurrence, regenerating processes of special type, estimation of event occurrence probability".

To demonstrate Alexander Dmitrievich's profound ideas, let us analyze the article "One General Model of Redundancy with Restoration," written jointly with D.B. Gnedenko (Izvestia of the Academy of Sciences of the USSR. Technical Cybernetics, No. 6, 1974).

In this paper we investigate a system consisting of *n+1th* element. During failures the elements are restored. There are r repair crews for repair, the duration of repair are independent random variables with an arbitrary distribution *G(x)*. In addition, with respect to the structure of the system, it is assumed that there are *n-r* places to wait for repairs. If at time *t* there are *k* failed elements in the system, $\xi(t) = k$, then the next failure appears after a random time distributed by the exponential law with the parameter λ_k . The failure of the system occurs at the moment of failure *of n+1th* element, $\xi(t) = n + 1$.

It is easy to see that the described reliability model coincides completely with a mass service system having r serving devices, *n-r* queue places, and for which the intensity of the input flow depends only on the number of demands in the system. An adverse event is the first loss of a demand.

Since service times are distributed arbitrarily, and several demands can be served simultaneously, the random process $\xi(t)$ - number of demands in the system at time t does not have good properties (such as the Markov property), which allows to involve known mathematical methods. The only property that can be used is the regeneration property. Moments of regeneration are moments of release of the system from requirements. Regeneration periods have two components – a free (random) period with an exponential distribution and an occupancy period when there are requirements in the system.

For such a regenerating process the theorem is proved

$$\lim_{\lambda_0 T_1 \to 0} P\{\lambda_0 q\tau > x\} = e^{-x},$$

where q *is the* probability of claim loss at one regeneration period, T_1 is the mathematical expectation of the employment period,τ is the moment of the first claim loss.

However, it is difficult to use the theorem in the presented form because it is necessary to express through the initial characteristics the mathematical expectation of the employment period and the probability of claim loss at one regeneration period.

Therefore, simpler sufficient conditions for convergence to the exponential distribution are formulated:

$$\text{If } T = \int_0^\infty x dG(x) \to 0 \;\; \text{then} \;\; P\{\lambda_0 q\tau > x\} \to e^{-x}$$

Note the original method of proving this theorem - the construction of a *majority process*. We construct a random process $\bar{\xi}(t)$ - number of demands at time t for a single-channel mass service system with an infinite queue, which receives a Poisson flow of demands with parameter $\lambda = max_{0\le k\le n} \lambda_k$. It is argued that the process $\bar{\xi}(t)$ majorizes the process$\xi(t)$ in the sense that any realization of the process $\xi(t,\omega)$ is not superior to the *corresponding* realization of the process $\bar{\xi}(t,\bar{\omega})$,$\xi(t,\omega) \le \bar{\xi}(t,\bar{\omega})$. Here we need to clarify what the correspondence of the realizations of the two processes means. The realizations of random processes $\xi(t,\omega)$ *u* $\bar{\xi}(t,\bar{\omega})$ are defined by the intervals between neighboring moments of arrival of demands $\vec{t} = \{t_k, \;\; k \ge 1\}$ and the service times of each demand $\vec{\tau} = \{\tau_k, \;\; k \ge 1\}$. If these sequences are the same, then the above inequality is fulfilled by in the second model there is one servicing device. The same inequality holds for the occupancy periods $\nu_1(\vec{t},\vec{\tau}) \le \nu_2(\vec{t},\vec{\tau})$ of the first and second models. Obviously, the occupancy period is a nonincreasing function of the intervals t_k. Therefore, similar inequalities are true for the mathematical expectations of the occupancy periods, provided that the distributions of the intervals between the moments of arrival of demands have exponential distributions with parameters $\lambda \ge \lambda_k$. This proves the theorem under simply testable conditions.

However, the question remains open about determining the probability q of the loss of the claim on the regeneration period. And here an original solution is proposed. It is proved that under certain conditions the probability q is equivalent to the probability q_0 of loss of a claim along a monotone trajectory when no claim has been served during the regeneration period (monotone trajectory method). We end up with a very nice result

$$lim_{\frac{m_{n+1}}{m_1^n} \to 0} P\{\lambda_0 I\tau > x\} = e^{-x},$$

where $I = \lambda_1\lambda_2 \ldots \lambda_n \int_0^\infty \frac{\{\int_x^\infty [1-G(y)]dy\}^{r-1} x^{n-r}}{(r-1)!(n-r)!} [1-G(x)]dx$, $m_k = \int_0^\infty x^r dG(x)$ and easily verifiable conditions for its fulfillment. Some special cases are also given at *r=n* the equality $P\{\lambda_0 q\tau > x\} \to e^{-x}$ is valid, at *r=1* the equality$I = \frac{\lambda_1\lambda_2 \ldots \lambda_n}{n!} m_n$ is valid.

This detailed analysis shows the depth of the ideas proposed by A. D. Soloviev, and the use of these ideas in the asymptotic analysis of other models testifies to their effectiveness.

Concluding these notes, I would like to say that in life Alexander Dmitriyevich Soloviev was a cheerful and benevolent man, treating any interlocutor with respect. He played the guitar, knew many stories and anecdotes, in the company was the soul of society. He liked to joke around. In 1967, at a Central Asian market in Tashkent, he would offer everyone a taste of bitter green pepper and immediately offer some fruit to eat to anyone who fell for his joke.

Time flies inexorably forward. It's been twenty-one years since Alexander Dmitrievich left us. Let's keep the memory of this wonderful man and be grateful to him for everything he did for us.

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Failure Criteria and Time over Thresholds in Them

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Abstract

A failure is one of the key concepts in dependability. Therefore, it is very important to distinguish whether a failure has occurred or not. To do this, a failure criterion is formulated. This article describes main approaches to determining failure criteria. Special attention is paid to the parametric approach, in which a failure is an event when one of the parameters characterizing the functioning of an item goes beyond the specified limits. In addition, a time over threshold can also be set. This means that short-term disruptions in item's operation are not considered as failures. The meaning of setting such a threshold is explained and examples of its use in telecommunications are given. For a parallel system with a time over threshold in a failure criterion, calculation formulas for dependability measures are derived. The errors that the use of traditional formulas gives in this situation are estimated.

Keywords: failure criterion, parametric approach, time over threshold, parallel system, MTBF, MTTR, availability

1. Introduction

The concept of a failure is one of the most important in the dependability theory. A failure of an item is defined as the loss of its ability to perform as required [1] (the terminology used in this paper mainly follows this basic international standard). In other words, a failure of an item is an event that transfers it from up to down state. Usually these two states are considered for an item in the reliability analysis: up or available state, in which it is able to perform as required, and down or unavailable state, in which it is unable to perform as required due to internal reason. Therefore, it is very important to distinguish between these two states.

As a rule, a failure criterion is introduced for this, which means a pre-defined condition for acceptance as conclusive evidence of failure [1]. The importance of the correct choice and formulation of failure criteria for reliability engineering is undeniable. In particular, one of the first popular books on reliability theory says [2, p. 14]: "We have placed great emphasis on the need for a clear-cut definition of the function of the device and its adequate performance on the one hand, and of failure or malfunction on the other".

In many works on reliability, it is assumed that the failure criterion has already been established in some way, but its exact formulation remains outside the scope of consideration. However, this can be done in various ways. Nevertheless, until now, insufficient attention has been paid to this issue, there is no sufficiently complete and clear description and analysis of possible approaches to determining a failure criterion in the literature. Perhaps the only field in which there are many publications devoted to this issue is materials science. It is easy to see by doing a Google search for the phrase "failure criterion". Even a special website is dedicated to this (https://www.failurecriteria.com).

This paper is devoted to eliminating this gap. It is organized as follows. Section 2 considers how a failure criterion can be determined. First, the two main approaches used for this are described. Then a time over threshold is introduced and explained. This means that short-term disruptions in item's operation are not considered as failures. The situations in which this may be appropriate are pointed out. The presence of such a threshold requires the correction of some well-known mathematical expressions and formulas used in reliability theory. They are discussed in the following sections. In section 3, the general mathematical model introduced in the classical monograph [3] is considered and its modification is proposed for the case of a time over threshold in the failure criterion. Section 4 explains why corrections in calculation formulas for dependability measures of a parallel system with time over threshold are required and the corrected formulas are derived. In this connection, the errors that the use of traditional formulas gives in this situation are estimated. At last, section 5 summarizes the main findings.

The presentation in section 2 is illustrated with specific examples from the field of telecommunications in which the author works. However, to understand them, the reader does not need to be an expert in this field; they will be understandable and useful to specialists in other industries. These examples are taken from the ITU-T documents. ITU-T is the Telecommunication Standardization Sector of the International Telecommunication Union (ITU). The International Organization for Standardization (ISO), the International Electrotechnical Commission (IEC), and ITU form the World Standards Cooperation. ITU standards (called Recommendations) are fundamental to the operation of today's information and communication networks.

2. How a Failure Criterion Can Be Determined

2.1. Two Approaches to Determining a Failure Criterion

There are two approaches to the formulation of a failure criterion. They have been known for a long time and were mentioned in [2, p. 14]: "…In some simple cases, where devices of the "go–no go" type are involved, the distinguish between adequate performance and malfunction is a very simple matter. <…> But there are many more cases of a nature such that a clear-cut decision cannot be made so easily and a number of performance parameters and their limits must first be specified; operation within the limits is considered adequate or satisfactory, and outside of the specified limits it is considered inadequate".

Similar considerations are presented in the classical monograph [3, p. 71]. As a typical example of an item having a well-defined failure, an electric light bulb was given in it: "The operation of light bulb has, as a rule, two states: either it gives normal illumination or it gives no illumination at all". As an example of an item with a parametric failure assignment, a resistor was considered "for which the basic parameter determining quality is the magnitude of the resistance expressed in ohms".

Thus, these two approaches to determining the failure criterion can be called "go/no-go" and parametric. Similar two approaches exist when defining the general concept of "dependability" [4]. There is also an analogy here with two inspection methods in statistical quality control: inspection by attributes and inspection by variables [5].

It is worth mentioning that formally the go/no-go failure criterion can also be set parametrically. In this case, a binary parameter is used, which takes the value 1 in up state and the value 0 in down state. This is widely used, in particular, to describe the state of a system depending on the states of its elements by means of the structural function of the system [6]. In this case, the states of the elements and the entire system are characterized by binary variables (1 or 0).

In many cases, the failure criterion can be defined as a set of several conditions connected by a logical "or", i.e. the fulfillment of any of them is regarded as a failure. Some of these conditions may be "go/no-go", others may be parametric.

As the first example, consider analogue cable transmission systems and associated equipments. According to [7], a failure of such a system is considered to occur when there is:

1) complete loss of signal;

2) one in which the pilot level drops by 10 dB below nominal value;

3) when the total unweighted noise power, measured or calculated with an integrating time of 5 ms exceeds 1 million pW on the 2500 km hypothetical reference circuit.

The first condition has the go/no-go type. The second and third conditions are parametric. Each of them uses a specific parameter (the pilot level and the total unweighted noise power, the meaning of these parameters is not important for this consideration), for which a threshold value is set.

The Recommendation [7] is quite old. For more modern digital telecommunication networks, a parametric approach is used to determine a failure criterion. In general, it was formulated in [8]. Exactly, it says that the transitions between the available (up) and the unavailable (down) states based on events which are defined as occurring when the value of a function of a primary performance parameter(s) crosses a particular threshold.

2.2. A Time over Threshold in a Failure Criterion

When determining a failure criterion, a threshold value for time can also be used. As an example, consider again the failure criterion from [7]. In its above wording, the last phrase was omitted. However, it is very important. It reads: "In all instances, this condition must last at least 10 seconds". Thus, a time over threshold is introduced here. A similar situation takes place for other telecommunication systems. Often a threshold of 10 seconds is also used for them.

In general, there may be the following reasons to use a time over threshold:

- An item may have certain inertia, and a short-term disruption in its operation has no serious negative consequences.
- Using time over threshold allows reducing the number of alarms in fault management systems [9].
- The parameter used in the failure criterion may be statistical in nature, and obtaining a representative sample for its evaluation requires some time.

The latter situation is typical for modern telecommunications, where the main performance parameters used to formulate failure criteria are statistical in nature. For example, these are parameters such as the bit error rate, frame loss ratio, packet loss ratio, etc. In many cases, such a parameter is evaluated within a one second, the resulting value is compared with a certain threshold, in case of crossing which the second is regarded as "bad" (in each case, there is a special formal name for such a second). The failure criterion is the appearance of a certain number of "bad" seconds in a row.

As an example, consider technology Ethernet, which is widely used in computer networks. In such networks, data is transmitted in units called frames. A "bad" second occurs for a block of frames observed during a one-second interval when the corresponding frame loss ratio (i.e., the ratio of lost frames to total frames in the block) exceeds 0.5 [10].

Ten consecutive "bad" seconds are considered as a failure, i.e. the transition from the available state to the unavailable state. The corresponding 10-second period of time is considered to be part of unavailable time. The reverse transition from the unavailable state to the available state occurs when ten consecutive "not bad" seconds appear. The corresponding 10-second period of time is considered to be part of available time. All this is depicted in Fig. 1.

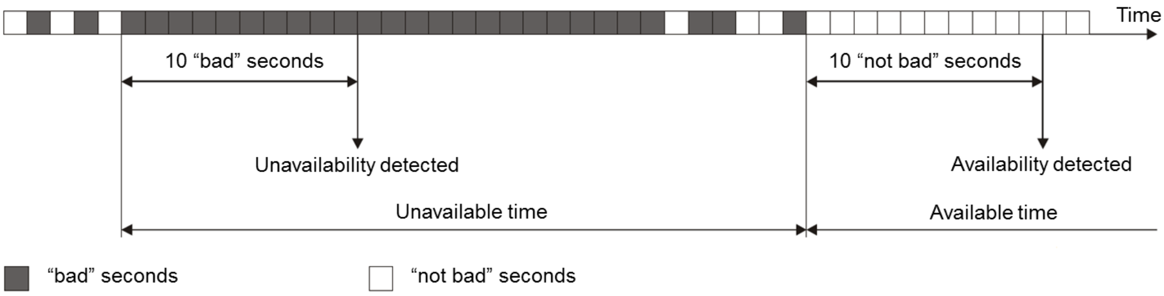

Figure 1: *Available and unavailable times with 10-second time over threshold*

3. A Time over Threshold in the General Set-Theoretic Model

In [3], a very general set-theoretic mathematical model was proposed to define and evaluate reliability measures. It is conceptual in nature and formed the basis for many further studies. It can be described as follows. Firstly, for the item under consideration, a set $S = \{x\}$ of states x is introduced that differ from each other in terms of reliability. It is called the phase space. For example, for the analogue cable transmission system discussed above, $x = (x_1, x_2, x_3)$, where x_1 is a binary variable that characterizes the presence ($x_1 = 1$) or loss ($x_1 = 0$) of the signal, x_2 is a non-negative variable equal to the pilot level, x_3 is a non-negative variable equal to the total unweighted noise power.

Then a random process with values in the phase space $x(t)$ is determined, which describes the change in the states of the item over time. Finally, the phase space S is divided into two disjoint subsets: S_1 and S_0 ($S_1 \cup S_0 = S$, $S_1 \cap S_0 = \emptyset$). If $x(t) \in S_1$, then at the moment t the item is in up state; if $x(t) \in S_0$, then at the moment t the item is in down state.

The moment of time $t^* > 0$ is the moment of failure, if and only if the following criterion is met:

$$(\exists \varepsilon > 0 \;\; \forall t \in (t^* - \varepsilon, t^*) \;\; x(t) \in S_1) \wedge (x(t^*) \in S_0). \tag{1}$$

The first condition in (1) means that immediately before the moment t^* an item was in up state, the second condition means that at the moment t^* it is in down state.

In this model, a reliability measure can be defined as the mathematical expectation of some functional $\Phi[x(t)]$ assigning numerical values to trajectories of the random process $x(t)$ [3]. For example, let

$$\Phi_1[x(t)] = \min \{ t^* > 0 \mid t^* \text{ satisfies (1)}\}. \tag{2}$$

Then, $\mathbf{E}\Phi_1[x(t)]$ is the mean operating time to the first failure ($\mathbf{E}$ is the symbol of mathematical expectation).

Another widely used measure is the reliability in the interval (t_1, t_2) (i.e., the probability of failure-free operation in this interval) $R(t_1, t_2)$. It is usually assumed that the item is in up state at the beginning of the time interval. $R(t_1, t_2) = \mathbf{E}\Phi_2[x(t)]$, where $\Phi_2[x(t)]$ is defined as

$$\Phi_2[x(t)] = \begin{cases} 1, \text{ if } \forall t \in (t_1, t_2) \;\; x(t) \in S_1; \\ 0, \text{ if } \exists t \in (t_1, t_2) \;\; x(t) \in S_0. \end{cases}$$

If there is a time over threshold in the failure criterion, the situation becomes more complicated. Indeed, the presence of the process $x(t)$ at the moment t at one or another point of the phase space no longer determines whether the item is currently in up or down state. This state depends on both the previous and future behavior of the process $x(t)$. The following are the appropriate formulations for this situation.

The phase space S is also divided into two disjoint subsets S_1 and S_0. However, in this case $x(t) \in S_1$ only means that at the moment t all the parameters used in the failure criterion are within the limits specified for them; $x(t) \in S_0$ means that at least one of these parameters has gone beyond these limits.

Denote the time over threshold by θ. Then the moment of time $t^* > \theta$ is the moment of failure, if and only if the following criterion is met:

$$(\exists t' < t^* - \theta \ (\forall t \in [t', t' + \theta] \ x(t) \in S_1) \ \wedge \ \overline{(\exists t'' \in [t' + \theta, t^*] \ \forall t \in [t'', t'' + \theta] \ x(t) \in S_0)} \ ) \ \wedge \ \wedge \ (\forall t \in [t^*, t^* + \theta] \ x(t) \in S_0). \tag{2}$$

The first and the second conditions in (2) together mean that immediately before the moment t^* an item was in up state (the overline means negation), the third condition means that starting from the moment t^* it is in down state.

The expectation of the functional $\Phi_1[x(t)]$, defined similarly to (2) with the replacement of (1) by (3), is equal to the mean operating time to the first failure. To determine $R(t_1, t_2)$, the functional $\Phi_2[x(t)]$ in this case takes the form

$$\Phi_2[x(t)] = \begin{cases} 1, \text{ if } \forall t' \in (t_1, t_2 - \theta) \ \exists t \in (t', t' + \theta) \ x(t) \in S_1; \\ 0, \text{ if } \exists t' \in (t_1, t_2 - \theta) \ \forall t \in (t', t' + \theta) \ x(t) \in S_0. \end{cases}$$

4. Calculation of Dependability Measures for a Parallel System with a Time over Threshold

The time over threshold also leads to the fact that adjustments have to be made to some well-known and widely used calculation formulas. In particular, this concerns formulas for mean operating time between failures (MTBF), mean time to restoration (MTTR) and availability of a parallel system.

Let the time over threshold θ be set for all elements and for the system as a whole. A parallel system is in down state if all its elements are in down state. However, periods of coincidence of down times of the elements can have different durations, both longer and shorter than θ. In the latter case, a system failure does not occur and such a short-term coincidence should not be considered as down time for the system. For the simplest example of a system having two parallel elements, this is shown in Fig. 2.

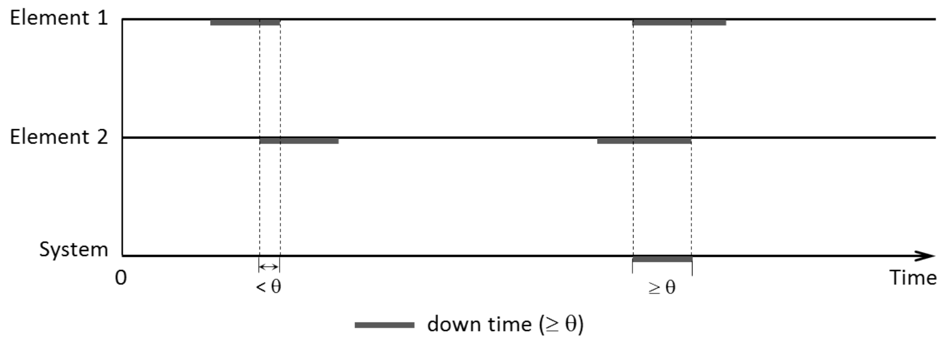

Figure 2: *Down time for a system of two parallel elements*

The traditional formulas do not take into account this circumstance. For most cases encountered in practice, the error in the calculation results will be very small. However, in order to evaluate it, it is necessary to be able to calculate dependability measures taking into account this circumstance, that is, to exclude short coincidences of downtime from consideration. The corresponding formulas will be derived below. To do this, a heuristic approximation is used, which gives good results for highly reliable systems [11, 12]. The higher the reliability, the more precise will be the result.

Consider a system of two independent parallel elements (as in Fig. 2). Let T_i and τ_i denote, respectively, the MTBF and the MTTR of the ith element, $i = 1, 2$. They are determined taking into account the time over threshold θ in the failure criteria. To apply the heuristic approximation, it is assumed that $T_i >> \tau_i$. In practice, this condition is usually met. The distribution function for the time to restoration of the ith element is denoted by $G_i(t)$. When $t \leq \theta$, $G_i(t) = 0$.

Denote by T_0 and τ_0 the MTBF and the MTTR of the system, calculated without taking into account the time over threshold, and the same measures determined taking the threshold into account are denoted by T and τ. For T_0 and τ_0 there are the following formulas [11]:

$$T_0 \approx \frac{T_1 T_2}{\tau_1 + \tau_2}, \tag{3}$$

$$\tau_0 \approx \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}. \tag{4}$$

The duration of a coincidence of elements' down times is the residual restoration time of the element that failed first, starting from the moment of failure of another element. This residual restoration time of the ith element has the density function $[1 - G_i(t)]/\tau_i$ [13]. Therefore, the probability q_i that this time for the ith element is less than θ can be calculated as follows:

$$q_i = \int_0^\theta \frac{1 - G_i(t)}{\tau_i} dt = \frac{\theta}{\tau_i}.$$

The total flow of coincidences has the rate $\lambda_0 \approx (\tau_1 + \tau_2)/(T_1 T_2)$ [9]. The coincidences in which the ith element fails first form a flow with the rate $\lambda_{0i} \approx \tau_i/(T_1 T_2)$. So, the probability that the ith element initially fails when a coincidence occurs, $\pi_i = \lambda_{0i}/\lambda_0 \approx \tau_i/(\tau_1 + \tau_2)$. Hence, for the probability that the duration of a coincidence is less than θ, we get:

$$q = \pi_1 q_1 + \pi_2 q_2 = \frac{\tau_1}{\tau_1 + \tau_2} \cdot \frac{\theta}{\tau_1} + \frac{\tau_2}{\tau_1 + \tau_2} \cdot \frac{\theta}{\tau_2} = \frac{2\theta}{\tau_1 + \tau_2}\ . \tag{5}$$

Using (3) and (5), the MTBF of the system can be calculated as follows:

$$T = \frac{T_0}{1 - q} \approx \frac{T_1 T_2}{\tau_1 + \tau_2} \cdot \frac{\tau_1 + \tau_2}{\tau_1 + \tau_2 - 2\theta} = \frac{T_1 T_2}{\tau_1 + \tau_2 - 2\theta}\ . \tag{6}$$

To compare T and T_0, their ratio is calculated. It follows from (3) and (6) that

$$\frac{T}{T_0} \approx \frac{\tau_1 + \tau_2}{\tau_1 + \tau_2 - 2\theta}\ . \tag{7}$$

Hence when $\theta << \tau_i$

$$\frac{T}{T_0} \approx 1 + \frac{2\theta}{\tau_1 + \tau_2} . \tag{8}$$

The mean duration of a short (less than θ) coincidence is θ/2. Therefore, the following equality holds:

$$\tau_0 = q(\theta/2) + (1 - q)\tau. \tag{9}$$

Expressing τ from (9) and substituting τ_0 from (4) and q from (5), we get:

$$\tau \approx \frac{\tau_1\tau_2 - \theta^2}{\tau_1 + \tau_2 - 2\theta} . \tag{10}$$

From (4) and (10), under the same condition $\theta \ll \tau_i$, an expression similar to (8) can be obtained:

$$\frac{\tau}{\tau_0} \approx 1 + \frac{2\theta}{\tau_1 + \tau_2}.$$

For example, if $\tau_i/\theta \approx 100$, not taking into account the time over threshold when calculating the MTBF and the MTTR of the system gives a relative error of about 1 %. However, if $\tau_i/\theta \approx 5$, the error will be about 20…25 %.

Availability and unavailability of the system can be calculated based on its MTBF and MTTR. In this case, it is advisable to compare the unavailability, which using (6) and (10) is expressed as

$$U = \frac{\tau}{T + \tau} \approx \frac{\tau_1\tau_2 - \theta^2}{T_1T_2 + \tau_1\tau_2 - \theta^2} . \tag{11}$$

Since $\theta < \tau_i \ll T_i$, it follows from (11) that

$$U \approx \frac{\tau_1\tau_2 - \theta^2}{T_1T_2} = \frac{\tau_1\tau_2}{T_1T_2} - \frac{\theta^2}{T_1T_2}.$$

In the traditional calculation, $U_0 = U_1U_2$, where $U_i = \tau_i/(T_i + \tau_i)$ is the unavailability the ith element. When $\tau_i \ll T_i$, $U_i \approx \tau_i/T_i$. Therefore $U_0 \approx (\tau_1\tau_2)/(T_1T_2)$, from which it follows that

$$U \approx U_0 - \frac{\theta^2}{T_1T_2} . \tag{12}$$

It can be seen from (12) that the difference between the values of unavailability U and U_0 is significantly less than for MTBF and MTTR. This is quite natural, since, as was shown above, the relative errors from not taking into account the time over threshold when calculating MTBF and MTTR are approximately the same, and the unavailability depends only on the ratio MTBF/MTTR.

Similar formulas can be derived for parallel systems with a number of elements greater than two, although rather cumbersome expressions are obtained. For example, for a system of three elements, they have the form:

$$T \approx \frac{T_1T_2T_2}{\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3 - 2\theta(\tau_1 + \tau_2 + \tau_3) + 3\theta^2},$$

$$\tau \approx \frac{\tau_1\tau_2\tau_3 - \theta^2(\tau_1 + \tau_2 + \tau_3) + 2\theta^3}{\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3 - 2\theta(\tau_1 + \tau_2 + \tau_3) + 3\theta^2},$$

$$U \approx \frac{\tau_1\tau_2\tau_3 - \theta^2(\tau_1 + \tau_2 + \tau_3) + 2\theta^3}{T_1 T_2 T_2}.$$

These formulas were obtained as follows. Initially, the first and second elements were replaced by one element, MTBF and MTTR for which were taken in accordance with (6) and (10), respectively. This element was then combined with the third element.

5. Conclusion

The main findings of this article are as follows.

- When specifying quantitative dependability requirements for an item, a failure criterion should be formulated for it. In particular, this can be done in a parametric way. This means that some performance parameters are selected and acceptable limits are set for them. When one of the parameters drifts out of its limits, a failure is fixed.
- A time over threshold can also be set in a failure criterion. This means that short-term disruptions in item's operation lasting less than this threshold are not considered as failures.
- The presence of a time over threshold requires correction in calculation formulas for dependability measures of parallel systems. However, when this threshold is much less than the mean times to restoration of elements, the error from applying traditional formulas will be insignificant.

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Analysis of risks in the modelling of material consumption trends in the production process

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Abstract

Quantitative risk analysis approaches in today's technologically advanced age represent a suitable process for mathematical investigation, revealing the context of the origin and existence of risks and their possible effects on ensuring reliability. Today, manufacturing, and industrial companies, with the growing pressure of globalization, must deal with vast amounts of data that evaluate various processes in maintenance management, warehouse and inventory management, or quality evaluation processes. One way to ensure objective collection, analysis and evaluation of robust data is to use Bootstrapping principles and modules. Many companies use these tools and are now becoming available to a wider range of users. Bootstrap principles, with which it is possible to enter the calculation of robust estimates, e.g., standard errors and confidence intervals based on the bootstrap method is therefore suitable for estimating statistics such as mean, median, correlation coefficient or regression coefficients. In this article, we will take a closer look at what bootstrapping is, show you how to enter the calculation of bootstrap estimates, and what types of output are then displayed. Logistic forecasting of spare parts with sporadic consumption are difficult because of problems associated with obtaining data inscrutable demand, which is usually characterized by long periods of zero demand. The presented contribution presents the possibilities of using the method, which is the starting point for the stochastic forecast of future consumption. Based on this method, we can determine the minimum order stock level. The results of the simulations are also presented in graphical outputs

Keywords: bootstrap, simulation, inventory management

1. Introduction

The method used makes it possible to modify the investigated data set. It will generate a number of usable and simulated samples. Based on this principle, the method makes it possible to determine standard errors, perform hypothesis testing for numerous types of statistics, and construct confidence intervals. Empirical values about sporadic consumption may contain random zero values. As a result, variable results can be affected to determine the desired quantity. Due to imponderability of input data naturally random distribution of variable / consumption / do not correspond to standard theoretical probability distributions. Suitable alternative is non-parametric method using past data of sporadic consumption known as bootstrapping [1]. We include it in the simulation statistical methods - MC, which are based on stochastic prediction of future consumption from data of past consumption. Matlab software can also be considered a suitable simulation tool, with which we can determine the prediction of individual indicators and we can thus prevent losses.

The used method analyzes the system based on the assessment of data with random consumption data, from which the simulation processes the experimental function PDF, CDF allocation, random variable / consumption / useful for determining the modeling parameters of surface management of supplies. The probability of a phenomenon or observed event that is equal to or less than a given value is defined by a function that has the character of a cumulative distribution. In technical terminology, abbreviated CDF. The inverse value of the CDF is defined as a function of the percentage value and gives a discrete result that is less than or equal to the probability of the phenomenon.

- PDF: Probability Density Function, returns the probability of a given continuous outcome.
- CDF: Cumulative Distribution Function, returns the probability of a value less than or equal to a given outcome.

Bootstrapping software products are now offered as software products of companies dealing with information technology, consulting in the areas of demand forecasting, inventory planning and optimization, for example Smart Software [2]. The method used is one of the possible application methods. It offers possibilities for simulating what states might occur if repeating data sets from a base file were followed. The principle then is that repeated random selections from the available data. Such selections of a random nature can have a small-er dimension than the dimension of the available data and therefore can be created without repetition or with repetition. The random sampling forecast is based on long-term, statistical monitoring of consumption, which may not be representative. They are therefore universal and suitable for use in many areas.

This article presents the possibilities of use and the basics of bootstrapping. The paper offers an example using real data to create confidence intervals. With suitable process prediction and programming tools with knowledge of bootstrapping principles, it is possible to use the possibilities of simulation modeling in processes that are demanding, and we can create a simulated model. We can then use the simulation together with the application in practice.

Bootstrapping is applied Monte Carlo method, where we make no parametric assumptions about the studied set n of data x = (x1, x2,.,xn), from which are randomly generated samples y. Monte Carlo simulation is a well-known simulation method using random sampling. Monte Carlo permutation is a method developed by Dr. The master who used it for testing. The statistical simulation method developed by Smart and Willema in is based on bootstrapping prediction [3]. It is a random selection based on a long consumption history, but may not be representative. If in practice these data are not available for a sufficiently long time, it is possible to use information on the consumption of a stock item for a shorter previous period, usually a month, a year. In a study that was carried out in practice, we assessed selected parameters in 50 monitored periods of data collection. The following figure graphically shows the basic scheme of the frequency of the monitored quantity.

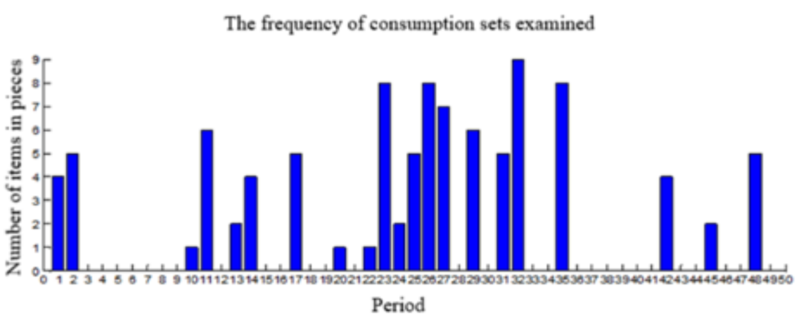

Figure 1: Bar graph of consumption item in 50 monitored period

The input data can be characterized as a set of data from the assessed, monitored consumption. From the given file we can determine the numerical characteristics of consumption. These are expressed deterministically. The given characteristics do not have the stochastic character of the consumption process. As an example, for the items of Fig. 1: min = 0, max = 9, average = 1.9600, default value = 2.8209. Bootstrapping is a statistical method based on repetitive selections from a single data set. This means that in this way it generates a large random selection of data from the input data and thus calculates the specified statistics for each of these selections [4]. The used method provides results about the numerical characteristics of the assessed system and further offers outputs of statistical characteristics, namely histograms and data sampling. Bootstrap random selections y are generated from the examined set several thousand times by se-lection with repetition or replacement of selected data from the examined data set x = (x1, x2, ..., xn, y = (y1, y2, ..., ym) required number of m (n) data The selected numerical values of yi are independent of each other and collected when selecting the Bootstrap with the same probability (even distribution). The monitored samples often have characteristics of differences. Since a recurring selection, some xi data may appear repeatedly or not at all. Appropriate use of the method is to determine future consumption for inventory / delivery time. Based on a random selection of data and a simulated number of selections, we can calculate and compile a histogram of the frequencies of the sum of the consumption of items. This will allow us to determine a signal level that will alert us to an impending shortage of items in stock. We can use the principles of lean maintenance. The following figure shows a histogram of frequencies from a given selection of monitored periods.

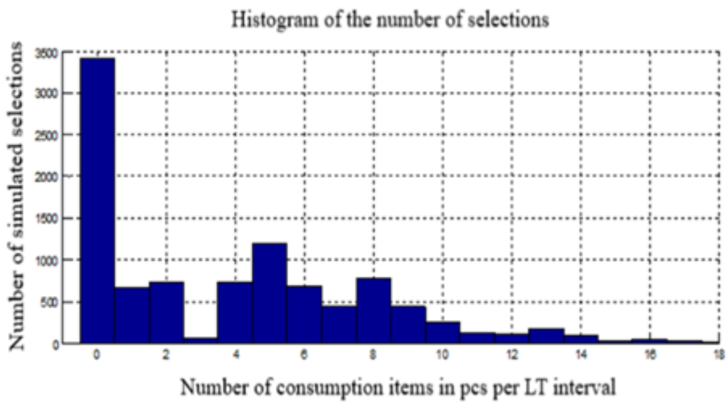

Figure 2: Histogram of sum frequency for 10000 selections

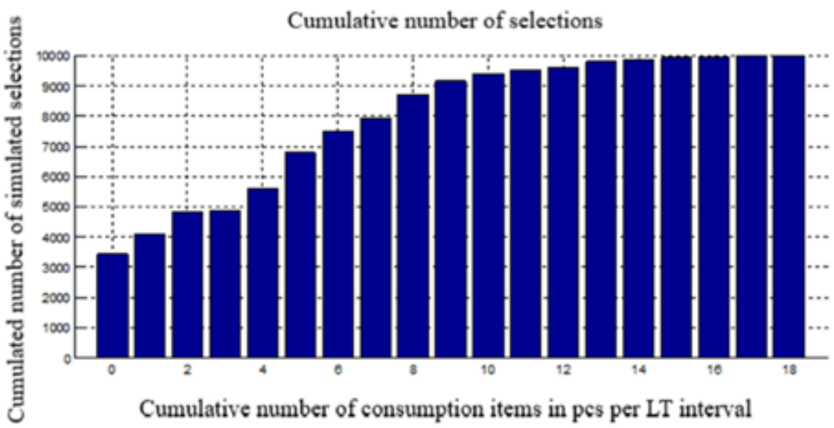

Figure: 3 Histogram of sum frequency for 10000 selections

Statistical data such as average, standard deviation, variance of consumption for lead-time are calculated directly from the data needed to create the histogram, not to generate a derived empirical distribution function. This provides the basic assumptions for the design and representation of the cumulative frequency or probability of future consumption of an item [5].

2. Experiment and simulation model

The presented simulation model was built from available algorithms and MATLAB software commands with the following sequence. MATLAB is a simulation language developed for scientific and technical calculations, modelling, algorithm design, simulation, data analysis and presentation, measurement and processing signals, design of control and communication systems. Therefore, its use for modelling bootstrapping processes is advantageous. The basis is the computational core, which is focused on operations with matrices and is therefore considered the strongest aspect of the MATLAB simulation language with its optimal algorithms. The kernel is extended by a number of extensions (Toolboxes = application libraries), which are intended for solving tasks from almost all areas of technical practice. The process simulation and prediction model is designed from simple MATLAB algorithms and commands with the following sequence, so that it is easily defined and usable as a universal procedure [6]. Initial sample for the determination of the consumption data vector. Determination of the sequence of calculation of numerical characteristics of samples of initial consumption.

Determination of the sequence in compiling the simulation algorithm:

- the model takes data of a random nature from the bootstrapping module options,
- defining the time interval requirement of the model,
- inventory decline and analysis / blue,
- the signal level informs about reaching more states than the order level, insufficient stocks.

Experiments created for the simulation of the given process should have the character of confirmation of the validity of the value of the optimal inventory for the tracked items defined by bootstrapping depending on the delivery date, the chosen probability of providing the item and the total cost of the inventory. The aim of the experiment is to assess the impact of changing the delivery time with different input data: the number of repetitions of the bootstrapping simulation, the number of time periods of the simulation, financial costs for storage, financial costs per warehouse unit of material tracked per day, costs for transportation and delivery of material. Determination of simulation input data. We have selected monitored variables such as the number of selected delivery time periods, the number of bootstrap selections of the quantile required logistic delivery support [7]. Subsequently, using the Matlab simulation tool, we generate matrices of bootstrap indexes of evenly distributed selections. The final steps are to convert the index matrix to the bootstrap selection consumption matrix and the sum of the bootstrap selection values. Graphic and statistical processing of output data are shown in the following figures. Simulation model was created from simple algorithms and MATLAB commands with the following sequence. Given the results of simulation experiments for determining the safety inventory in the likelihood of assuring 0.95 / starred / for periods for 100000 Bootstrap selections.

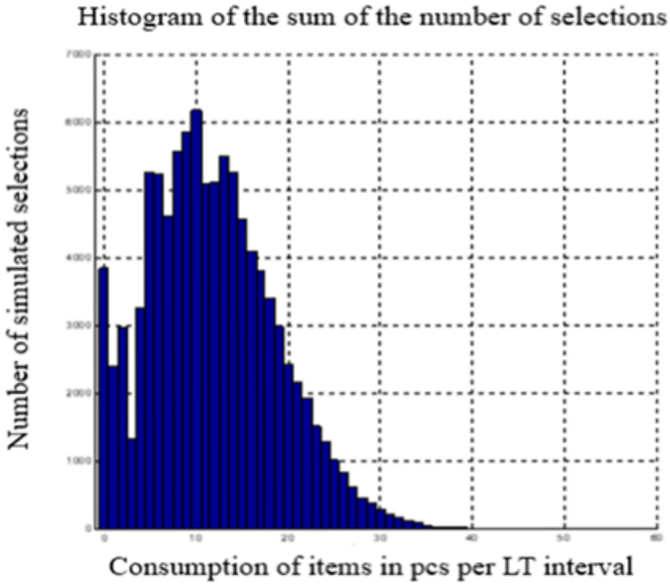

Figure 4: Result of simulation bootstrap experiments

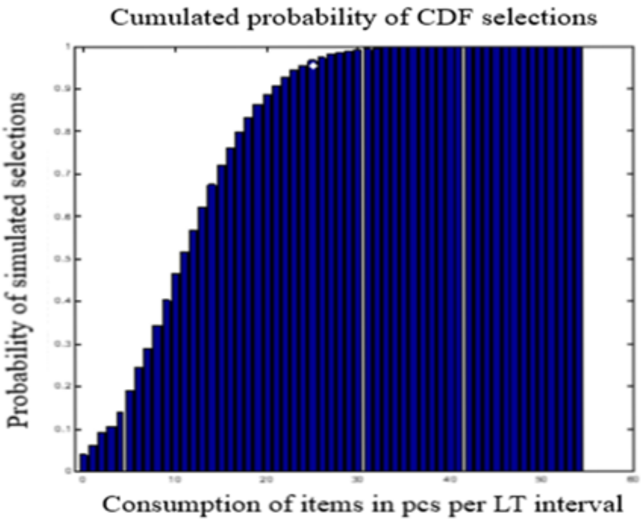

Figure 5: Result of simulation bootstrap experiments

The empirical CDF is interesting, but also as the fundamental component of a statistical approach called the bootstrap [8]. The use of the empirical CDF curve gives us a picture of the statistical sample. The cumulative distribution function of the fair value of the random variable X is a given function $F_x(x) = P(X \leq x)$.If we made a simulation experiment with hundreds of thousands of selections of item consumption data for 50 monitored periods, we would see that sums of selections according to index of item oscillate approximately at the level of 2000, which confirms that selections are made with uniform distribution probability.

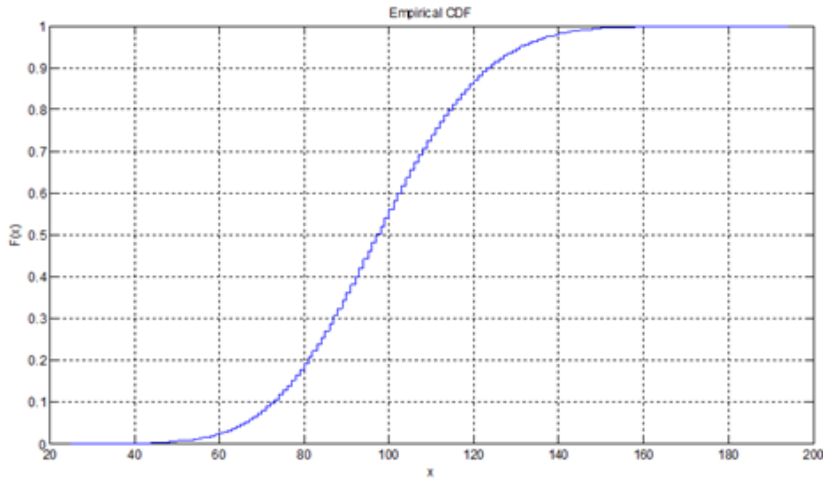

Figure 6: Simulation of sum of 100000 whole initial sample selections

The presented analysis and model combines the simulation algorithm of the supply process based on the principle of simulation with a variable time step.

We can characterize the parameters of the model:

- Lead Time, the number of time units from sending the order to the delivery of the item.
- Stock order level – Reorder Level is set as an optimal level with regard to delivery time and security probability.
- Probability of provision – The level of service provided. Demand during implementation will not exceed supply with a specified probability.
- Safety Stock – Safety Stock. Inventory created due to fluctuating demand and/or lead time to protect against item shortages.
- Determining the stock level - the level is defined as the optimal level depending on the delivery time. The optimal order level is modelled by bootstrapping selection with the demand forecast requirement during the supplier's lead time rounded to the nearest higher order quantity. Fig. 7. at the time when the stock ordering level is reached, the software will generate an order request to the supplier marked with a red star.

The descriptive approach allows tracking the level of the order as well as the time to draw the offer to replenish the stock with the requirements for the specified level of logistics security.
Protection against item shortages is implemented through insurance stocks, which are dimensioned due to unstable demand. The safety stock is not necessary if we define the optimal stock using the bootstrapping definition of the optimal stock. Curves in graphic form then evaluate the current state of supplies of materials to warehouses.

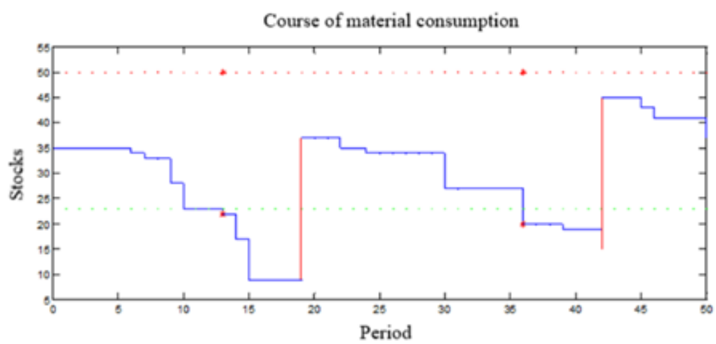

Figure 7: Development of simulation experiment with level inventory management

3. Results

The above approach allows to set the ordering level ordering and the time of issue of requirement to resupply according to determined level of logistic support [9]. At the point of intersection with ordering level information system generates an order to the supplier. Ordered quantity is determined by bootstrap forecasts of consumption during the delivery period of the supplier and rounded to the next higher order quantity. Level management and course of simulation through the simulation model for its use is shown in Fig.7. The initial inventory is current consumption is gradually reduced to the optimum level of inventory (bottom green signal level).

The model allows to change the levels of input values / number of simulation periods, the level of probability of logistic support, initial inventory, the level of ordering /. The logic of model is useful for setting up automatic level management of inventory for items with sporadic consumption. The signal level set by bootstrap method ensures maintenance of required logistics service. From the presented simulation of the processes, we can determine that the increased demand for logistical support of the optimal level of supply can cause an increase in the level of the optimal stock level and also an increase in costs. Interestingly, procurement costs are about the same, shipping costs are going down and storage costs are going up. Simulation experiments are intended to demonstrate the validity of determining the optimal inventory of an item determined by bootstrapping, depending on the lead time of the order, the selected probability of securing the item and the total cost of inventory. The presented model makes it possible to change the levels of input values / number of simulation periods, required level of probability of logistics security, initial stock level, order level/. The logic of the model can be used to set up automatic stock level control in the case of items with sporadic consumption

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Methodical rationale of system solutions to reduce risks and retain them within acceptable limits for knowledge management process

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Abstract

An approach to the formalization of the standard knowledge management process is proposed, taking into account the requirements for information protection. The approach has been developed to the level of methodical approach for estimation and rationale system solutions to reduce risks and/or retain risks within acceptable limits for various threats scenarios. The use of the approach allows to estimate the impact of various threats on knowledge management process performance by probabilistic measures (including threats to the violation of information protection requirements). The usability of the proposed methodical approach is demonstrated by examples.

Keywords: analysis, engineering, information protection, knowledge, model, prediction, risk, system

1. Introduction

Modern enterprises widely use the standard system process of knowledge management (see, for example, the descriptions of the standard process in ISO/IEC/IEEE 15288 "Systems and software engineering — System life cycle processes"). This concerns to both developing and operating systems, their subsystems and processes. In particular, the enterprise may be considered as a system interested in knowledge management about itself. The purpose of the system knowledge management process is to improve the quality and/or security and/or effectiveness of the system or related systems through the acquisition, creation, distribution, timely application and storage of useful knowledge in lifecycle. In turn, the knowledge itself serves as the basis for extracting latent effects and preventing possible errors during creation, operation of systems and their decommissioning.

Note. Knowledge means the volume of comprehensions and skills that are invented by people.

In the process of knowledge management, new knowledge is created and acquired, knowledge bases and centers (KnC) are formed. This explains the importance of the problem of storing acquired knowledge in the conditions of heterogeneous threats, including threats to the information protection. There are many works on risk analysis, see for example [1-22]. In [20] a qualitative risk assessment is carried out using a general method of analogy, the essence of which is to analyze a set of data on similar projects. In [21] risk analysis, risk factors identification and systematization are based on spatial structuring in the coordinate plane, including development trends and features of the territory, taking into account its own possibility, the ability for innovations and competent risk management. In [22] the risks identification, the definition of priority strategies for reducing risks in supply chain is carried out using the supply chain Performance Standard (SCOR), Fuzzy Failure mode and Effect analysis (Fuzzy FMEA) and Fuzzy Analytical Hierarchical Process (FAHP). According to ISO Guide 73 risk is understood as effect of uncertainty on objectives considering consequences (an effect is a deviation from the expected — positive and/or negative). However despite a lot of works, the issue of risks prediction, choosing system solutions to reduce risks and retain them within acceptable limits continues to remain relevant.

In comparison with the existing approaches, the proposed approach allows to estimate the impact of various threats on the effectiveness of the knowledge management process (including threats to the violation of information protection requirements), allows us to predict risks taking into account the complexity of the modelled system and measures to counter threats in each element, determine the reliability of the process and required information protection. It is expected that the use of the proposed approach in knowledge management processes in modern KnC will help both to increase the effectiveness of the process itself, and to choose and apply the rational measures to reduce risks and/or retain risks within acceptable limits for various threats scenarios.

2. General

It is proposed the approach to assess the integral risk of violation of the knowledge management process performance, taking into account the requirements for information protection and the particular risks (concerning the actions performance and the generalized risk of unreliability of knowledge management process performance).

It is proposed to characterize particular risks by the probabilities of corresponding events (in comparison with possible consequences):

- the probability of violating the reliability of the knowledge acquisition process performance without taking into account the requirements for information protection;
- the probability of violating the reliability of creating useful knowledge without taking into account the requirements for information protection;
- the probability of violating the reliability of the distribution of acquired or created useful knowledge without taking into account the requirements for information protection.

In turn, the reliable distribution of acquired or created useful knowledge means their application in time.

The generalized risk of unreliability of the system knowledge management process performance takes into account all the listed particular risks .

Possible ways to reduce risks that can be quantitatively justified are itself the mechanisms for directly managing risks in the knowledge management process performance:

- for the risk of violating the reliability of the knowledge management process performance without taking into account the requirements for information protection, this is the fulfillment of the necessary conditions with the completion of all the actions taken by the processes of acquiring knowledge and creating useful knowledge (compliance with the delivery dates of acquired knowledge and created useful knowledge and the acceptable level of defects in them);
- the risk of violating the requirements for information protection in the process of knowledge management – this is a reduction in the frequency of occurrence of sources of threats to the violation of information protection requirements in the process of knowledge management (if possible), an increase in the time of threat development before the violation (if possible), optimization of the time

period between system diagnostics, reducing the duration of system diagnostics and system recovery time after a violation, choosing a prognostic period when effective preventive management actions are possible;

- the integral risk of the violation of the knowledge management process performance, taking into account the requirements for information protection, is a balanced action to ensure the reliability of the knowledge management process performance and information protection in the process, aimed at risks retention within acceptable limits.

The following statement are to be considered:

- analyzed objects for risks prediction in the knowledge management process performance;
- propositions on formalization;
- measures;
- the procedure for risks prediction;
- calculation methods, examples, interpretations.

3. Analyzed objects for risks prediction in the knowledge management process performance

To predict the risks it is proposed to define:

- the composition of the output results and performed actions of the knowledge management process and the assets used in this process;
- a list of potential threats and possible scenarios of the occurrence and development of threats for the output results, the actions performed by the process and the assets used in this process;
- technologies for countering threats used in the process of managing knowledge in given system application environment;
- formalized requirements or conditions for completing the necessary actions of the knowledge management process, meeting the deadlines for the delivery of knowledge, the absence of defects in the acquired and created knowledge, the distribution and application of useful knowledge.

To calculate typical risk measures, the analyzed entities are considered as a modelled system of simple or complex structure. In the models and methods of system analysis, in relation to such modelled systems, data obtained after the occurrence of events, according to the identified prerequisites for the occurrence of events, and data collected and accumulated statistics on the process and possible conditions for its implementation are used [1-8], [13-14].

Depending on the goals of risk prediction, models are presented in the form of a «Black box» or in the form of a complex structure. For separate elements of a complex system or for its rough modeling, a «Black box» models are used. To obtain more accurate results of risk prediction, a complex modelled system is decomposed to the level of composite system elements characterized by their parameters and operating conditions and combined to describe the integrity of the modelled system by the logical conditions "AND" and "OR". At the same time, the integrity of the modelled system (or system element) during specified prognostic period means such a state of this system (or system element) that during this prognostic period corresponds to the intended purpose of the applied model.

Notes

1 The logical condition " AND" for two elements connected by this condition is interpreted as follows: the modelled system of two sequentially connected elements is in a state of integrity when " AND" the first element, " AND" the second element are in a state of integrity.

2 The logical condition "OR" for two elements connected by this condition is interpreted as follows: a system of two elements connected in parallel is in a state of integrity when "OR" the first element, "OR" the second element is in a state of integrity (in particular, when the execution of separate actions is duplicated to increase reliability).

4. Proposition for formalization

To solve the problems of system analysis, a modelled system can be: a set of output results and/or assets, a set of actions of the knowledge management process, united by a purpose in the interested system.

For each of the elements of the modelled system, depending on the goals set, their own system analysis tasks can be solved. In general, the modelled system is represented as a «Black box» or as a complex system, the elements of which are combined sequentially or in parallel. At the same time, each element may be characterized by its own heterogeneous threats and the technologies used to control, monitor and recovery the violated integrity – see, for example [1-8], [13-14].

For each of the elements and for the modelled system as a whole, a space of elementary states is introduced (taking into account the logical relationships of the elements with the conditions " AND", "OR").

For example, in the application to predicting the risk of violation of information protection requirements, the space of elementary states on the time axis can be formally defined by two basic states:

- "Compliance with the requirements for information protection in the process of knowledge management is ensured", if the requirements for information protection are met during entire prognostic period, i.e. from the point of view of mathematical modeling, their non-compliance leads to damage;
- "Compliance with the requirements for information protection in the process of knowledge management is violated" - otherwise.

In the application to the prediction of the integral risk of violation of the process performance, taking into account the requirements for information protection, the space of elementary states on the time axis can be formally defined by the other two basic states:

- "The reliability of the knowledge management process performance and the fulfillment of the requirements for information protection in the system are ensured", if during entire prognostic period the reliability of performing certain actions of the process for obtaining output results and the fulfillment of certain requirements for information protection are ensured;
- "The reliability of the system knowledge management process performance and/or the fulfillment of the requirements for information protection in the system is violated" – otherwise.

In general, it is possible to expand or rename the elementary states themselves, the main thing is that they form a complete set similar to the sets presented above.

The use of the risk prediction helps to justify acceptable risks. In fact, for each analyzed object there are its own conditions of acceptability in the intended use. The priority is to choose the criterion of acceptable risk based on the precedent principle. The essence of the precedent principle is that as a norm for information protection, such a value of acceptable risk is established, which was chosen as acceptable based on the results of modeling various past events. For the specified prognostic period, the calculated risk values that are characteristic of the violations that have taken place are determined as unacceptable, and those that are smaller than those that are unacceptable are determined as acceptable (these risk values correspond to the precedent absence of violations of information protection requirements).

As measures to counter threats that can reduce the calculated risks when they are applied, more frequent (compared to the time of threat development) system diagnostics or control with the restoration of normal operation (of the system, process, system element) can act. When using the specified limits of acceptable risk, predictions for real cases of violations of the norm "before" and "after" the occurrence of violations allow (when using the quantified limits of acceptable risk) to perform an analytical rationale of proactive measures to reduce or retain risks within acceptable limits and/or reduce costs and / or possible damages under the specified restrictions. The reasoned determination of balanced system measures and actions that prevent the occurrence of damage under restrictions on resources and acceptable risks, as well as the assessment and rationale of effective short-, medium - and long-term security plans are carried out by solving independent optimization tasks using the calculated values of the predicted risks.

5. Measures

In application to modeled system, which can be represented as a «Black box» or a complex logical structure, the next measures are proposed:

$R_{\text{gen}}\,(T_{\text{spec}})$ — the risk of unreliability of the knowledge management process performance during specified prognostic period T_{spec} without taking into account the requirements for information protection;

$\boldsymbol{R_{\text{sec}}\,(T_{spec})}$ — the risk of violating the requirements for information protection in the process of knowledge management during specified prognostic period $\boldsymbol{T_{\text{spec}}}$;

$R_{int}\,(T_{spec})$ — integral risk of the violation of the knowledge management process performance, taking into account the requirements for information protection during specified prognostic period T_{spec}.

The integral risk of the violation of the knowledge management process performance depends on unreliability of the process performance or on the violation of requirements for information protection, or both, with the severity of possible consequences.

6. The procedure of risks prediction

To predict the risks, it is proposed to perform the following steps:

1) to define the modelled system and set the analyzed objects of risk prediction;

2) to set the specific goals of risk prediction;

3) to create a list of possible threats. The decision is made to represent the modelled system in the form of a «Black box» or in the form of a complex structure decomposed to composite elements. They form the space of elementary events for each element and the modelled system as a whole;

4) to select calculated measures and suitable mathematical models and methods (including methods to increase their adequacy).

7. Calculation methods, examples, interpretations

The proposed methods to rationale system solutions, to reduce risks and/or retain them within acceptable limits are presented in combination with examples and the practical interpretations of the calculation results concerning some problems of Arctic development.

To achieve the main goals in the Arctic development for the period up to 2035, numerous problems must be systematically solved in the areas of social and economic development, development of the infrastructure of the Arctic zone, development of science and technology in the economic interests, environmental safety, development of international cooperation, ensuring the protection of the population and territories of the Arctic zone from natural and man-made emergencies, ensuring information protection. The system solution of the entire set of tasks is based on knowledge management, based on the analytical processing of heterogeneous monitoring data and providing for the improvement, accumulation and timely application of emerging knowledge.

Unavoidable uncertainties in the specifics of applications for a given prognostic period are taken into account when solving practical problems using mathematical modeling, risk prediction, system analysis and optimization at various meta-levels.

Given the complexity and versatility of the practical tasks being solved for the development of the Arctic region, the creation of one or more KnC is inevitable. In the conditions of real and potential threats to the security of critical information infrastructure, information protection in the KnC is of priority importance. Without going into the details and specifics of the heterogeneous knowledge to be integrated and applied, some practical problems to the use of this methodic approach are concerning:

- to solve the profile tasks of ensuring environmentally safe marine exploration, production and transportation of various types of minerals in extreme natural and climatic conditions (profile tasks of the 1st type);
- to solve specialized tasks of ensuring integrated safety of operations on the continental shelf, including monitoring and forecasting of extreme situations of natural and man-made nature (profile tasks of the 2nd type);
- to solve the specialized tasks of preventing and eliminating emergency oil spills in ice conditions, including the creation of technologies for detecting oil under ice (profile tasks of the 3rd type);
- to solve the profile tasks of developing technologies for integrated hydrometeorological and environmental monitoring of natural hazards in the Arctic regions (profile tasks of the 4th type);
- to solve the profile tasks of developing technologies for remote sensing of the Earth, including environmental monitoring, resource estimation and forecasting of the state of the Arctic environment (profile tasks of the 5th type).

The methodic approach is illustrated by the examples of the predictions:

- the risk of unreliability of the knowledge management process performance without taking into account the requirements for information protection;
- the risk of violating the requirements for information protection;
- the integral risk of the violation of the knowledge management process performance, taking into account the requirements for information protection.

For certainty from the point of view of system engineering for information protection, two options are considered: the creation and operation of five autonomous specialized KnC, each of which specializes in solving its own profile tasks (option 1), and the addition of a single KnC integrating the capabilities of all autonomous KnC (option 2). Taking into account possible consequences, the objectives of risk prediction are formulated as follows. In the conditions of existing uncertainty:

- to quantify the risk of unreliability of the knowledge management process performance without taking into account the requirements for information protection;
- quantify the risk of violating requirements for information protection (both piecemeal for each KnC, and for a complex of all KnC);
- identify critical conditions in the development of various threats;
- to quantify the integral risk of violating the reliability of the knowledge management process performance, taking into account the requirements for information protection;
- to determine such a period in which guarantees of non-excess of acceptable risks are maintained.

Examples 1-3 show an assessment of the risk of unreliability of the knowledge management process performance (without taking into account the requirements for information protection). Assuming the commensurability of possible consequences, the examples assess the probabilities of unreliability of acquiring and creating useful knowledge and the probability of unreliability of the distribution of acquired or created useful knowledge and their timely application.

7.1. Example 1

The example shows an assessment of the risks of unreliability of the knowledge acquisition process performance.

When assessing the risks of unreliability of the knowledge acquisition process performance, the methods of system analysis are adapted in terms of assessment:

- the risk of incomplete performance of the necessary actions for the supply of acquired knowledge;
- the risk of violation of the delivery dates of acquired knowledge;
- the risk of an unacceptable defects level in the acquired knowledge (analytical errors, descriptions, unsubstantiated conclusions and/or recommendations).

From the point of view of calculations, the models for assessing the above risks are identical, since when assessing each of the risks, the calculated probabilistic measures are compared with the possible consequences proper due to non-fulfillment of the conditions for acquiring knowledge.

The example below shows an estimation of the violation of the reliability of the timely delivery of acquired knowledge. The estimation of the incompleteness of performing the necessary actions to supply the acquired knowledge and the presence of an unacceptable defect in the acquired knowledge (analytical errors, descriptions, unsubstantiated conclusions and/or recommendations) is done by analogy.

The probability $R_{\mathrm{td}\,i}(T_{\mathrm{spec}\,i})$ of violation of the terms of a single delivery for knowledge of i-th type for a given time $T_{\mathrm{spec}\,i}$ is calculated by the formula

$$R_{\mathrm{td}\,i}(T_{\mathrm{spec}\,i}) = N_{\mathrm{sec}\,i}(T_{\mathrm{spec}\,i})/N_i(T_{\mathrm{spec}\,i}), \qquad (1)$$

where $N_{\mathrm{sec}\,i}(T_{\mathrm{spec}\,i})$ and $N_i(T_{\mathrm{spec}\,i})$ – accordingly, the number of violations and the total number of deliveries in a given time $T_{\mathrm{spec}\,i}$ to the knowledge of i-th type statistics.

The delivery time fulfillment indicator for k-type knowledge is defined as follows

$$Z_{\mathrm{term}\,i}\,(T_{\mathrm{spec}\,i}) = \begin{cases} 0, & \text{if the conditions of delivery terms are met;} \\ R_{\mathrm{td}\,i}(T_{\mathrm{spec}\,i}) & \text{according to the formula (1), if the conditions are not met or not specified.} \end{cases} \qquad (2)$$

The condition for fulfilling the terms of the knowledge delivery of k-th type is defined as the condition for not exceeding the maximum acceptable level $R_{\mathrm{add.св}\,i}(T_{\mathrm{spec}\,i})$, set for the probability of violating the terms of a single delivery. This condition is expressed in the form: $R_{\mathrm{td}\,i}(T_{\mathrm{spec}\,i}) \leq R_{\mathrm{add.td}\,i}(T_{\mathrm{spec}\,i})$. In the expression for the generalized risk the execution rate of the delivery terms for the acquisition of knowledge of i-th type $Z_{\mathrm{term}\,i}\,(T_{\mathrm{spec}\,i})$ is marked as $Z(\mathrm{acq})_{\mathrm{term}\,i}\,(T_{\mathrm{spec}\,i})$.

The probability of violation of delivery dates for the entire set of knowledge of various types implemented in the process according to statistical data, taking into account the multiplicity of deliveries characterized by the input data for each of the types of knowledge, is calculated by the formula

$$R_{\mathrm{td}}\,(T_{\mathrm{spec}}) = 1 - \sum_{i=1}^{I} M_i[1 - R_{\mathrm{td}\,i}\,(T_{\mathrm{spec}\,i})] \Big/ \sum_{i=1}^{I} M_i; \qquad (3)$$

where T_{spec} – is the specified total delivery time of the entire set of knowledge of various types, including all the particular values of $T_{\mathrm{spec}\,i}$ taking into account their overlaps, M_i – is the number of deliveries of knowledge of the i – th type taken into account for multiple deliveries, in the expression for the generalized risk in relation to the acquisition process , the designation M(acq) i , i = 1,…,I(acq) is used.

In accordance with the tasks set for the development of the Arctic region, it is planned to acquire several types i of knowledge. The acquisition of all types of knowledge, with the exception of one, takes place without violating the delivery dates, i.e. in this case $Z_{\mathrm{term}\,i}\,(T_{\mathrm{spec}}) = 0$. Therefore, the risk assessment takes into account only the type of acquired knowledge for which the delivery dates are violated.

Taking into account the statistical data on the development of the Arctic, for certainty, it is conditionally assumed that for a given time $\boldsymbol{T}_{\mathbf{spec}\,\boldsymbol{i}}$ = 1 year for type *i* of knowledge, the total number of deliveries N_i = 100, the number of violations of delivery dates $\boldsymbol{N}_{\mathbf{sec}\,\boldsymbol{i}}$ = 3, which is 3% of the total number of deliveries, and the number of multiple deliveries M_i = 1.

The results of the estimation of the violation of the reliability process performance of creating useful knowledge are completely identical to this example.

7.2. Example 2

The example illustrates the assessment of the risks of violating the reliability process performance of distributing useful knowledge. The methods for calculations see in [13-14].

Let's assume that, taking into account statistical data, the frequency of a significant change in the usefulness of knowledge about Arctic conditions in the system's knowledge base will be no more than one change per 10 years, i.e. **ξ** = 10 years. The average time for acquiring or creating and placing new knowledge in the knowledge base of the system (from the creators or distributors of knowledge) will be about three months, i.e. $T_{\text{knowledge base}}$ = 3 months, which, translated to the same units of measurement, is 0,25 years. Updates from the KnC are delivered to the system consumers on a monthly basis, i.e. q = 1 month or 0,083 years. In addition, a restriction is imposed on the probability of violating the reliability of the distribution of useful knowledge from above: this probability should not exceed the maximum allowable level $R_{\mathbf{add.dist}}(\boldsymbol{T}_{\mathbf{spec}})$ = 0,10.

Thus, the risk assessment for the discipline of knowledge distribution immediately after its acquisition or creation is determined by the formula

$$R_{\text{dist}} = 1 - \frac{\xi}{\xi + T\text{knowledge base}} = 1\text{-}10/(10+0{,}25) = 0{,}024. \quad (4)$$

The risk assessment for discipline periodic distribution of knowledge regardless of the dates of their acquisition or creation, i.e. regulation (confirming the usefulness of existing stored knowledge in the absence of changes) is determined by the formula

$$R_{\text{dist}} = 1 - \frac{\xi^2}{q(\xi + T\text{knowledge base})}\left[1 - \exp\left(-\frac{q}{\xi}\right)\right] =$$
$$=1\text{-}\ 102\cdot[1\text{-}\exp(\text{-}0{,}083/10)]/0{,}083\cdot(10+0{,}25)=0{,}060. \quad (5)$$

Since the condition of not exceeding the maximum acceptable level of $\mathrm{R}_{\text{dist}}(T_{\text{spec}}) \le \mathrm{R}_{\text{add.dist}}(T_{\text{spec}})$, is met, this indicator can be neglected in further calculations, i.e. $\mathrm{Z}_{us}\ (T_{\text{spec}}) = 0$, the conditions for the distribution of knowledge are met, see formula (6). For the period T_{spec}, for which the input data ξ, $T_{\text{knowledge base}}$, **q**, is determined, the indicator of the reliability of the distribution of useful knowledge, assuming the timeliness of their subsequent application, is defined as follows

$$Z_{us}\ (T_{\text{spec}}) = \begin{cases} 0, \ \text{if the conditions for the distribution and application of knowledge are met;} \\ R_{\text{dist}}\ (T_{\text{spec}}) \ \text{according to formulas (4)and(5), if the conditions are not met or not specified.} \end{cases} \quad (6)$$

7.3. Example 3

The example presents an assessment of the generalized risk of the unreliability of the knowledge management process performance, which is determined by the formula

$$R_{\text{spec}}\,(T_{\text{spec}}) = 1 - [1 - Z_{us}\,(T_{\text{spec}})] \cdot$$

$$\cdot\{\sum_{k=1}^{K(acq)} W(acq)_k[1-Z(acq)_{act\,k}(T_{\text{spec}\,k})] + \sum_{k=1}^{K(creat)} W(cr)_k[1-Z(cr)_{act\,k}(T_{\text{spec}\,k})] + \tag{7}$$

$$+\sum_{i=1}^{I(acq)} M(acq)_i[1-Z(acq)_{term\,i}(T_{\text{spec}\,i})] + \sum_{i=1}^{I(cr)} M(cr)_i[1-Z(cr)_{term\,i}(T_{\text{spec}\,i})] +$$

$$+\sum_{j=1}^{J(acq)} L(acq)_j[1-Z(acq)_{def\,j}(T_{\text{spec}\,i}) + \sum_{j=1}^{J(cr)} L(cr)_j[1-Z(cr)_{def\,j}(T_{\text{spec}\,i})]\}$$

$$/\,[\sum_{k=1}^{K(acq)} W(acq)_k + \sum_{i=1}^{I(acq)} M(acq)_i$$

$$+\sum_{j=1}^{J(acq)} L(acq)_j + \sum_{k=1}^{K(cr)} W(cr)_k + \sum_{i=1}^{I(cr)} M(cr)_i + \sum_{j=1}^{J(cr)} L(cr)_j],$$

where $\boldsymbol{T}_{\mathbf{spec}}$ – is the specified total time, including all the partial values $\boldsymbol{T}_{\mathbf{spec}\,\boldsymbol{k}}$, $\boldsymbol{T}_{\mathbf{spec}\,\boldsymbol{i}}$, $\boldsymbol{T}_{\mathbf{spec}\,\boldsymbol{j}}$

$$R_{\text{spec}}(T_{\text{spec}}) = 1\text{-}[1\cdot(1\text{-}0,03)+1\cdot(1\text{-}0,03)+1\cdot(1\text{-}0,03)+1\cdot(1\text{-}0,03)+1\cdot(1\text{-}0,03)+1\cdot(1\text{-}0,03)]/(1+1+1+1+1+1) = =0,03.$$

As a calculation result, the risk of unreliability of the system knowledge management process performance in the prognostic period of 1 year will be approximately 0,03.

7.4. Example 4

The example demonstrates the prediction of the risk of violation of information protection requirements in several autonomous KnC. Elements of the modelled system are elements 1-5, formally associated with assets and output results of solving profile problems of the 1st-5th types, respectively.

By definition, the absence of violations of information protection requirements in the modelled system is considered to be ensured during a given prognostic period if there are no violations in all autonomous data centers during this period. The prognostic period itself for an separate element can be interpreted as referring to the stage of creation (for threats inherent in this stage), and to the stage of operation in the future (for potentially possible threats).

Performing step 3 of this methodic approach (see section 6), many critical threats were identified that affect the information protection of each of the structural elements of the modelled system. Hypothetical input data for each of the five elements of the modelled system with a brief rationale in the comments are presented in Table 1.

Table 1: *Hypothetical input data for predicting the risk of violation of information protection requirements*

Input data	Element #	Values and comments
σ – the frequency of occurrence of sources of threats to the violation of information protection requirements	1	four times a year, which is commensurate with the occurrence of threats associated with subjective factors and errors of intermediate-qualified IT specialists in solving problems of ensuring environmentally safe offshore exploration, production and transportation of various types of minerals in extreme natural and climatic conditions
	2	twice a year, which is commensurate with the time of failure of software and technical equipment to ensure comprehensive safety of operations on the continental shelf, including monitoring and forecasting of extreme situations of a natural and man-made nature
	3	once a year, which is commensurate with the emergence of threats related to the causes of human errors at the decision-making levels for

		the prevention and elimination of emergency oil spills in ice conditions, including the creation of technologies for detecting oil under ice
	4	once in two years, which is comparable with the emergence of threats from the use of undeclared capabilities of the software technology integrated hydrometeorological and environmental monitoring of natural hazards in the Arctic regions
	5	once in two years, which is comparable with the emergence of threats from the use of undeclared capabilities of the software in the technologies of remote sensing, including environmental monitoring, resource estimation and forecasting of the Arctic environment
β – the average time of threat development from the moment of the occurrence of threat sources to the violation of information protection requirements	1–5	1 day (it is assumed that due to the source of threats, they are activated not immediately, but with a certain delay of at least a day) – this is the time before possible damage after the occurrence of threat signs
T_{av}– the average time between the end of the previous and the beginning of the next diagnostics of the system's capabilities to meet the requirements for information protection	1	1 hour - it is determined by the regulations for monitoring the integrity of the KnC software and assets during shift work in terms of marine exploration, production and transportation of various types of minerals in extreme natural and climatic conditions
	2	1 hour - is determined by the regulations for monitoring the integrity of software and assets when monitoring extreme situations of a natural and man-made nature
	3	2 hours - it is determined by the regulations for monitoring the integrity of the KnC software and assets during shift work in terms of preventing and eliminating emergency oil spills in ice conditions
	4	1 hour - is determined by the regulations for monitoring the integrity of the KnC software and assets during complex hydrometeorological and environmental monitoring of natural hazards in the Arctic regions
	5	8 hours - it is determined by the regulations for monitoring the integrity of the KnC software and assets during shift work in terms of remote sensing of the Earth, including environmental monitoring, resource estimation and forecasting of the state of the Arctic environment
T_{diag}– the average time for diagnosing the state of assets and the system itself	1–5	30 seconds which is commensurate with the duration of automatic integrity control of the software and assets of the KnC
T_{rec} – the average recovery time of the required norm of information protection effectiveness after detection of violations	1–5	5 minutes including rebooting the software and restoring the KnC data
T_{spec}– the specified duration of the prognostic period	1–5	From 1 month up to 2 years (to determine the period during which guarantees are maintained that the acceptable risk of violating information protection requirements will not be exceeded)

The analysis of modeling results showed that in probabilistic terms, the risk of violating the requirements for information protection during year will be about 0,222 for the entire complex of knowledge centers, see Figure 1, amounting to 0,080 for the 1st element ("bottleneck"), not exceeding 0,041 for the 2nd-4th elements, and 0,072 for the 5th element ("bottleneck"). If the duration of the prognostic period changes from one to four months, the risk increases from 0.020 to 0.080. For an acceptable risk level of 0,050, a period of up to 2.5 months is justified, in which guarantees are maintained that the acceptable risk will not be exceeded for the entire complex of KnC, characterized by the conditions of the example from Table 1-see Figure 2.

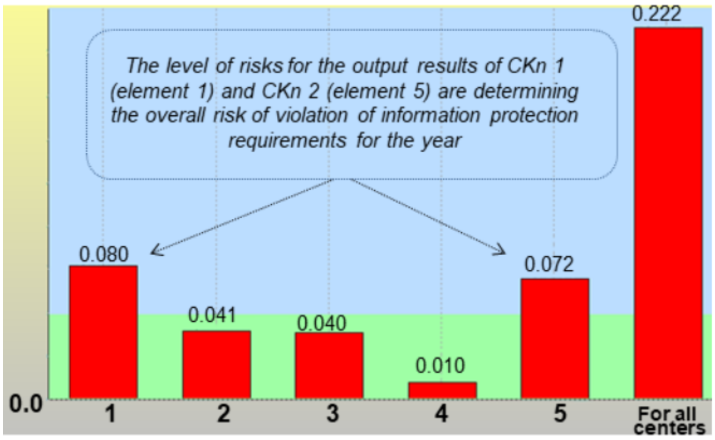

Figure 1: *Assessment of the risk of violation of information protection requirements*

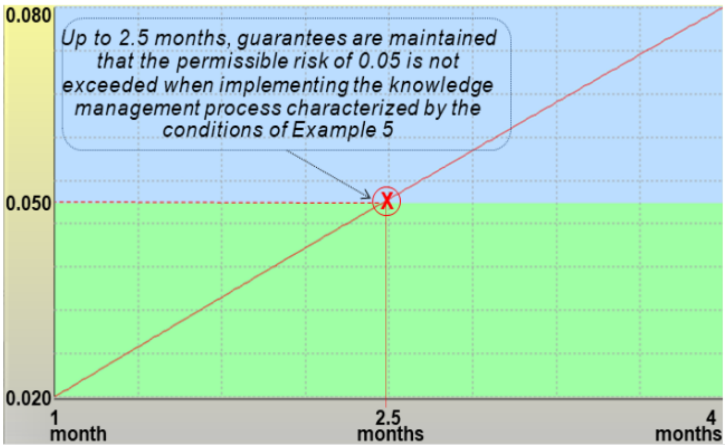

Figure 2: *Dependence of the risk for all knowledge centers on the prognostic period of one to four months*

The risk levels for threats to the output results of the KnC 1 (related to subjective factors and errors of intermediate-qualified IT specialists in solving problems of ensuring environmentally safe offshore exploration, production and transportation of various types of minerals in extreme natural and climatic conditions - element 1) and threats to the output results of the KnC 2 (related to the use of undeclared software capabilities in Earth remote sensing technologies, including environmental monitoring, resource estimation and forecasting of the state of the Arctic environment-element 5) are determining the overall risk of violating information protection requirements for the year. Moreover, the reason that element 1 is a kind of" bottleneck " in the KnC complex is the relatively high frequency of occurrence of sources of threats to commit human errors (4 times a year). And for element 5, the reason is the relatively long average time between the end of the previous one and the beginning of the next diagnostics of the system's capabilities in terms of meeting information protection requirements (after 8 hours) – see Table 1.

7.5. Example 5

The example demonstrates the prediction of the risk of violation of information protection requirements with the addition of a single KnC that integrates the capabilities of all autonomous KnC and performs the functions of a backup center for various types of failures in specialized KnC (option 2) – see Figure 5.

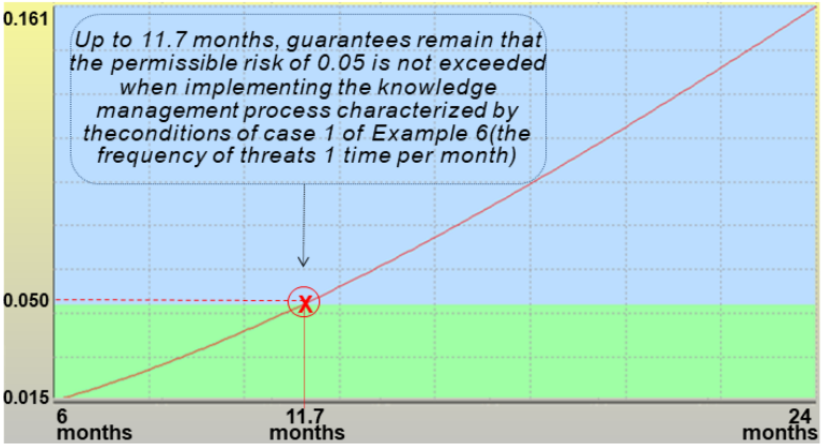

Figure 3: *The Dependence of risk for all KnC from the prognostic period lasting from 6 to 24 months (for case 1)*

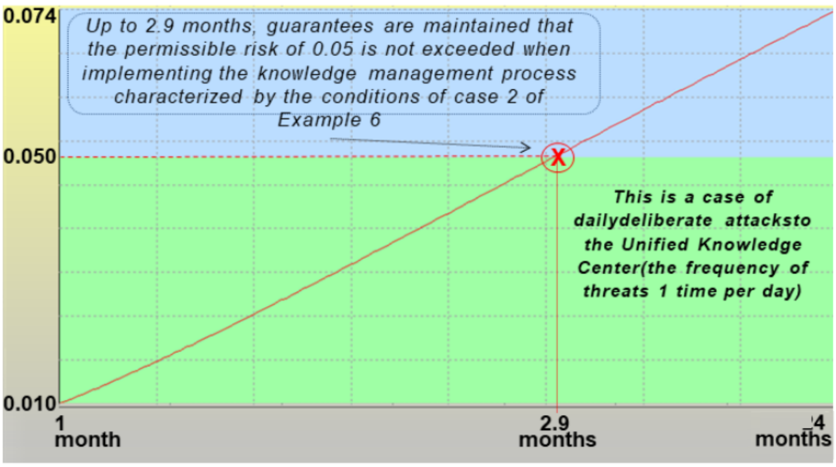

Figure 4: *Dependence of the risk for all KnC on the prognostic period lasting from 1 to 4 months (for case 2 – deliberate attacks)*

Two cases are considered:

- case 1: the frequency of occurrence of threat sources increases to 1 time per month, which is not much higher than the total frequency of occurrence of various threat sources for KnC 1 – KnC 5 according to Table 1;

- case 2: the frequency of occurrence of threat sources increases to 1 time per day, which is 30 times higher than the frequency compared to case 1 and is comparable to deliberate computer attacks on a single KnC.

For both cases, the average time between the end of the previous and the beginning of the next diagnosis of the system's capabilities to meet the requirements for information protection is 1 hour, which is typical for most specialized KnC.

The analysis of the simulation results for the complex structure shown in Figure 5 showed the following.

For case 1, in probabilistic terms, the total risk of violating information protection requirements during year will be about 0,051 for the entire complex of knowledge centers, i.e. it will decrease by more than 4 times compared to Example 4. This is achieved by reserving the operation of specialized knowledge centers with the capabilities of a single KnC. If the duration of the prognostic period changes from 6 to 24 months, the risk increases from 0,015 to 0,161. And for an acceptable risk at the level of 0,050, a period of up to 11,7 months is justified, in which guarantees are maintained that the acceptable risk is not exceeded for the entire complex of KnC characterized by the conditions of case 1 of Example 4 (see Figure 3).

For case 2, associated with daily deliberate attacks on a single KnC, the total risk of violating information protection requirements during year will be about 0,222 for the entire complex of knowledge centers, i.e. the same as for example 5 with a frequency of threat sources 30 times less. If the duration of the prognostic period changes from 1 to 4 months, the risk increases from 0,010 to 0,074. And for acceptable risk level 0,050 justified period to 2,9 months, which retain guarantee not to exceed acceptable risk to the whole complex of knowledge, characterized by the conditions of case 2 of example 4 (see Figure 4).

7.6. Example 6

Given that the prognostic period $\boldsymbol{T}_{\mathbf{spec}}$ = 1 year, year, according to the results of calculations of examples 1─3 takes place $\boldsymbol{R}_{\mathbf{gen}}$ ($\boldsymbol{T}_{\mathbf{spec}}$)= 0,030, and according to the results of calculations of the 5th example (case 2-deliberate attacks on a single KnC) $\boldsymbol{R}_{\mathbf{sec}}$ ($\boldsymbol{T}_{\mathbf{spec}}$) = 0,051, then

$$\boldsymbol{R}_{\mathbf{int}}\ (\boldsymbol{T}_{\mathbf{spec}}) = 1 - (1-0{,}030)\cdot(1-0{,}051) \approx 0{,}080.$$

As a result, the integral risk of the violation of the knowledge management process performance during year, taking into account the requirements for information protection, will be 0,080. At the same time, the risk of violating the requirements for information protection (0,051) is 1,57 times less than the generalized risk of unreliability of the knowledge management process performance without taking into account the requirements for information protection.

CONCLUSION

Within the framework of the proposed methodical approach, the knowledge management process is formalized taking into account the requirements for information protection. The approach allows to estimate the impact of various threats (including threats to the violation of information protection requirements) on the effectiveness of process implementation. The measures of integral risk of the violation of knowledge management process performance, taking into account requirements for information protection, particular risks (covering knowledge acquisition, creating useful knowledge, distribution of acquired or created useful knowledge), and generalized risk taking into account all particular risks are proposed. Recommendations on methods of risk prediction are interpreted, taking into account the complexity of the modelled system and measures to counter threats in each element. The examples illustrate the proposed methodical rationale of system solutions to reduce risks and retain them within acceptable limits with a practical interpretation of the results obtained. This methodical approach is implemented on the level of national standard GOST R 59333-2021.

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Improving Dijkstra's algorithm for Estimating Project Characteristics and Critical Path

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Abstract

Developing a project planning structure for all industries is a technological challenge involving evaluating several restrictions for each activity's respective task and its planning tools. Any restriction affects the completion time, operating costs, and overall project performance. Programme Evaluation Review Technique (PERT) and Critical Path Method (CPM) processes made many researchers study the possible ways of finding the critical paths and activities in the network. The advancement of the CPM and PERT towards a probabilistic environment is still a long way off. However, Artificial intelligence approaches such as the Genetic Algorithm, Dijkstra's algorithm, and others are utilized for network analysis within the project management framework. This study is to help the project manager plan schedule for a construction project to determine the expected completion time. In this research paper, we describe a method for obtaining the earliest and latest times of a critical path using modified Dijkstra's algorithm with triangular fuzzy numbers. Forward pass and backward pass algorithms are designed to find the optimal path for the proposed method. Numerical examples are also illustrated for the same. Simulation results are included by the use of the "C" program. Finally, a comparison is made with the traditional method PERT.

Keywords: Critical Path, Dijkstra's Algorithm, Earliest and latest times, modified Dijkstra's algorithm, PERT.

1. Introduction

In the project network development process, the project controller's objective is to develop a primary plan. The critical path method is one of the most significant approaches in network study. The concept of the critical path allows the decision-maker to control the project's cost and schedule, and it can improve the quality of the work. This method is commonly used in various industries to analyze and improve the efficiency of a project. Many cases have been discussed where the activity times are deterministic; the PERT method applies to a probabilistic environment. Different methods and various working techniques are applied in project management. Every procedure has its own time to complete the task. Gantt chart, network diagram, CPM and PERT are a few strategies commonly used to handle projects.

Every activity's length is considered when estimating the longest path in CPM. In PERT, the activity's estimated time is assumed. Due to that, PERT handles by taking three-point estimates (most likely [m], optimistic [a], and pessimistic [b] and if the activity allocation fits the beta distribution. Malcom first proposed a PERT approximation using the Beta distribution in 1959 [7]. An efficient critical path analysis algorithm (CHAN) based on the automatic test pattern generation (ATPG) approach PODEM was presented by Chang in 1993[1]. Traditionally, the beta distribution was used in the PERT. Solomon Sackey et al., 2018 proposed that an altered PERT was enlarged and utilized to model scheduling risk. The proposed PERT model was based on suspicion of a 95% certainty level. According to the due date and the probability of lag, the project completion probability is computed in five perspectives for both approaches. A sample is taken to assess the error rate for every example. The average error rate was calculated using the traditional PERT technique and the updated PERT method for all cases. The revised PERT version improved the average error rate by 2.46%, correlated with 3.31% of the traditional PERT approach. This way of considering has confirmed that the revised PERT approach can more precisely assess the completion probability better than the traditional PERT. By the way, because the new PERT was entirely based on suspicion, it is ambitious to decide with confidence that it is far superior to the conventional PERT model [8]. Li et al., 2007; The Monte Carlo simulation studied the real-world project's network program. The outcome revealed that the stochastic network program gave significant scheduling data better than the conventional network program [4]. Lee, 2005 proposed Stochastic Project Scheduling Simulation (SPSS) software approach affects the possibility of finishing a project with a deadline set by the software's operator. The SPSS program can simulate activity time using several probability distributions and uniform, triangular, and normal distributions. SPSS also computes the CI for entire project activities [3]. In general, project network simulation is utilized to improve the feasibility and reliability of the PERT study. Cheng et al., 2004 explained the use of MonteCarlo simulation in PERT to achieve a stochastic period of activity [2]. A simulation model that generates a 20 histogram for the distribution of the completed activity network. CPM/PERT simulation technique that adds discrete event simulation plan technique and the critical path determination process. According to the authors, "for each activity, Earliest start ($E\tilde{S}_{ij}$), Latest start ($L\tilde{S}_{ij}$), Earliest finish ($E\tilde{F}_{ij}$), Latest finish ($E\tilde{F}_{ij}$), and Total float ($T\tilde{F}_{ij}$) times should be included in the CPM analysis." The $E\tilde{S}_{ij}$ and $E\tilde{F}_{ij}$ of the project network are computed during its forward pass, whereas the $L\tilde{S}_{ij}$,$L\tilde{F}_{ij}$ and $T\tilde{F}_{ij}$ are determined throughout its backward pass. The $T\tilde{F}_{ij}$ is utilized to evaluate the project's criticality by Lu et al., 2000[5]. Mac Crimmon et al., 1964; one drawback of the PERT is that though many pathways must follow to finish a project, the project time is reduced and does not exceed the average project time [6]. Shankar et al. 2010 used modified Dijkstra's algorithm to estimate project duration [9]. Xiaokang Han et al. proposed in 2021 an improved ant colony algorithm to determine the critical path by setting the path distance and time as negative, while the transition probability remains unchanged, is proposed [10].

In 1965, Zadeh introduced [11] the concept of fuzzy set theory. In today's highly competitive world, many problems in fuzzy mathematics have been produced. When the activity periods in a project environment are deterministic, many real-life events change faster by utilizing the idea of fuzziness.

In this paper, with the help of the program evaluation review technique (PERT) and critical path method (CPM) along with Dijkstra's algorithm approached with an example to formulate the critical path and project duration. The main objective values are taken in fuzzy numbers; we can rank the fuzzy number to find the best alternative.

2. Methodology

2.1 Proposal Algorithm

An altered Dijkstra's Algorithm determines the maximum time between a start node (referred to as the "source node") and further nodes in a network. In this technique, the weights of the edges are utilized to determine the path that optimizes the overall distance (weight) within the start node and further nodes. Modified Dijkstra's Algorithm is only suitable for positive weighted graphs because the weights of the edges must be added during the procedure to determine the longest path.

Basic concepts in an altered Dijkstra's Algorithm

- An altered Dijkstra's Algorithm starts at the node you choose (the source node) and analyzes the graph to identify the longest path within that node and all further nodes in the network.
- The model considers the currently known longest path within an individual node and the origin node, and it modifies the values if the longest path is identified.
- The model finds the maximum distance from one event to another event; the node is labeled as "visited" and adds to the path.
- The procedure is continued till all the nodes in the graph are connected to the path. In the process, we have a path that adds the source node to all further nodes by taking the longest possible path to an individual node.
- The source node is at zero distance from itself. Initially gives '0' labels to all vertices.
- Use the infinity sign to indicate the distance from the source node to all other nodes for the time being because it has not yet been estimated.
- We will find the earliest times in modified Dijkstra's algorithm using the forward pass algorithm and the latest times using the backward pass algorithm.

2.2 Forward pass calculations in Dijkstra's algorithm

Step1: In sequence$v_1 = 1, v_2 = 2, \dots\dots.. v_n = n$, allocate n vertices.
Step2: Assign permanent label '0' to the primary vertex $v_1 = 1$ and provisional label '0' to the rest of $n - 1$ edges.
Step3: Every vertex j that is not permanently labeled would receive a new provisional label.

$$\text{i.e, } E_j = \max\{oldlabelofj, (oldlabelofi + t_{ij})\}$$

Where i is permanently labeled with the new vertex and t_{ij} is the duration of activity between vertices i and j, if an edge is not connected to i and j,$t_{ij} = \infty$.
Step4: The next vertex turns into the fixed (visited) label.
Step3 and step4 repeated until $v_n = n$ gets a fixed label. The $E_j's$ permanently labeled values are the earliest times as $E_1 = 0$.

2.3 Backward pass calculations in Dijkstra's algorithm

Step1: Set n vertices to$v_n = n, v_{n-1} = n - 1, \dots\dots v_1 = 1$.
Step2: Allocate fixed label $L_n = E_n$to the vertex $v_n = n$ and temporary labels to remains $n - 1$ vertices.
Step3: Any node j that does not get a constant label gets a new provisional label.

$$\text{i.e., } L_j = \min[oldlabelofi, (oldlabelofj + t_{ij})]$$

Where j is the fixed labeled with the new vertex $t_{ij}$is the duration
of activity among vertices i and$j$
Step4: as per step1, the next vertex will become a fixed label or permanent label.
Repeated step3 and step4 until then the initial vertex $v_1 = 1$ gets a fixed label.

2.4 The proposal ranking in a Triangular fuzzy number

Let$\tilde{A} = (a, b, c)$be the Triangular fuzzy number and consider the Triangle centroid as the ranking in the Triangular fuzzy number and its diagram expressed in Figure 1.

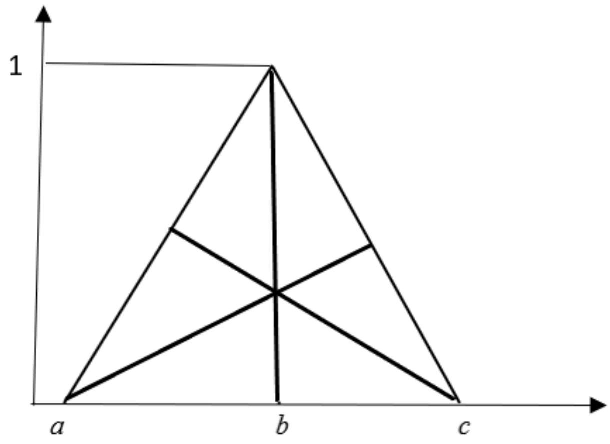


Figure 1: *Diagram representation of centroid of TFN*

The centroid of the Triangle is$\frac{a+b+c}{3}$. Consider, the centroid of triangle is a new ranking in Triangular fuzzy number.
Therefore, the new ranking in Triangular fuzzy number is;

$$\mathcal{R}(\tilde{A}) = \frac{a+b+c}{3}.$$

3 Numerical Analysis

Here I collected applications from Network sources presented in Table1. Moreover, a related network diagram is presented in Figure 2.

Table1: *Application Problem*

Activity	Code	Predecessor	*a*	*m*	*b*
1→2	P	-	5	6	7
1→3	Q	-	1	3	5
1→4	R	-	1	4	7
2→5	S	P	1	2	3
3→6	T	Q	1	2	9
4→6	U	R	1	5	9
4→7	V	R	2	2	8
6→7	W	T, U	4	4	10
5→8	X	S	2	5	8
7→8	Y	W, V	2	2	8

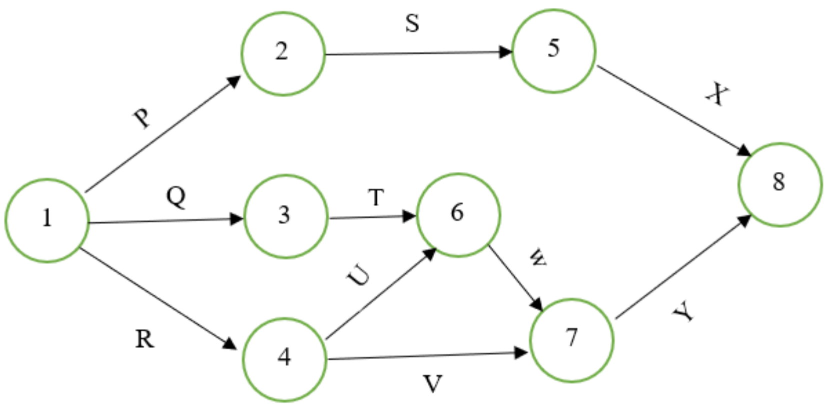


Figure 2: *Diagram representation of application problem*

3.1 Duration of activities calculated by adopting with TFN

The duration of activities taken as the Triangular fuzzy number is depicted in Table 2. Moreover, a related diagram is represented in Figure 3.

Table 2: *Expected time of activities with TFN*

Activity	*a*	*m*	*b*	*TFN*
1→2	5	6	7	(5,6,7)
1→3	1	3	5	(1,3,5)
1→4	1	4	7	(1,4,7)
2→5	1	2	3	(1,2,3)
3→6	1	2	9	(1,2,9)
4→6	1	5	9	(1,5,9)
4→7	2	2	8	(2,2,8)
6→7	4	4	10	(4,4,10)
5→8	2	5	8	(2,5,8)
7→8	2	2	8	(2,2,8)

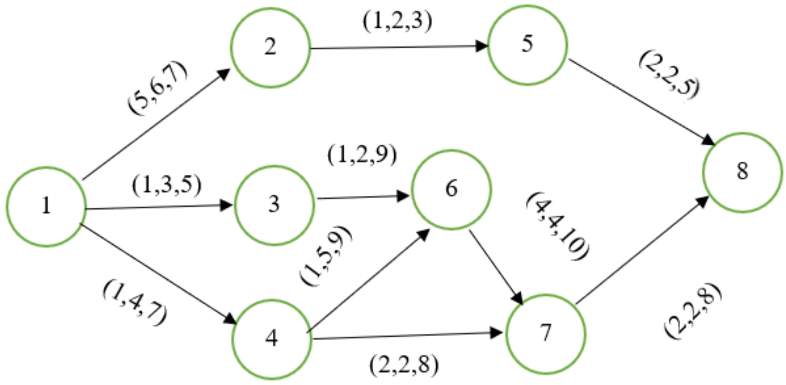


Figure3: *Activities of project network with TFN*

3.2 Earliest times

Earliest times of every node in the project network using forward pass algorithm with TFN seen in Table 3.

Table 3: *Earliest times of every node with a TFN*

Vertex Number								Earliest time
1	2	3	4	5	6	7	8	
(0,0,0)	(0.,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	$E_1 = (0,0,0)$
(0,0,0)(F)	(5,6,7)	(1,3,5)	(1,4,7)	∞	∞	∞	∞	
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)	(1,4,7)	∞	∞	∞	∞	$E_2 = (5,6,7)$
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)	(1,4,7)	(6,8,10)	∞	∞	∞	
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)	(6,8,10)	∞	∞	∞	$E_3 = (1,3,5)$
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)	(6,8,10)	(2,5,14)	∞	∞	
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10)	(2,5,14)	∞	∞	$E_4 = (1,4,7)$
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10)	(2,9,16)	(3,6,15)	∞	
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10)(F)	(2,9,16)	(3,6,15)	∞	$E_5 = (6,8,10)$
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10) (F)	(2,9,16)	(3,6,15)	(8,10,15)	
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10) (F)	(2,9,16)(F)	(3,6,15)	(8,10,15)	$E_6 = (2,9,16)$
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10) (F)	(2,9,16)(F)	(6,13,26)	(8,10,15)	
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10) (F)	(2,9,16)(F)	(6,13,26)(F)	(8,10,15)	$E_7 = (6,13,26))$
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10)(F)	(2,9,16)(F)	(6,13,26)(F)	(8,15,34)	
(0,0,0)(F)	(5,6,7)(F)	(1,3,5)(F)	(1,4,7)(F)	(6,8,10)(F)	(2,9,16)(F)	(6,13,26)(F)	(8,15,34)(F)	E_8= (8,15,34)

3.3 Latest times

The latest times of every node in the project network using backward pass algorithm with Triangular fuzzy ranking formula are seen in Table 4.

Table 4: *Latest times of every node with a TFN*

Vertex number								Latest time
8	7	6	5	4	3	2	1	
(8,15,34)(F)	(8,15,34)	(8,15,34)	(8,15,34)	(8,15,34)	(8,15,34)	(8,15,34)	(8,15,34)	$L_8 = E_8$
(8,15,34)(F)	(6,13,26)	(8,15,34)	(6,13,29)	(8,15,34)	(8,15,34)	(8,15,34)	(8,15,34)	
(8,15,34)(F)	(6,13,26)(F)	(8,15,34)	(6,13,29)	(8,15,34)	(8,15,34)	(8,15,34)	(8,15,34)	$L_7 = (6,13,26))$
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)	(6,13,29)	(4,11,18)	(8,15,34)	(8,15,34)	(8,15,34)	
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)	(4,11,18)	(8,15,34)	(8,15,34)	(8,15,34)	$L_6 = (2,9,16)$
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)	(1,4,7)	(1,7,7)	(8,15,34)	(8,15,34)	
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)	(1,7,7)	(8,15,34)	(8,15,34)	$L_5 = (6,13,29)$
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)	(1,7,7)	(5,11,26)	(8,15,34)	
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)(F)	(1,7,7)	(5,11,26)	(8,15,34)	$L_4 = (1,4,7)$
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)(F)	(1,7,7)	(5,11,12)	(0,0,0)	
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)(F)	(1,7,7)(F)	(5,11,12)	(0,0,0)	$L_3 = (1,7,7)$
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)(F)	(1,7,7)(F)	(5,11,12)	(0,0,0)	
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)(F)	(1,7,7)(F)	(5,11,12)(F)	(0,0,0)	$L_2 = (5,11,12)$
(8,15,34)(F)	(6,13,26)(F)	(2,9,16)(F)	(6,13,29)(F)	(1,4,7)(F)	(1,7,7)(F)	(5,11,12)(F)	(0,0,0)(F)	L_1=(0,0,0)

From the above two tables,
$E_1 = L_1 = (0,0,0), E_4 = L_4 = (1,4,7), E_6 = L_6 = (2,9,16),\ E_7 = L_7 = (6,13,26) E_8 = L_8 = (8,15,34)$.
As a result, the critical path is 1→4→6→7→8, and the project duration is (8, 15, 34).

Now, the TFN (8, 15, 34) is converted into normal time using Triangular fuzzy ranking formula $\frac{a+b+c}{3}$. The defuzzified value of (8, 15, 34) is 19.
Therefore, the project ends in 19 days.

4 Traditional methods

4.1 Program Evaluation Review Technique (PERT)

Program Evaluation Review Technique (PERT) is a project management method for estimating how long it will take to complete a project successfully. There is an approach breakdown structure in project management that split a project into minor projects or activities. Every activity has its timeframe; it demands requirements and gives a result. Much of the time, these activity times are non-deterministic. In specific circumstances, the traditional PERT obtains three-point estimates; optimistic, pessimistic, and most likely. It is a simple strategy that uses a beta distribution mechanism.

The estimation duration for every activity can be predicted by the beta distribution means of the following weighted average:

$$ET = \frac{(Optimistic + 4 * Most\ likely + Pessimistic)}{6}$$

Here we are calculating activity durations using the mean of probabilistic times and presented them in Table 5. The related network diagram is presented in Figure 4.

Table 5: *Duration of activities with probabilistic mean*

Activity	a	m	b	$ET = \frac{a+4m+b}{6}$
1→2	5	6	7	6
1→3	1	3	5	3
1→4	1	4	7	4
2→5	1	2	3	2
3→6	1	2	9	3
4→6	1	5	9	5
4→7	2	2	8	3
6→7	4	4	10	5
5→8	2	5	8	5
7→8	2	2	8	3

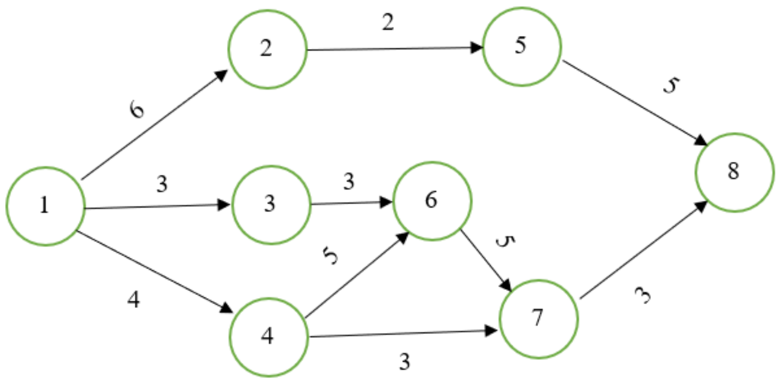


Figure 4: *Activity duration with probabilistic time's network diagram*

4.2 Procedure to find the Critical path:

Step 1: Establish a project network *G (V, E).*

Step 2: Express every activity time as probabilistic time.

Step 3: Determine the earliest start time of activity using forward pass calculations. Let the earliest time as zero for the initial event, $\tilde{E}_1 = 0$.

Then $\tilde{E}_j = max\{\tilde{E}_i + \tilde{t}_{ij}\}$ where *i*= number of preceding events

Step 4: Compute the earliest finish time of activity;

$$(E\tilde{F}_{ij}) = Earliest\ start\ time + the\ activity\ duration$$

$$i.e.\ E\tilde{F}_{ij} = E\tilde{S}_{ij} + \tilde{t}_{ij} = \tilde{E}_i + \tilde{t}_{ij}$$

Step 5: Estimate the latest finish time of activity using backward pass calculations.

$\tilde{E}_n = \tilde{L}_n$. So that $\tilde{L}_i = L\tilde{F}_{ij} = min\{\tilde{L}_j - \tilde{t}_{ij}\}, i = n-1, n-2, \dots\dots, 2, 1.$

Step 6: Calculate the latest start time of activity $(L\tilde{S}_{ij}) = L\tilde{F}_{ij} - \tilde{t}_{ij}$

Step 7: Total float $(T\tilde{F}_{ij}) = L\tilde{F}_{ij} - E\tilde{F}_{ij}\ \ or\ \ L\tilde{S}_{ij} - E\tilde{S}_{ij}$

Table 6: *Earliest and Latest tomes of project activities with probabilistic mean*

Activity	Node	Activity duration	$E\tilde{S}_{ij}$	$E\tilde{F}_{ij}$	$L\tilde{S}_{ij}$	$L\tilde{F}_{ij}$	$T\tilde{F}_{ij}$
1→2	P	6	0	6	4	10	4
1→3	Q	3	0	3	3	6	3
1→4	R	4	0	4	0	4	0*
2→5	S	2	6	8	10	12	4
3→6	T	3	3	6	6	9	3
4→6	U	5	4	9	4	9	0*
4→7	V	3	4	7	11	14	7
6→7	W	5	8	13	12	17	4
5→8	X	5	9	14	9	14	0*
7→8	Y	3	14	17	14	17	0*

The critical activities are 1→4, 4→6, 5→8, 7→8.
Therefore, the critical path is 1→4→6→7→8, and the project completion time is 17.

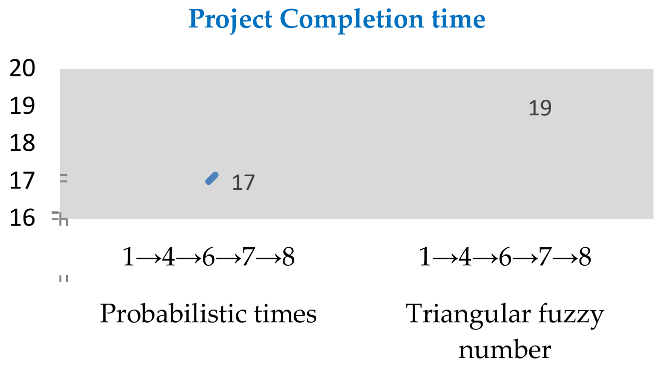
5. Results

Table 7 presents, Critical path and Project duration with Probabilistic and Triangular fuzzy activity times, respectively.

Table 7: *Results*

Activity times	Critical Path	Project Completion time
Probabilistic times	1→4→6→7→8	17
Triangular fuzzy number	1→4→6→7→8	19

Graph 1 presents results that correlate between Probabilistic and Triangular fuzzy mean.



Graph 1: *Correlates the project completion time with Probabilistic mean and Triangular fuzzy mean*

6. Discussion

This article determines the project's earliest and latest times by Modified Dijkstra's algorithm with a triangular fuzzy number and probabilistic times. The network's critical path is identified using project activities earliest and latest times. Moreover, the entire project time is calculated. The project critical path is the same in both cases, but the project completion time is different. Probabilistic mean gives less time compare to fuzzy triangular mean. However, in a non-academic example, this number is affected by various circumstances such as the availability of analysts, the type of activity.

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MLE OF A 3-PARAMETER GAMMA DISTRIBUTION ANALYSIS OF RAINFALL INTENSITY DATA SETS

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Abstract

This research presents the maximum likelihood estimation of a three-parameter Gamma distribution with application to four types of average rainfall intensities in Nigeria. These data sets are average half-yearly, yearly, quarterly and monthly rainfall intensities. The fitted three-parameter Gamma is compared to a two-parameter Gamma distribution using empirical distribution function (EDF) tests. The tests used are Cramér-von Mises, Anderson-Darling and Kolmogorov-Smirnov statistics. Based on the results obtained at 10% significance level both the two-parameter and three-parameter Gamma distributions are of good fit to only the average yearly rainfall intensity data. A kernel density plot revealed that the average half-yearly, quarterly and monthly rainfall intensity data sets are multi-modal in nature hence a reason for both Gamma distributions poor fit to the data sets. Also, the PDF, CDF and Q-Q plots are presented which supported the outcome of the analysis.

Keywords: Gamma distribution, Anderson-Darling, Cramér-von Mises, Kernel density, Kolmogorov-Smirnov, Maximum likelihood estimation

1. Introduction

Classical analysis of statistical data in most fields including meteorology and hydrology has assumed that the data being analyzed may be reasonably modeled by distribution with somewhat *light tailed* where the tail of the density function approaches zero like some kind of exponential function (Arshad, Rasool & Ahmad, [1]). One of the most difficult problems in rainfall modeling is often the fitting of theoretical models to rainfall data (Richard, [10]). According to Hughes [8], the primary objective of modeling is frequently to generate a long representative time series of stream flow volumes from which water supply schemes can be designed. Wolfram [14] stated that Gamma distribution is a general type of statistical distribution that is related to the Beta distribution and arises naturally in processes for which the waiting times between Poisson distributed events are relevant. According to Alghazali & Alawadi [2], the two-parameter Gamma distribution is widely known and used in hydrological analysis. However, Chow *et al.*, [4] stated that the two-parameter Gamma distribution has a lower bound at zero, this condition handicaps its application to hydrological variables with lower bound larger than zero.

In the theory of probability and statistics, the gamma distribution is a two-parameter family of continuous probability distributions. It has a shape and scale parameters, say α and β respectively. If β is an integer, then the distribution represents the sum of β independent exponentially distributed random variables, each of which has a mean of α [which is equivalent to a rate parameter of α^{-1}] (Wackerly *et al.*, [12]). It often appear as solution to problems in Statistical Physics, for example, the energy density of classical ideal gas or the Wien (Vienna) distribution is an approximation to the relative intensity of black radiation as a function of the frequency (Crooks, [5]). The disadvantage of Gamma distribution is that the cumulative distribution function cannot be plotted. The 1-parameter gamma distribution is very limited in hydrological analysis due to its relative inflexibility in fitting to frequency distributions of hydrologic variation (Aksoy, [5]). Gamma distribution is widely used in many fields like reliability, survival analysis, hydrology, ecology, etc. (Dikko, *et al.*, [7]) Many variant of the gamma distribution exist and different estimation techniques have been used for estimating the gamma distribution parameters. These estimation techniques include methods of moment (MOM), percentile method, graphical estimation technique, maximum likelihood estimation (MLE), etc, with different modifications of the estimating techniques. The objective of this research is to present the estimation of the three parameters Gamma distribution using MLE and its application to four average rainfall intensity data sets for Nigeria.

2. Methods

In this section, the Gamma distribution assumptions for its applicability are presented. The probability density function (PDF) for the Gamma distribution is presented and its parameter estimation is presented using the maximum likelihood estimation technique. Four average rainfall intensity data sets which span for 115 years (1901 – 2015) are fitted for this research. The first data set is a quarterly data while the second data set used was obtained by collapsing the quarterly data to first half (FH) of the year and second half (SH) of the year, that is, average of first and second quarters to produce FH and average of third and fourth quarters to produce SH. The yearly rainfall intensity was used as the third data set and the monthly rainfall intensity data was used as the fourth. Data used was obtained from climate knowledge portal, https://climateknowledgeportal.worldbank.org.

2.1. PDF for A 3-Parameter Gamma Distribution

According to Aksoy [5], the Gamma distribution function is of three different types, 1-parameter, 2-parameters and 3-parameters Gamma distributions. If the continuous random variable x fits to the probability density function of:

$$f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}; x \geq 0 \tag{1}$$

it is said that the variable x is 1-parameter Gamma distributed, with the shape parameter k. The Gamma function $\Gamma(k)$ in equation (1) is generally expressed as:

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx \tag{2}$$

when k = 1, equation (1) becomes a simple exponential distribution function. If x is replaced by x/β in equation (1) the 2-parameter Gamma distribution (2-PGD) with k being the shape parameter and β being the scale parameter is obtained as:

$$g(x;k,\beta) = \frac{1}{\beta^k \Gamma(k)} x^{k-1} e^{-x/\beta} \quad ; \ x \geq 0 \tag{3}$$

which can easily return to 1-parameter Gamma distribution for β =1. Gamma distribution with two parameters k and β denoting the shape and the scale parameters respectively are commonly used in hydrological studies (Alghazali & Alawadi, [2]). The shape of the rainfall distribution is regulated by the shape parameter and the scale parameter controls the variation of rainfall intensity series which is specified in the same unit as the random variable x (Suhaila, & Jemain, [11]). If x is replaced by $(x-\lambda)/\beta$ in equation (1) the 3-parameter Gamma distribution (3-PGD) with k, β and λ being the shape, scale and location parameters respectively is obtained as:

$$g(x,\theta)=\frac{1}{\beta^k\Gamma(k)}(x-\lambda)^{k-1}e^{-(x-\lambda)/\beta} \quad ;\ \theta=(\beta,\lambda,\mathrm{k})>0 \tag{4}$$

2.2. Parameter Estimation with Maximum Likelihood Estimation Technique

The likelihood function:

$$Lg(x,\theta)=\prod_{i=1}^{n}g(x,\theta) \tag{5}$$

Applying the likelihood function of equation (5) to equation (4) we have

$$\prod_{i=1}^{n}g(x,\theta)=\frac{1}{\beta^{nk}\left[\Gamma(k)\right]^n}\sum_{i=1}^{n}(x-\lambda)^{n(k-1)}e^{-\sum_{i=1}^{n}(x-\lambda)/n\beta} \tag{6}$$

Taking the logarithm (ln) of equation (6) we get

$$\begin{aligned} ln\prod_{i=1}^{n}g(x,\theta)&=ln\left[\beta^k\Gamma(k)\right]^{-n}+n(k-1)ln\sum_{i=1}^{n}(x-\lambda)-n\beta^{-1}\sum_{i=1}^{n}(x-\lambda)\\ &=-nk\,ln\,\beta-n\,ln\Gamma(k)+n(k-1)ln\sum_{i=1}^{n}(x-\lambda)-n\beta^{-1}\sum_{i=1}^{n}(x-\lambda)\\ &=-nk\,ln\,\beta-n\,ln\Gamma(k)+n(k-1)ln\sum_{i=1}^{n}x_i-n^2(k-1)ln\,\lambda\\ &\quad-n\beta^{-1}\sum_{i=1}^{n}x_i+n^2\beta^{-1}\lambda \end{aligned} \tag{7}$$

Differentiating equation (7) with respect to β and setting the derivative to zero, we have

$$\frac{d\,ln\,Lg(x,\theta)}{d\beta}=-nk\beta^{-1}+n\sum_{i=1}^{n}x_i\left(n\beta\right)^{-2}-\lambda\beta^{-1}$$

$$-nk\beta^{-1}+n\sum_{i=1}^{n}x_i\left(n\beta\right)^{-2}-\lambda\beta^{-2}=0$$

$$-nk\beta^{-1}+\bar{x}\beta^{-2}-\lambda\beta^{-2}=0$$

$$\bar{x}\beta^{-2}-\lambda\beta^{-2}=nk\beta^{-1} \tag{8}$$

Multiply both sides of equation (8) by $(nk)^{-1}\beta^2$ we have

$$\beta=\bar{x}(nk)^{-1}-\lambda(nk)^{-1}$$

$$\hat{\beta}=(nk)^{-1}\left[\bar{x}-\lambda\right] \tag{9}$$

Differentiating equation (7) with respect to λ and setting the derivative to zero, we have

$$\frac{d\,ln\,Lg(x,\theta)}{d\lambda}=-n^2(k-1)\lambda^{-1}+n^2\beta^{-1}$$

$$-n^2(k-1)\lambda^{-1}+n^2\beta^{-1}=0$$

$$n^2\beta^{-1}=n^2(1-k)\lambda^{-1}$$

(10)

Multiply both sides of equation (10) by $\left(n^{-2}\right)\lambda\beta$ we have

$$\hat{\lambda} = \hat{\beta}\left(1-k\right) \tag{11}$$

Differentiating equation (7) with respect to k and setting the derivative to zero, we have

$$\frac{d\, ln\, Lg(x,\theta)}{dk} = -n\, ln\, \beta - n\mathrm{D}\left[ln\,\Gamma\left(k\right)\right] + n\, ln\sum_{i=1}^{n} x_i - n^2\, ln\lambda$$

$$-n\, ln\, \beta - n\mathrm{D}\left[ln\,\Gamma\left(k\right)\right] + n\, ln\sum_{i=1}^{n} x_i - n^2\, ln\lambda = 0$$

$$-n\, ln\, \beta + n\, ln\sum_{i=1}^{n} x_i - n^2\, ln\lambda = n\mathrm{D}\left[ln\,\Gamma\left(k\right)\right]$$

$$\mathrm{D}\left[ln\,\Gamma\left(k\right)\right] = ln\sum_{i=1}^{n} x_i - n\, ln\lambda - ln\, \beta$$

$$\mathrm{D}\left[ln\,\Gamma\left(k\right)\right] = ln\left[\sum_{i=1}^{n} x_i - n\hat{\lambda} - \hat{\beta}\right] \tag{12}$$

where D in equation (12) is the derivative, this implies

$$\mathrm{D}\left[ln\,\Gamma\left(k\right)\right] = \frac{d}{dk} ln\,\Gamma\left(k\right) = \frac{\Gamma'\left(k\right)}{\Gamma\left(k\right)}$$

$$\frac{\Gamma'\left(k\right)}{\Gamma\left(k\right)} = -\gamma + \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+k-1}\right) \tag{13}$$

where γ is the Euler-Mascheroni constant and it is given as

$$\gamma = \lim_{n\to\infty}\left(-ln\left(n\right) + \sum_{i=1}^{n}\frac{1}{i}\right) \approx 0.58 \tag{14}$$

Substituting the value of γ in equation (14) into equation (13) we have

$$-0.58 + \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+k-1}\right) \tag{15}$$

Substituting equation (15) into equation (12) and inserting the estimates of $\hat{\beta}$ and $\hat{\lambda}$ we have

$$-0.58 + \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+k-1}\right) = ln\left[\sum_{i=1}^{n} x_i - n\hat{\beta}k + n\hat{\beta} - \left(nk\right)^{-1}\bar{x}_i + \left(nk\right)^{-1}\hat{\lambda}\right] \tag{16}$$

Equation (16) does not exist in a closed form hence the estimation of k can only be obtained through numerical solution. This can be accomplished using any statistical software. In this research, Statistical Analytical System (SAS) version 9.4 is used to fit both the 2-PGD and 3-PGD.

2.3. Goodness of Fit Test

The goodness-of-fit tests based on empirical distribution function (EDF) are used in this research work. The EDF tests offer advantages over traditional chi-square goodness-of-fit test, including improved power and invariance with respect to the histogram midpoints (D'Agostino and Stephens, [6]). The empirical distribution function is defined for a set of n independent observations $X_1, \ldots, X_n$ with a common distribution function $F(x)$. If we Denote the observations ordered from smallest to largest as $X_{(1)}, \ldots, X_{(n)}$. The empirical distribution function, $F_n(x)$, is defined as:

$$F_n(x) = \begin{cases} 0; x < X_{(1)} \\ \frac{i}{n}, X_{(i)} \le x \le X_{(i+1)}; i = 1,2,\ldots,n \\ 1; x \ge X_{(n)} \end{cases} \tag{17}$$

Note that $F_n(x)$ is a step function that jump [$1/n$] in height at each observation, but in the case where two observations or more are equal, that is, when there are n_j observations at x_j, *then* $F_n(x)$ becomes a step function that jump [n_j/n] in height at each observation x_j. This function estimates the distribution function $F(x)$. At any value x, $F_n(x)$ is the proportion or fraction of observations less than or equal to x, while $F(x)$ is the probability of an observation less than or equal to x. EDF statistics measure the discrepancy between $F_n(x)$ and $F(x)$ which are used to conclude whether the empirical distribution $F_n(x)$ fit the hypothesize distribution $F(x)$. In this research, three EDF tests are used in testing the goodness of fit of each distribution fitted to the average monthly, quarterly, half-yearly and yearly rainfall intensity data. The EDF are Kolmogorov-Smirnov, Anderson-Darling and Cramer-von Mises. These GOF tests are presented below as follows.

2.3.1. Kolmogrov-Smirnov (D) Statistic

According to Wilks [13], the Kolmogorov-Smirnov (D) Statistic is defined as

$$D = Sup_x \left| F_n(x) - F(x) \right| \tag{18}$$

The Kolmogorov-Smirnov statistic belongs to the supremum class of empirical distribution function (EDF) statistics. This class of statistics is based on the largest vertical difference between $F(x)$ and $F_n(x)$. The Kolmogorov-Smirnov statistic is computed as the maximum of D^+ and D^-, where D^+ is the largest vertical distance between the EDF and the distribution function when the EDF is greater than the distribution function, and D^- is the largest vertical distance when the EDF is less than the distribution function.

$$D = \max(D^+, D^-) \tag{19}$$

D represents the maximum difference between the empirical and theoretical distributions over all real numbers x, and is referred to as the Kolmogorov-Smirnov value. $F_n(x)$ is the empirical cumulative probability of observing a value less than or equal to y and $1/n_p$ is added for each observation (x_i) that is greater than zero and less than or equal to y. $F(x)$ is the theoretical cumulative probability at x described by the estimated gamma distribution parameters $(\beta, \lambda, \mathrm{k})$. $F_n(x)$ and $F(x)$ are given as (Husak *et al.*, [9])

$$F_n(x) = \frac{\left(\{i \in \{1,\ 2,\ \ldots,\ n\} : y_i \le y\}\right)}{n} \tag{20}$$

$$F(x) = \int_0^x f(y)dy = \frac{1}{\hat{\beta}\hat{\alpha}\Gamma(\hat{\alpha})} \int_0^x y^{\hat{\alpha}-1} e^{-y/\hat{\beta}} dy \tag{21}$$

A smaller value of D implies a better fit between the observed and theoretical distributions for a fixed number of observations, n.

2.3.2. Anderson-Darling Statistic

The Anderson-Darling statistic and the Cramér-von Mises statistic belong to the quadratic class of EDF statistics. This class of statistics is based on the squared difference $\left(F_n(x) - F(x)\right)^2$. Quadratic statistics have the following general form:

$$Q = n\int_{-\infty}^{\infty} \left(F_n(x) - F(x)\right)^2 \psi(x) dF(x) \tag{22}$$

where, $\psi(x)$ is the weight function for the squared differences $\left(F_n(x)-F(x)\right)^2$.

When the weight function $\psi(x)=\left[F(x)(1-F(x)\right]^{-1}$, then the Anderson-Darling Statistic denoted by A^2 is defined as:

$$A^2 = n\int_{-\infty}^{\infty}\left(F_n(x)-F(x)\right)^2\left[F(x)(1-F(x)\right]^{-1}dF(x) \tag{23}$$

The Anderson-Darling statistic (A^2) is computed as follows.

$$A^2 = -n-\frac{1}{n}\sum_{i=1}^{n}\left[(2i-1)\log U_{(i)}+(2n+1-2i)\log(1-U_{(i)})\right] \tag{24}$$

where, $U_{(i)}$ is the *ith* order Statistic.

2.3.3. Cramer-von Mises Statistic

Explained the Cramér-von Mises statistic as similar to Anderson-Darling Statistic, but in the case of Cramér-von Mises statistic, the weights function $\psi(x)=1$. The Cramér-von Mises statistic denoted by (W^2) is defined by:

$$W^2 = n\int_{-\infty}^{\infty}\left(F_n(x)-F(x)\right)^2 dF(x) \tag{25}$$

The Cramér-von Mises Statistic (W^2) is computed as:

$$W^2 = \sum_{i=1}^{n}\left(U_{(i)}-\frac{2i-1}{2n}\right)^2+\frac{1}{12n} \tag{26}$$

where, $U_{(i)}$ is the *ith* order Statistic.

3. Results

Results from the fitted distributions are presented below. Table 1 presents the empirical 2-PGD and 3-PGD mean and standard deviation (Std. Dev) values for the average half-yearly rainfall intensity, average yearly rainfall intensity, average quarterly rainfall intensity, and average monthly rainfall intensity data sets, that is, **AHYRI**, **AYRI**, **AQRI**, and **AMRI** respectively. It is observed that for all the data sets, the 2-PGD and 3-PGD estimates for the mean is the same as the empirical mean estimate. However, both fitted distributions estimates for the standard deviation are different from the empirical standard deviation for each data set except for the **AYRI** data set. Therefore, both the 2-PGD and 3-PGD estimated equivalent mean and standard deviation values to the that of the empirical mean and standard deviation values of 96.4014 and 7.7945 respectively.

Table 1: *Summary Statistics for the Rainfall Data*

Data Type	Statistic	Observed	2-Gamma Estimate	3-Gamma Estimate
AHYRI	Mean	96.401398	96.4014	96.4014
	Std. Dev	32.162844	32.74474	35.61137
AYRI	Mean	96.401398	96.4014	96.4014
	Std. Dev	7.7944973	7.87561	7.860996
AQRI	Mean	96.401398	96.4014	96.4014
	Std. Dev	78.321223	86.39379	90.5222
AMRI	Mean	96.401398	96.4014	96.4014
	Std. Dev	85.00559	110.8792	113.5311

The results from the summary statistics clearly give a clue that both the 2-PGD and 3-PGD will fit the average yearly rainfall intensity data better. However, such conclusion cannot be for certain

RT&A, No 4 (71)
Volume 17, December 2022

until the fitted distributions are subjected to goodness of fit tests described earlier in section 2.3. The results for the parameter estimates from the 2-PGD and 3-PGD are presented in Table 2 and Figure 1, 2, 3, and 4 shows the histogram plots, the 2-PGD, and 3-PGD curves with the kernel density curve as well for the **AHRI**, **AYRI**, **AQRI**, and **AMRI** data sets.

Table 2: *Maximum Likelihood Parameter Estimates Results*

Data Type	Parameter	2-PGD Estimate	3-PGD Estimate
	Location	****	41.0887
AHYRI	Scale	11.12243	22.92728
	Shape	8.667296	2.412528
	Location	****	-10.213
AYRI	Scale	0.643406	0.579615
	Shape	149.8298	183.9402
	Location	****	4.125437
AQRI	Scale	77.42508	88.80176
	Shape	1.245093	1.039123
	Location	****	0.4245
AMRI	Scale	127.5312	134.296
	Shape	0.755904	0.714667

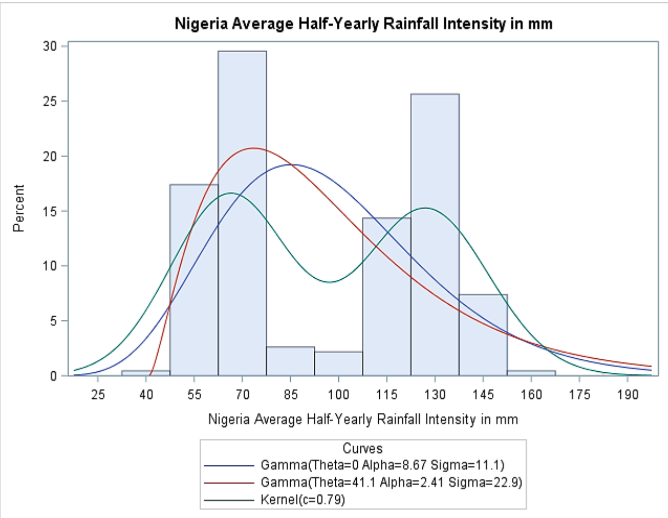

Figure 1: *Fitted Curve for AHYRI Data set*

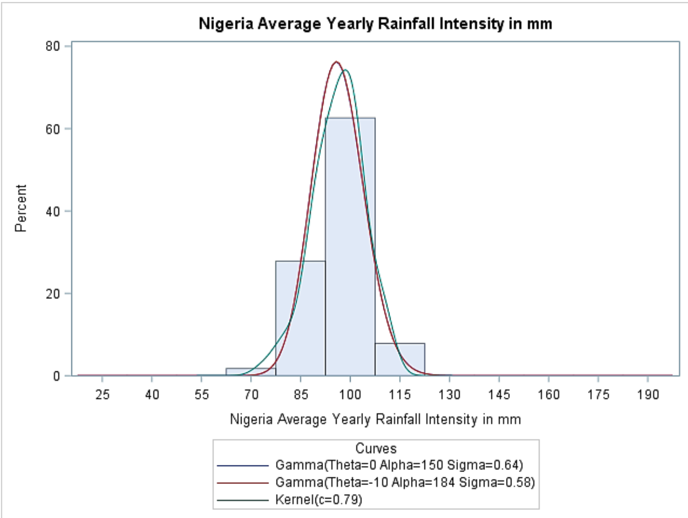

Figure 2: *Fitted Curve for AYRI Data set*

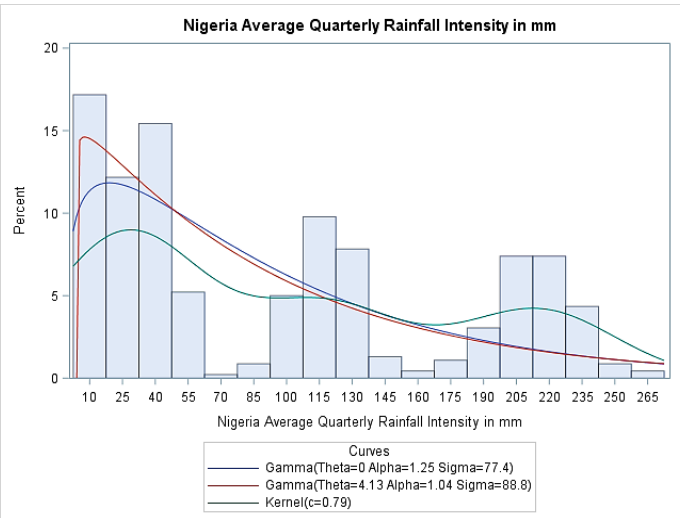

Figure 3: *Fitted Curve for AQRI Data set*

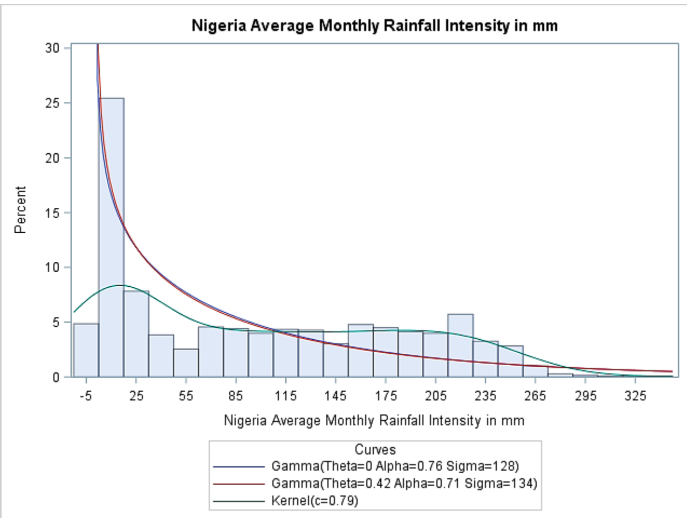

Figure 4: *Fitted Curve for AMRI Data set*

From the figures displayed, it can be seen that the two and three parameter Gamma distributions fits the **AYRI** data set (Figure 2) better compared to the **AHYRI**, **AQRI**, and **AMRI** data sets. Figure 2 shows a peaked shape with one mode compared to Figure 1, 2, and 3 with two modes, three modes and two modes respectively as depicted by the kernel density curve. To ascertain the 2-PGD and 3-PGD goodness of fit for all data sets, Table 3 presents Cramér-von Mises (W^2), Anderson-Darling (A^2), and Kolmogorov-Smirnov (D) statistics results for assessing the fitted distributions.

Table 3: *Criterion for Assessing Goodness of Fit*

Data Type and GOF Methods		Goodness of Fit Estimate (P-Values)	
		2-PGD	**3-PGD**
AHYRI	D	0.1809663(<*0.001*)	0.1972426(<*0.001*)
	W^2	2.3407958(<*0.001*)	2.0138826(<*0.001*)
	A^2	12.7906188(<*0.001*)	11.0297705(<*0.001*)
AYRI	D	**0.06071233(>*0.250*)**	**0.05959971(>*0.250*)**
	W^2	**0.08224762(*0.194*)**	**0.07871282(*0.217*)**
	A^2	**0.54769583(*0.161*)**	**0.52514845(*0.184*)**
AQRI	D	0.1095454(<*0.001*)	0.1179290(<*0.001*)
	W^2	1.9423611(<*0.001*)	1.6980158(<*0.001*)
	A^2	12.1566899(<*0.001*)	10.5830877(<*0.001*)
AMRI	D	4.18050(<*0.001*)	4.78650(<*0.001*)
	W^2	6.45624(<*0.001*)	6.00624(<*0.001*)
	A^2	38.67804(<*0.001*)	37.57614(<*0.001*)

Bold p-values imply good fit

From Table 3 above, it is clearly seen that the 2-PGD and 3-PGD are poor fit to Nigeria average half-yearly, quarterly, and monthly rainfall intensity data sets. The reason is that D, W^2 and A^2 statistic values produced p-values less than 0.01 but they produced p-values greater than 10% significance level for average yearly rainfall intensity. Therefore, it is clear from the goodness of fit statistics p-values that both the 2-PGD and 3-PGD are good fit to only the average yearly rainfall intensity data. To buttress the results discussed thus far, the cumulative density function (CDF), quantile estimates, and quantile plots (Q-Q plots) are presented. The CDF plots presented in Figure 5, 6, 7, 8, 9, 10, 11 and 12 clearly shows that only the 2-PGD and 3-PGD CDF plots for the AYRI data has a well fitted *S*-shape as seen in figure 7 and 8 respectively.

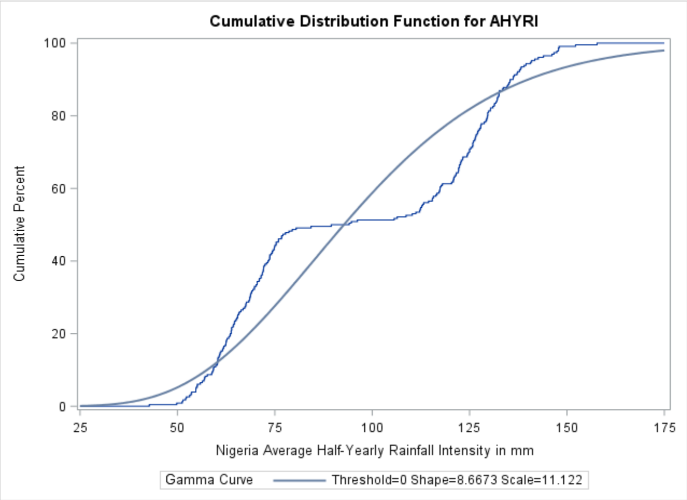

Figure 5: *2-Parameter Gamma CDF Curve*

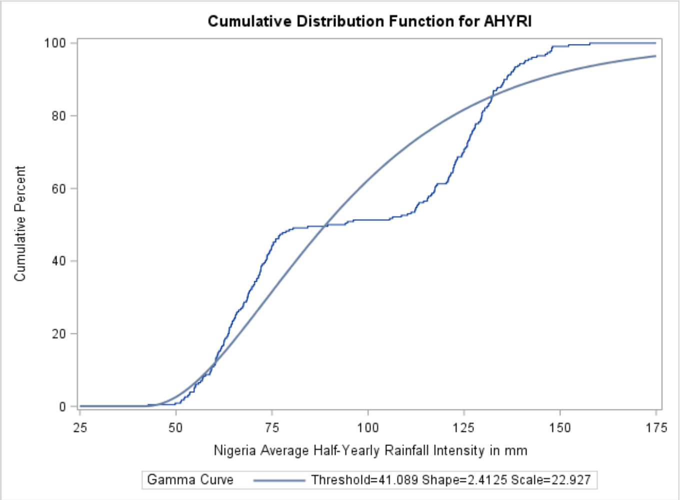

Figure 6: *3-Parameter Gamma CDF Curve*

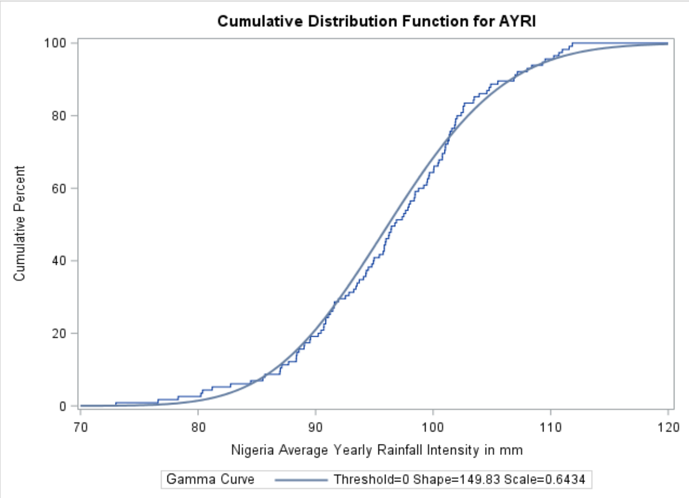

Figure 7: *2-Parameter Gamma CDF Curve*

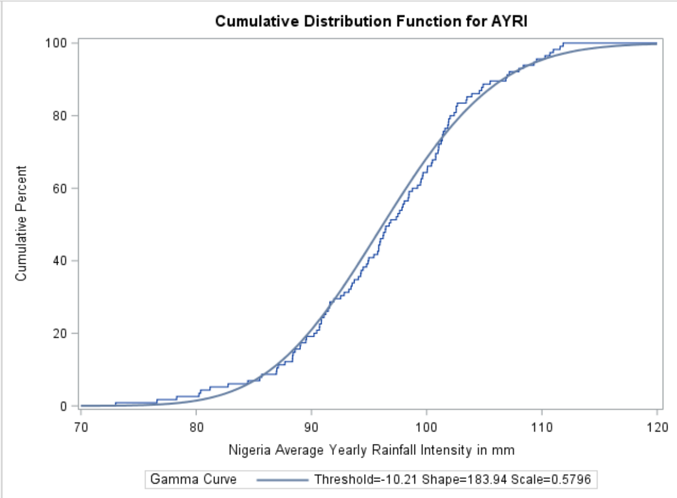

Figure 8: *3-Parameter Gamma CDF Curve*

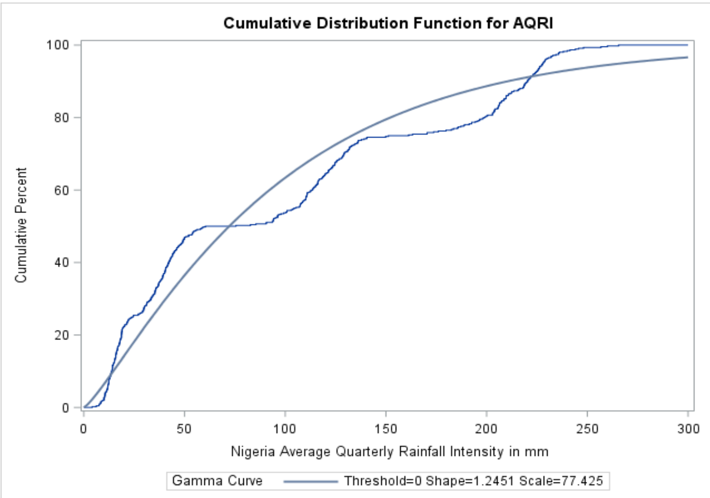

Figure 9: *2-Parameter Gamma CDF Curve*

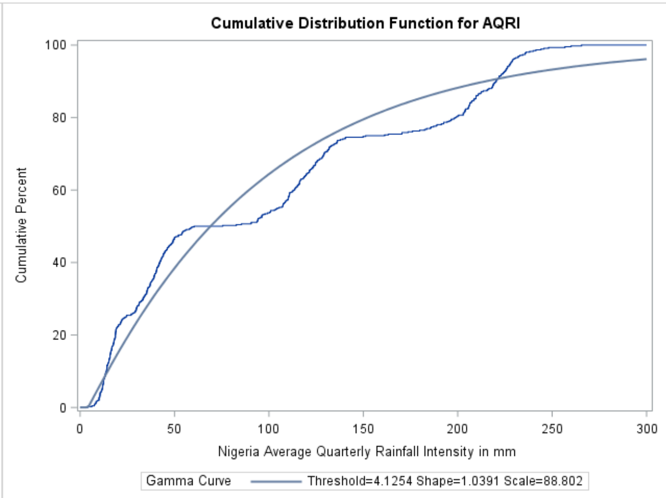

Figure 10: *3-Parameter Gamma CDF Curve*

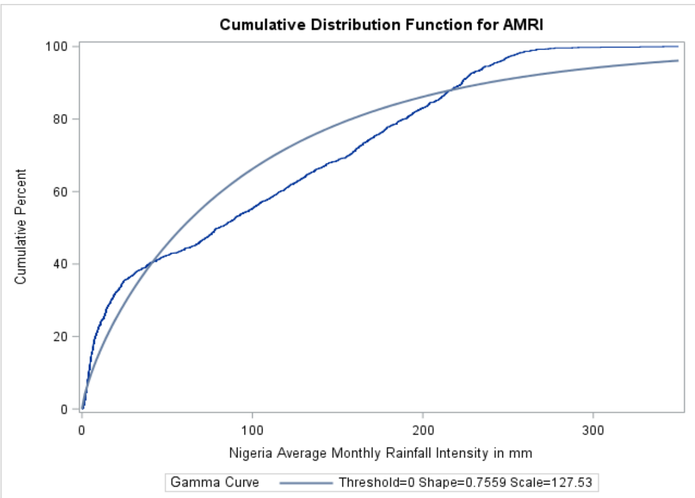

Figure 11: *2-Parameter Gamma CDF Curve*

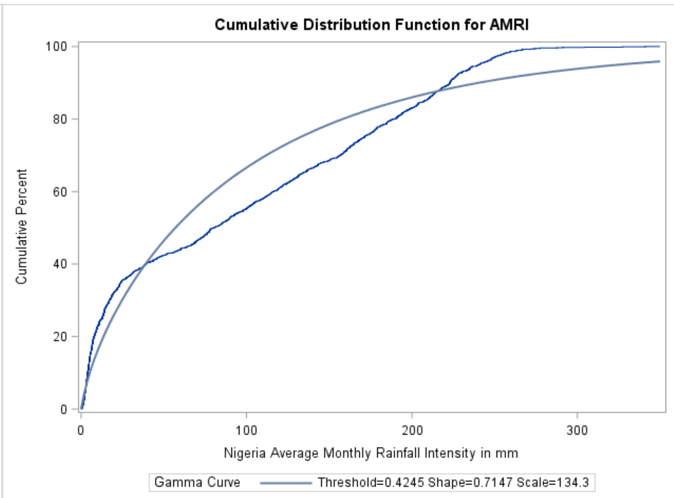

Figure 12: *3-Parameter Gamma CDF Curve*

The estimated quantile presented in Table 4 shows that the 2-PGD and 3-PGD estimated quantiles are similar to the empirical quantiles for AYRI compared to AHRI, AQRI and AMRI data sets.

Table 4: *Quantile Estimates from the three Distributions for the four Quarters*

	PERCENTAGE	OBSERVED	2-Gamma	3-Gamma
	1.0	51.2246	36.7578	46.8728
	5.0	54.7701	49.5577	53.2972
	10.0	59.2066	57.5328	58.4083
	25.0	65.4879	72.7802	70.2148
AHYRI	50.0	91.8013	92.7203	88.9753
	75.0	126.9564	116.0211	114.5791
	90.0	135.8106	140.0208	144.0947
	95.0	140.9320	155.8089	164.8805
	99.0	147.9636	188.3987	210.4818
	1.0	76.5994	79.0310	78.9703
	5.0	81.2038	83.8231	83.8090
	10.0	86.9932	86.4562	86.4594
	25.0	91.0947	90.9790	90.9992
AYRI	50.0	96.7386	96.1870	96.2083
	75.0	101.4315	101.5901	101.5931
	90.0	106.8712	106.6221	106.5916
	95.0	109.5241	109.7110	109.6526
	99.0	111.5782	115.6631	115.5364
	1.0	7.98870	2.14038	5.20545
	5.0	11.63405	8.06373	9.32423
	10.0	13.95009	14.59845	14.54737
	25.0	24.26035	33.88568	31.70857
AQRI	50.0	67.92154	72.16883	69.04598
	75.0	156.24420	133.05217	132.00585
	90.0	220.33728	210.27723	214.59713
	95.0	228.02246	267.51252	276.84191
	99.0	245.61700	398.36593	420.96745
	1.0	0.97407	0.25860	0.61226
	5.0	2.58519	2.19311	2.22199
	10.0	3.91610	5.56936	5.22762
	25.0	12.62992	19.93095	18.76462
AMRI	50.0	80.51468	58.62833	56.97441
	75.0	171.20598	132.96517	132.42207
	90.0	223.01040	237.61879	240.17013
	95.0	240.39757	319.16261	324.67099
	99.0	266.14557	512.60240	526.04011

The quantile plots for the 2-PGD and 3-PGD are presented in Figure 13, 14, 15, 16, 17, 18, 19 and 20. The 2-PGD and 3-PGD Q-Q plots for the AYRI data set showed almost all points fall on the reference straight line. This implies that the quantiles of the theoretical and data distribution agree for AYRI data set only.

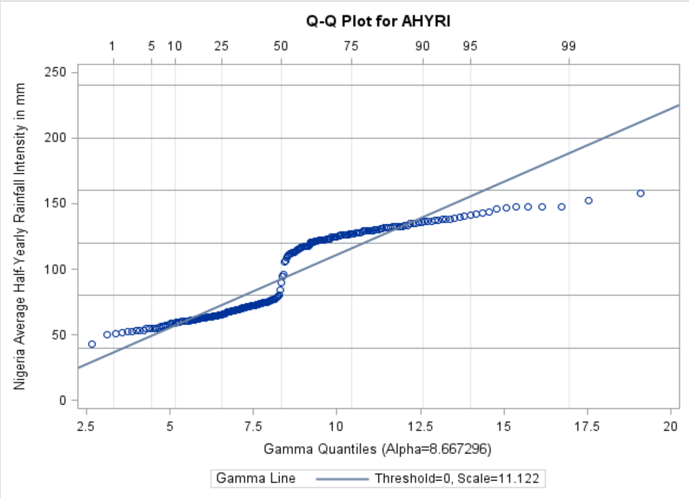

Figure 13: *AHYRI 2-P-Gamma Q-Q Plot*

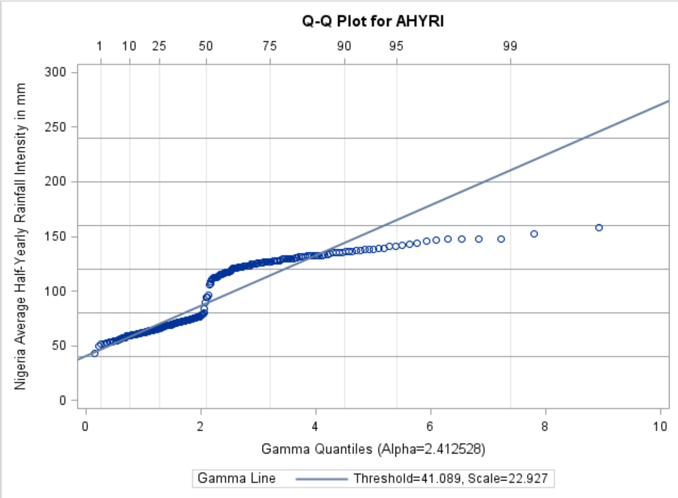

Figure 14: *AHYRI 3-P-Gamma Q-Q Plot*

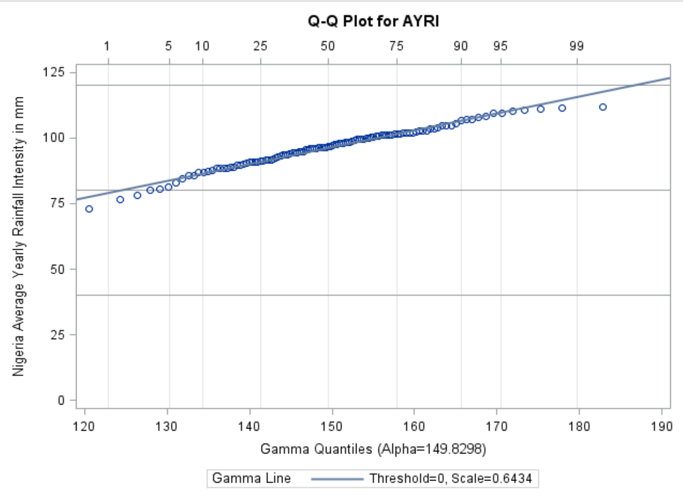

Figure 15: *AYRI 2-P-Gamma Q-Q Plot*

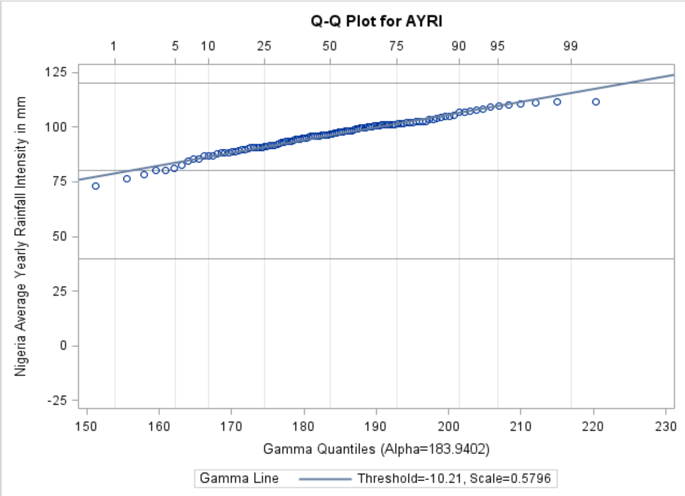

Figure 16: *AYRI 3-P-Gamma Q-Q Plot*

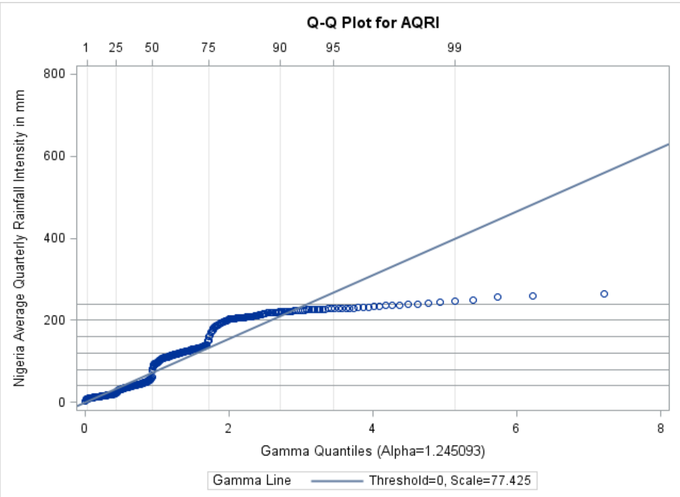

Figure 17: *AQRI 2-P-Gamma Q-Q Plot*

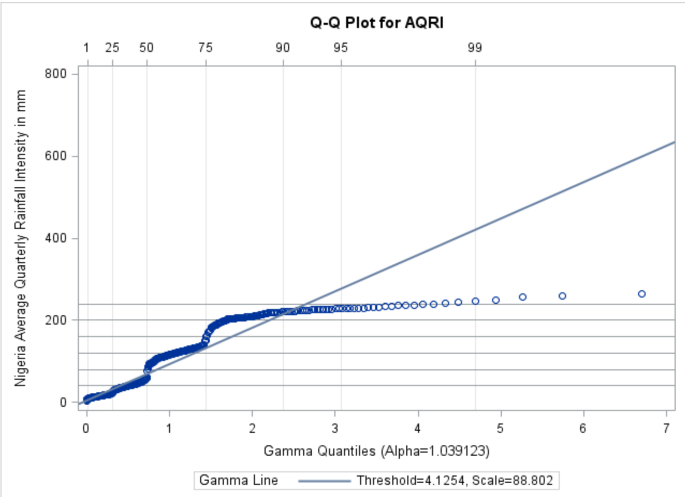

Figure 18: *AQRI 3-P-Gamma Q-Q Plot*

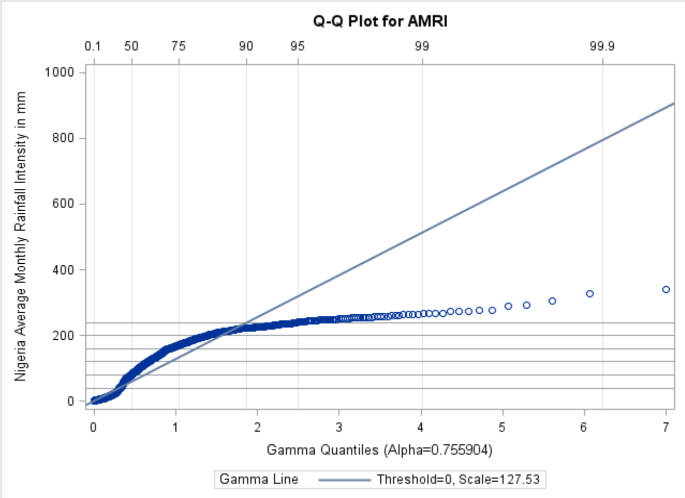

Figure 19: *AMRI 2-P-Gamma Q-Q Plot*

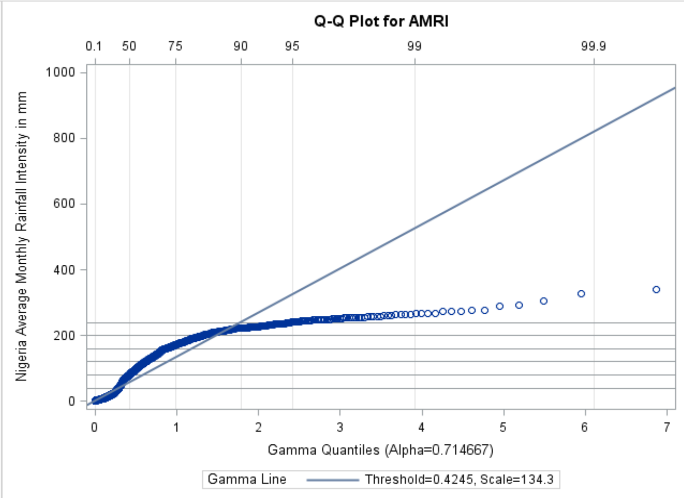

Figure 20: *AMRI 3-P-Gamma Q-Q Plot*

4. Conclusion

In this research, the maximum likelihood parameter estimation of a 3-PGD is presented. Also, its application to four different average rainfall intensity data sets was performed and compared to a 2-PGD. A goodness of fit test was performed using three criterions, that is, Cramér-von Mises (W^2), Anderson-Darling (A^2) and Kolmogorov-Smirnov (D) statistics. Based on the results obtained it is concluded that among the four data sets fitted, the 2-PGD and 3-PGD are good fit to Nigeria yearly rainfall intensity data set only. The PDF curves with kernel density curves, CDF curves and Q-Q plots showed supporting evidence as the goodness of fit statistics (W^2, A^2 and D) results. The kernel density curves showed that AHYRI, AQRI and AMRI data sets are multi-modal data sets and it is a major reason both the 2-PGD and 3-PGD fitted the data sets poorly. Hence, distributions that handle multi-modal data will be more suitable for fitting the AHYRI, AQRI and AMRI data sets.

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Designing of Inventory Management for Determining the Optimal Number of Objects at the Inventory Grouping Based on ABC Analysis

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Abstract

The most appropriate procedures in the inventory organization area are inventory arrangements based on ABC investigation, a well-known technique for establishing the objects in a different collection, giving their status and principles. This research Bi- A mathematical goal to advance the inventory group founded on the ABC. The Planned model instantly improves the amenity level, the amount of inventory grouping, and the number of due things. An Arithmetical model is available in this study to categorize inventory objects, considering significant revenue and rate decrease catalogues. The model aims to maximize the net gain of available items. Economic and inventory constraints are also taken into account. The Benders decay and Lagrange reduction procedures respond to classical arithmetical stands. The outcomes of the two answers are then equated. TOPSIS and numerical examinations estimate the planned answers and choose the best. Later, numerous sensitivity studies on the classic were completed, which assists inventory control executives in regulating the outcome of inventory administration rates configured for optimum verdict production and element grouping. The Arithmetical diagram was run for ten different arithmetic instances, and the results of the two suggested explanations were statistically equated using a t-test. As a result, the TOPSIS technique was appropriate; the Lagrangean approach was chosen as the more fabulous technique.

Keywords: ABC analysis, Bi-Goal optimization, inventory control, decomposition procedures, TOPSIS

1. Introduction

Given the intense business competition in today's manufacturing world, it is vital to repay the kindness to inventory control and appropriate regulation of altogether sorts of administrations, particularly industrial singles. Many establishments' entire investment has been completed up of their inventories in recent centuries. In established and emerging nations, investment properties of altogether periods are tall. The ABC study is the utmost frequently utilized technique for preparation and supervising the inventory [1]. The cataloging of stocks founded on the ABC investigation allows the establishments to classify their stocks into expressive collections.

Overall, the ABC technique trails the Pareto law, implying thatlone30% of catalogs make 70% of the overall revenue, and the remaining 70% of records yield 30% of revenue [2]. Session A has the invention by the maximum worth, and session C has the product with the bottom worth [3]. The stated ABC technique has specific difficulties, like the absence of suitable strategies to regulate the service level, supervising the relative amongst facility levels and group results, and deserting to deliberate the economic restriction in all steps [4]. These difficulties make investigators project an established ABC technique.

This arrangement is emphasized, particularly for the association in evolving and established countries, as a significant proportion of their venture is grounded on catalogs. This article plans a multiple-criteria decision analysis (MCDA) ABC investigation to exploit the entire clear revenue at different planes for a construction business below the economic restriction. Other approaches exist in current centuries to categorize ABC multiple-criteria decision analysis (MCDA). In this esteem, it may mention the analytic hierarchy process (AHP), non-natural intelligence methods, arithmetical study, information envelopment investigation (IEI) [5], emotional Euclidean detachment, standard measure matrix classical, collection investigation classical, meta-heuristic procedures, optimization measures, ABC-FUZZY organization method. However, they have been utilized in the current study and even compared. This arrangement typically too serves inventory and procedure executives to optimize several objects instantaneously, counting(a) the amount of catalog grouping, (b) their facility stages, and (c) the distribution of individual pieces toward a piece collection below a partial economical.

2. Literature review

2.1. ABC study for the inventory controller of stock

In 1915, Ford Harris from the Westinghouse Foundation gave an unpretentious formulation for inventory control [5]. Then, this autonomous formula was verified by some investigators [6, 7]. The old-style ABC technique categorizes the objects founded on sole standards (i.e., rate), and numerous lessons are absorbed in multi-criteria cataloging [8- 11]. Ng and Ramanathan [12] also [13] deliberated the multiple-criteria decision analysis (MCDA) ABC technique toward regulating the account by calculating the average price of each component, yearly in getting worth, then head period. [14] too offered multiple-criteria decision analysis (MCDA) ABC technique then applied the combined standards medium.

Supplementary investigators study the grouping of the ABC- investigation procedure by different thoughts toward the account organization's progress. As an initial move, additional rational then inclusive pointers for account organization were planned. Furthermore, Douissa and Jabeur [15] designed an original ABC investigation classical and utilized it characterized by compensation. Accumulation procedure toward organizing the account objects. The outcome established that the organization of things shaped the lowermost inventory rate related to additional models.

Moreover, Yu [16] connected the group techniques founded on old-style numerous prejudiced analyses (MDA). Based on their consequences, AI procedures were specific in the account group, and the SVM was more precise among the AI procedures. Furthermore, Gong et al. [17] utilized an involuntary knowledge technique (IKT), which involves the TOPSIS method for analyzing the slash of individual account items. Similarly, Mehdizadeh [18] planned a combined ABC analysis technique for circulation systems of auto replacement fragments with the reflection of the account controller procedure. The planned ABC investigation additional the economic morals to strains then classifies the replacement fragments found and arranged their effects scheduled supplier presentation. Gong et al. [19] utilized the disappointment method result and criticalness investigation (DMRCI) to plan a multiple-criteria decision analysis (MCDA) ABC investigation system for replacement fragment manufacturing. They found a massive development in ABC investigation and a substantial decrease in chief mechanisms percentage. Most of the studies cited the ABC investigation.

2.2. Lagrange and benders procedures in the evaluation

Optimization sums consist of various matrix constructions arranged in the plan of medium chunks and their association [20]. The procedures that apply this problematic medium construction are frequently additional effective and find the correct response to the sum at the suitable period. Overall, the structure of optimization sums often contains compound restrictions or compound variables. These restrictions and variables typically replicate the communal usage of sum chunks on one or other infrequent bases. Then additional arrangements are utilized to such sums. Indirectly explaining sum using decomposition methods, it is compulsory to recognize the sum construction. Many investigators have used those analytical procedures, some of which are presented below.

Adaptable get-together lines often occur in businesses fabricating extensive goods. Numerous workforces are allocated toward a similar position toward accomplishing multiple errands on identical merchandise instantaneously. The effective Arithmetical formulas obtainable can only resolve a few minor examples, while greater ones are resolved by empirical approaches that organize non-require assurance optimality. Wang et al. [21]'s article introduced an original short-interest LP construction by substantial equilibrium disruption restrictions.

2.3. Investigation gap

Inventories are a dynamic component for altogether establishments in today's manufacturing world. In fresh periods, establishments have met thousands of unlike kinds of accounts, then account managing and making has stayed the focus of planning's in this esteem. Appropriate account controller arrangements have developed a substantial contest for altogether establishments, which is essential for Investigation in this zone. The absence of proper inventory control schemes generates numerous glitches for administrations. Initially, they face inventory-related rates for fields, collection, and famines.

3. Explanation of the Arithmetical classical

The Arithmetical problem is founded on dominant stock and w constituencies. There is supposed to be only one significant stock and numerous divisions with special requests and scar city rates in this model.

Table 1: *Summary of the literature review*

Reference	Segment A					Segment B		
ABC investigation						Decomposition procedures		
	Goal function		Criteria		Solution techniques	Lagrange	Benders	Comparison
	Single	Multi	Single	Multi				
Hadi-Vencheh [19]	Yes	No	No	Yes	Extended version of NG-model	No	No	No
Massart [16]	Yes	No	Yes	No	-	No	No	No
Flores and Whybark [17]	No	No	No	Yes	A simple mechanical procedure	No	No	No
Kaabi *et al*. [5]	No	No	No	Yes	Automatic learning technique	No	No	No
Douissa & Jabeur [13]	Yes	No	No	Yes	PROFT technique	No	No	No
Liu *et al*. [7]	Yes	No	No	No	Clustering examination	No	No	No
Hajbabaie [8]	No	No	No	No	-	No	No	No
Mardan *et al*. [4]	No	No	No	No	-	No	Yes	No
Jaglarz *et al*. [9]	No	No	No	No	-	Yes	No	No
Li *et al*. [11]	No							
Zetina *et al*. [25]	No							-
Sudhakar *et al*. [23]	-							
Wang *et al*. [21]	No	No	No	No		Yes	No	Yes
Li and Jia [10]	No	No	No	No	-	No	Yes	No

Where
i = Established inventory objects
j =Established inventory grouping
t = Establishedtime
Strictures
$d_{(i,t)}$ =Average of monthly calls for SKU I in time$t$

σ_i =Typical deviation of the once-a-month direction of SKU i
$h_{(i,t)}$ = Headperiod of SKU i
$\pi_{(i,t)=}$ Un civilized revenue per component of SKUI in time t
$e_{(i,t)=}$ Inventory field price per component of SKU I in time t
θ_j = Variable above control rate for inventory objects j
M = Amount of inventory objects (SKUs)
D = Existing in expensive
β_j = Facility level related to inventory objects j
Z_J = Z – value related to the facility level β of inventoryobjects j
$O_{(i,t)}$ = Set direct rate for SKU i from the central stock to traders in time t
$CL_{(I,j,t)}$ = Set rate of shortage for SKU I at inventory group j in time t

Decision variables
$V_{(i,t)}$ = The inventory level of SKU $i$in the dominant stock in time t
$La_{(i,j,t)}$ = The total of lack SKU$i$at inventory group $j$in time t
$X_{(i,j,t)}$ = If SKU$i$is allocated to group j in time t,one and otherwise 0
$Y_{(i,t)}$ = If inventory group $j$is specific in time$t$, one and otherwise 0

$$\delta_{(i,j,t)} = d_{(i,t)}h_{(i,t)} + z_j\sigma_i\sqrt{h_{(i,t)}} \quad (1)$$

$$MaxZ = \sum_i\sum_j\sum_t \pi_{(i,t)}d_{(i,t)}\beta_j X_{(i,j,t)} - \sum_j\sum_t \theta_j Y_{(i,t)} - \sum_i\sum_j\sum_t CL_{(i,j,t)}\, La_{(i,j,t)} \quad (2)$$

Subject to:

$$\sum_j X_{(i,j,t)} \leq 1 \quad (3)$$

$$\sum_j X_{(i,j,t)} \leq MY_{(i,t)} \quad (4)$$

$$V_{(i,t)} + \sum_j La_{(i,j,t)} = \sum_j d_{(i,t)}h_{(i,t)}X_{(i,j,t)} + \sum_j z_j\sigma_i\sqrt{h_{(i,t)}}\; X_{(i,j,t)} + V_{(i,t)-1} + \sum_j La_{(i,j,t)-1} \quad (5)$$

$$\sum_i e_{(i,t)}V_{(i,t)} \leq D\forall t \quad (6)$$

$$\sum_i V_{(i,t)}, La_{(i,j,t)} \geq 0 \quad (7)$$

$$X_{(i,j,t)}, Y_{(i,t)} \in [0,1] \quad (8)$$

4. Planned decomposition procedures Head time of SKU

In several Arithmetical replicas, with the problematic magnitude, the computational difficulty of the perfect also grows exponentially so that the particular answers cannot be intended in a sensible date [23, 7]. Subsequently, investigators have planned several systems that use a specific method to pursue estimated and near-optimal answers. These approaches are usually separated into two group's experiential and meta-heuristic procedures. Disintegration events are among the experiential techniques that aim to shorten compound Arithmetical models to attain an estimated response in a reasonable time. Frequent requests for these procedures have commanded their application in many optimization difficulties. Educations through Yolmeh and Saif [24], Wang et al. [25], Naderi et al. [26], and Aydin and Ta¸skin [27] are instances of the request for these procedures toward hard restraint complications.

This segment presents the explanations for the planned classical. In this respect, situations are completed to Lagrange and Bender's disintegration procedures. Then, they are associated with choosing the most satisfactory answer to the model. As stated, disintegration explanations are envisioned to shorten the Arithmetical model planned in this Investigation. These explanations are familiarized in the subsequent subdivisions.

4.1. Lagrange reduction procedure

The Lagrange reduction process is one of the advanced approaches that employ the Lagrange proposition to explain composite Arithmetical representations to find an estimated solution in a reasonable amount of time. This method has been used to solve a variety of optimization problems. Diabat et al. [16], Kang and Kim [28], and Ahmadi-Javid and Hoseinpour [29] are examples of applications of the Lagrange approach to such challenges.

$Min\ c^T x$

Subject to

$Ax \leq \text{b}, x \in X$

$Min\ c^T x + \mu^T(Ax - b)$

Subject to $x \in X$

$$MaxZ + u\left(-\sum_i \sum_t V_{(i,t)} - \sum_i \sum_j \sum_t La_{(i,j,t)} + \sum_i \sum_j \sum_t d_{(i,t)} h_{(i,t)} X_{(i,j,t)} + \sum_i \sum_j \sum_t z_j \sigma_i \sqrt{h_{(i,t)}}\ X_{(i,j,t)} + \sum_i \sum_t V_{(i,j)-1} - \sum_i \sum_j \sum_t La_{(i,j,t)-1}\right)$$

$$u^{c+1} = max\left[0, \left\{u^c + \pi^c . \left(-\sum_i \sum_t V_{(i,t)} - \sum_i \sum_j \sum_t La_{(i,j,t)} + \sum_i \sum_j \sum_t d_{(i,t)} h_{(i,t)} X_{(i,j,t)} + \sum_i \sum_j \sum_t z_j \sigma_i \sqrt{h_{(i,t)}}\ X_{(i,j,t)} + \sum_i \sum_t V_{(i,j)-1} - \sum_i \sum_j \sum_t La_{(i,j,t)-1}\right)\right\}\right]$$

$$\pi^c = \frac{v^c(BUB^c - LB^c)}{\left(-\sum_i \sum_t V_{(i,t)} - \sum_i \sum_j \sum_t La_{(i,j,t)} + \sum_i \sum_j \sum_t d_{(i,t)} h_{(i,t)} X_{(i,j,t)} + \sum_i \sum_j \sum_t z_j \sigma_i \sqrt{h_{(i,t)}}\ X_{(i,j,t)} + \sum_i \sum_t V_{(i,j)-1} - \sum_i \sum_j \sum_t La_{(i,j,t)-1}\right)^2}$$

4.2. Bender's decomposition algorithm

Benders [30] planned the Benders decomposition algorithm to resolve compound number complications.

$$Min\ Z = -\sum_i \sum_j \sum_t \pi_{(i,t)} d_{(i,t)} \beta_j X_{(i,j,t)} + \sum_j \sum_t \theta_j Y_{(i,t)} + \sum_i \sum_j \sum_t CL_{(i,j,t)} La_{(i,j,t)} \tag{9}$$

Subject to: $1 \geq \sum_j X_{(i,j,t)}\ \forall\ i, t$ (10)

$$0 \geq -MY_{(i,t)} \sum_i X_{(i,j,t)}\ \forall\ j, t \tag{11}$$

$$\sum_j d_{(i,t)} h_{(i,t)} X_{(i,j,t)} + \sum_j z_j \sigma_i \sqrt{h_{(i,t)}}\ X_{(i,j,t)} + V_{(i,t)-1} + \sum_j La_{(i,j,t)-1} - V_{(i,t)} + \sum_j La_{(i,j,t)} = 0 \tag{12}$$

$$\sum_j e_{(i,t)} V_{(i,t)} \geq -D\ \forall\ i, t \tag{13}$$

5. Comparison of decomposition procedures

This section decides whether to approve the anticipated approach for resolving the current Arithmetical model to manage the optimal amount of inventory grouping in stock. There are two decomposition processes available: Lagrangean and Benders. The classical is then resolved, and the effects are equal. The Arithmetical classic for each technique is applied in 10 different arithmetical illustrations in the direct repetition and comparison of these two intended ways. The results of this procedure are then equated in all mathematical instances using t-tests using the arithmetical presumption investigation. The TOPSIS method is also used to determine the optimal approach. It is worth noting that with the GAMS software version 24.1.3 and CPLEX problem solver, all arithmetical samples are used to resolve the intended Arithmetical classic.

5.1. Numerical instances

In this section, the indicators are defined first to approximate the anticipated Arithmetical construction and technique of solution. As a result, two indices are determined. They include the value of the Goal purposes premeditated by the model with each planned technique and the time spent. In addition, the planned technique is equated to generating some mathematical drawings.

Table 2: *The consequences of the application of the classical by dissimilar mathematical instances*

Arithmetical instances	Bender's decomposition		Lagrangean reduction	
	OBJ(RS)	Time(s)	OBJ(RS)	Time(s)
1	3.49040D+7	2.14	3.468116D+7	9.51
2	2.62670D+7	3.03	2.624825D+7	6.23
3	4.22314D+7	7.67	3.601867D+7	9.64
4	4.41611D+7	47.88	4.413372D+7	12,12
5	1.00500D+7	3.04	1.003803D+7	7.07
6	3.23075D+7	19.67	3.228605D+7	8.53
7	1.31380D+7	4.43	1.312534D+7	4.28
8	2.83043D+7	8.31	2.827428D+7	8.64
9	3.33035D+7	7.32	3.327178D+7	9.06
10	3.41170D+7	9.17	3.410455D+7	90
Average	3.08563D+7	12.27	3.020718D+7	8.31

5.2. Statistical examination of the outcomes

Some t-tests are utilized to investigate the outcomes of the 2 planned approaches of resolving the Arithmetical model and equating them. Specified the 95% confidence equal, the numerical evaluation of the resources of the outcomes of the 2proposedapproaches is achieved for individually of the definite estimation directories. In individual assessment, the supposition of nothing (H0) is equivalent to the unkindness of the consequences of the 2 planned techniques, and the conflicting hypothesis (H1) pursues to contest this supposition. This theory examination is stretched for counted directories; these resources at 95% self-possession level; nearby is no expressive change among the answers of the two planned explanations concerning the Goal role worth power. Similarly, the insignificant hypothesis concerning the CPU Period power of the classical is recognized since its P-value is more advanced than 0.05, which incomes there is no considerable modification among the replies of the 2planned key methods about the CPU Period index.

Table 3: *The outcome of numerical examination for the OBJ function*

OBJ function	N	Mean	St. Dev	SE Mean	P-Value
Bender's breakdown	10	309562500	110663870	24020014	0.783
Lagrangean reduction	10	302071820	103420821	22044473	

Table 4: *The outcome of numerical examination for the CPU period*

OBJ function	N	Mean	St.Dev	SE Mean	P-Value
Bender's breakdown	10	12.3	15.7	4.2	0.368
Lagrangean reduction	10	8.31	1.01	0.56	

Table 5: *Summary of the old-style ABC examination*

Class	Proportion of objects	The ratio of things worth	Facilities level (%)	The worth of each type ($)
A	8.45 7.34	65.82 54.71	93 82	102038493853.37 102028382742.26
B	19 18	24.33 13.22	90 80	37722210667 26611100556
C	61.44	8.24	70	141578166681
Total	90	90	90	144018521201.26

Table 6: *The preparation has got the best facility level and inventory grouping answers*

The group with service level (%)	No. of SKUs (%)	Inventory spending ($)	Gross revenue ($)	ROI
99	15(13.72)	524664040.4	231800800	3.33
95	4(3.52)	606106844.3	226030700	2.63
90	6(5.38)	681011602.0	2167043000	2.08
87	2(1.67)	744766781.0	2073800000	1.67
80	3(2.60)	821125077.1	1980535000	1.31
75	3(2.60)	1041313306	1747400000	1.56
70	5(4.45)	659732677.2	1584206000	1.30
60	15(13.71)	783055104.4	1351071000	1.62
50	9(8.15)	823782025.1	1118037000	1.25
40	8(7.23)	906 697 438.5	885801300	0.98
30	7(6.30)	620761028	652666500	1,05
	19(17.41)	0	0	-

5.3. Regulate the best algorithm using the TOPSIS method

Founded on the outcomes of the mathematical instances and the statistical judgments, it is not conceivable to determine an explanation technique greater than the others in both circumstances. Therefore, the Technique for Direct of Favorite by Comparison to Ideal Solution (TOPSIS) method selects the most proper technique. The word TOPSIS means partiality founded on resemblance to the perfect key. The model recognized by Hwang and Yoon [21] is a unique method for ranking possibilities.

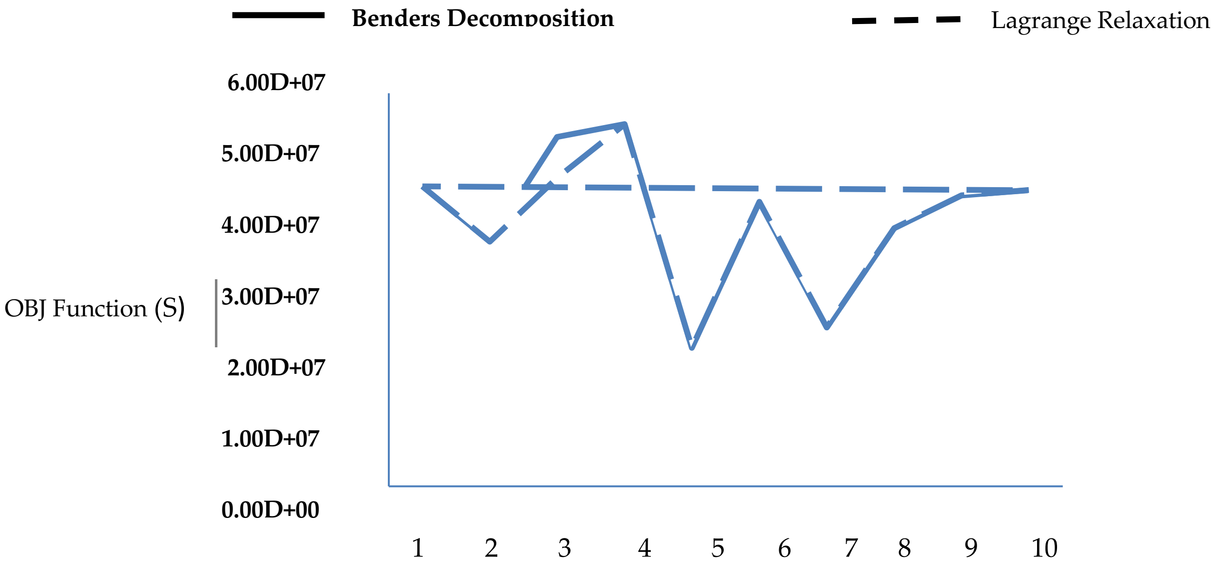

Figure1: *Numerical example*

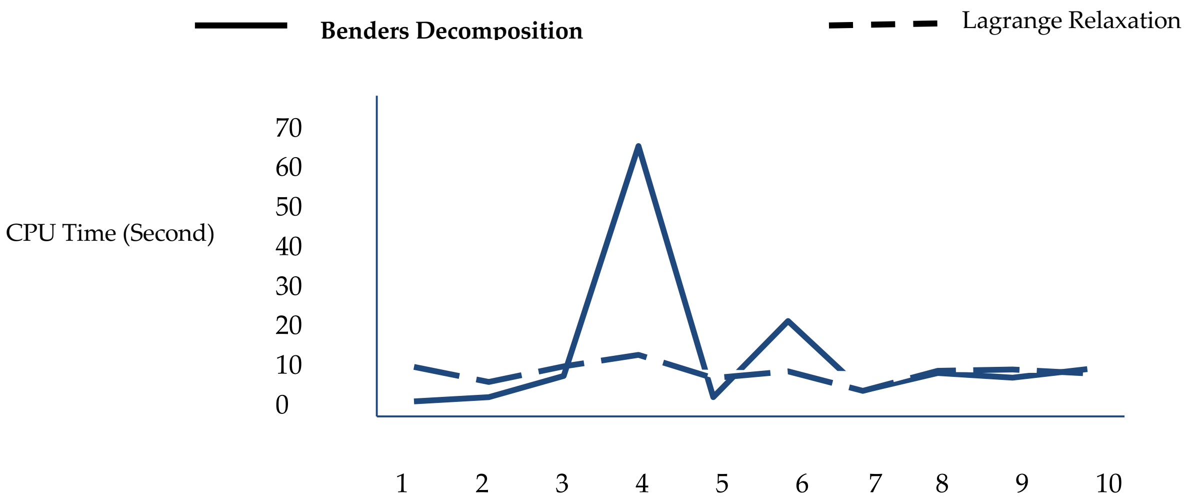

Figure2: *Numerical example*

6. Conclusion

Currently, inventory management and controller constructions are major issues presented by developing administrations. This research identified an optimization Diagram to categorize inventory groups, regulate their facility stages instantly, and assign objects to those assemblies. This strategy improves the inventory group founded on the ABC examination by integrating automatic and optimal replies. The Diagram designed in this education differs from existing optimization Diagrams in two ways. Initially, the model observes and exploits the company's income rather than minimizing inventory rates. It also optimizes the trade-off between inventory rates and payments and allocates inventory-to-inventory items. The Arithmetical Diagram used in this article was to maximize the residual income from stock items.

The interpretation occupied boundaries such as low cost and a lack of inventory. Disintegration events and their proportionate scrutiny are other elements that distinguish this Investigation from other papers on the subject. Two indices, counting the Goal purpose value and CPU while approaching the anticipated solution, approach tenaciously. The Arithmetical diagram was then running for ten distinct arithmetic occurrences, and the results of the two suggested explanations were statistically equated using a t-test. In terms of superiority and response time, the responses were fairly close. Choose one of these. As a result, the TOPSIS technique was appropriate, whereas the Lagrangean technique was chosen as the more spectacular technique.

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BAYESIAN INTERVAL ESTIMATION FOR THE PARAMETERS OF POISSON TYPE RAYLEIGH CLASS MODEL

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Abstract

In this article, two sided Bayesian interval is proposed for the parameters of Poisson type Rayleigh class software reliability growth model. In this work, the failure intensity function, mean time to failure function and likelihood function of this model have been derived by considering parameter total number of failures i.e. γ_0 and scale parameter γ_1. The mathematical expressions of Bayesian interval for the parameters have been obtained by considering non informative priors. The performance of proposed Bayesian interval is studied on the basis of average length and coverage probability. Average length and coverage probability is obtained by using Monte Carlo simulation technique after generating 1000 random samples. From the obtained results, it is concluded that Bayesian interval of parameters perform better for appropriate choice of execution time and certain values of parameters.

Keywords: Rayleigh distribution, Non informative prior, Software reliability growth model, Bayesian interval, average length, coverage probability.

1. Introduction

Software reliability is the quality characteristic of operation system which can measure, predict and estimate quality of software system. In last several decades various model have been proposed to assess software reliability. Most of them are probabilistic models. Software modeling techniques can be divided into two categories: Prediction and estimation models. Estimation models determines the current software reliability by applying statistical inference techniques to failure data while the prediction models determines future software reliability based upon available software metrics and measures. The parameters included in software reliability models can be estimated by using some basic procedures like maximum likelihood, least square estimation and Bayesian point estimation, etc.

The research in this area of software reliability has been started since 1950's and a large number of researchers have done work in this field. Most of the past research work in software reliability modeling has concentrated on the point estimation of the parameters. The uncertainty of the estimates by using interval estimation has not been fully discussed. The most commonly applied interval estimation technique is based on the central limit theorem assuming large sample size. However, in real world testing the number of software failures observed is usually not large enough. Whereas Bayesian approaches produce interval estimates even in the case of small sample size, by utilizing prior knowledge.

This paper considers Poisson type Rayleigh model as per classification scheme of Musa and Okumoto [10] Rayleigh distribution has wide application in life time data especially in reliability theory and survival analysis. A specific case of Weibull distribution exhibiting aging effect with an integer valued shape parameter is known as "Rayleigh distribution." Dey and Dey [3] presented Bayes estimators for the parameter of Rayleigh model on the basis of loss function. Also provided Highest Posterior Density (HPD) for the unknown parameter of Rayleigh model. Lee et al [8] proposed software reliability growth model (SRGM) and obtained confidence interval using Obha's inflection S shaped model that can assess software developers optimal release time of software testing tasks. Rabie and Li [12] has studied Bayesian and E-Bayesian approaches under squared error loss function LINEX loss functions and constructed confidence interval for maximum likelihood and credible interval. Xie et al [18] estimated software reliability using Goel-Okumoto model and obtained confidence interval for failure intensity. Wu et al [17] obtained Bayes estimates and credible interval for Rayleigh distribution for the parameters and reliability function. Shrestha and Kumar [13] have obtained Bayes parameter estimates such as reliability function, hazard function under loss function for lomax distribution. Also provides Bayesian credible interval and Highest Posterior Density (HPD) interval for the corresponding parameters. Ogura and Yanagimoto [11] proposed a novel credible interval of the binomial proportion by improving the Highest Posterior Density (HPD) interval using logit transformation. The 100(1-α)% confidence interval through MLE is compared with corresponding level of credible interval. The reason for this is that MLE is preferred by researchers and Bayesian inference is effective for small sample size. Cunha and Rao [2] estimated credible interval and confidence interval through MLE for lognormal distribution also compared average length and coverage probability of the calculated interval. Fang and Yeh [4] proposed a software reliability estimation process that uses stochastic differential equations (SDEs) with fault detection function to construct confidence interval of mean value function *m(t)* of SRGMs. Song et al [16] proposed a new NHHP software reliability model and estimated confidence interval.

The association of the paper is such that section 2 presents derivation of failure intensity and expected number of failures using Rayleigh distribution. Section 3 presents selection of priors and posterior distribution of model. Section 4 presents two sided Bayes interval for the parameters γ_0 and γ_1. Results are discussed in the section 5 while concluding remarks are provided in section 6.

2. Model Formulation

Considering that software failure time of a system following Rayleigh distribution with scale parameter γ_1 and software failures occurred in Poisson manner. Let t be the positive random variable having Rayleigh distribution then its probability density function is given by

$$f(t) = \{t\,\gamma_1^{-2} e^{-\frac{1}{2}\left[\frac{t}{\gamma_1}\right]^2} \quad ,\ t > 0, \gamma_1 > 0 \tag{1}$$

Assuming that the total number of failures remaining in the program at time t = 0 is a Poisson random variable with mean γ_0 then the failure intensity λ (t) = γ_0 f (t) can be obtained as follows (cf Musa et al [9])

$$\lambda(t) = \gamma_0 \gamma_1^{-2} t e^{-\frac{1}{2}\left[\frac{t}{\gamma_1}\right]^2} \quad , \ t > 0, \gamma_1 > 0, \gamma_0 > 0 \tag{2}$$

The mean time to failure function i.e. expected number of failures at time t_e using equation (2)

comes to be

$$\mu(t_e) = \gamma_0 \gamma_1^{-2} \int_0^{t_e} x e^{-\frac{1}{2}\left[\frac{x}{\gamma_1}\right]^2} dx$$

On simplification it can be written as

$$\mu(t_e) = \eta_0 [1 - e^{-\frac{1}{2}\left(\frac{t_e}{\eta_1}\right)^2}] \tag{3}$$

Now assuming that m_e software failures for a system are experienced at times t_i , i = 1,2,……,m_e up to execution time t_e ($\geq t_{me}$) and the likelihood function of parameters γ_0 and γ_1 can be obtained as $L(\Theta, \underline{t}) = \left[\prod_{i=1}^{m_e} \lambda(t_i)\right] exp\ exp\ [-\mu(t_e)]$ (cf. Singh et al [14]). Using failure intensity function given in (2) and mean time to failure function given in (3) the likelihood function is obtained as

$$L(\Upsilon_0, \Upsilon_1) = \Upsilon_0^{m_e} \Upsilon_1^{-2m_e} \left[\prod_{i=1}^{m_e} t_i\right] e^{-\frac{1}{2} T \Upsilon_1^{-2}} e^{-\Upsilon_0} exp \left\{ \Upsilon_0\, e^{-\frac{1}{2}\left(\frac{t_e}{\Upsilon_1}\right)^2} \right\} \tag{4}$$

3. Choice of priors and Posterior distribution

In Bayesian estimation appropriate choice of the prior(s) for the parameter is necessary. However Bayesian analyst pointed out that there is no perfect technique from which one can conclude that one prior is better than the other. Very often, priors are chosen according to one's subjective knowledge and beliefs. However, in case of adequate information about the parameter is available one can use informative prior(s) otherwise it is preferable to use non informative prior(s). Bayesian estimation is a method that combines prior information with information obtained from sample data. While testing the software, the experimenter have very little knowledge relative to the total number of failures present in the software i. e. γ_0 and γ_1. Here insufficient prior information is available about parameters γ_0 and γ_1, hence non-informative priors are considered.

Jeffrey's [6] has suggested the use of non-informative priors. Jeffrey's prior is widely used because it is proper under slight conditions. It requires likelihood function from which the prior is then derived using Jeffrey's rule. More discussion properties of Jeffrey's prior has been studied by Chen et al [1].The following non informative prior distributions g (γ_0) and g (γ_1) are considered for parameters γ_0 and γ_1 which are as follows:

$$g\,(\gamma_0)\ \propto\ \{\gamma_0{}^{-1} \quad ,\gamma_0 \epsilon\ [0,\infty) \tag{5}$$

and

$$g(\gamma_1)\ \propto\ \{\gamma_1{}^{-1} \quad ,\gamma_1 \epsilon\ [0,\infty) \tag{6}$$

The joint posterior of Υ_0 and Υ_1 given $\underline{t}$ (= t_i, i=1, 2,……….m_e) is obtained by using equations (4),(5) and (6) is as follows:

$$\pi(\underline{t}) = D^{-1}\gamma_0^{m_e-1}\gamma_1^{-2m_e-1}e^{-\frac{1}{2}T\gamma_1^{-2}}e^{-\gamma_0}exp\left\{\gamma_0\, e^{-\frac{1}{2}\left(\frac{t_e}{\gamma_1}\right)^2}\right\} \qquad m_e < \gamma_0 < \infty\,, 0<\gamma_1<\infty \tag{7}$$

Where D is normalizing constant

$$D = \sum_{j=1}^{\infty}\frac{\Gamma(\,m_e + j,\ m_e)\Gamma(2m_e)}{j!}\ (2/S)^{(2m_e)}$$

Where, $S = (T + jt_e^2)$, $T = \sum_{i=1}^{m_e} t_i^2$

The marginal posterior distribution of γ_0 given $\underline{t}$ is obtained by integrating equation (8) over the whole range of γ_1 i.e.

$$\pi(\underline{t}) = D^{-1}\sum_{j=0}^{\infty}\left[\frac{(2/S)^{(2m_e)}\Gamma(2m_e)}{j!}\right]\left[\gamma_0^{m_e+j-1}e^{-\gamma_0}\right]\,, \qquad m_e<\gamma_0<\infty \tag{8}$$

Similarly, the marginal posterior distribution of γ_1 given $\underline{t}$ is as

$$\pi(\underline{t}) = D^{-1}\sum_{j=0}^{\infty}\left[\frac{\Gamma(m_e+j,m_e)}{j!}\right]\left[\gamma_1^{-2m_e-1}e^{-\frac{1}{2}S\gamma_1^{-2}}\right]\ ,0<\gamma_1<\infty \tag{9}$$

4. Bayesian interval estimation of parameters γ_0 and γ_1

The equal tailed 100(1- α)% Bayes probability interval is given as:

$\int_{-\infty}^{\gamma_*}\pi(\underline{t})dt = \alpha/2$ and $\int_{\gamma*}^{\infty}\pi(\underline{t})dt = \alpha/2$

Where $(\underline{t})$ is the marginal posterior distribution and γ_* lower limit and γ^*upper limit of the Bayesian interval respectively. For details see Martz and Waller [9], B.K. Kale [7], S. K. Sinha [15].

Now by integrating equations (8) and (9) w.r.t. γ_0 and γ_1 respectively 100(1- α)% two sided Bayesian interval for the parameter γ_0 and γ_1 can be obtained as follows:

$$\widetilde{\gamma_{0l}} = D^{-1}\sum_{j=1}^{\infty}\frac{(2/S)^{(2m_e)}\Gamma(2m_e)}{j!}\,\Gamma(\,m_e+j,\gamma_{0*}\,)$$

$$\widetilde{\gamma_{0u}} = D^{-1}\sum_{j=1}^{\infty}\frac{(2/S)^{(2m_e)}\Gamma(2m_e)}{j!}\,\Gamma(\,m_e+j,\gamma_0^*\,)$$

$$\widetilde{\gamma_{1l}} = D^{-1}\sum_{j=1}^{\infty}\frac{\Gamma(\,m_e+j,me\,)}{j!}\left(\frac{2}{S}\right)^{(2m_e)}\Gamma(2m_e,S/2\,\gamma_{1*}\,)$$

$$\widetilde{\gamma_{1u}} = D^{-1}\sum_{j=1}^{\infty}\frac{\Gamma(\,m_e+j,me\,)}{j!}\left(\frac{2}{S}\right)^{(2m_e)}\Gamma(2m_e,S/2\,\gamma_1^*)$$

Where, $\widetilde{\gamma_{0l}}$ and $\widetilde{\gamma_{0u}}$ is the Bayes lower limit and upper limit of parameter γ_0 i.e. total number of failures, $\widetilde{\gamma_{1l}}$ and $\widetilde{\gamma_{1u}}$ is the Bayes lower limit and upper limit of parameter γ_1. And $\Gamma(\,m_e+j,\gamma_0^*\,)$ and $\Gamma(2m_e,S/2\,\gamma_1^*)$ are incomplete gamma functions.

The details about the incomplete gamma function can be seen from Gradshteyn and Ryzhik [5].

5. Discussion

Table (1) to (8) represents average length and coverage probability of the Bayesian two sided interval is obtained for the parameter γ_0 i.e. total number of failures and parameterγ_1. The Bayesian interval depends upon the values of execution time i.e. t_e and m_e failures experienced at times t_i, i = 1,2………..,m_e. Bayesian interval is studied by calculating the average length and coverage probability of the simulated interval. To study the performance, a sample size was generated from the Rayleigh distribution and it is repeated 1000 times. Average length and coverage probability is calculated for Bayes two sided interval for different execution time t_e for different values of parameters. Monte Carlo simulation is used to study the performance of Bayesian interval. Average length and coverage probability have been calculated by assuming parameter γ_0 (=1(1)5) and $\gamma_1(=0.25(0.25)1.25)$ using 1000simulations.

From tables (1) to (4) it is observed that Bayesian interval's average length decreases as $\gamma_0$increases and it is increases as γ_1 increases for different execution time i.e. t_e. It can be seen that as execution time increases average length also increases. Here assumes that Bayesian interval maintains the credible level if the estimated coverage probability is in between the range of 0.940 to 0.960 i.e. (1-α)$\pm$0.01 where, α = 0.05. It was found that the coverage probabilities of the interval decreases as $\gamma_0$increases and coverage probability increases as γ_1 increases.

Here, table (5) to (8) represents average length and coverage probability for Bayesian two sided interval for the parameter γ_1.It can be seen that average length computed for Bayesian interval is increases as total number of failures i.e. γ_0 increases. And as $\gamma_1$increases average length also increases. It also observes that average length decreases as execution time i.e. t_e increases. From tables it can be seen that coverage probability increases as $\gamma_0$and γ_1 increases. When average length will be shorter coverage probability will decreases. As execution time t_e increases coverage probability decreases.

Table 1: *Average length and coverage probability of Bayesian interval $\widetilde{\gamma_0}$ calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 5$*

γ_1 \ γ_0	1	2	3	4	5
0.25	2.14713 (0.996)	2.023915 (0.996)	1.574669 (0.995)	1.02334 (0.995)	0.295229 (0.994)
0.50	2.14714 (0.996)	2.058673 (0.996)	1.604102 (0.995)	1.04157 (0.995)	0.375861 (0.994)
0.75	2.14715 (0.996)	2.066402 (0.996)	1.621196 (0.995)	1.30856 (0.995)	0.624627 (0.994)
1	2.14723 (0.996)	2.078305 (0.996)	1.683277 (0.995)	1.40131 (0.995)	0.674976 (0.994)
1.25	2.14768 (0.997)	2.081441 (0.996)	1.751818 (0.995)	1.41164 (0.995)	0.718004 (0.994)

*The values in the parenthesis are coverage probability.

Table 2: *Average length and coverage probability of Bayesian interval $\widetilde{\gamma_0}$ calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 10$*

γ_1 \ γ_0	1	2	3	4	5
0.25	2. 42186 (0.998)	2.14698 (0.996)	2.13883 (0.996)	2.051196 (0.995)	1.664166 (0.994)
0.50	2.46526 (0.998)	2.22720 (0.996)	2.147932 (0.996)	2.06754 (0.995)	1.673916 (0.994)
0.75	2.48655 (0.998)	2.247208 (0.996)	2.147905 (0.996)	2.07153 (0.995)	1.67429 (0.994)
1	2.48658 (0.998)	2.327472 (0.997)	2.14932 (0.997)	2.07210 (0.996)	1.67455 (0.994)
1.25	2.49144 (0.999)	2.348076 (0.998)	2.15390 (0.997)	2.074465 (0.996)	1.68610 (0.994)

Table 3: *Average Length and Coverage Probability of Bayesian interval $\widetilde{\gamma_0}$calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 15$*

γ_1 \ γ_0	1	2	3	4	5
0.25	2.43761 (0.998)	2.24616 (0.998)	2.18678 (0.997)	2.151995 (0.997)	2.08162 (0.996)
0.50	2.43786 (0.998)	2.26656 (0.998)	2.18701 (0.997)	2.152415 (0.997)	2.082083 (0.996)
0.75	2.43791 (0.998)	2.271643 (0.998)	2.20720 (0.997)	2.15321 (0.997)	2.083737 (0.996)
1	2.44656 (0.998)	2.33112 (0.998)	2.21472 (0.998)	2.153597 (0.997)	2.084265 (0.996)
1.25	2.48656 (0.999)	2.35643 (0.998)	2.22476 (0.998)	2.15383 (0.998)	2.102458 (0.997)

*The values in the parenthesis are coverage probability.

Table 4: *Average Length and Coverage Probability of Bayesian interval $\widetilde{\gamma_0}$ calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 20$*

γ_1 \ γ_0	1	2	3	4	5
0.25	2.46155 (0.998)	2.25882 (0.998)	2.14701 (0.996)	2.14309 (0.995)	2.10662 (0.994)
0.50	2.46237 (0.998)	2.26956 (0.998)	2.14701 (0.996)	2.14382 (0.995)	2.10883 (0.994)
0.75	2.46256 (0.998)	2.27436 (0.998)	2.14720 (0.996)	2.14445 (0.995)	2.11537 (0.995)
1	2.46506 (0.998)	2.34452 (0.998)	2.147682 (0.996)	2.14894 (0.995)	2.12780 (0.995)
1.25	2.49164 (0.999)	2.36436 (0.998)	2.14807 (0.996)	2.14979 (0.995)	2.12858 (0.995)

*The values in the parenthesis are coverage probability.

Table 5: *Average length and coverage probability of Bayesian interval $\tilde{\gamma}_1$ calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 5$*

γ_1 \ γ_0	**1**	**2**	**3**	**4**	**5**
0.25	0.001098 (0.995)	0.003211 (0.996)	0.006615 (0.997)	0.009148 (0.998)	0.011176 (0.998)
0.50	0.001114 (0.996)	0.003322 (0.996)	0.007081 (0.997)	0.009530 (0.998)	0.012966 (0.998)
0.75	0.001116 (0.996)	0.003406 (0.996)	0.007276 (0.997)	0.010573 (0.998)	0.013338 (0.998)
1	0.001116 (0.996)	0.003485 (0.996)	0.007954 (0.998)	0.013072 (0.998)	0.014761 (0.998)
1.25	0.001125 (0.996)	0.003821 (0.997)	0.008098 (0.998)	0.013388 (0.998)	0.015302 (0.998)

*The values in the parenthesis are coverage probability.

Table 6: *Average length and coverage probability of Bayesian interval $\tilde{\gamma}_1$ calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 10$*

γ_1 \ γ_0	**1**	**2**	**3**	**4**	**5**
0.25	0.001071 (0.995)	0.001090 (0.995)	0.001723 (0.996)	0.002980 (0.996)	0.002112 (0.997)
0.50	0.001074 (0.995)	0.001102 (0.995)	0.002021 (0.996)	0.00302 (0.996)	0.002406 (0.997)
0.75	0.001074 (0.995)	0.001102 (0.995)	0.002023 (0.996)	0.003133 (0.996)	0.002988 (0.997)
1	0.001078 (0.995)	0.001103 (0.996)	0.002036 (0.996)	0.003106 (0.996)	0.003079 (0.997)
1.25	0.001079 (0.995)	0.001103 (0.996)	0.002221 (0.996)	0.003374 (0.996)	0.003684 (0.997)

*The values in the parenthesis are coverage probability.

Table 7: *Average Length and Coverage Probability of Bayesian interval $\tilde{\gamma}_1$ calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 15$*

γ_1 \ γ_0	1	2	3	4	5
0.25	0.001069 (0.995)	0.001082 (0.995)	0.001091 (0.996)	0.001114 (0.996)	0.001116 (0.997)
0.50	0.001071 (0.995)	0.001085 (0.995)	0.001113 (0.996)	0.001115 (0.996)	0.001119 (0.997)
0.75	0.001072 (0.995)	0.001086 (0.995)	0.001115 (0.996)	0.001115 (0.996)	0.001120 (0.997)
1	0.001075 (0.995)	0.001086 (0.995)	0.001116 (0.996)	0.001116 (0.996)	0.001122 (0.997)
1.25	0.001077 (0.995)	0.001088 (0.995)	0.001119 (0.996)	0.001117 (0.997)	0.001122 (0.997)

*The values in the parenthesis are coverage probability.

Table 8: *Average Length and Coverage Probability of Bayesian interval $\tilde{\gamma}_1$ calculated for different values of parameters γ_0 and γ_1 when execution time $t_e = 20$*

γ_1 \ γ_0	1	2	3	4	5
0.25	0.001068 (0.994)	0.001079 (0.995)	0.001082 (0.995)	0.001085 (0.995)	0.001090 (0.995)
0.50	0.001068 (0.994)	0.001080 (0.995)	0.001084 (0.995)	0.001088 (0.995)	0.001092 (0.995)
0.75	0.001067 (0.995)	0.001081 (0.995)	0.001085 (0.995)	0.001089 (0.995)	0.001093 (0.996)
1	0.001066 (0.995)	0.001083 (0.995)	0.001085 (0.995)	0.001092 (0.995)	0.001094 (0.996)
1.25	0.001064 (0.995)	0.001086 (0.995)	0.001088 (0.995)	0.001093 (0.996)	0.001095 (0.996)

*The values in the parenthesis are coverage probability.

6. Conclusion

In this research paper, Bayesian interval has been proposed considering Poisson type Rayleigh class software reliability growth model as function with parameters total number of failures i.e. γ_0 and scale parameter γ_1. Bayesian analysis is carried out by considering non informative prior. The performance of two sided Bayesian interval is studied using Monte Carlo simulation technique. Average length and coverage probability of Bayesian interval is calculated for both the parameters γ_0 and γ_1 for different execution time t_e. From study it is concluded that proposed Bayesian interval has shorter average length for both parameters. Bayesian interval maintained coverage probability for both the parameters γ_0 and γ_1 for different execution time for different values of parameters. In future, confidence interval will be obtained for proposed model and will be compared with Bayesian interval on the basis of average length and coverage probability.

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AN INFERENTIAL STUDY OF DISCRETE BURR-HATKE EXPONENTIAL DISTRIBUTION UNDER COMPLETE AND CENSORED DATA

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Abstract

In this article, a new one-parameter discrete distribution called discrete Burr-Hatke exponential distribution is introduced and its mathematical characteristics are thoroughly investigated. The proposed distribution is capable of modelling over-dispersed, positively skewed, decreasing failure rate, and randomly right-censored data. We have also introduced many statistical properties including moments, skewness, kurtosis, mean residual life and mean past lifetime, index of dispersion, coefficient of variation, stress strength parameter, quantile function, and order statistics. Method of maximum likelihood is used to estimate unknown model's parameter under complete and censored data. In addition, a technique for generating randomly right-censored data from the proposed model is provided. To evaluate the behaviour of the estimator with complete and censored data, two simulation studies are presented. Two complete and two censored datasets from various disciplines are studied to demonstrate the significance of the suggested distribution in comparison to the existing discrete probability distributions.

Keywords: Burr-Hatke exponential distribution, Method of maximum likelihood, Discrete distribution, Random censoring, Simulation study

1. Introduction

Many continuous lifetime models have been proposed and investigated in reliability theory. However, measuring the life of a component on a continuous scale is frequently impossible or inconvenient. For example, in reliability engineering, the lifetime of an on/off switching device, in survival analysis, the survival times for those suffering from diseases such as lung cancer or the period from remission to relapse may be recorded as the number of days/weeks etc. Furthermore, the count phenomenon arises in a wide range of practical scenarios, including the number of earthquakes that occur in a calendar year, the number of absences, the number of accidents, the number of species kinds in ecology, the number of insurance claims, the number of deaths/daily cases due to the COVID-19 pandemic observed over a specified duration and so on. In all of these circumstances, it is more appropriate to measure these characteristics on a discrete scale rather than a continuous analogue.

Although there are several conventional discrete distributions such as the Binomial, Poisson, Geometric etc and recently developed discrete models to analyse above discussed characteristics. The research for new discrete distributions that are appropriate under various scenarios is still underway. One prominent area of study in this field is the development of discrete distributions by discretizing suitable continuous probability distributions. Discretization of continuous distribution can be accomplished by a variety of methods. Out of which one of the most widely used methods is [1]. In this approach, he proposed discrete normal distribution using the survival function of its continuous counterpart. Chakraborty in [2] named this technique the survival discretization method. One of the most important advantages of this method is that the produced discrete distribution has the same functional form of the survival function as its continuous version. As a result of this feature, many of the reliability characteristics of the distribution remain unchanged. According to this methodology, for a given continuous random variable (RV) $'Y'$ with survival function (SF) $S_Y(y) = P(Y \geq y)$, the random variable $X = [Y] =$ largest integer less than or equal to Y will have the probability mass function (PMF),

$$\begin{aligned} \mathrm{P}(X = x) &= P(x \leq Y \leq x+1) \\ &= P\left(Y \geq x\right) - P\left(Y \geq x+1\right) \\ &= S_Y(x) - S_Y(x+1);\ x = 0, 1, 2, ... \end{aligned} \tag{1}$$

Many scholars have discretized various well-known continuous distributions using this approach. For instance, [3] investigated the discrete Rayleigh distribution, [4] researched the discrete Maxwell distribution. In addition, [5] investigated the discrete Burr and discrete Pareto distribution. Discrete inverse Weibull distribution developed by [6] . Discrete-continuous Burr III distribution defined by [7]. For more studies on discrete distribution, one can refer to [8], [9], [10], [11] and the references cited therein. Recently, [12] developed a discrete analogue of the odd Weibull-G family of distributions: properties, classical and Bayesian estimation with applications to count data of the number of new coronavirus cases.

In many circumstances, data collection is restricted by constraints such as time or money, making it hard to obtain the entire dataset. This form of incomplete data is referred to as censored data. Various censoring mechanisms are available in the literature to examine these datasets. One of the greatly applicable censorship is random censoring. This scheme consists of studies in which subjects can be censored at any time during the experiment period. Random censoring can be seen in clinical trials or medical studies where patients do not finish the course of treatment and leave before the endpoint. Randomly censored lifetime data are common in many applications such as medical science, biology, reliability studies, and so on, and must be properly analysed to make correct inferences and appropriate research conclusions. Random censoring has been widely investigated in the literature for continuous models see [13]. The censoring technique has also been studied merely under discrete models, namely [14] and [15]. Recently, [16] developed discrete inverted Nadarajah-Haghighi distribution and estimated its parameters under complete and random right-censored censored data.

The majority of existing discrete models were developed to assess count data and, in most cases, they do not accurately analyse the censored data. These situations motivate us to develop a more appropriate discrete distribution that is not capable only of analysing count data but also well enough for modelling censored data. Therefore, in this article, we have proposed a discrete analogue of the Burr-Hatke exponential model by using approach (1) and named it as discrete Burr-Hatke exponential (DBHE) distribution. Hence the ultimate objectives of developing the DBHE model is as follows, a) To construct a discrete model capable of modelling both complete and censored data, b) To design a discrete model with more flexibility and fewer parameters so that the form of diverse distributional properties can be easily handled, c) Numerous practical studies, such as newly developed engineering systems and infant mortality, have shown decreasing failure rate; consequently, we wish to construct a discrete model with a decreasing failure rate function, d) To develop a model that can fit positively skewed, leptokurtic and over-dispersed real data, e) To produce a discrete model that can provide consistently better fits than other well-known discrete models in the existing statistical literature.

The rest of the article is structured as follows: Section 2 introduces the DBHE distribution. Some significant distributional and survival features are investigated in Section 3. In Section 4, we use the maximum likelihood estimation approach to estimate the parameter of the DBHE distribution with complete data and also present numerical illustrations based on empirical and real-world datasets. Section 5 discusses the maximum likelihood estimator (MLE) for the model's parameter under randomly right-censored data and it also includes the technique for generating censored observations from the proposed model. The numerical examples using randomly right-censored empirical and real data have also been presented in section 5. Section 6 concludes with some final observations.

2. The DBHE distribution

The Burr–Hatke exponential (BHE) distribution was proposed by [17].The probability density function (PDF) and SF of the BHE distribution are given as

$$f(y,\theta) = \frac{\theta(2+\theta y)}{(1+\theta y)^2}\exp(-\theta y); y \geq 0, \theta > 0, \tag{2}$$

$$S(y,\theta) = P(Y > y) = \frac{\exp(-\theta y)}{(1+\theta y)}; y \geq 0, \theta > 0, \tag{3}$$

respectively. The BHE distribution is rightly skewed with decreasing hazard rate function (HRF). This model is very useful to analyse reliability/medical data which have the pattern of decreasing hazard rate. Since it has been generalized by exponential baseline distribution so it may be regarded as an alternative to the several one-parameter exponential families of distributions. Now, using a methodology (1) the PMF of the DBHE model can be obtained as

$$P_X(x,\theta) = \left(\frac{1}{(1+\theta x)} - \frac{\exp(-\theta)}{(1+\theta+\theta x)}\right)\exp(-\theta x), x = 0,1,2...; \theta > 0. \tag{4}$$

The CDF corresponding to Equation (4) is given by,

$$F_X(x,\theta) = 1 - \frac{\exp(-\theta(x+1))}{(1+\theta+\theta x)}, x = 0,1,2,...; \theta > 0. \tag{5}$$

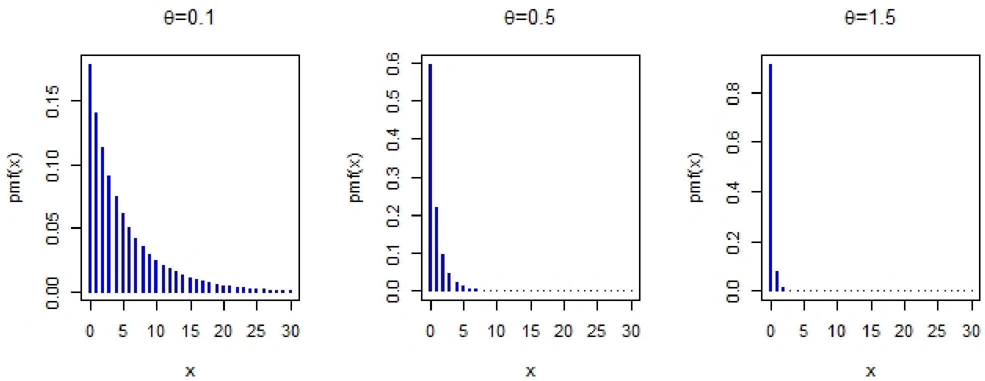

Figure 1: *The PMF plots of the DBHE model for different values of θ.*

Figure 1 shows the PMF plots for different values of the model parameter. From Figure 1, we can conclude that the PMF of the DBHE distribution is unimodal and right-skewed. Also, the behaviour of the PMF at endpoints are as follows:

- $\lim_{x\to 0} P_X(x,\theta) = 1 - \frac{\exp(-\theta)}{(1+\theta)}$,
- $\lim_{x\to\infty} P_X(x,\theta) = \lim_{\theta\to 0} P_X(x,\theta) = \lim_{\theta\to\infty} P_X(x,\theta) = 0.$

3. Distributional Properties

3.1. Recurrence Relation for Probabilities

To obtain the probability mass on various values of X, we can use the following recursive relation

$$P_X\left(x+1,\theta\right)=\left(\frac{1}{(1+\theta+\theta x)}-\frac{\exp\left(-\theta\right)}{(1+2\theta+\theta x)}\right)\left(\frac{1}{(1+\theta x)}-\frac{\exp(-\theta)}{(1+\theta+\theta x)}\right)^{-1}\exp\left(-\theta\right)P_X\left(x,\theta\right).$$

It is observable that $\{P_X\left(x+1\right)\}^2<P_X\left(x\right)P_X\left(x+1\right)$ for all x. As a result, the DBHE distribution is log-convex. Due to this convexity, the proposed distribution has a non-increasing failure rate [18].

3.2. Moments, Skewness and Kurtosis

Moments of a probability distribution are an important tool for measuring its different properties such as mean, variance, skewness, kurtosis, etc. If $F\left(x\right)$ is the CDF of a discrete random variable, then the r^{th} raw moments of this random variable can be obtained by using the following formula:

$$E(X^r)=\sum_{x=0}^{\infty}\left\{\left((x+1)^r-x^r\right)\left(1-F\left(x\right)\right)\right\}.$$

Using the above expression, the r^{th} raw moment denoted by $\mu_r^{'}$ of the DBHE distribution can be written as

$$\mu_r^{'}=E(X^r)=\exp\left(-\theta\right)\sum_{x=0}^{\infty}\frac{\left((x+1)^r-x^r\right)}{(1+\theta+\theta x)}\exp\left(-\theta x\right). \tag{6}$$

Using the ratio test, we can easily observe that, the expression in Equation (6) is convergent. It implies the existence of the r^{th} moment of the proposed distribution.
Now, using Equation (6), the first four-row moments of the DBHE distribution are

$$\mu_1^{'}=E(X)=\exp\left(-\theta\right)\sum_{x=0}^{\infty}\frac{\exp\left(-\theta x\right)}{(1+\theta+\theta x)}, \tag{7}$$

$$\mu_2^{'}=E(X^2)=\exp\left(-\theta\right)\sum_{x=0}^{\infty}\frac{(2x+1)}{(1+\theta+\theta x)}\exp\left(-\theta x\right), \tag{8}$$

$$\mu_3^{'}=E(X^3)=\exp\left(-\theta\right)\sum_{x=0}^{\infty}\frac{\left(3x^2+3x+1\right)}{(1+\theta+\theta x)}\exp\left(-\theta x\right), \tag{9}$$

$$\mu_4^{'}=E(X^4)=\exp\left(-\theta\right)\sum_{x=0}^{\infty}\frac{\left(4x^3+6x^2+4x+1\right)}{(1+\theta+\theta x)}\exp\left(-\theta x\right). \tag{10}$$

The variance of the DBHE distribution is given by ,

$$\begin{aligned}Var(X)&=E\left(X^2\right)-E(X)^2\\&=\left(\sum_{x=0}^{\infty}\frac{(2x+1)\exp\left(-\theta x\right)}{(1+\theta+\theta x)}\right)-\left(\exp\left(-\theta\right)\sum_{x=0}^{\infty}\frac{\exp\left(-\theta x\right)}{(1+\theta+\theta x)}\right)^2.\end{aligned}$$

Using above raw moments in (7)-(10), we can easily find the skewness and kurtosis from the following relations

$$K=\frac{E(X^4)-4E(X^2)E\left(X\right)+6E(X^2)(E\left(X\right))^2-3(E\left(X\right))^4}{(Var(X))^2}.$$

Table 1 presents some numerical results of the mean, variance, skewness and kurtosis for the DBHE distribution for different values of θ.

Table 1: *Mean, Variance, Skewness and kurtosis for different values of θ.*

Measure ↓ θ→	0.1	0.2	0.3	0.5	0.7	0.9	1	1.5	2
Mean	4.6575	1.8326	0.9674	0.3692	0.1701	0.0863	0.0629	0.0149	0.0041
Variance	52.5614	13.4619	5.7640	1.7577	0.7144	0.3336	0.2356	0.0505	0.0130
Skewness	4.8141	5.4455	6.9073	11.9474	20.6287	34.9582	45.2208	156.6311	517.1785
Kurtosis	10.7294	10.8777	12.1065	17.1201	26.0157	40.5689	50.8788	160.0551	504.2940

From Table 1, it is clear that:

1. As the parameter's value increases, the values of mean and variance of the DBHE distribution decrease, whereas the values of skewness and kurtosis increase.

2. The proposed model is appropriate for modelling positively skewed and leptokurtic data.

3.3. Index of Dispersion and Coefficient of Variation

The index of dispersion (IOD) is a measure used to determine the possibility of over-dispersion (under-dispersion) of the model under study. An IOD greater than one indicates over-dispersion, whereas an IOD lower than one indicates under-dispersion. Equi-dispersion is indicated when the IOD is equal to one. The expression for IOD of the DBHE distribution is

$$IOD\left(X\right)=\frac{Var(X)}{E\left(X\right)}=\frac{\left(\sum\limits_{x=0}^{\infty}\frac{(2x+1)\exp(-\theta x)}{(1+\theta+\theta x)}\right)-\left(\exp\left(-\theta\right)\sum\limits_{x=0}^{\infty}\frac{\exp(-\theta x)}{(1+\theta+\theta x)}\right)^{2}}{\exp\left(-\theta\right)\sum\limits_{x=0}^{\infty}\frac{\exp(-\theta x)}{(1+\theta+\theta x)}}. \tag{11}$$

Furthermore, the coefficient of variation (COV) is a measure of data variability. The COV measure is commonly used to compare the variability of independent samples. The larger the coefficient of variation (COV), the more erratic the data. If X follows DBHE model, the COV of DBHE may be represented as

$$COV\left(X\right)=\frac{\left(Var(X)\right)^{1/2}}{E\left(X\right)}=\frac{\left(\left(\exp\left(-\theta\right)\sum\limits_{x=0}^{\infty}\frac{(2x+1)\exp(-\theta x)}{(1+\theta+\theta x)}\right)-\left(\exp\left(-\theta\right)\sum\limits_{x=0}^{\infty}\frac{\exp(-\theta x)}{(1+\theta+\theta x)}\right)^{2}\right)^{1/2}}{\exp\left(-\theta\right)\sum\limits_{x=0}^{\infty}\frac{\exp(-\theta x)}{(1+\theta+\theta x)}}. \tag{12}$$

The numerical values of IOD and COV are shown in Table 2 for a variety of model parameter values.

Table 2: *Index of dispersion and coefficient of variation of DBHE for different values of θ.*

Measure ↓ θ→	0.1	0.2	0.3	0.5	0.7	0.9	1	1.5	2
IOD	11.2853	7.3457	5.9583	4.7613	4.1991	3.8674	3.7481	3.3862	3.2143
COV	1.5566	2.0021	2.4818	3.5913	4.9680	6.6959	7.7212	15.0746	28.1402

From Table 2, it is observable that, when the parameter's value increases, the IOD decreases and the COV increases. Since, IOD>1 indicating that the proposed model is appropriate for modelling over-dispersed data.

3.4. Quantile Function

The point x_q is known as the q^{th} quantile of a discrete random variable X if it satisfies $P\left(X\leq x_q\right)\geq q$ and $P\left(X>x_q\right)>1-q$ that is $F\left(x_q-1\right)<q\leq F\left(x_q\right)$ (See, [19]).

Using this result, the q^{th} quantile of DBHE distribution can be obtained by

$$x_q = \left\lceil \frac{1}{\theta} \left\{ \log \left(\frac{1}{(1+\theta x_q)} - \frac{\exp(-\theta)}{(1+\theta+\theta x_q)} \right) - \log q \right\} \right\rceil , \tag{13}$$

where $\lceil . \rceil$ is the ceiling function that returns the smallest integer greater than or equal to its argument.
A random number (integer) can be easily sampled from the proposed distribution by using Equation (13) when q be a uniform random number drawn from a Uniform distribution on the unit interval, i.e. U(0,1). In particular, if we put $q = 0.5$, we will get the value of the median of the proposed distribution.

3.5. Order Statistics

Order statistics have several applications in reliability engineering and life testing. Let $X_1, X_2, ..., X_n$ be a random sample from DBHE distribution. Also, let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, denote the corresponding order statistics. Then, the CDF of r^{th} order statistic, say, $Z = X_{(r)}$, is given by

$$\begin{aligned} F_r(z,\theta) &= \sum_{i=r}^{n} \binom{n}{j} F^i(z) \left[1 - F(z,\theta)\right]^{n-i} \\ &= \sum_{i=1}^{r} \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} \left\{ 1 - \frac{\exp(-\theta(z+1))}{(1+\theta+\theta z)} \right\}^{(i+k)}. \end{aligned} \tag{14}$$

The corresponding PMF of r^{th} order statistic is

$$\begin{aligned} f_r(z) &= F_r(z) - F_r(z-1) \\ &= \sum_{i=1}^{r} \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} \left[\left\{ 1 - \frac{\exp(-\theta(z+1))}{(1+\theta+\theta z)} \right\}^{(i+k)} - \left\{ 1 - \frac{\exp(-\theta z)}{(1+\theta z)} \right\}^{(i+k)} \right]. \end{aligned} \tag{15}$$

Particularly, by putting $r = 1$ and $r = n$ in Equation (15), we can obtain the PMF of minimum$\left(X_{(1)}, X_{(2)}, ..., X_{(n)}\right)$ and the PMF of maximum$\left(X_{(1)}, X_{(2)}, ..., X_{(n)}\right)$, respectively.

3.6. Survival Characteristics

The Survival function of the proposed distribution is

$$S(x,\theta) = P(X > x) = \frac{\exp(-\theta x)}{(1+\theta x)} \; ; x = 0, 1, 2, ...$$

The hazard rate is a reliability characteristic that describes the system's failure behaviour over time. The discrete HRF for the DBHE distribution is given by

$$h(x,\theta) = P(X = x | X \geq x) = \frac{P(X = x)}{S(x-1,\theta)} = \frac{(1+\theta+\theta x - \exp(-\theta)(1+\theta x))}{(1+\theta+\theta x)}; x = 0, 1, 2, ..., \tag{16}$$

provided that $S(x-1,\theta) > 0$.

Figure 2 shows the HRF plots of the DBHE distribution for different values of θ. It is noted that the shape of the HRF is decreasing.
The reverse hazard rate function of the DBHE distribution is given by

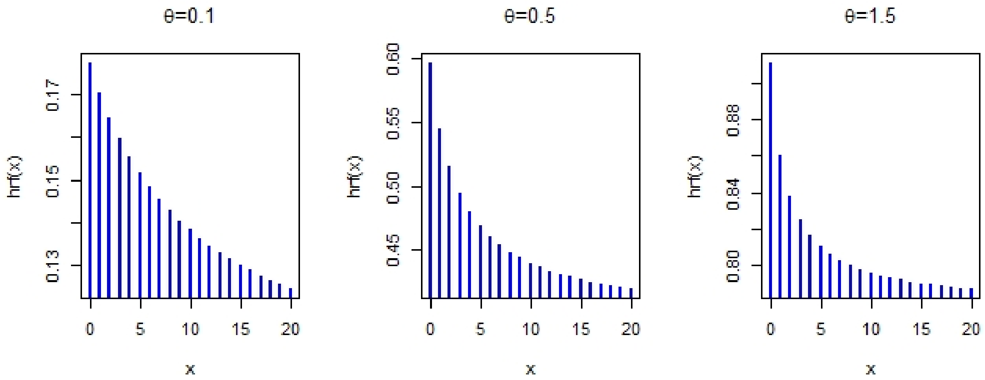

Figure 2: *The HRF plots of the DBHE model for different values of θ.*

$$h^{*}(x,\theta) = P(X = x|X \leq x) = \frac{P(X = x)}{F(x,\theta)} = \frac{\left(\frac{1}{(1+\theta x)} - \frac{\exp(-\theta)}{(1+\theta+\theta x)}\right)}{\left(1 - \frac{\exp(-\theta(x+1))}{(1+\theta+\theta x)}\right)} \exp(-\theta x). \tag{17}$$

The second rate of failure of the proposed model is given by

$$h^{**}(x,\theta) = \log\left\{\frac{S(x-1)}{S(x)}\right\} = \theta + \log(1+\theta+\theta x) - \log(1+\theta x). \tag{18}$$

3.7. Mean Residual and Mean Past Lifetime

The mean residual life (MRL) function, which represents the ageing mechanism, is broadly used in a wide variety of fields, including reliability engineering, survival analysis, biomedical research, and among others. In the literature, it is widely established that the MRL function uniquely characterises the distribution function F since it comprises all of the model's data. In discrete setup, the MRL, represented by the symbol $m(i)$, may be defined as follows:

$$m(i) = E(Y - i|Y \geq i) = \frac{1}{S(i)} \sum_{j=i+1}^{\infty} S(j);\ i = 0,1,2,...,$$

where $S(.)$ is SF. If X has DBHE distribution with parameter θ, then the MRL function of X is

$$m(i) = \frac{(1+\theta i)}{\exp(-\theta i)} \sum_{j=i+1}^{\infty} \frac{\exp(-\theta j)}{(1+\theta j)}.$$

A function is known as the mean past life (MPL) function or expected inactivity time function (EITF) denoted by $m^{*}(i)$, is used to estimate the amount of time since the failure of X if the system has failed at some point before 'i'. In a discrete setting, the MPL function can be defined as

$$m^{*}(i) = E(i - X|X < i) = \frac{1}{F(i-1)} \sum_{k=1}^{i} F(k-1);\ i = 1,2,....$$

By replacing the CDF (5) in the expression of $m^{*}(i)$, we can easily obtain the MPL for the proposed model.

3.8. Stress–Strength Parameter

Stress–strength analysis has been extensively used in reliability modelling. Suppose the random variable X and Y denotes the strength and stress of a system (both X and Y are in the positive domain), respectively, then the stress strength reliability $R = P[X > Y]$ can be defined as

$$R = P[X > Y] = \sum_{x=0}^{\infty} P_X(x) F_Y(x),$$

where $P_X(x)$ and $F_Y(x)$ respectively, denote the PMF and CDF of the independent discrete random variables X and Y. Let $X \sim DBHE(\theta_1)$ and $Y \sim DBHE(\theta_2)$, then R of the DBHE is,

$$R = \sum_{x=0}^{\infty} \left\{ \left(1 - \frac{\exp(-\theta_1(x+1))}{(1+\theta_1+\theta_1 x)}\right) \left(\frac{1}{(1+\theta_2 x)} - \frac{\exp(-\theta_2)}{(1+\theta_2+\theta_2 x)}\right) \exp(-\theta_2 x) \right\}. \tag{19}$$

Since, it is difficult to obtain the expression of R in explicit form therefore we perform a numerical analysis of R for different values of θ_1 and θ_2. The numerical outputs of R are presented in Table 3.

Table 3: *The numerical values of R for different combinations of θ_1 and θ_2.*

$\theta_1 \downarrow \theta_2 \rightarrow$	0.05	0.1	0.25	0.5	1	2	5
0.05	0.51381	0.35905	0.18823	0.09638	0.03710	0.00830	0.00020
0.1	0.66849	0.51372	0.30089	0.16324	0.06501	0.01478	0.00036
0.25	0.81902	0.69564	0.46859	0.27722	0.11696	0.02740	0.00067
0.5	0.87830	0.77949	0.56485	0.35267	0.15502	0.03718	0.00092
1	0.90091	0.81440	0.61111	0.39304	0.17724	0.04320	0.00107
2	0.90559	0.82202	0.62219	0.40351	0.18342	0.04496	0.00112
5	0.90593	0.82258	0.62304	0.40435	0.18394	0.04511	0.00112

From this table, we observe that for any fixed value of θ_1, R decreases as θ_2 increases, whereas for a fixed value of θ_2, as θ_1 increases, the value of R also increases.

4. Analysis of complete data under DBHE distribution

In this section, we estimate the unknown parameter of the DBHE distribution using the MLE method. An algorithm for generating random data is presented. We also present numerical examples based on empirical and real-world datasets to demonstrate the utility of the proposed approach for evaluating complete data.

4.1. Maximum Likelihood Estimation with Complete Data

Suppose $\underline{x} = (x_1, x_2, ..., x_n)$ be a random sample from DBHE distribution then the log-likelihood function can be written as

$$\log L(\underline{x}; \theta) = -\theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\left(\frac{1}{(1+\theta x_i)} - \frac{\exp(-\theta)}{(1+\theta+\theta x_i)}\right). \tag{20}$$

By differentiating Equation (22) with respect to the parameter θ, we get the non-linear likelihood equation as follows

$$\sum_{i=1}^{n} \left[\left(\frac{\exp(-\theta)}{(1+\theta+\theta x_i)}\right) \left(\frac{1+x_i}{(1+\theta+\theta x_i)} + 1\right) - \frac{x_i}{(1+\theta x_i)^2} \right] \left[\frac{1}{(1+\theta x_i)} - \frac{\exp(-\theta)}{(1+\theta+\theta x_i)}\right]^{-1} - \sum_{i=1}^{n} x_i = 0. \tag{21}$$

The solution of Equation (21) gives the MLE of θ. However, there is no explicit form for the solution of Equation (21). Therefore, Equation (21) has to be solved by using iterative methods such as Newton-Raphson, Nelder-Mead etc.

4.2. Numerical Illustration Using Simulated Data

In this subsection, we perform a Monte Carlo simulation study to show how well the MLE can estimate the unknown parameter of the DBHE distribution. Therefore, we conduct a simulation study with replication number 1,000. The true parameter values are used as $\theta = 0.05$, $\theta = 0.25$, and $\theta = 0.5$. There is no stated reason for using these parameter values. It may be used in several different ways. Random samples from the DBHE distribution are generated with $n = 15, 20, 25, ..., 100$ sample sizes using Equation (13). The simulation results are interpreted based on the mean square errors (MSEs) and absolute biases (ABs) where

$$MSE = \frac{1}{1000}\sum_{j=1}^{1000}\left(\hat{\theta}^j - \theta\right)^2 \text{and } AB = \frac{1}{1000}\sum_{j=1}^{1000}\left|\hat{\theta}^j - \theta\right|,$$

here, $\hat{\theta}$ is an estimate of θ.
The simulation results are graphically summarized and displayed in Figure 3.

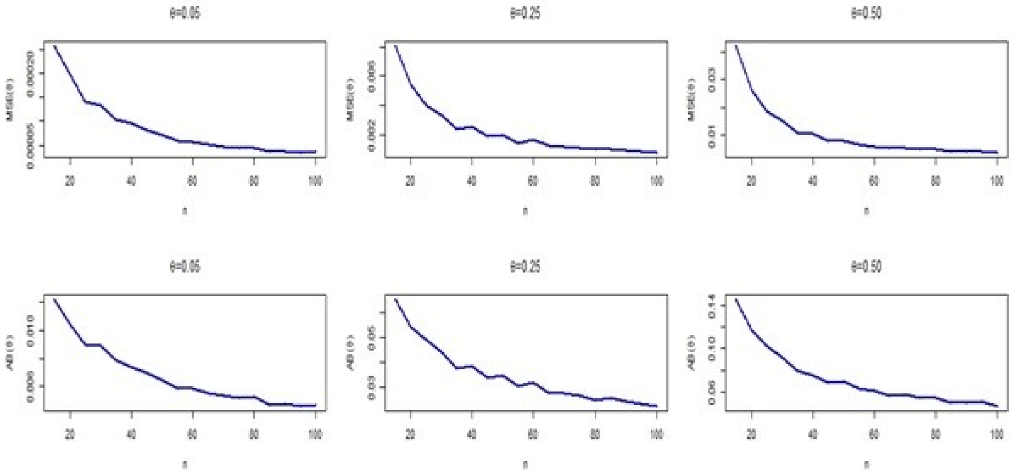

Figure 3: *Plots for MSEs and ABs for different values of θ for complete data.*

Figure 3 illustrates that the MSEs of the MLEs tend to zero as n approaches infinity. This demonstrates the consistency of the estimator. Furthermore, when n increases, the ABs is also declined to zero.

4.3. Real Data Analysis

In this section, we illustrate the utility of the DBHE distribution by examining two real-world datasets. Several criteria are used to compare fitted models, including the -logL, the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the Hannan Quinn information criterion (HQIC), and the Chi-square (χ^2) statistic with its associated P-value. The descriptive summaries of the datasets are shown in Table 4. From this table, we can see that the IOD for all datasets is greater than 1, indicating that the considered datasets can only be modelled by discrete distributions with overdispersion phenomena. The comparing models to DBHE distribution are listed in Table 5.

Table 4: *Descriptive Statistics of the Datasets.*

Data	n	Mean	Variance	Skewness	Kurtosis	IOD	COV
Dataset I	100	0.67	1.1526	2.4697	4.532	1.7203	1.6024
Dataset II	400	0.5475	1.1256	9.7478	15.6829	2.0558	1.9378

Table 5: *The competitive models of the DBHE distribution.*

Distribution	Abbreviation	Parameter(s)	Author(s)
Geometric	Geo	θ	-
Discrete Lindley	DLi	λ	[20]
Discrete Lindley-Two Parameter	DLi-II	p, β	[21]
Discrete Pareto	DPa	β	[5]
Discrete linear failure rate	DLFR	λ_1,λ_2	[22]
Discrete inverse Weibull	DIW	α,β	[6]
Discrete log-logistic	DLogL	δ,λ	[23]
Discrete Nielsen	DN	p, θ	[24]
Negative Binomial	NB	μ,Θ	-
Zero-Inflated Negative Binomial	ZINB	μ, Θ, ω	-
Poisson- Lindley	PL	θ	[25]
Generalized Poisson-Lindley	GPL	θ, α	[26]

Dataset I: The first dataset, consists of the recordings of the total number of carious teeth among the four deciduous molars in a sample of 100 children 10 and 11 years old [5]. The expected frequency of the fitted models along with their MLE, standard error (SE), -logL, and goodness of fit measures are presented in Table 6. Since, the values of -logL, χ^2 test statistic, AIC, BIC, CAIC, and HQIC of DBHE distribution are smallest among those of other considered models, hence this new distribution appears to be a very suitable model for this dataset. Similarly, the higher P-value corresponding to χ^2 statistic for DBHE distribution show its dominance on other candidate models in terms of model fitting.

Table 6: *The MLE (SEs) and goodness of fit statistics for different models under dataset I.*

X	Observed Frequency	DBHE	Geo	DLi	DLi-II	DPa	DLFR	DIW	DLogL
0	64	62.80	59.88	57.13	59.88	69.04	59.9	63.3	62.73
1	17	21.37	24.02	26.88	24.02	15.37	24.01	22.48	22.42
2	10	8.60	9.64	10.45	9.64	6.01	9.63	6.44	7.01
3	6	3.78	3.87	3.71	3.87	3.01	3.86	2.76	2.98
>=4	3	3.45	2.59	1.83	2.59	6.57	2.6	5.02	4.86
Total	100	100	100	100	100	100	100	100	100
MLE (SE)		0.55043 (0.064)	0.59879, (0.038)	0.274 (0.029)	0.401 (0.269), 0.478 (0.529)	0.184 (0.032)	0.401 (0.056), 1.0 (0.044)	0.633 (0.049), 1.576 (0.251)	0.745 (0.101), 1.768 (0.267)
$-\log$ L		112.328	112.474	113.68	112.475	116.83	112.470	116.275	115.470
χ^2		1.575	3.347	6.638	3.347	3.225	3.340	3.503	2.783
D.F.		2	2	2	1	2	1	1	1
P-value		0.455	0.188	0.036	0.067	0.199	0.068	0.061	0.095
AIC		226.656	226.947	229.36	228.950	235.66	228.940	236.550	234.940
BIC		229.261	229.552	232.96	234.160	238.27	234.150	241.760	240.150
CAIC		226.697	226.988	229.39	229.073	235.70	229.063	236.673	235.063
HQIC		227.710	228.001	230.41	231.058	236.72	231.048	238.658	237.048

Dataset II: The second dataset represents the number of chromatid aberrations in 24 hours [28]. The expected frequency of the fitted models along with their MLE, SE, -logL, and goodness of fit measures are presented in Table 7. On comparison of the values of -logL, χ^2 test statistic, P-value, AIC, BIC, CAIC, and HQIC, we again found that the DBHE distribution is the best model than the other five models understudy for this dataset.

Table 7: *The MLE (SEs) and Goodness of fit statistics for different models under dataset II.*

X	Observed Frequency	DBHE	DN	NB	ZINB	PL	GPL
0	268	269.36	270.14	270.18	270.18	257.02	269.24
1	87	80.48	79.40	78.55	78.55	93.39	78.70
2	26	29.28	29.21	29.84	29.84	32.76	30.86
3	9	11.76	11.88	12.22	12.22	11.21	12.55
4	4	5.01	5.11	5.19	5.19	3.77	5.13
5	2	2.22	2.28	2.25	2.25	1.25	2.09
6	1	0.90	1.05	0.99	0.99	0.41	0.85
7	3	0.47	0.93	0.78	0.78	0.13	0.35
Total	400	400	400	400	400	400	400
MLE (SEs)		0.63026 (0.037)	0.5301 (0.0601), 1.1089 (0.2179)	0.5475 (011539), 0.6200 (0.1270)	0.5475 (0.1701), 0.6200 (0.3383), 0.00008 (0.2989)	2.379 (0.169)	1.576 (0.259), 0.473 (0.159)
− log L		399.342	399.410	399.860	399.860	399.857	400.553
χ_2		1.781	1.924	2.416	2.416	6.283	2.940
D.F.		3	2	2	1	3	2
P-value		0.619	0.382	0.299	0.120	0.098	0.229
AIC		800.683	802.820	803.720	805.720	801.714	805.106
BIC		804.675	810.803	811.703	817.694	805.706	813.089
CAIC		800.693	802.850	803.750	805.781	801.724	805.136
HQIC		802.264	805.981	806.881	810.462	803.295	808.267

5. Analysis of randomly censored data under DBHE distribution

In this section, we derive the MLE of the unknown parameter of the DBHE distribution for random rightly-censored data. For the DBHE model, an algorithm for generating random right-censored data is presented. We also present numerical examples based on empirical and real-world datasets to show the usefulness of the proposed approach for evaluating random censored data.

5.1. Maximum Likelihood Estimation with Randomly Censored Data

Due to the availability of right-censored observations, the contribution of the i^{th} individual for the likelihood function based on a random sample (x_i, d_i) of size n is given by

$$L_i = [f(x_i)]^{d_i}[S(x_i)]^{1-d_i},$$

where d_i is a censoring indicator variable, that is, $d_i = 1$ for an observed lifetime and $d_i = 0$ for a censored lifetime $(i = 1, 2, 3, ..., n)$. Assuming the DBHE model, the likelihood function for θ is given by

$$L(\theta|\underline{x}, \underline{d}) = \prod_{i=1}^{n} \left\{ \left(\frac{1}{(1+\theta x_i)} - \frac{\exp(-\theta)}{(1+\theta+\theta x_i)} \right) \exp(-\theta x_i) \right\}^{d_i} \left\{ \frac{\exp(-\theta x_i)}{(1+\theta x_i)} \right\}^{1-d_i}, \tag{22}$$

where $\underline{d} = (d_1, d_2, ..., d_n)$. The corresponding log-likelihood function is

$$\log L(\theta|\underline{x}, \underline{d}) = \sum_{i=1}^{n} d_i \log \left\{ \frac{1}{(1+\theta x_i)} - \frac{\exp(-\theta)}{(1+\theta+\theta x_i)} \right\} + \sum_{i=1}^{n} (d_i - 1)\log(1+\theta x_i) - \theta \sum_{i=1}^{n} x_i. \tag{23}$$

Taking the first derivative of Equation (23) w.r.t. θ and setting this derivative equal to zero, we can obtain the likelihood equation for the parameter θ. Although, it is hard to find a closed-form expression of MLE for the parameter θ using this likelihood equation, therefore, we can use an appropriate numerical methodology such as the Newton-Raphson iteration method to obtain the MLE of θ.

5.2. Algorithm to Simulate Random Right-Censored Data

We present a simple approach in this part for generating random right-censored data from the suggested model. The algorithm is as follows:

Step 1: Fix the values of the parameter θ.

Step 2: Draw n random pseudo from $Uniform(0,1)$ i.e. $u_i \sim U(0,1); i = 1,2,...,n$.

Step 3: Obtain $x_i^{/} = F^{-1}(u_i;\theta); i = 1,2,...,n$, where $F^{-1}(\bullet)$ is defined in Equation (13).

Step 4: Draw n random pseudo from $c_i \sim U(0, \max(x_i^{/})); i = 1,2,...,n$. This is the distribution that controls the censorship mechanism.

Step 5: If $x_i^{/} \leq c_i$, then $x_i = [x_i^{/}]$ and $d_i = 1, i = 1,2,...,n$, else, $x_i = [c_i]$ and $d_i = 0, i = 1,2,...,n$. Hence, pairs of values $(x_1, d_1), (x_2, d_2), ..., (x_n, d_n)$ are obtained as the random right-censored data.

5.3. Numerical Illustration Using Simulated Random Right-Censored Data

This subsection portrays a simulation study to evaluate the performance of the MLE using randomly right-censored data. The whole study is based on randomly chosen samples from the DBHE distribution of sizes 20, 25, ...,100. The values of θ are set to 0.05, 0.25, and 0.50. The procedure described above is used to generate the requisite random right-censored data. All simulation findings are based on 1000 replications for different settings of parameter values and sample sizes. Based on these 1000 values, we estimated the MSE and AB of the parameter estimate, and the resultant graphs are given in Figure 4.

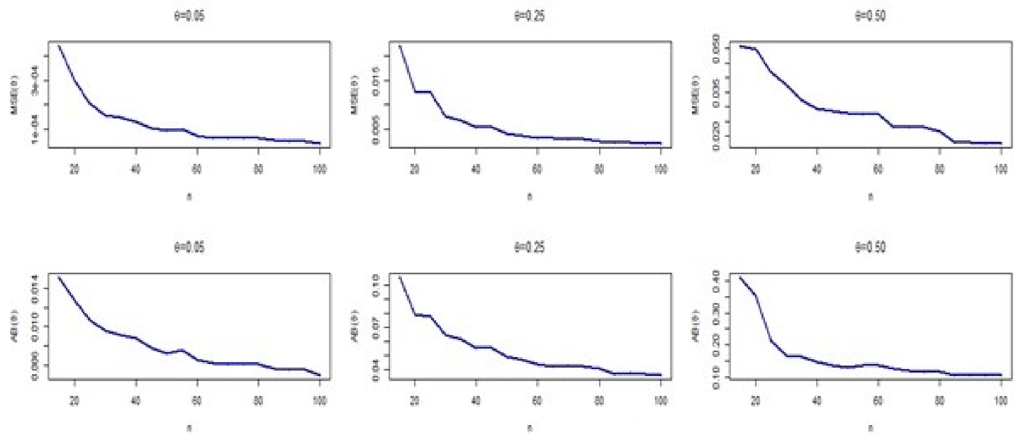

Figure 4: *Plots for MSEs and ABs for different values of θ under censored data.*

As seen in Figure 4, the MSEs of the MLE approach θ as n approaches infinity. This illustrates the estimator's consistency. Additionally, when n increases, the ABs is also tending to zero.

5.4. Application to Real Data Analysis

Here, we examine two real datasets to illustrate the applicability of the DBHE model to randomly censored data. The following datasets and their fitting are described as follows:
Dataset III: This dataset is obtained from [29]. The data below are remission times, in weeks, for a group of 30 patients with leukaemia who received similar treatment.
1, 1, 2, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31*, 42, 45*, 50*, 57, 60, 71*, 85*, 91.
The observations with asterisks indicate censored times. The MLE (SE) of the θ for the given dataset is 0.0201 (0.0008). Now, we have been used Kolmogorov-Smirnov (K-S) test to check whether the given data follows DBHE distribution or not. The calculated value of the K-S test is 0.13333 and P-value is equal to 0.9525. These values announce that the DBHE distribution can be used to model this data.

Dataset IV: Here, we analyze another real dataset obtained from [29]. The data below show survival times (in months) of patients with Hodgkin's disease who were treated with nitrogen mustards.
1.05, 2.92, 3.61, 4.20, 4.49, 6.72, 7.31, 9.08, 9.11, 14.49*, 16.85, 18.82*, 26.59*, 30.26*, 41.34*.
The asterisks observations represent censored times. For the provided dataset, the MLE (SE) of the θ is 0.0311 (0.0027). We have also performed the K-S test to see whether the data distribution fits the DBHE distribution or not, and it is found that the K-S test has a value of 0.2 and a P-value of 0.9383. So, it can be seen that the DBHE distribution fits the data very well.

6. Conclusions

In this paper, we have proposed discrete Burr-Hatke exponential distribution. It is observed that with one parameter, this model has great flexibility in terms of fitting as it is capable of modelling right-skewed, decreasing failure rate, and over-dispersed counts datasets. Some of its fundamental properties have been discussed in detail. The unknown parameter of the DBHE distribution with complete and censored data has been estimated by using the maximum likelihood approach. We have provided an algorithm to generate randomly right-censored data. Additionally, the performance of the estimator under complete and censored data have been examined through an extensive simulation study. Finally, the flexibility of the DBHE distribution has been empirically proven by using four real-life applications consisting of two complete and two censored datasets. Hence, we can conclude that the proposed model will serve a wide spectrum of applications in various domains such as medical, reliability, survival analysis, etc.

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CONSTRUCTION AND SELECTION OF SKIP LOT SAMPLING PLAN OF TYPE SKSP-V FOR LIFE TESTS BASED ON PERCENTILES OF EXPONENTIATED RAYLEIGH DISTRIBUTION

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Abstract

This study uses percentiles under the exponentiated Rayleigh distribution to build a skip lot sampling plan of the SkSP-V type for a life test. A truncated life test may be carried out to determine the minimum sample size to guarantee a specific percentage life time of products. In particular, this paper highlights the construction of the Skip lot Sampling Plan of the type SkSP-V by considering the Singe Sampling Plan as reference plans for life tests based on percentiles of Exponentiated Rayleigh Distribution. Calculations are made for various quality levels to determine the minimum sample size, prescribed ratio, and operational characteristic values. The proposed sampling plan, which is appropriate for the manufacturing industries for the selection of samples, is also analyzed in terms of its parameters and metrics. The curve is produced after tabulating the operating characteristic data of the plan. Illustrations are provided to help you comprehend the plan. In addition, it addresses the feasibility of the new strategy.

Keywords: Exponentiated Rayleigh Distribution, Percentiles, Life tests, Single Sampling Plan, Double Sampling Plan, SkSP –V.

I. Introduction

The word used for statistical quality control (SQC) describes the collection of qualitative statistical methods used by Professionals in manufacturing. One of the main industries of Acceptance sampling is the regulation of statistical consistency. A sampling of acceptance is a methodology that addresses Procedures under which an approval or rejection decision is focused on sample inspection. Acceptance sampling plans in statistical quality control are concerned with accepting or rejecting a submitted lot of large-size products based on the quality of products inspected through the sample taken from the lot, whereas reliability is the "probability of performing without failure a specified function under the given condition for a specified period." Therefore, reliability testing usually involves the simulation of conditions under which the item will be used during its lifespan.

Dodge [5] proposed skip-lot sampling plans based on the principle of continuous sampling plan of type CSP-1 for a series of lots or consignments of material. Balamurali and Chi-Hyuck Jun [4] derived the new skip-lot sampling plan, which is designated as SkSP-V. It is based on the principles of a continuous sampling plan of type CSP-V and derived the cost model for the SkSP-V plan. Huge authors like Epstein [6], Sobel and Tischendr [14], Goode and Kao [7], Gupta and Groll [8], Kantam et al. [9,10], Baklizi [2], Tsai and Wu [16], Balakrishnan et al. [3], Aslam and Shahbaz [1], Rao et.al.[13], Pradeepa Veerakumari and Ponneeswari [11], Pradeepa Veerakumari et.al [12], Suganya and Pradeepa Veerakumari [15] have fascinated the methodology of time-truncated acceptance sampling plans.

The paper focuses on building an SkSP-V life-test plan with single sampling plan as a percentile-based comparison plan with Exponential Rayleigh Distribution. Rayleigh Exponential Distribution is Important distribution of life testing and analysis of reliability. It has some of the essential structural properties and exhibits great mathematical consistency. Most features of the Exponentiated Rayleigh distribution are close to those of gamma, Weibull, and exponential distribution. ERD's functions for distribution and density are in similar forms. As a result, this is quickly extended to the truncated plans. The cumulative function of the ERD distribution is given by,

$$F(t;\tau,\theta) = \left[1 - e^{-1/2(t/\tau)^2}\right]^{\theta},\ t>0,\ 1/\tau>0,\ \theta>0 \tag{1}$$

Where, τ and θ are the scale and shape parameters respectively. The first derivative of any cumulative distribution function is its probability density function. Hence the probability density function of ERD can be written as,

$$f(t;\tau,\theta) = \theta\left[1 - e^{-1/2(t/\tau)^2}\right]^{\theta-1}\left[\frac{t}{\tau^2}e^{-1/2(t/\tau)^2}\right] \tag{2}$$

Pradeepa Veerakumari and Ponneeswari [11] proposed SSP and DSP for life testing based on the percentiles of ERD. Subsequently, Pradeepa Veerakumari et.al [12] developed Skip-lot sampling plans for life testing based on the percentiles of ERD.

II. Operating procedure for Skip-lot sampling plan of type SkSP-V

The operating procedure for skip-lot sampling plan of type SkSP-V is given by

- The procedure begins with a normal inspection of samples using a suitable reference sampling plan procedure.
- Under the normal inspection, if i number of successive lots are accepted discontinue normal inspection and switch on to skipping inspection.
- Under skipping inspection, fraction f of lots are randomly selected and inspected based on the conditions of the assigned reference plan. Continue the skipping inspection until a sampled and inspected lot is nonconforming.
- Again under skipping inspection if fraction f of the lots are rejected before k consecutively sampled lots are accepted, go to the normal inspection step (1) as above.

- When k consecutive lots are accepted under skipping inspection then go to normal inspection with reduced clearance number x as per step (6) given below.
- Under normal inspection with clearance number x, lots are inspected one by one in the order of being submitted to inspection.
- When a lot is rejected, immediately return to normal inspection with clearance number i as per (1) given above.
- When x lots are accepted on normal inspection mode, immediately stop the normal inspection and switch to skipping inspection as per (3) above.
- When a lot is rejected, perform screening inspection and substitute all the non-conforming units found with conforming units in the rejected lots in the case of non-destructive testing.

III. Operating procedure of SkSP-V with SSP as a reference plan based on percentiles of ERD

A random sample of size n is drawn and put Draw a random sample of size and place on test for time t_0.

- The numbers of defectives d are counted and a comparison is made with the acceptance number c.
- If d>c , then reject the lot.
- If d≤c, then accept the lot.
- If d>c, is obtained before the specified time t_0, terminate the test, and reject the lot.

IV. Operating characteristic function for SkSP-V using Single Sampling Plan

OC function is the most applied technique to measure the efficiency of the sampling plan and from where the probability of acceptance is derived. It provides the probability that the lot can be accepted. The OC function of SSP for life tests based on the percentiles of ERD is as follows,

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i} \tag{3}$$

Where $P = F(t, \delta_0)$ represents the failure probability at time t given a determined 100q^th percentile of the lifetime t_q^0 and p depends only on $\delta_0 = t/t_q^0$. The OC values are tabulated in Table 3 of Pradeepa Veerakumari [11].

The OC function of SkSP-V for the lot quality p is given by,

$$P_a(p) = \frac{fP+(1-f)P^i+fP^{k+1}(P^i-P^x)}{(1+P^{i+k}-P^{2k})+(1-f)P^i} \tag{4}$$

Then, the Average Sample number is

$$ASN\ (p) = ASN\ (R)F \tag{5}$$

Where, R- represents the Average Sample number of the reference plan, P represents the probability of acceptance of the reference plan.

4.1. Illustration

Presume that the lifetime of the electric goods follows ERD. Skip lot sampling plan of type SkSP-V with SSP as a reference plan based on the 10th percentile is applied for testing. The parameters for the life testing is as follows: θ=2,t=40hrs, $t_{0.1}$=20hrs, c=0, α=0.05 and β=0.05 then η = 0.871929 from the equation and the ratio is found to be $t/t_{0.1}$ =2.00 by applying the minimum sample size according to the requirements is n=3 and the corresponding OC values L(p) for the Single Sampling plan for the life tests based on percentiles of ERD (n,c, $t/t_{0.1}$ = 3,0,0.7921) with $P^* = 0.95$. $L(p)$ is the P-value for SkSP-V with SSP as a reference plan for life tests based on the percentiles of ERD. For i=1,k=2, and f=1/3; the probability of acceptance $L(p)$ values of SkSP-V with SSP for life tests grounded on percentiles of ERD is derived from Eqn. 4 as,

$t/t_{0.1}^0$	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
$L(p)$	0.0285	0.5350	0.8802	0.9678	0.9909	0.9973	0.9991	0.9997	0.9999

From the illustrations, it is indicated that the actual 10th percentile is almost equal to the required 10th percentile ($t/t_{0.1}$ = 1.00) the producer's risk is approximately 0.9715 (1-0.0285). Also, the producer's risk is nearly equal to 0.05 or less and the actual producer risk is large or nearly equal to 1.5 times the required percentile. The OC curve is provided for the illustration as fig.1.

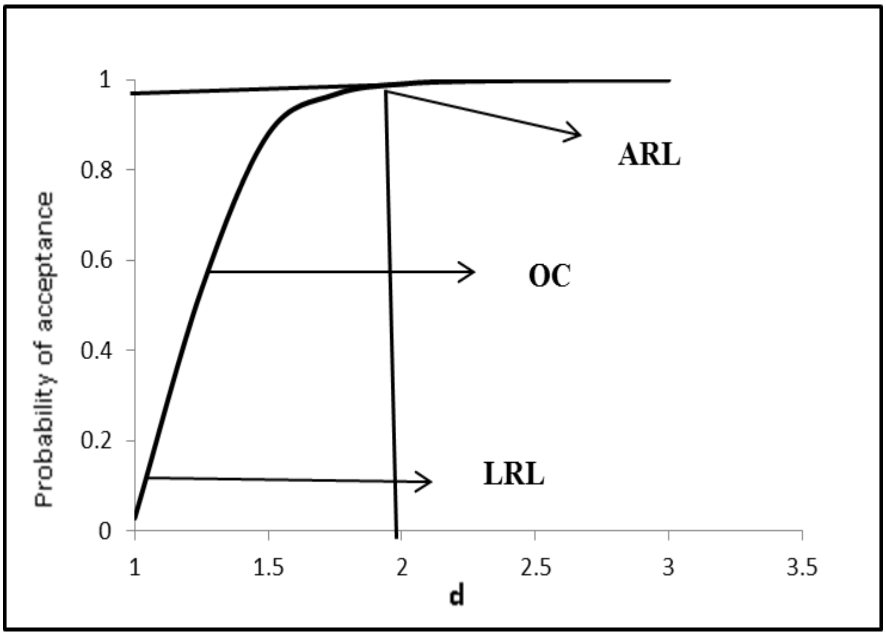

Figure 1. *OC Curve for i=1,k=2, f=1/3, P*=0.95, d=$d_{0.1}$ and θ=2*

Figure 1 clearly says that the plan attains ARL when the actual lifetime percentile is in close proximity to 2 times greater than the specified 10th percentile and attains LRL when the actual lifetime percentile is roughly equal to the specified lifetime percentile. For the purpose of convenience OC values of the table are constructed and tabulated with parameters *i=1,k=2, f=1/3,* and *c=0* in Table 1

Table 1: *Gives the OC values for sampling plan (n,c = 2, $t/t_{0.1}$) for a given P^* under ERD when θ=2*

P^*	$t/t_{0.1}^0$	$t_{0.1}/t_{0.1}^0$								
		0.7	**0.9**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**
0.75	0.7	0.5025	0.9070	0.9814	0.9960	0.9990	0.9997	0.9999	0.9999	0.9999
0.75	0.9	0.5016	0.9006	0.9791	0.9953	0.9988	0.9997	0.9999	0.9999	0.9999
0.75	1	0.4870	0.8932	0.9768	0.9946	0.9986	0.9996	0.9999	0.9999	0.9999
0.75	1.5	0.4907	0.8757	0.9690	0.9919	0.9978	0.9993	0.9998	0.9999	0.9999
0.75	2	0.3738	0.8070	0.9440	0.9830	0.9947	0.9983	0.9994	0.9998	0.9999
0.75	2.5	0.3420	0.7623	0.9220	0.9734	0.9908	0.9967	0.9988	0.9995	0.9998

0.9	0.7	0.2489	0.8205	0.9614	0.9910	0.9977	0.9994	0.9998	0.9999	0.9999
0.9	0.9	0.2413	0.8051	0.9559	0.9893	0.9972	0.9992	0.9998	0.9999	0.9999
0.9	1	0.2434	0.8001	0.9536	0.9885	0.9970	0.9991	0.9997	0.9999	0.9999
0.9	1.5	0.2135	0.7469	0.9326	0.9810	0.9944	0.9983	0.9994	0.9998	0.9999
0.9	2	0.2214	0.7167	0.9149	0.9733	0.9914	0.9971	0.9990	0.9996	0.9998
0.9	2.5	0.1160	0.5668	0.8473	0.9456	0.9801	0.9926	0.9971	0.9989	0.9995
0.95	0.7	0.1360	0.7467	0.9434	0.9863	0.9965	0.9990	0.9997	0.9999	0.9999
0.95	0.9	0.1330	0.7284	0.9363	0.9839	0.9957	0.9988	0.9996	0.9999	0.9999
0.95	1	0.1342	0.7214	0.9331	0.9827	0.9953	0.9986	0.9996	0.9998	0.9999
0.95	1.5	0.1275	0.6700	0.9094	0.9738	0.9921	0.9975	0.9992	0.9997	0.9999
0.95	2	0.1204	0.6165	0.8801	0.9613	0.9871	0.9956	0.9984	0.9994	0.9998
0.95	2.5	0.1160	0.5668	0.8473	0.9456	0.9801	0.9926	0.9971	0.9989	0.9995
0.99	0.7	0.0294	0.5609	0.8928	0.9723	0.9924	0.9978	0.9993	0.9998	0.9999
0.99	0.9	0.0285	0.5350	0.8802	0.9678	0.9909	0.9973	0.9991	0.9997	0.9999
0.99	1	0.0288	0.5248	0.8743	0.9655	0.9901	0.9970	0.9990	0.9997	0.9999
0.99	1.5	0.0216	0.4274	0.8202	0.9449	0.9824	0.9942	0.9980	0.9993	0.9997
0.99	2	0.0143	0.3240	0.7441	0.9127	0.9690	0.9887	0.9958	0.9984	0.9993
0.99	2.5	0.0081	0.2214	0.6406	0.8631	0.9467	0.9787	0.9914	0.9964	0.9985

V. Operating procedure of SkSP-V with DSP as a reference plan based on percentiles of ERD

The modus operandi of SkSP-V with DSP as a reference plan based on percentiles of ERD are as follows:
Step 1: A random sample of size n_1 is drawn and put on a life test.
Step 2: The number of defectives $d_1$is counted and a comparison is made with the acceptance number c.
i. If $d_1>c_1$, then reject the lot.
ii. If $d_1\leq c_1$, then accept the lot.
Step 3: If $d_1<c_2$, is obtained before the specified time t_0, terminate the test, and reject the lot.
Step 4: If $c_1< d_1 \leq r_1$, take a second sample of size n_2 from the remaining lot and put them on test for time $t_0$and count the number of non-conformities (d_2).
Step 5:
If $d_1+d_2 \leq r_1$, accept the lot.
If $d_1+d_2 > r_1$, reject the lot.

VI. Operating characteristic function for SkSP-V using Double Sampling Plan

OC function is the most applied technique to measure the efficiency of the sampling plan and from where the probability of acceptance is derived. It provides the probability that the lot can be accepted. The OC function of DSP for life tests based on the percentiles of ERD is as follows,

$$L(p) = \sum_{d_1}^{c_1} \binom{n_1}{d_1} p^{d_1}(1-p)^{n_1-d_1} . \sum_{d_1+c_{1+1}}^{c_2} \binom{n_1}{d_1} p^{d_1}(1-p)^{n_1-d_1} . \sum_{d_2=0}^{c_2-d_2} \binom{n_2}{d_2} p^{d_2}(1-p)^{n_2-d_2} \tag{6}$$

Where $P = F(t, \delta_0)$ represents the failure probability at time t given a determined 100q^th^ percentile of the lifetime t_q^0 and p depends only on $\delta_0 = t/t_q^0$. The ASN Value of DSP is calculated from the equation,

$$ASN = n_1 p_1 + (n_1 + n_2)(1 - p_1) = n_1 + n_2(1 - p_1) \tag{7}$$

The OC function of SkSP-V for the lot quality p is given by,

$$P_a(p) = \frac{fP + (1-f)P^i + fP^{k+1}(P^i - P^x)}{(1 + P^{i+k} - P^{2k}) + (1-f)P^i} \tag{8}$$

Then, the Average Sample number is

$$ASN\ (p) = ASN\ (R)F \tag{9}$$

Where ASN (R) represents the Average Sample number of the reference plan; P represents the probability of acceptance of the reference plan.

6.1 Illustration

Presume that the lifetime of the electric goods follows ERD. Skip lot sampling plan of type SkSP-V with DSP as a reference plan based on the 10th percentile is applied for testing. The parameters for the life testing is as follows θ=2,t=40hrs, $t_{0.1}$=20hrs, c=0, α=0.05 and β=0.05 then η = 0.871929 from the equation and the ratio is found to be $t/t_{0.1}$ =2.00 by applying the minimum sample size according to the requirements is *n_1=9,n_2=11* and the corresponding OC values L(p) for the Double Sampling plan for the life tests based on percentiles of ERD $n_1, n_2, c_1, c_2, {}^{t}/_{t_{0.1}}$ = (9,11,0,3,0.9379) with P^* = 0.99. *L (p)* is the *P*-value for SkSP-V with DSP for life tests as a reference plan defined on the percentiles of ERD. For *i=1,k=3,* and *f=1/5;* the probability that SkSP-V with DSP will consider *L(p)* values for life tests based on percentiles of ERD is found from Eqn; 8 For,

$t/t_{0.1}^0$	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
L(p)	0.0434	0.3454	0.7235	0.9158	0.9678	0.9957	0.9991	0.9998	0.9999

From the illustrations, it is indicated that the actual 10th percentile is almost equal to the required 10th percentile $t/t_{0.1}^0$ the producer's risk is approximately 0.9566 (1-0.0434). Moreover, the producer's risk is closely equal to 0.05 or less and the actual producer risk is large or nearly equal to 2 times the required percentile. The OC curve is provided for the illustration as fig 2.

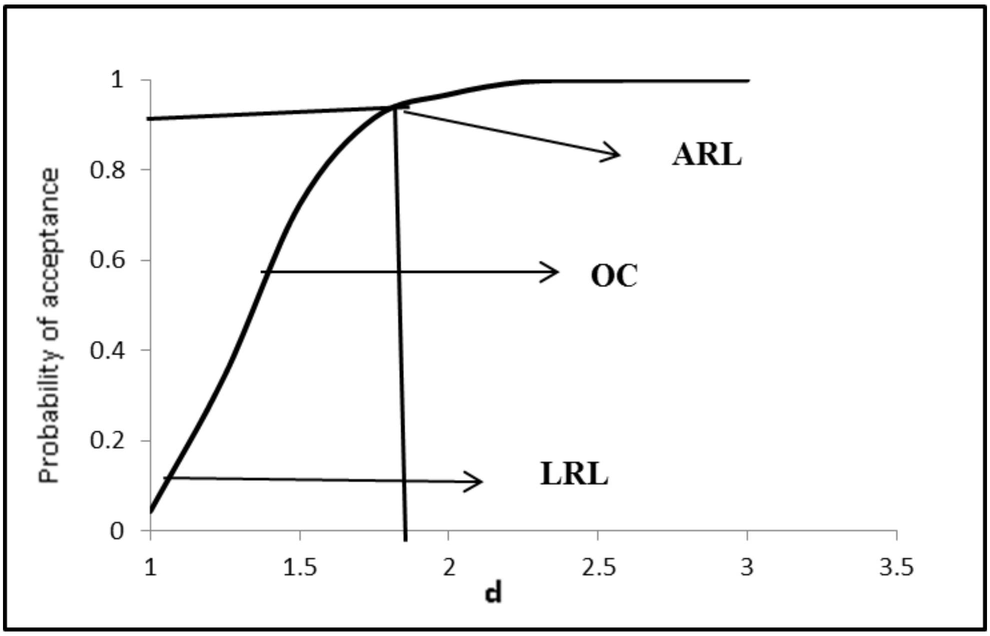

Figure 2. *OC Curve for i=1,k=3, f=1/5, P*=0.99, d=d0.1 and θ=2*

Figure 2.clearly says that the plan attains ARL when the actual lifetime percentile is in close proximity to 1.85 times greater than the specified 10th percentile and attains LRL when the actual lifetime percentile is approximately equal to the specified lifetime percentile. For the purpose of convenience OC values of the table are constructed and tabulated with parameters *i=1,k=3, f=1/5* and c_1=0,c_2 =3 in Table 2.

Table 2 : *Gives the OC values for Sampling Plan (n, c_1=0, c_2=1, $t_{0.1}/t^0_{0.1}$) for a given P* under ERD when θ=2*

P^*	$t/t^0_{0.1}$	$t_{0.1}/t^0_{0.1}$								
		1	**1.25**	**1.5**	**1.75**	**2**	**2.25**	**2.5**	**2.75**	**3**
0.75	0.7	0.5047	0.8428	0.9569	0.9880	0.9964	0.9988	0.9995	0.9998	0.9999
0.75	0.9	0.4943	0.8408	0.9559	0.9874	0.9961	0.9986	0.9995	0.9998	0.9999
0.75	1	0.4928	0.8411	0.9557	0.9871	0.9959	0.9986	0.9994	0.9998	0.9999
0.75	1.5	0.4787	0.8325	0.9499	0.9843	0.9947	0.9980	0.9992	0.9996	0.9998
0.75	2	0.4777	0.8322	0.9468	0.9820	0.9935	0.9975	0.9989	0.9995	0.9998
0.75	2.5	0.3021	0.7179	0.8951	0.9578	0.9819	0.9918	0.9961	0.9980	0.9989
0.9	0.7	0.2495	0.6922	0.9070	0.9718	0.9909	0.9968	0.9988	0.9995	0.9998
0.9	0.9	0.2478	0.6895	0.9051	0.9706	0.9903	0.9966	0.9986	0.9994	0.9997
0.9	1	0.2493	0.6741	0.8954	0.9667	0.9889	0.9960	0.9985	0.9993	0.9997
0.9	1.5	0.2379	0.6615	0.8870	0.9617	0.9864	0.9948	0.9979	0.9991	0.9996
0.9	2	0.2095	0.6615	0.8841	0.9580	0.9838	0.9933	0.9971	0.9986	0.9993
0.9	2.5	0.1505	0.5815	0.8422	0.9385	0.9746	0.9890	0.9950	0.9976	0.9987

0.95	0.7	0.1352	0.5971	0.8770	0.9620	0.9874	0.9954	0.9982	0.9992	0.9996
0.95	0.9	0.1327	0.6009	0.8781	0.9615	0.9868	0.9951	0.9980	0.9991	0.9996
0.95	1	0.1330	0.5902	0.8714	0.9587	0.9857	0.9946	0.9978	0.9990	0.9995
0.95	1.5	0.1206	0.5561	0.8487	0.9476	0.9806	0.9923	0.9967	0.9985	0.9993
0.95	2	0.0966	0.5297	0.8286	0.9355	0.9738	0.9887	0.9948	0.9975	0.9987
0.95	2.5	0.1034	0.4777	0.7873	0.9165	0.9657	0.9854	0.9935	0.9969	0.9985
0.99	0.7	0.0297	0.4122	0.8088	0.9387	0.9783	0.9916	0.9965	0.9984	0.9992
0.99	0.9	0.0291	0.3965	0.7954	0.9326	0.9756	0.9904	0.9959	0.9981	0.9991
0.99	1	0.0297	0.3823	0.7832	0.9275	0.9735	0.9895	0.9955	0.9979	0.9990
0.99	1.5	0.0259	0.3365	0.7393	0.9059	0.9633	0.9847	0.9932	0.9968	0.9984
0.99	2	0.0265	0.2432	0.6105	0.8397	0.9340	0.9718	0.9874	0.9941	0.9971
0.99	2.5	0.0210	0.2620	0.6464	0.8533	0.9366	0.9711	0.9862	0.9931	0.9964

VII. Conclusion

In this study, life testing plans based on percentiles of ERD for Skip-lot Sampling plan of type-V with SSP and DSP as reference plan are developed. Skip-lot Sampling plan of type-V with SSP and DSP as reference plan requires minimum sample size and also has better-operating characteristics values. Thus, results in a reduction of inspection cost and better efficiency. The proposed plan can be further extended to other sampling plans for instance SkSP and other probability distribution.

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STOCHASTIC ANALYSIS OF A COLD STANDBY COMPUTER SYSTEM WITH UP-GRADATION PRIORITY AND FAILURE OF SERVICE FACILITY

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Abstract

We describe the development of a stochastic model for a computer system with cold standby redundancy, priority and failure of service facility. A computer system (called a single unit) means the simultaneous working of its hardware and software components. The system has one more unit (called computer system) that can be used as and when required at the failure of any of the hardware/software components of the initially operative computer system. A single repair facility is made available to rectify the faults which occur due to the failure of hardware and software components. The failed hardware component undergoes for repair immediately while failed software is up-graded. The service facility is subjected to failure during hardware repair. The provision of perfect treatment has been made for the failed service facility. The components work as new after repair and up-gradation with the same life time distribution. The priority is given to the software up-gradation over the hardware repair. In steady state, the expressions for some important reliability measures have been derived using the well known semi-Markov process and regenerative point technique. The behavior of some useful reliability characteristics has been observed for particular values of the parameters related to failure times, repair and up-gradation times and treatment time which follow negative exponential distribution.

Keywords: Computer System, Unit Wise Redundancy, Priority, Failure of Service Facility and Stochastic Modelling

I. Introduction

Over the years an overwhelming transformation of the modern society into the digitalization World has been observed with the advent of advanced technology and frequent use of computer systems. As a result of which we are now in a position to complete the assigned jobs within time limits and perfectness. In the modern World of today the use of computer systems cannot be ignored completely or partially in order to survive in the competitive markets. On the other hand, the burden for the heavy use of computer systems grabs the attention of reliability engineers and scientists to identify all possible ways and means to improve the reliability and performance of these systems. The researchers in the field of reliability have succeeded somehow in identifying the reliability improvement techniques. The provisions of standby redundancy in both parallel and cold standby have been frequently being used by the system developers. The other means such as priority in repair disciplines and proper repair facility have also been suggested by the researchers while analyzing profit of repairable and non repairable systems. The reliability can also be improved by giving priority to repair activities. Many researchers including Goel et al. [3], Leung et al. [11] and Malik [13] explained the model with the help of priority concept. Kumar and Saini [6] three models are developed under different priority policies. Kumar and Yadav [5] described a computer system with priority given to software up-gradation over hardware repair. Kumar et al. [7] the reliability of single unit system is calculated subject to arrival time of server. Kumar et al. [8] assumed the single server to handle the repair activities of computer system. Subramanian and Anantharaman [19] described the reliability analysis of a complex redundant system where standby unit is in cold state for a certain amount of time before it is allowed to become warm.

In most of the research work authors have analyzed the system models of repairable systems under a common assumption that the service facility cannot fail while performing jobs. This assumption seems to be unrealistic in case system has some complex failures and the service facility is very careless. In that situation the treatment to the failed service facility may be given in order to resume the jobs with full efficiency and perfectness. Kuo and Ke [9] compared system availability among three configurations with unreliable server and switching failure. Meng et al. [14] described a two unit cold standby system with switch failure and equipment maintenance. Nandal and Malik [15] evaluated reliability of a single unit system subject to arrival time of the server. Singh [18] evaluated the expected profit by taking repair man appearance and disappearance for a two unit cold standby system. Sridharan and Mohanavadivu [17] analyzed the two unit cold standby redundant system, two types of repairmen (regular and expert). It is also proved that component wise redundancy is better than that of unit wise redundancy so far as reliability of the system is concerned. Friedman and Tran [2] used the combined hardware/software systems. Gupta et al. [4] gave an idea of single server to determine the profit of two unit standby system model in which priority unit is in operation and ordinary unit is in cold standby. Lai et al. [10] determined the system availability for distributed hardware/software system. Mahmoud and Moshref [12] had taken the human error failure with hardware failure for cold standby system. Bhardwaj and Singh [1] considered the failure of server in steady state behavior of cold standby system. Poonam and Malik [16] analyzed a stochastic parallel system with the assumption of failure of service facility. Yadav and Malik [20] analyzed the computer system with unit wise cold standby redundancy.

In view of the above facts and observations here we describe the stochastic modeling of a computer system with cold standby redundancy (unit wise), priority in repair discipline and failure of service facility. A computer system (called a single unit) means the simultaneous working of its hardware and software components. The system has one more unit (called computer system) that can be used as and when required at the failure of any of the hardware/software components of the initially operative computer system. A single repair facility is made available to rectify the faults which occur due to the failure of hardware and software components. The failed hardware component undergoes for repair immediately while failed software is up-graded. The service facility is subjected to failure during hardware repair. The provision of perfect treatment has been made for the failed service facility. The components work as new after repair and up-gradation with the same life time distribution. The priority is given to the software up-gradation over the hardware repair. In steady state, the expressions for some important reliability measures including MTCSF, availability and profit function have been derived using the well known semi-Markov process and regenerative point technique. The behavior of some useful reliability characteristics has been observed for particular values of the parameters related to failure times, repair and up-gradation times and treatment time which follow negative exponential distribution.

In section 2, notations and abbreviations are explained. In section 3, assumptions and state descriptions are described. In section 4, the reliability measures are calculated. Section 5 determines the profit analysis. The particular values are given in section 6. Section 7 describes the graphical behavior of reliability measures. The numerical example is illustrated in section 8. Section 9 comprises of conclusion of the present study. In final, the relevant references are incorporated.

II. Assumptions and State Descriptions

1. There is a computer system comprises hardware & software components which function independently.
2. The hardware and software components fail independently.
3. The system is a cold standby in which one unit (called computer system) is initially operative and the other unit (computer system) is kept as spare.
4. There is a single service facility that repairs the hardware and upgrades the software.
5. The service facility (server) can fail during hardware repair.
6. The h/w repairs, s/w up-gradation and treatments are perfect.
7. The h/w and s/w failures (s/w failure occurs when it fails to furnish the jobs as per the instructions) are assumed to be constant.
8. The distributions for repair, up-gradation and treatment rates are considered as arbitrary.
9. S_0 is an initial state in which one unit (computer system) is in operation and another unit (computer system) is in cold standby.
10. S_1 is the operative state in which one unit is in operation and second unit's failed h/w component is under repair.
11. S_2 is the failed state in which one unit's h/w component is continued under repair from state S_1 while second unit's h/w component is waiting for repair.
12. S_3 is the operative state in which one unit is in operation and second unit's failed s/w component is under up-gradation.
13. S_4 is the operative state in which the failed server is under treatment, one unit is in operation and second unit's h/w component is waiting for repair.
14. S_5 is the failed state in which the failed server is under treatment, one unit's h/w component is continued waiting for repair from state S_2 while second unit's h/w component is waiting for repair.
15. S_6 is the failed state in which the failed server is under continued treatment from state S_4 while one unit's h/w component is continued waiting for repair from state S4 and second unit's h/w component is waiting for repair.
16. S_7 is the failed state in which one unit's h/w component is continued waiting for repair from state S_5 and second unit's h/w component is under repair.
17. S_8 is the failed state in which the failed server is under continued treatment from state S_4 while one unit's h/w component is continued waiting for repair from state S_4 and second unit's s/w component is waiting for up-gradation.
18. S_9 is the failed state in which one unit's h/w component is waiting for repair from while second unit's s/w component is under up-gradation.
19. S_{10} is the failed state in which one unit's s/w component is under continued up-gradation from state S_3 while second unit's s/w component is waiting for up-gradation.
20. S_{11} is the failed state in which one unit's s/w component is under up-gradation while h/w

component of second unit is continued waiting for repair from state S_8.

21. S_{12} is the failed state in which one unit's h/w component is waiting for repair while second unit's s/w component is under continued up-gradation from state S_3.

The state transition diagram shown in the figure 1 as:

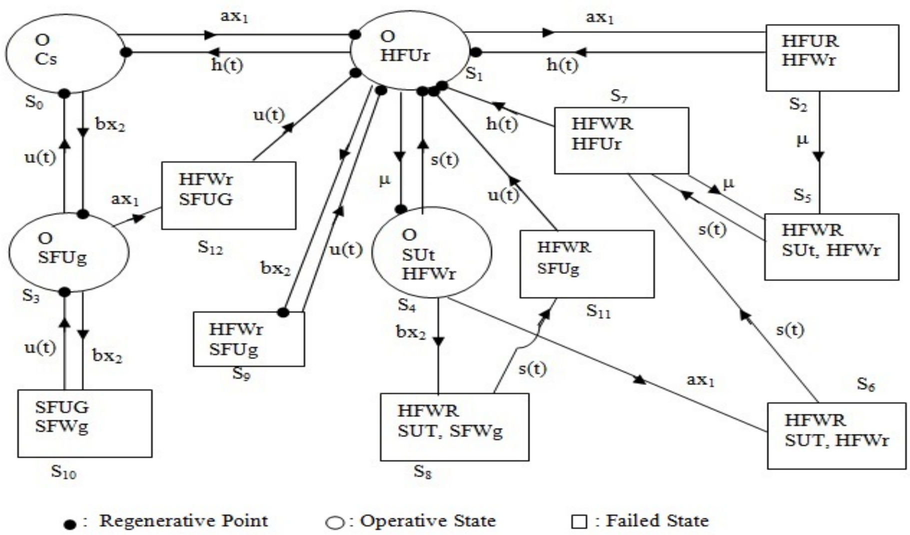

Figure 1: *State Transition Diagram*

a) Notations and Abbreviations

MTCSF	Mean Time to Computer System Failure
SMP	Semi-Markov Process
RPT	Regenerative Point Technique
MST	Mean Sojourn Time
O/Cs	The unit is operative/ in cold standby
a/b	Probability of hardware/software failure
x1/x2/ μ	Hardware/software/ server failure rates
HFUr/HFWr	The failed hardware is under/waiting for repair
HFUR/HFWR	The failed hardware is continuously under/waiting for repair from prior state
SFUg/SFWg	The failed software is under/waiting for up-gradation
SFUG/SFWG	The failed software is continuously under/waiting for up-gradation from prior state
SUt	The failed server (service facility) is under treatment
SUT	The failed server (service facility) is continuously under treatment from prior state
h(t)/H(t)	pdf/cdf of hardware repair time
u(t)/U(t)	pdf/cdf of software repair time
s(t)/S(t)	pdf/cdf of server treatment time
m(t)/M(t)	pdf/cdf of hardware preventive maintenance time
q_{ij}/Q_{ij}	pdf/cdf of first passage time
m_{ij}	Contribution to MST (μi) in state Si when system transits directly to state Sj
$M_i(t)$	Probability that the system up initially in regenerative state Si is up at time t without visiting any other regenerative state
$\mathrm{W_i^H}(\mathrm{t})$	Probability that the server is busy in the state Si due to hardware failure up to time 't' without making any transition to any other regenerative state or returning to the same state via one or more non-regenerative states

$W_i^S(t)$	Probability that the server is busy in the state Si due to software up-gradation up to time 't' without making any transition to any other regenerative state or returning to the same state via one or more non-regenerative states
Ⓢ/©	Standard notation for Laplace-Stieltjes convolution/Laplace convolution
*/**	Symbol for Laplace Transform (LT)/Laplace Stieltjes Transform (LST)
P	Profit function by considering busy period cost of the server per unit time due to hardware repair/ software up-gradation and treatment cost of the server per unit time
Z_1	System revenue per unit up-time
Z_2/Z_3	Busy period cost of the server per unit time due to hardware repair/ software up gradation
Z_4	Treatment cost of the server per unit time

III. Reliability Measures of the System

a) Transition Probabilities

The differential transition probabilities for state S_0 are given by

$$dQ_{01}(t) = ax_1 e^{-(ax_1+bx_2)t}dt \text{ , } dQ_{02}(t) = bx_2 e^{-(ax_1+bx_2)t}dt$$

Taking LST of above equations and using the following results

$$p_{ij} = \lim_{s\to 0} \emptyset_{ij}^{**}(s) = \emptyset_{ij}^{**}(0) = \int_0^\infty \mathrm{d}Q_{ij}(t) = \int_0^\infty q_{ij}(t)\mathrm{dt}, \text{ we get}$$

$$p_{01} = \int_0^\infty ax_1 e^{-(ax_1+bx_2)\mathrm{t}}\,dt = \frac{ax_1}{ax_1+bx_2}, p_{02} = \int_0^\infty bx_2 e^{-(ax_1+bx_2)\mathrm{t}}\,dt = \frac{bx_2}{ax_1+bx_2}$$

Similarly, the other transition probabilities for remaining states are given by

$$p_{10} = h^*(ax_1 + bx_2 + \mu), p_{12} = \frac{ax_1}{ax_1+bx_2+\mu}\{1 - h^*(ax_1 + bx_2 + \mu)\}, p_{21} = p_{71} = h^*(\mu)$$

$$p_{14} = \frac{\mu}{ax_1+bx_2+\mu}\{1 - h^*(ax_1 + bx_2 + \mu)\}, p_{19} = \frac{bx_2}{ax_1+bx_2+\mu}\{1 - h^*(ax_1 + bx_2 + \mu)\},$$

$$p_{21} = p_{71} = h^*(\mu),\ p_{25} = p_{75} = 1 - h^*(\mu), p_{30} = u^*(ax_1 + bx_2), p_{41} = s^*(ax_1 + bx_2)$$

$$p_{3,10} = p_{33.10} = \frac{bx_2}{ax_1+bx_2}\{1 - u^*(ax_1 + bx_2)\}, p_{3,12} = p_{31.12} = \frac{ax_1}{ax_1+bx_2}\{1 - u^*(ax_1 + bx_2)\}, p_{41} =$$

$$s^*(ax_1 + bx_2), p_{46} = \frac{ax_1}{ax_1+bx_2}\{1 - s^*(ax_1 + bx_2)\}, p_{57} = p_{8,11} = p_{67} = s^*(0)$$

$$p_{48} = p_{41.8,11} = \frac{bx_2}{ax_1+bx_2}\{1 - s^*(ax_1 + bx_2)\}, p_{91} = p_{10,3} = p_{11,1} = p_{12,1} = u^*(0),$$

$$p_{11.2} = p_{12}p_{21}, p_{41.67} = p_{46}p_{71}, p_{11.2(5,7)^n} = p_{12}p_{25}, p_{41.67(5,7)^n} = p_{46}p_{75}$$

From the above transition probabilities, the following relations are obtained as follows:

$$p_{01} + p_{03} = p_{10} + p_{12} + p_{14} + p_{19} = p_{21} + p_{25} = p_{30} + p_{3,10} + p_{3,12} = p_{41} + p_{46} + p_{48} = 1, p_{71} + p_{75} =$$

$$p_{57} = p_{67} = p_{8,11} = p_{9,1} = p_{10,3} = p_{11,1} = p_{12,1} = p_{30} + p_{33.10} + p_{31.12} = 1,$$

$$p_{10} + p_{14} + p_{11.2} + p_{11.2(5,7)^n} + p_{19} = p_{41} + p_{41.67} + p_{41.67(5,7)^n} + p_{41.8,11} = 1$$

b) Mean Sojourn Times

The expected time taken by the system in a particular state before transiting to any other state is known as mean sojourn time or mean survival time in the state. If T_i be the sojourn time in the state i, then the mean sojourn time in the state i is
The MST (μ_i) in state S_i are calculated by the following relations

$$m_{ij} = \left|-\frac{d}{ds}Q_{ij}^{**}(s)\right|_{s=0} = -Q_{ij}^{**\prime}(0) \text{ and } \mu_i = \sum_j m_{ij} \text{ where } Q_{ij}^{**}(s) = \int_0^\infty \mathrm{e}^{-\mathrm{st}}\mathrm{d}Q_{ij}(t).$$

Thus, we have

$$\mu_0 = m_{01} + m_{03}, \mu_1 = m_{10} + m_{12} + m_{14} + m_{19}, \mu_3' = m_{30} + m_{33.10} + m_{31.12},$$

$$\mu_3 = m_{30} + m_{3,10} + m_{3,12}, \mu_4 = m_{41} + m_{46} + m_{48},\ \mu_9 = m_{91},$$

$$\mu_1' = m_{10} + m_{11.2} + m_{11.2(5,7)^n} + m_{14} + m_{19}, \mu_4' = m_{41} + m_{41.67} + m_{41.67(5,7)^n} + m_{41.8,11}$$

c) Reliability and MTCSF

Let $\emptyset_i(t)$ be the c.d.f. of first passage time from regenerative state S_i to a failed state. Regarding the failed state as absorbing state, we have following recursive relations for $\emptyset_i(t)$:

$$\emptyset_i(t) = \sum_j Q_{ij}(t)Ⓢ\emptyset_j(t) + \sum_k Q_{ik}(t) \qquad (1)$$

where S_j is an un-failed regenerative state to which the given regenerative state S_i can transit and S_k is a failed state to which the state S_i can transit directly. Thus, the following equations are obtained by using (1) as:

$$\emptyset_0(t) = Q_{01}(t)Ⓢ\emptyset_1(t) + Q_{03}(t)Ⓢ\emptyset_3(t)$$
$$\emptyset_1(t) = Q_{10}(t)Ⓢ\emptyset_0(t) + Q_{12}(t) + Q_{14}(t)Ⓢ\emptyset_4(t) + Q_{19}(t)$$
$$\emptyset_3(t) = Q_{30}(t)Ⓢ\emptyset_0(t) + Q_{3,10}(t) + Q_{3,12}(t)$$
$$\emptyset_4(t) = Q_{41}(t)Ⓢ\emptyset_1(t) + Q_{46}(t) + Q_{48}(t)$$

Taking LST of above equations, we get

$$\emptyset_0^{**}(s) = Q_{01}^{**}(s)\emptyset_1^{**}(s) + Q_{03}^{**}(s)\emptyset_3^{**}(s)$$
$$\emptyset_1^{**}(s) = Q_{10}^{**}(s)\emptyset_0^{**}(s) + Q_{12}^{**}(s) + Q_{14}^{**}(s)\emptyset_4^{**}(s) + Q_{19}^{**}(s)$$
$$\emptyset_3^{**}(s) = Q_{30}^{**}(s)\emptyset_0^{**}(s) + Q_{3,10}^{**}(s) + Q_{3,12}^{**}(s)$$
$$\emptyset_4^{**}(s) = Q_{41}^{**}(s)\emptyset_1^{**}(s) + Q_{46}^{**}(s) + Q_{48}^{**}(s)$$

By using Cramer Rule, $\emptyset_0^{**}(s)$ is calculated as

$$\emptyset_0^{**}(s) = \frac{\Delta_1}{\Delta}$$

Where
$$\Delta_1 = \begin{vmatrix} 1 & -Q_{01}^{**}(s) & -Q_{03}^{**}(s) & 0 \\ -Q_{10}^{**}(s) & 1 & 0 & -Q_{14}^{**}(s) \\ -Q_{30}^{**}(s) & 0 & 1 & 0 \\ 0 & -Q_{41}^{**}(s) & 0 & 1 \end{vmatrix}$$

and

$$\Delta = \begin{vmatrix} 0 & -Q_{01}^{**}(s) & -Q_{03}^{**}(s) & 0 \\ Q_{12}^{**}(s) + Q_{19}^{**}(s) & 1 & 0 & -Q_{14}^{**}(s) \\ Q_{3,10}^{**}(s) + Q_{3,12}^{**}(s) & 0 & 1 & 0 \\ Q_{46}^{**}(s) + Q_{48}^{**}(s) & -Q_{41}^{**}(s) & 0 & 1 \end{vmatrix}$$

Now, we have
$$R^*(s) = \frac{1-\emptyset_0^{**}(s)}{s}$$

The reliability of the computer system model can be obtained by

$$R(t) = L^{-1}[R^*(s)]$$

The MTCSF is given by

$MTCSF = \lim_{s\to 0} R^*(s) = R^*(0) = \frac{N_1}{D_1}$, where $N_1 = (1-p_{14}p_{41})(p_{03}\mu_3 + \mu_0) + p_{01}(p_{14}\mu_4 + \mu_1)$ and $D_1 = (1-p_{14}p_{41})(1-p_{03}p_{30}) - p_{01}p_{10}$

d) Steady State Availability

Let $A_i(t)$ be the probability that the system is in up-state at epoch 't' given that the computer system entered regenerative state S_i at $t = 0$. The recursive relations for $A_i(t)$ are given as

$$A_i(t) = M_i(t) + \sum_j q_{ij}^{(n)}(t)©A_j(t) \qquad (2)$$

where S_j is any successive regenerative state to which the regenerative state S_i can transit through n transitions. Thus, the following equations are obtained by using (2) as:

$$A_0(t) = M_0(t) + q_{01}(t)©A_1(t) + q_{03}(t)©A_3(t)$$
$$A_1(t) = M_1(t) + q_{10}(t)©A_0(t) + \left[q_{11.2}(t) + q_{11.2(5,7)^n}(t)\right]©A_1(t) + q_{14}(t)©A_4(t) + q_{19}(t)©A_9(t)$$
$$A_3(t) = M_3(t) + q_{30}(t)©A_0(t) + q_{31.12}(t)©A_1(t) + q_{33.10}(t)©A_3(t)$$
$$A_4(t) = M_4(t) + \left[q_{41}(t) + q_{41.67}(t) + q_{41.67(5,7)^n}(t) + q_{41.8,11}(t)\right]©A_1(t)$$
$$A_9(t) = q_{91}(t)©A_1(t)$$

Where,

$M_0(t) = e^{-(ax_1+bx_2)t}$, $M_1(t) = e^{-(ax_1+bx_2+\mu)t}\bar{H}(t)$, $M_3(t) = e^{-(ax_1+bx_2)t}\bar{U}(t)$, $M_4(t) = e^{-(ax_1+bx_2)t}\bar{S}(t)$

Taking LT of above equations and solving for $A_0^*(s)$, the steady state availability is calculated by

$$A_0(\infty) = \lim_{s\to 0} sA_0^*(s) = \frac{N_2}{D_2}$$

Where $N_2 = (p_{14}\mu_4 + \mu_1)(1 - p_{3,10} - p_{03}p_{30}) + p_{10}[\mu_0(1 - p_{3,10}) + \mu_3 p_{03}]$

$$D_2 = (p_{14}\mu_4' + \mu_1' + p_{19}\mu_9)(1 - p_{3,10} - p_{03}p_{30}) + p_{10}[\mu_0(1 - p_{3,10}) + \mu_3' p_{03}]$$

and

$$\mu_i = M_i^*(0), i = 1,2,3,4$$

e) Busy Period of the Repairman Due to Repairs

Let $B_i^R(t)$ be the probability that server is busy in repairing the unit at epoch 't' given that the system entered state S_i at $t = 0$. The recursive relations for $B_i^R(t)$ are given as:

$$B_i^R(t) = W_i^R(t) + \sum_j q_{ij}^{(n)}(t)©B_j^H(t) \quad (3)$$

where S_j is any successive regenerative state to which the regenerative state S_i can transit through n transitions. Thus, the following equations are obtained by using (3) as:

i)Repair of Hardware

$$B_0^H(t) = q_{01}(t)©B_1^H(t)+q_{03}(t)©B_3^H(t)$$
$$B_1^H(t) = W_1^H(t) + q_{10}(t)©B_0^H(t)+[q_{11.2}(t) + q_{11.2(5,7)^n}(t)]©B_1^H(t) + q_{14}(t)©B_4^H(t) + q_{19}(t)©B_9^H(t)$$
$$B_3^H(t) = q_{30}(t)©B_0^H(t)+q_{31.12}(t)©B_1^H(t) + q_{33.10}(t)©B_3^H(t)$$
$$B_4^H(t) = [q_{41}(t) + q_{41.67}(t) + q_{41.67(5,7)^n}(t) + q_{41.8,11}(t)]©B_1^H(t)$$
$$B_9^H(t) = q_{91}(t)©B_1^H(t)$$

Where, $W_1^H(t) = [e^{-(ax_1+bx_2+\mu)t} + (ax_1 e^{-(ax_1+bx_2+\mu)t}©\mu e^{-\mu t}©s(t)©1) + (ax_1 e^{-(ax_1+bx_2+\mu)t}©1)]\bar{H}(t)$

Taking LT of above equations and solving for $B_0^{H^*}(s)$, then busy period of server due to h/w repair is given by

$$B_0^H(\infty) = \lim_{s\to 0} sB_0^{H^*}(s) = \tfrac{N_3}{D_2}, \text{ where } N_3 = (1 - p_{3,10} - p_{03}p_{30})W_1^{H^*}(0) \text{ and}$$
$$D_2 = (p_{14}\mu_4' + \mu_1' + p_{19}\mu_9)(1 - p_{3,10} - p_{03}p_{30}) + p_{10}[\mu_0(1 - p_{3,10}) + \mu_3' p_{03}]$$

ii)Software Up-gradation

$$B_0^S(t) = q_{01}(t)©B_1^S(t)+q_{03}(t)©B_3^S(t)$$
$$B_1^S(t) = q_{10}(t)©B_0^S(t)+[q_{11.2}(t) + q_{11.2(5,7)^n}(t)]©B_1^S(t) + q_{14}(t)©B_4^S(t) + q_{19}(t)©B_9^S(t)$$
$$B_3^S(t) = W_3^S(t) + q_{30}(t)©B_0^S(t)+q_{31.12}(t)©B_1^S(t) + q_{33.10}(t)©B_3^S(t)$$
$$B_4^S(t) = [q_{41}(t) + q_{41.67}(t) + q_{41.67(5,7)^n}(t) + q_{41.8,11}(t)]©B_1^S(t)$$
$$B_9^S(t) = W_9^S(t) + q_{91}(t)©B_1^S(t)$$

Where, $W_3^S(t) = [e^{-(ax_1+bx_2)t} + (ax_1 e^{-(ax_1+bx_2)t}©1) + (bx_2 e^{-(ax_1+bx_2)t}©1)]\bar{U}(t)$ and $W_9^S = \bar{U}(t)$

Taking LT of above equations and solving for $B_0^{S^*}(s)$ (same as 4.3), busy period of server due to s/w up-gradation is given by

$$B_0^S(\infty) = \lim_{s\to 0} sB_0^{S^*}(s) = \tfrac{N_4}{D_2}, \text{ where } N_4 = W_3^{S^*}(0)p_{03}p_{10} + W_9^{S^*}(0)p_{19}(1 - p_{3,10} - p_{03}p_{30}) \text{ and}$$
$$D_2 = (p_{14}\mu_4' + \mu_1' + p_{19}\mu_9)(1 - p_{3,10} - p_{03}p_{30}) + p_{10}[\mu_0(1 - p_{3,10}) + \mu_3' p_{03}]$$

f) Expected Number of Server Treatment

Let $T_i^R(t)$ be the expected number of repairs of the unit by the server in (0, t] such that the system entered regenerative state$i$ at t = 0. The recursive relation for$T_i^R(t)$ are given as:

$$T_i^R(t) = \sum_j Q_{i,j}^{(n)}(t)Ⓢ[\delta_j + T_i^R(t)] \quad (4)$$

Where j is any regenerative state to which the given regenerative state i transits and $\delta_j = 1$ if j is the regenerative state where the server does job afresh, otherwise, $\delta_j = 0$. Thus, the following equations are obtained by using (4) as:

$$T_0(t) = Q_{01}(t)Ⓢ T_1(t) + Q_{03}(t)Ⓢ T_3(t)$$
$$T_1(t) = Q_{10}(t)Ⓢ T_0(t) + \left[Q_{11.2}(t) + Q_{11.2(5,7)^n}(t)\right]Ⓢ T_1(t) + Q_{14}(t)Ⓢ T_4(t) + Q_{19}(t)Ⓢ T_9(t)$$
$$T_3(t) = Q_{30}(t)Ⓢ T_0(t) + Q_{31.12}(t)Ⓢ T_1(t) + Q_{33.10}(t)Ⓢ T_3(t)$$
$$T_4(t) = [Q_{41}(t) + Q_{41.67}(t) + Q_{41.67(5,7)^n}(t) + Q_{41.8,11}(t)]Ⓢ[1 + T_1(t)]$$
$$T_9(t) = Q_{91}(t)Ⓢ T_1(t)$$

Taking LST of above relation and solving for $T_0^{**}(s)$ (same as 4.3). The expected no. of the server treatments is given by

$$T_0(\infty) = \lim_{s\to 0} sT_0^{**}(s) = \frac{N_5}{D_2} \text{ where } N_5 = \left(1 - p_{3,10} - p_{03}p_{30}\right)(p_{12}p_{25} + p_{14}) \text{ and}$$
$$D_2 = (p_{14}\mu_4' + \mu_1' + p_{19}\mu_9)\left(1 - p_{3,10} - p_{03}p_{30}\right) + p_{10}\left[\mu_0\left(1 - p_{3,10}\right) + \mu_3' p_{03}\right]$$

IV. Profit Analysis

The profit function in time 't' of the computer system is given by
P (t) = Expected revenue in (0, t] – expected total cost in (0, t]
In steady state, the profit of the computer system model can be obtained by the following formula:

$$P = Z_1 A_0(\infty) - Z_2 B_0^H(\infty) - Z_3 B_0^S(\infty) - Z_4 T_0(\infty) \tag{5}$$

V. Particular Cases

Let us assume $h(t) = \alpha e^{-\alpha t}$, $u(t) = \beta e^{-\beta t}$ and $s(t) = \gamma e^{-\gamma t}$ then reliability measures are determined as follows:

$$p_{10} = \frac{\alpha}{ax_1+bx_2+\mu+\alpha}, p_{12} = \frac{ax_1}{ax_1+bx_2+\mu+\alpha}, p_{14} = \frac{\mu}{ax_1+bx_2+\mu+\alpha}, p_{19} = \frac{bx_2}{ax_1+bx_2+\mu+\alpha},$$
$$p_{25} = \frac{\mu}{\mu+\alpha}, p_{30} = \frac{\beta}{ax_1+bx_2+\beta}, p_{3,10} = \frac{bx_2}{ax_1+bx_2+\beta}, p_{41} = \frac{\gamma}{ax_1+bx_2+\gamma}, \mu_0 = \frac{1}{ax_1+bx_2},$$
$$\mu_1 = \frac{1}{ax_1+bx_2+\mu+\alpha}, \mu_3 = \frac{1}{ax_1+bx_2+\beta}, \mu_4 = \frac{1}{ax_1+bx_2+\gamma}, \mu_1' = \frac{\gamma\alpha+ax_1(\mu+\gamma)}{\gamma\alpha(ax_1+bx_2+\mu+\alpha)},$$
$$\mu_3' = \frac{1}{\beta} = W_3^{S^*}(0) = W_9^{S^*}(0), W_1^{H^*}(0) = \frac{(\mu+\alpha)(\alpha+\gamma)(\alpha+ax_1)+\mu\gamma ax_1}{\alpha(ax_1+bx_2+\mu+\alpha)(\mu+\alpha)(\alpha+\gamma)},$$
$$\mu_4' = \frac{\beta\alpha(ax_1 + bx_2 + \gamma) + \gamma\alpha + \beta ax_1(\mu + \gamma)}{\beta\gamma\alpha(ax_1 + bx_2 + \gamma)}$$
$$MTSF = \frac{N_1}{D_1}, A_0(\infty) = \frac{N_2}{D_2}, B_0^H(\infty) = \frac{N_3}{D_2}, B_0^S(\infty) = \frac{N_4}{D_2}, T_0(\infty) = \frac{N_5}{D_2}$$

where

$$N_1 = \frac{\{(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2+\gamma)-\gamma\mu\}\{ax_1+2bx_2+\beta\}+ax_1(ax_1+bx_2+\mu+\gamma)(ax_1+bx_2+\beta)}{(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2+\gamma)(ax_1+bx_2)(ax_1+bx_2+\beta)}$$
$$D_1 = \frac{\begin{array}{c}\{(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2+\gamma)-\gamma\mu\}\{(ax_1+bx_2)(ax_1+bx_2+\beta)-bx_2\beta\}\\ -ax_1\alpha(ax_1+bx_2+\gamma)(ax_1+bx_2+\beta)\end{array}}{(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2+\gamma)(ax_1+bx_2)(ax_1+bx_2+\beta)}$$
$$N_2 = \frac{(ax_1+bx_2+\mu+\gamma)+\alpha(ax_1+bx_2+\gamma)}{(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2+\gamma)(ax_1+bx_2)}$$
$$D_2 = \frac{\begin{array}{c}ax_1(ax_1+bx_2+\beta)[(ax_1+bx_2+\gamma)\{\alpha\beta\gamma+\beta ax_1(\mu+\gamma)+\alpha\beta\mu+\alpha\gamma bx_2\}+\alpha\gamma\mu+\beta\mu ax_1(\mu+\gamma)]\\ +\alpha^2\gamma(ax_1+bx_2+\gamma)\{\beta(ax_1+bx_2+\beta)+bx_2(ax_1+bx_2)\}\end{array}}{\alpha\beta\gamma(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2+\gamma)(ax_1+bx_2)(ax_1+bx_2+\beta)}$$
$$N_3 = \frac{ax_1\{(\mu+\alpha)(\gamma+\alpha)(\alpha+ax_1)+\gamma\mu ax_1\}}{\alpha(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2)(\mu+\alpha)(\gamma+\alpha)},$$
$$N_4 = \frac{bx_2(\alpha+ax_1)}{\beta(ax_1+bx_2+\mu+\alpha)(ax_1+bx_2)},$$
$$N_5 = \frac{\mu ax_1(\alpha + ax_1 + \mu)}{(ax_1 + bx_2 + \mu + \alpha)(ax_1 + bx_2)(\mu + \alpha)}$$

VI. Graphical Presentation

The graphical representation of MTCSF, availability and profit function has been shown in figures 2, 3 and 4 respectively to check their behavior with respect to the values of the parameters associated with failure and repair rates. From Figure 2, it is observed that the MTCSF of the system decreases when failure rate of hardware and software is increased from 0.01 to 0.1. Also, MTCSF increases with an increase in hardware repair rate, software up-gradation rate and treatment rate of the server.

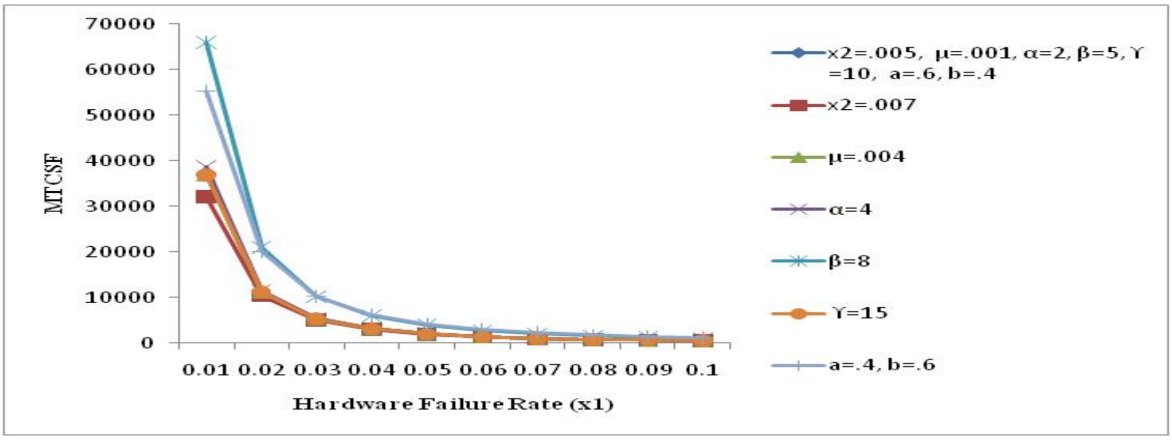

Figure 2: *MTCSF Vs Hardware Failure Rate (X_1)*

From Figure 3, it is clearly seen that the availability of the system decreases rapidly with increase of failure rate of hardware and software. Also, availability of the system increases with an increase of hardware repair, software up-gradation and treatment rate of the server.

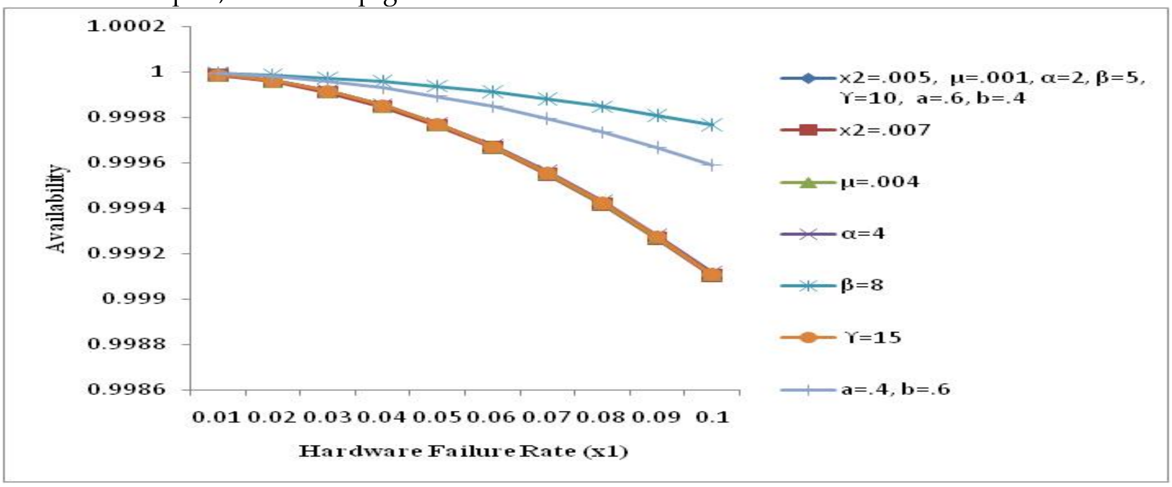

Figure 3: *Availability Vs Hardware Failure Rate (X_1)*

From Figure 4, it is observed that the profit decreases when failure rate of the hardware and software increases. Also, the profit of the system is increases with an increase of hardware repair rate, software up-gradation rate and treatment rate of the server.

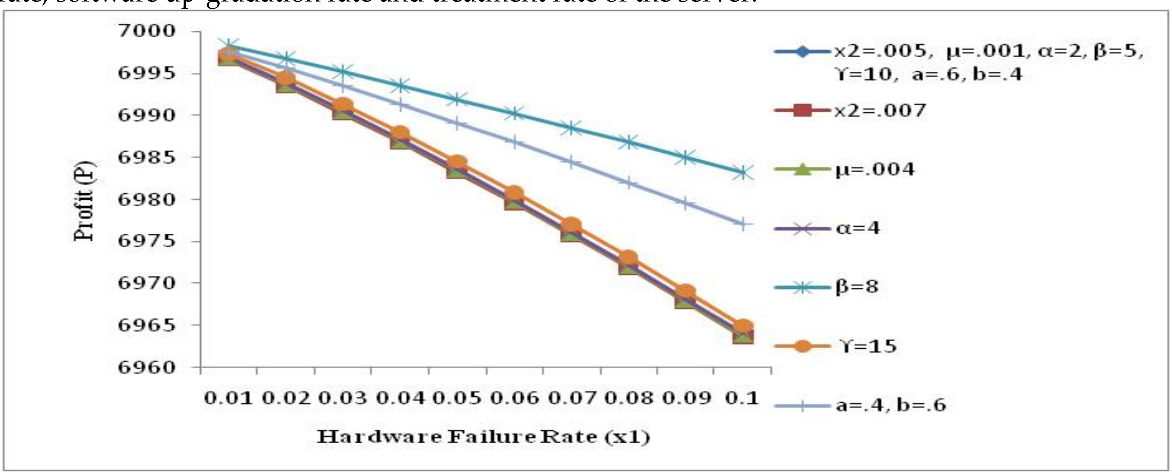

Figure 4: *Profit Vs Hardware Failure Rate (X_1)*

VII. Conclusion

The present study mainly focuses on MTCSF, availability and profit analysis of a computer system with unit wise redundancy and failure of service facility. The preference is given to the software up-gradation over hardware repair. The graphical behavior of some important measures such as MTCSF, availability and profit has been observed w.r.t. hardware failure rate (x_1) and for the fixed values of server failure rate, repair rates of components and server's treatment rate as shown in the respective figures (Fig.2, Fig.3 and Fig.4). From these figures, it is concluded that MTCSF (Fig.2), availability (Fig.3) and profit (Fig.4) decrease with increase in hardware failure rate (x_1) & software failure rate (x_2) and increase with increase of hardware repair rate (α) and software up-gradation rate (β) and treatment rate (Υ) of server. It is also examined that the provision of priority to software up-gradation of one unit over the hardware repair of other unit can only be helpful in increasing the profit of the system model provided the software up-gradation rate is increased.

VIII. Illustration

Suppose the department office has two computers for furnishing day to day assigned jobs. The official starts the jobs initially at a single computer (unit) and the other computer system is kept as spare in order to makes its use as and when required at any type of problems which occur in the initial operative computer system. The computer can have problems in both hardware and software like damage of RAM, defects in CPU and short-circuit in the monitor as the hardware problems while software can fail to follow the instructions due to malware in the system and failure of drivers. In that situation it becomes necessary to take the help of another computer system in order to complete the assigned jobs in time. In order to secure the data from any kind of malware attack the priority to up-grade the software is required instead of repair of any type of hardware faults. On the other hand, it is not necessary that the service facility can be made available immediately to rectify the faults and in that case we can consider the failure of the service facility. On the basis of the experience and practices the present study is illustrated on a computer system by considering the ideas of unit wise redundancy, priority to software up-gradation and failure of the service facility. The reliability characteristics such as MTCSF, availability and profit have been obtained by taking the hypothetical values for the parameters as:

$$\text{Here, } x_1 = 0.04, x_2 = 0.007, \mu = 0.001, \alpha = 2, \beta = 5, \Upsilon = 10, a = 0.6 \text{ and } b = 0.4\,,$$
$$Z_1 = 7000,\ Z_2 = 1000, Z_3 = 800, Z_4 = 500.$$

We have

$$\text{MTCSF} = 3046.581, \text{Availability} = 0.999848 \text{ and Profit} = 6986.86$$

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M/M/∞ Queue with Catastrophes and Repairable Servers

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Abstract

An infinite server Markovian queueing system with randomly occurring breakdowns and non zero exponentially distributed repair time is proposed. Upon arrival, a catastrophes deactivate all the servers and system is under catastrophic failure. Immediately, a repair process is started and after successful repair the system is ready to serve the newly arrived customers. Continued fraction techniques have been used to obtain the time dependent probabilities of the studied model. The stationary probability distribution for the number of customers in the system is also derived. Some important stationary as well as transient moments are also determined. Further, The availability and reliability of the system under consideration are investigated. Finally, some graphical results are presented to visualize the model practically.

Keywords: M/M/∞ Queue, Server Breakdown, Transient Analysis,Steady State Solution, Confluent Hypergeometric Function, Reliability and Availability.

1. Introduction

Here, we consider a classical M/M/∞ queueing model subjected to randomly occurring breakdowns (catastrophes). Upon arrival, a catastrophes deactivate all the servers and system is under disasters breakdown. Immediately a repair process is started and after successful repair the system is again restart their functioning and provide service to a newly arrived customer. We analyze this model and provide steady state and time dependent solution.

During the last four decades the interest in catastrophic queueing model has been increased by a rapid phase. Therefore queueing models in the presence of catastrophes has been analyzed by many researchers.(see e.g., [1],[2],[3], [12],[17],[19],[21][22]). Occurrence of catastrophes destroys all present customers and also breakdown the servers. Some authors assumes that whenever catastrophes occurs, it flush out all present customers and immediately the server is ready for service for a newly arrived customer(see e.g. [1], [3],[8], [9]). And some assumes that the server or system may take a non zero repair time for their re-functioning whenever it is affected by a catastrophic failure(see e.g. [4], [18], [21], [25]).

Infinite servers queueing models are also analyzed by many researchers with the possibility of catastrophes. Gursoy *et* al. [15] analyzed an infinite server queue with randomly occurring interruption and provide steady state solution . Giorno *et* al. [26] have discussed the various properties of a bilateral birth-death process, affected by randomly occurring catastrophes. Linton and Purdue[6] have obtained the stationary and transient distribution of the probabilities for an M/G/∞ queue with catastrophes. Yechiali [25] considered an M/M/∞ queues with catastrophes and studied the impatient behavior of customers when server is down. The transient solution of an infinite servers Markovian queue subjected to catastrophes has been obtained by Krishna Kumar *et* al. [5] and Gulab Singh Bura [8].

In this work, we present an M/M/∞ queuing system with catastrophes, Server breakdown and

non-zero repair time. Although,the operating model has been already analyzed by Sophia and Murali [23]. Then, objective of this paper is to illustrate a different approach and provide some additional important measures of the system under consideration.
The M/M/∞ queueing system with repairable servers finds its application in telecommunication field. Our system under consideration is a University campus which provide free Wi-Fi service to their students. Within the campus, each mobile is considered as one queueing server. Whenever a breakdown occur i.e. (connectivity loss or signal failure), all the servers gets deactivated and none of them works until that breakdown is repaired. So, an M/M/∞ queue with system failure and repair is a suitable approximation.

The paper is arranged in the following way. Next section describe the formulation of the model. The transient solution of the model have been obtained in section 3. Under section 4, we have obtained some moments of the model in transient form. Section 5, gives the time independent solution of the model. In Section 6, we have discussed about the availability and reliability of the system. Section 7 presents some graphical illustrations to observe the system performance with the effect of various parameters. Conclusion is given in the last Section.

2. MATHEMATICAL MODEL

An infinite servers Markovian queueing system with server breakdown and repair is in operation. Arrivals occur one by one in a Poisson stream with mean rate λ. Service times are exponentially distributed with parameter θ. The system may fails due the disastrous breakdown occurs at a Poisson rate γ. Whenever a catastrophes occur all the servers are deactivated and the system is under disasters breakdown. Immediately a repair process is started and the repair time distribution is exponential with rate η. After successful repair the system again restart their functioning and provide service to a newly arrived customer. Also, it is assumed that, no customer is allowed to enter in to the system during the repair process of failed servers. Let the random variable $C(t)$ represents the number of customers present in the system at time t and $P_n(t)$ denotes its probability.

3. TRANSIENT ANALYSIS UNDER MARKOVIAN SETUP

This section provides the probability mass function of the random variable $C(t)$. For this, the differential-difference equations are given as:

$$F'(t) = \gamma(1 - F(t) - P_0(t)) - \eta F(t) \tag{1}$$

$$P_0'(t) = \theta P_1(t) + \eta F(t) - \lambda P_0(t) \tag{2}$$

$$P_n'(t) = (n+1)\theta P_{n+1}(t) + \lambda P_{n-1}(t) - (\lambda + n\theta + \gamma)\, P_n(t),\ \ n = 1, 2, 3, ... \tag{3}$$

Initially, at $t = 0$,

$$P_n(0) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{if } n \neq 0. \end{cases} \tag{4}$$

Taking Laplace transform of Eq.(1),Eq.(2),Eq.(3) and by the use of Eq.(4), we have

$$(s + \gamma + \eta)\, F^*(s) = \gamma(\frac{1}{s} - P_0^*(s)) \tag{5}$$

$$(s + \lambda)\, P_0^*(s) = 1 + \theta P_1^*(s) + \eta F^*(s) \tag{6}$$

$$(s + \lambda + n\theta + \gamma)\, P_n^*(s) = (n+1)\theta P_{n+1}^*(s) + \lambda P_{n-1}^*(s), \tag{7}$$

After some manipulation, Eq.(7), gives an expression

$$\frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\frac{\lambda}{\theta}}{\frac{s+\lambda+\gamma}{\theta} + n) - (n+1)\frac{P_{n+1}^*(s)}{P_n^*(s)}}$$

$$\frac{\lambda}{\theta}\frac{P_{n-1}^*(s)}{P_n^*(s)} = \left(\frac{s+\lambda+\gamma}{\theta}+n\right) - \cfrac{(n+1)\frac{\lambda}{\theta}}{\left(\frac{s+\lambda+\gamma}{\theta}+n+1\right) - \cfrac{(n+2)\frac{\lambda}{\theta}}{\left(\frac{s+\lambda+\gamma}{\theta}+n+2\right)-\cdots}} \tag{8}$$

Now using the identity given by Lorentzen and Waadeland [13]

$$\frac{{}_1F_1(q+1;r+1;z)}{{}_1F_1(q;r;z)} = \frac{r}{r-z+}\,\frac{(q+1)z}{r-z+1+}\,\frac{(q+2)z}{r-z+2+}\cdots$$

rewritten as

$$\frac{{}_1F_1(q;r;z)}{{}_1F_1(q+1;r+1;z)} = \frac{r-z}{r+}\,\frac{(q+1)z}{r-z+1+}\,\frac{(q+2)z}{r-z+2+}\ldots, \tag{9}$$

by using Eq.(9) in Eq.(8), we have

$$\frac{P_n^*(s)}{P_{n-1}^*(s)} = \frac{\lambda}{\theta}\frac{{}_1F_1(q+1;r+1,z)}{\left(\frac{s+\gamma}{\theta}+n\right)\ {}_1F_1(q;r;z)}, \tag{10}$$

therefore for $n \geq 1$,we have

$$P_n^*(s) = \left(\frac{\lambda}{\theta}\right)^n \frac{{}_1F_1(n+1;\frac{s+\gamma}{\theta}+n+1;-\frac{\lambda}{\theta})}{\prod_{j=1}^n\left(\frac{s+\gamma}{\theta}+j\right)\ {}_1F_1(1;\frac{s+\gamma}{\theta}+1;-\frac{\lambda}{\theta})}P_0^*(s), \tag{11}$$

$$P_n^*(s) = \zeta_n^*(s)P_0^*(s), \tag{12}$$

where

$$\zeta_n^*(s) = \left(\frac{\lambda}{\theta}\right)^n \frac{{}_1F_1(n+1;\frac{s+\gamma}{\theta}+n+1;-\frac{\lambda}{\theta})}{\prod_{j=1}^n\left(\frac{s+\gamma}{\theta}+j\right)\ {}_1F_1(1;\frac{s+\gamma}{\theta}+1;-\frac{\lambda}{\theta})}, \tag{13}$$

It is well known that

$$F^*(s) + \sum_{n=0}^{\infty} P_n^*(s) = \frac{1}{s}, \tag{14}$$

by the use of Eq.(12) and Eq.(5), we get

$$P_0^*(s) = (1+\frac{\eta}{s})\left[(s+\lambda+\eta) - \theta\zeta_1^*(s) + \eta\sum_{n=1}^{\infty}\zeta_n^*(s)\right]^{-1} \tag{15}$$

after simplification Eq.(15) reduces to

$$P_0^*(s) = (1+\frac{\eta}{s})\sum_{j=o}^{\infty}\frac{(-1)^j}{(s+\lambda+\eta)^{j+1}}\left[\sum_{k=1}^{\infty}(\eta-\delta_k\theta)\zeta_k^*(s)\right]^j \tag{16}$$

on inversion, we get

$$\begin{aligned} P_0(t) = &\sum_{j=0}^{\infty}(-1)^j\int_0^t e^{-(\lambda+\eta)(t-u)}(t-u)^j\left[\sum_{k=1}^{\infty}(\eta-\delta_k\theta)\zeta_k(u)\right]^{*J} du \\ &+\eta\sum_{j=0}^{\infty}(-1)^j\int_0^t e^{-(\lambda+\eta)x}\frac{x^j}{j!}\left[\sum_{k=1}^{\infty}(\eta-\delta_k\theta)\zeta_k(x)\right]^{*J} dx \end{aligned} \tag{17}$$

Now for $P_n(t)$, consider Eq.(12), which on inversion, gives

$$P_n(t) = \zeta_n(t) * P_0(t), \tag{18}$$

where the symbol * denotes the convolution and $P_0(t)$ given in Eq.(17).
Next we derive the expression for $\zeta_n(t)$, where $\zeta_n(t)$ represents the inverse Laplace transform of

$\zeta_n^*(s)$.
From Eq.(13)

$$\zeta_n^*(s) = \left(\frac{\lambda}{\theta}\right)^n \frac{{}_1F_1(n+1; \frac{s+\gamma}{\theta}+n+1; \frac{-\lambda}{\theta})}{\prod_{j=1}^{n}\left(\frac{s+\gamma}{\theta}+j\right)\ {}_1F_1(1; \frac{s+\gamma}{\theta}+1; \frac{-\lambda}{\theta})}.$$

We known that

$${}_1F_1(n+1; \frac{s+\gamma}{\theta}+n+1; \frac{-\lambda}{\theta}) = \sum_{k=0}^{\infty} \frac{(n+1)_k \left(\frac{-\lambda}{\theta}\right)^k}{(\frac{s+\gamma}{\theta}+n+1)_k k!}$$

where $(b)_k$ represents the Pochhammor symbol, i.e.

$$(b)_k = \begin{cases} 1 & \text{if } k=0; \\ b(b+1)(b+2)...(b+k+1) & \text{if } k=1,2,3,.... \end{cases}$$

Therefore

$$\frac{{}_1F_1(n+1; \frac{s+\gamma}{\theta}+n+1; \frac{-\lambda}{\theta})}{\prod_{j=1}^{n}\left(\frac{s+\gamma}{\theta}+j\right)} = \sum_{k=0}^{\infty} \frac{\binom{n+k}{k}\left(-\frac{\lambda}{\theta}\right)^k}{\prod_{j=1}^{n+k}\left(\frac{s+\gamma}{\theta}+j\right)}$$

Applying partial fraction expansion, the above equation can be written as

$$\begin{aligned}\frac{{}_1F_1(n+1; \frac{s+\gamma}{\theta}+n+1; \frac{-\lambda}{\theta})}{\prod_{j=1}^{n}\left(\frac{s+\gamma}{\theta}+j\right)} =& \theta\sum_{k=0}^{\infty}\binom{n+k}{k}\left(-\frac{\lambda}{\theta}\right)^k \\ & \sum_{j=1}^{n+k}\frac{(-1)^{j-1}}{(j-1)!\,(n+k-j)!\,(s+\gamma+j\theta)}.\end{aligned} \tag{19}$$

Also

$${}_1F_1\left(1; \frac{s+\gamma}{\theta}+1; \frac{-\lambda}{\theta}\right) = \sum_{k=0}^{\infty}(-\lambda)^k\, d_k^*(s),$$

where

$$d_k^*(s) = \frac{1}{\prod_{j=1}^{k}(s+\gamma+j\theta)} \quad and \quad d_0^*(s) = 1.$$

$$\frac{1}{{}_1F_1(1; \frac{s+\gamma}{\theta}+1; \frac{-\lambda}{\theta})} = \sum_{k=0}^{\infty}(\lambda)^k e_k^*(s), \tag{20}$$

where $e_0^*(s) = 1$, and for k=1,2,3,...

$$e_k^*(s) = \begin{vmatrix} d_1^*(s) & 1 & & . & . & . & & \\ d_2^*(s) & d_1^*(s) & 1 & . & . & . & & \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ d_{k-1}^*(s) & d_{k-2}^*(s) & d_{k-3}^*(s) & . & . & . & d_1^*(s) & 1 \\ d_k^*(s) & d_{k-1}^*(s) & d_{k-2}^*(s) & . & . & . & d_2^*(s) & d_1^*(s) \end{vmatrix}$$

$$= \sum_{l=1}^{k}(-1)^{l-1}e_{k-l}^*(s)d_l^*(s).$$

By substituting Eq.(19) and Eq.(20) in Eq.(13), we get

$$\zeta_n^*(s) = (\lambda)^n \sum_{i=0}^{\infty}(-\lambda)^i\binom{n+i}{i}d_{n+i}^*(s)\sum_{k=0}^{\infty}(\lambda)^k e_k^*(s).$$

On inversion, we obtain

$$\zeta_n(t) = (\lambda)^n \sum_{i=0}^{\infty} (-\lambda)^i \binom{n+i}{i} d_{n+i}(t) \sum_{k=0}^{\infty} (\lambda)^k e_k(t), \tag{21}$$

where

$$d_k(t) = \frac{1}{(\theta)^{k-1}} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{(k-j)!\,(j-1)!} e^{(-j\theta+\gamma)t}, k = 1, 2, 3, ...,$$

$$e_k(t) = \sum_{j=1}^{k} (-1)^{j-1} d_j(t) * e_{k-j}(t), \quad k = 2, 3, 4, ...; \quad e_1(t) = d_1(t)$$

Now from Eq(5) , we have

$$F^*(s) = \frac{\gamma}{s+\gamma+\eta} \left(\frac{1}{s} - P_0^*(s) \right)$$

On inversion,

$$F(t) = \gamma \int_0^t (1 - P_0(z)) e^{-(\gamma+\eta)(t-z)} dz \tag{22}$$

4. TIME DEPENDENT MOMENTS

4.1. MEAN

Let A(t) denote the mean value of the random variable C(t),therefore

$$A(t) = E(C(t)) = \sum_{n=1}^{\infty} n P_n(t) \tag{23}$$

Initially, at t=0, Eq(23) gives

$$A(0) = 0,$$

which implies

$$A'(t) = \sum_{n=1}^{\infty} n P_n'(t), \tag{24}$$

where $A'(t)$ denotes the derivative of $A(t)$. Application of Eq.(3) in Eq.(24),after some calculation gives

$$A'(t) = -(\theta+\gamma)A(t) + \lambda \tag{25}$$

which is a linear differential equation in $A(t)$, whose solution gives

$$A(t) = \frac{\lambda}{\theta+\gamma} [1 - e^{-(\theta+\gamma)t}] \tag{26}$$

4.2. VARIANCE

An average is not sufficient to understand completely the distribution of the random variable $C(t)$).Hence, variance is also needed for better understanding. Let $Var(C(t))$ represents the variance of the random variable $C(t)$, then

$$Var(C(t)) = E[C(t) - E(C(t))]^2$$

Which may be written as

$$Var(C(t)) = c(t) - [A(t)]^2, \tag{27}$$

and

$$c(t) = E(C^2(t)) = \sum_{n=1}^{\infty} n^2 P_n(t),$$

with

$$c(0) = 0,$$

and

$$c'(t) = \sum_{n=1}^{\infty} n^2 P_n'(t) \tag{28}$$

Substitution of $P_n'(t)$ in Eq.(28), after some calculation results in the form of a linear differential equation in $c(t)$ i.e.

$$c'(t) = -(2\theta + \eta)c(t) + (2\lambda + \theta)A(t) + \lambda \tag{29}$$

which after integration gives

$$\begin{aligned} c(t) = &\frac{(2\lambda + \theta)\lambda(\theta - e^{-(2\theta+\gamma)t}(3\theta + \gamma) + e^{-(\theta+\gamma)t}(2\theta + \gamma))}{(2\theta + \gamma)\theta(\theta + \gamma)} \\ &+ \frac{\lambda}{(2\theta + \gamma)}[1 - e^{-(2\theta+\gamma)t}]. \end{aligned} \tag{30}$$

subsitutation of Eq.(30) in Eq.(27), gives the expression of $Var(C(t))$.

5. STEADY STATE SOLUTION

Here, we derive an expression for the stationary probabilities of the operating model

Theorem 5.1. Stationary probabilities of the system under consideration are given as

$$F = \frac{\gamma}{\gamma + \eta}(1 - \eta\rho_1)$$
$$P_n = \eta\rho_n\rho_1$$
$$P_0 = \eta\rho_1$$

where

$$\rho_n = \left(\frac{\lambda}{\theta}\right)^n \frac{{}_1F_1(n+1; \frac{\gamma}{\theta} + n + 1; \frac{-\lambda}{\theta})}{\prod_{j=1}^{n}\left(\frac{\gamma}{\theta} + j\right)\ {}_1F_1(1; \frac{\gamma}{\theta} + 1; \frac{-\lambda}{\theta})}.$$

and

$$\rho_1 = \sum_{j=o}^{\infty} \frac{(-1)^j}{(\lambda + \eta)^{j+1}} \left[\sum_{n=1}^{\infty} (\eta - \delta_n\theta)\rho_n\right]^j$$

Proof. Multiplying by s on both side of Eq.(16) and taking limit as $s \to 0$, and using $\lim_{s\to 0} sP_0^*(s) = P_0$, we get

$$P_0 = \eta\rho_1 \tag{31}$$

where

$$\rho_1 = \sum_{j=o}^{\infty} \frac{(-1)^j}{(\lambda + \eta)^{j+1}} \left[\sum_{n=1}^{\infty} (\eta - \delta_n\theta)\rho_n\right]^j$$

For $n = 1, 2, ...,$
Multiplying by s on both side of Eq.(12) and taking limit as $s \to 0$, and using $\lim_{s\to 0} sP_n^*(s) = P_n$, we get

$$P_n = \eta\rho_n\rho_1, \tag{32}$$

where

$$\rho_n = \left(\frac{\lambda}{\theta}\right)^n \frac{{}_1F_1(n+1; \frac{\gamma}{\theta} + n + 1; \frac{-\lambda}{\theta})}{\prod_{j=1}^{n}\left(\frac{\gamma}{\theta} + j\right)\ {}_1F_1(1; \frac{\gamma}{\theta} + 1; \frac{-\lambda}{\theta})}.$$

The failure distribution is obtained by multiplying s on both sides of Eq(5) and using Tauberian theorem after taking the limit as $s \to 0$, we get

$$F = \frac{\gamma}{\gamma + \eta}(1 - \eta\rho_1) \tag{33}$$

■ It is observed that the stationary solution exist only if $\rho_1 < 1$.

5.1. Mean and Variance

Taking limit as $t \to \infty$ in Eq.(26) and in Eq.(27) after putting the values of $c(t) and A(t)$, we get directly an expression for steady state mean and variance i.e.

$$A = \frac{\lambda}{(\theta + \gamma)} \tag{34}$$

$$Var(C) = \frac{1}{2\theta + \gamma} \left[(2\lambda + \theta)A + \lambda - (2\theta + \gamma)A^2 \right] \tag{35}$$

6. RELIABILITY AND AVAILABILITY ANALYSIS

The probability that a system perform well without any failure for a given period of time is known as its reliability. In this section, we derive an expression for availability and reliability of the system. Let $Av(t)$ be the probability that a repairable system is available at a given point of time t. Therefore, from Eq(22), the availability of the system is obtained as

$$Av(t) = 1 - F(t)$$

$$= \frac{1}{(\eta + \gamma)}(\gamma + \eta e^{-(\gamma+\eta)t}) + \gamma \int_0^t P_0(x) e^{-(\eta+\gamma)(t-x)} dx, \tag{36}$$

where $P_0(t)$ is given by Eq(17).
Next, we obtain an expression for the average availability of the system i.e.

$$Av(t)^* = \frac{1}{t} \int_0^t Av(y) dy$$

$$= \frac{1}{(\eta + \gamma)} \left(\gamma + \frac{\eta}{(\eta + \gamma)t} [1 - e^{-(\eta+\gamma)t}] \right) + \frac{\gamma}{(\eta + \gamma)t} \int_0^t P_0(y)[1 - e^{-(\eta+\gamma)(t-y)}] dy, \tag{37}$$

If $\eta = 0$, then we get from Eq(22)

$$F(t) = 1 - e^{-\gamma t} - \gamma \int_0^t P_0(x) e^{-\gamma(t-x)} dx$$

Therefore $R(t)$, the system reliability is obtained as

$$R(t) = 1 - F(t)$$

$$= e^{-\gamma t} \left(1 + \gamma \int_0^t e^{\gamma x} P_0(x) dx \right) \tag{38}$$

7. NUMERICAL ANALYSIS

Here, some graphical results are presented to study the behavior of the probability P_0 and $E(C)$ with various parameters i.e. arrival rate λ, catastrophic rate γ and service rate θ.

In fig.(1 to 2) we have plotted the probability P_0 as a function of (λ, γ) and (θ, γ) respectively. We observe that the value of P_0 is decreasing with increasing value of λ and increasing with increasing value of θ. Also, in both the figures P_0 increases with increasing γ i.e. the probability of an empty system increases with the increase in catastrophic rate. Fig.(3 and 4), illustrates that the expected number of customers decreases with the increasing service and catastrophic rates and increases with the corresponding increase in arrival rate.

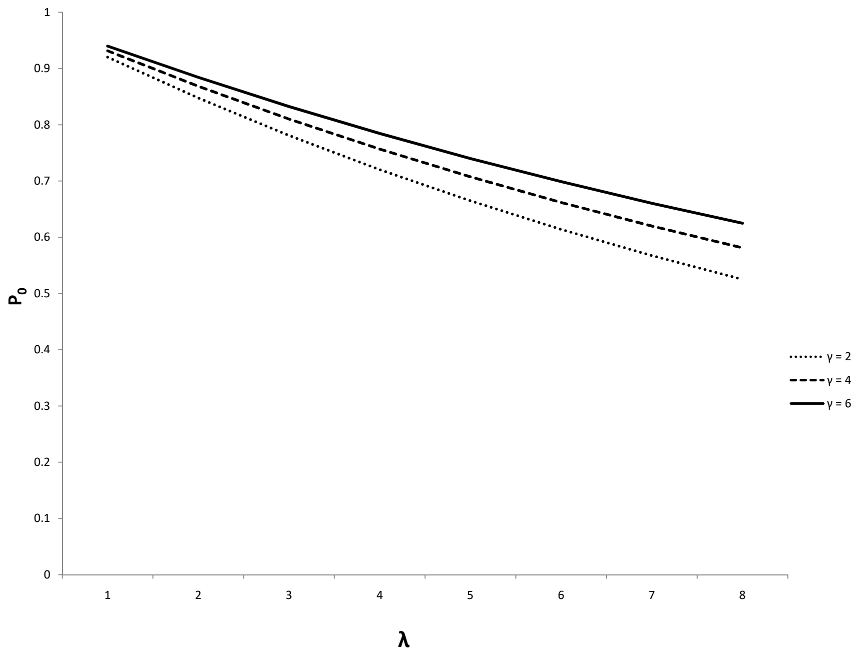

Figure 1: P_0 *as a function of* λ *for* $\theta = 10$

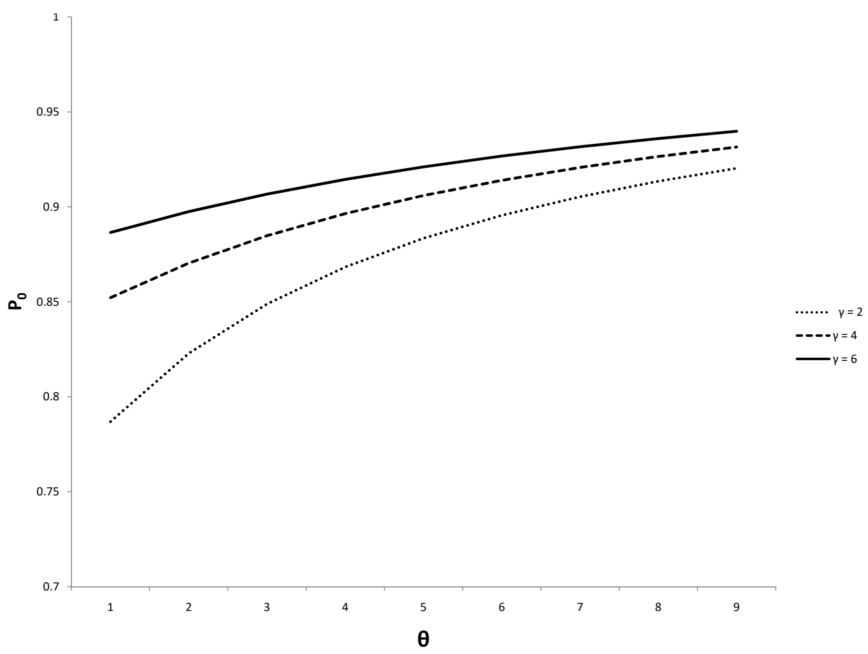

Figure 2: P_0 *as a function of* θ *for* $\lambda = 1$

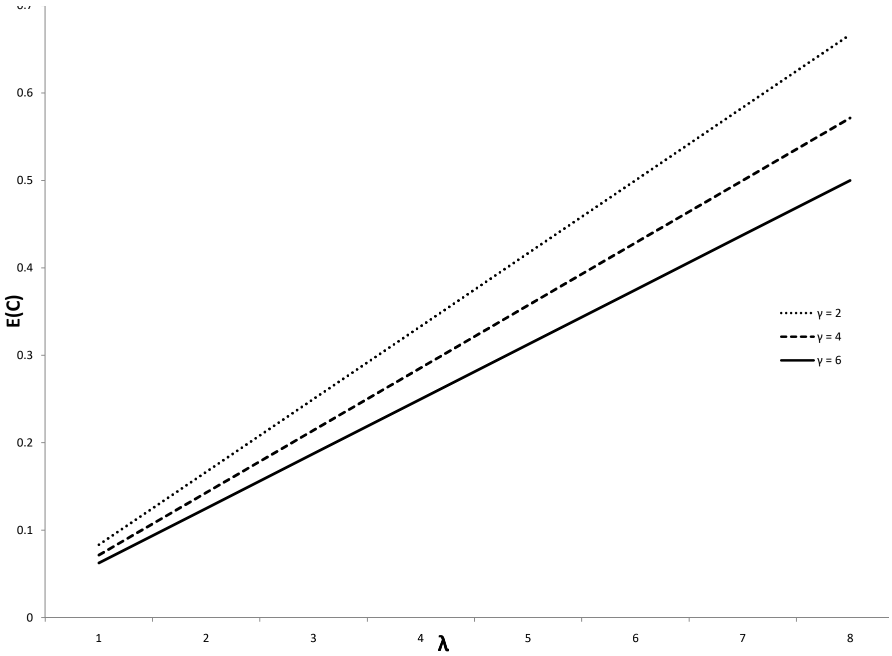

Figure 3: $E(C)$ *as a function of* λ *for* $\theta = 10$

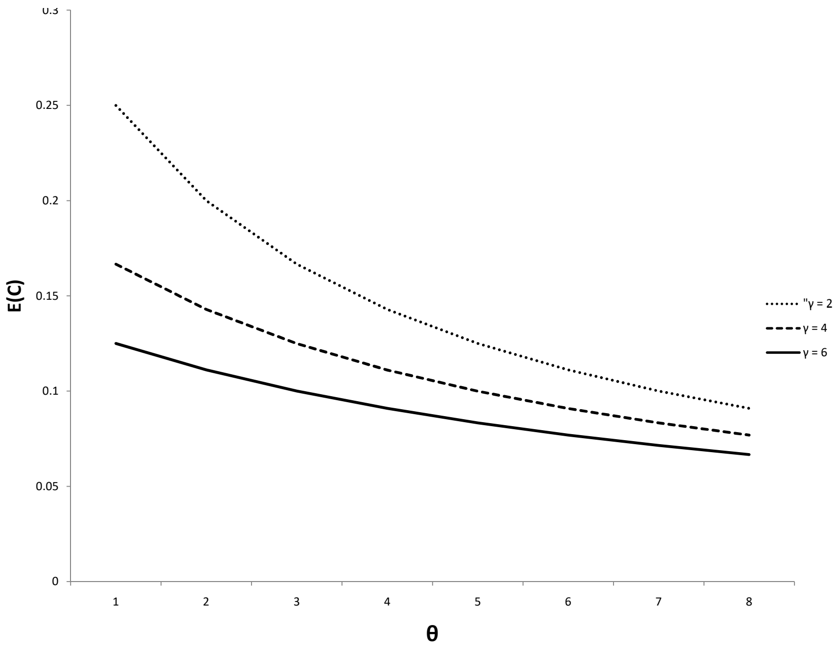

Figure 4: $E(C)$ *as a function of* θ *for* $\lambda = 1$

8. CONCLUSION

In this paper, we have considered an infinite servers Markovian queueing system with catastrophes and repairable servers. The transient and stationary probabilities are obtained analytically. The system availability and reliability are two important characteristics for those queueing system which are failed and repaired. Therefore, these two measures are also investigated for the system. Some graphical results are also added to visualize the model in practical situations.

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A NEW ALGORTHIM TO SOLVE FUZZY TRANSPORTATION MODELWITH L-R TYPE HEXAGONAL FUZZY NUMBERS USING RANKING FUNCTION

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Abstract

The transportation problems have much utilization in logistics and supply chains for minimizing costs. In real life circumstances, the limitations of transportation models may not be known absolutely because of unmanageable elements. In the several research papers the transportation costs, availability and demands of the commodity are shown as general fuzzy numbers and L-R flat fuzzy numbers for minimizing the transportation cost using different algorithms. But in this article, proposed the fuzzy costs, supply, and demands of the commodity at origins and destinations are taken as L-R type hexagonal fuzzy numbers for obtaining the optimal solution of unbalanced and balanced fuzzy transportation model by using ranking function to get minimum transportation cost. Here in, the numerical examples are also included. It is very simple to express and execute in real world transportation problem for decision maker.

Keywords: L-R HFN's, ranking function, Transportation problem, balanced transportation problem and unbalanced transportation problem.

1. Introduction

The transportation is the demand of proliferation. All societies are arriving closely not only by ability, tradition and custom but demand and inventory of goods and equipments are transferred from warehouses to retailers through identical vehicles. In actual life, the shipping problem and its optimal solution techniques are used to huge circumstances of practical fields, trades and manufacturing implementations but the constraints are ambiguous and imprecise. To accomplish the aim, the quantity of available supplies and the amount demanded must be known. The transportation models have vast utilizations in logistics and supply chain for minimize the cost. A fuzzy transportation problem contains fuzzy costs, supply and demand of the shipping algorithms. These are all characterized by L-R hexagonal fuzzy numbers. The main concept of the transportation problem is to obtain the optimal solution of the transportation model to minimize the transportation cost of a commodity for gratifying the demand at destinations using the supply at origins. Several literatures are discussed about the transportation problem to reduce the fully fuzzy transportation cost.

The modern process for obtaining fuzzy transportation problem in which the costs, availabilities and demands for the commodity are taken as non negative L-R flat fuzzy numbers by using standard transportation simplex algorithm in [1] and also the results are compare with other existing methods. A novel algorithm is introduced for finding fuzzy transportation problems in [2] by considered that the inventor is unpredictable about the accurate amount of the transportation costs, supply and demands of the commodity are taken as general fuzzy numbers using modern ranking function. A modern algorithm is introduced for finding the special cases of transportation problem in [3] by considering that the decision maker has doubt about the exact values of shipping cost. In the transportation problem the constraints are taken as general fuzzy numbers. The advanced method called as Mehar's method for finding fully fuzzy linear programming problems are presented in [4] in which the constraints are taken as L-R flat fuzzy numbers and the numerical example is also given. The permanent of both inter valued and triangular number fuzzy matrices are defined with examples and few properties, propositions to the permanent of inter valued and triangular number fuzzy matrices are proved in [5]. A new method for solving optimal solution of fully fuzzy linear programming problem is proposed in [6] by utilize the ranking technique with hexagonal fuzzy numbers. The permanent of square L-R hexagonal fuzzy matrix by using various techniques from partial derivatives and derived some properties and constant matrix of the permanent of non square L-R hexagonal fuzzy numbers are investigated in [7]. A fuzzy inventor model with allowable shortage with new L-R hexagonal fuzzy numbers are considered in [8] for obtaining the fuzzy optimal cost and optimal order amount in which the constraints are described by L-R hexagonal fuzzy numbers. The mathematical computation in novel arithmetic operations on α - cut of hexagonal fuzzy numbers, ranking function and their properties are proposed in [9]. Novel procedures of matrix inversion method with the hexagonal fuzzy number matrices for finding fuzzy linear system of equations are established in [10]. [11] has proposed the permanent of a square matrix from Ryser's formula or standard definition. It was calculated two formulas by using in several approaches. One is related to symmetric tensors another one is algebraic method. [12] have compared the optimal solution for reducing the minimum transportation cost of balanced and unbalanced fuzzy transportation problems which is solved by using ranking technique with hexagonal fuzzy numbers. The advanced ranking technique based on the hexagonal fuzzy numbers using centroid of the triangle and rectangle is proposed in [13]. In this method hexagonal fuzzy numbers transferred to crisp number. The constant type-2 triangular fuzzy matrices are proposed in [14]. It is extension of type-2 fuzzy sets whose membership function define in [0,1]. In this, the properties and the examples of constant type-2 triangular fuzzy matrices are verified with the help of type-2 fuzzy sets. Two identities for the estimation of permanents like the formulas of Binet and Minc and of Ryser are obtained in [15]. It is used to reduce in an easy approach. The notion of triangular fuzzy matrices are defined and their new properties, special cases like pure and fuzzy triangular, symmetric, skew-symmetric, singular, semi singular etc. using the elementary operations and main properties of triangular fuzzy matrices are given in [16]. [17] have proposed a new ranking technique for determining the fuzzy transportation model, in which the constraints are taken as trapezoidal fuzzy numbers. These numbers represents the costs, supply and demands for the product. A fuzzy linear programming with hexagonal fuzzy numbers by using simplex method is investigated in [18] for finding the optimal solution and compare with existing algorithms. The concept of the determinant of the permanent of a square matrix is introduced in [19]. In this, it is concentrated on graphs and theorem for the determinant. Also various auxiliary facts are proved.

In view of this article, a novel approach for obtaining the balanced and sun balanced fuzzy transportation problem using ranking system. The fuzzy transportation problem considering that the decision maker is undetermined about the accurate values of shipping cost only even so unpredictability about the supply and demands of the product are not there. In proposed method shipping costs, supply and demands are taken as L-R hexagonal fuzzy numbers. For illustration of proposed method a numerical example has been given.

The structure of this article organized as follows: 2.Preliminaries: A few fundamental definitions, arithmetic operators and Ranking function of L-R hexagonal fuzzy numbers are presented 3. Formulation of transportation model is discussed 4. Proposed method is explained 5. Illustration of the numerical examples are presented 6.Explained the conclusions.

2. Preliminaries

Definition 1: [8] A fuzzy number $\widetilde{P}_h = (a,b,c,d,e,f)$ is said to be hexagonal fuzzy numbers (HFN's). Which are belongs to real numbers and its membership function is as follows

$$\mu_{\widetilde{P}_h}(x) = \begin{cases} 0 & x < a \\ \frac{1}{2}\left(\frac{x-a}{b-a}\right) & a < x < b \\ \frac{1}{2} + \frac{1}{2}\left(\frac{x-b}{c-b}\right) & b < x < c \\ 1 & c < x < d \\ 1 - \frac{1}{2}\left(\frac{x-d}{e-d}\right) & d < x < e \\ \frac{1}{2}\left(\frac{f-x}{f-e}\right) & e < x < f \\ 0 & x > f \end{cases}$$

Definition 2: [8] A fuzzy number $\widetilde{P}_{hLR} = (a,b,c_1,c_2,d_1,d_2)$ is said to be L-R hexagonal fuzzy number. Where (a,b,c_1,c_2,d_1,d_2) are belongs to real numbers satisfying $a \le b,\ c_1 \ge c_2\ and\ d_1 \ge d_2$ and its membership function is given by

$$\mu_{\widetilde{P}_{hLR}} = \begin{cases} 0 & x \le a-(c_1+c_2) \\ 1-\left(\frac{a-x}{c_1+c_2}\right) & a-(c_1+c_2) \le x \le a-c_1 \\ 1-\frac{1}{2}\left(\frac{a-x}{c_1}\right) & a-c_1 \le x \le a \\ 1 & a \le x \le b \\ 1+\frac{1}{2}\left(\frac{b-x}{d_1}\right) & b \le x \le b+d_1 \\ 1+\frac{2}{3}\left(\frac{b-x}{d_1+d_2}\right) & b+d_1 \le x \le b+(d_1+d_2) \\ 0 & x \ge b+(d_1+d_2) \end{cases}$$

Here a and b are the points with membership value of 1 is known as the flat region of mean value and c_1,c_2,d_1,d_2 are the four different left and right shapes of $\widetilde{P}_{hLR}$ respectively.

Definition 3: [8] An L-R hexagonal fuzzy number is called as symmetric, if the addition of both its shapes are equal, i.e; if $c_1 + c_2 = d_1 + d_2$ and it is denoted as $\widetilde{P}_{hLR} = (a,b,c_1,c_2)_{LR}$

Definition 4: [7] Arithmetic operations on L-R hexagonal fuzzy numbers

Let $\widetilde{P}_{hLR} = (a,b,c_1,c_2,d_1,d_2)_{LR}$ and $Q_{hLR} = (p,q,r_1,r_2,s_1,s_2)_{LR}$ are two L-R hexagonal fuzzy numbers. Then

(i) Addition: $\widetilde{P}_{hLR} + \widetilde{Q}_{hLR} = (a+p, b+q, c_1+r_1, c_2+r_2, d_1+s_1, d_2+s_2)_{LR}$

(ii) Subtraction: $\widetilde{P}_{hLR} - \widetilde{Q}_{hLR} = (a-p, b-q, c_1+r_1, c_2+r_2, d_1+s_1, d_2+s_2)_{LR}$

(iii) Multiplication: $\widetilde{P}_{hLR}\ (\times)\ \widetilde{Q}_{hLR} = \left(\frac{a}{6}\sigma_v, \frac{b}{6}\sigma_v, \frac{c_1}{6}\sigma_v, \frac{c_2}{6}\sigma_v, \frac{d_1}{6}\sigma_v, \frac{d_2}{6}\sigma_v \right)$

Where $\sigma_v = (3p + 3q - r_1 - r_2 + s_1 + s_2)$

(iv) Division: $\widetilde{P}_{hLR}\ (\div)\ \widetilde{Q}_{hLR} = \left(\frac{6a}{\sigma_v}, \frac{6b}{\sigma_v}, \frac{6c_1}{\sigma_v}, \frac{6c_2}{\sigma_v}, \frac{6d_1}{\sigma_v}, \frac{6d_2}{\sigma_v} \right)_{LR}$

If $\sigma_v \neq 0$, Where $\sigma_v = (3p + 3q - r_1 - r_2 + s_1 + s_2)$

(v) Scalar Multiplication: If $k \neq 0$ is scalar, then $k\,\widetilde{P}_{hLR}$ is defined as

$$k\,\widetilde{P}_{hLR} = \begin{cases} (ka, kb, kc_1, kc_2, kd_1, kd_2) & if\ k \geq 0 \\ (kb, ka, -kc_1, -kc_2, -kd_1, -kd_2) & if\ k < 0 \end{cases}$$

Definition 5: [7] If $\widetilde{R}: F(R) \to R$ maps every numbers to real line $F(R)$ represented the set of all hexagonal fuzzy numbers. If R be any linear ranking function, then we write the ranking function is given below

$$\widetilde{R}(\widetilde{P}_{hLR}) = \left(\frac{3a + 3b - c_1 - c_2 + d_1 + d_2}{6} \right)$$

Definition 6: [7] An L-R hexagonal fuzzy number $\widetilde{P}_{hLR}$ is called Zero L-R hexagonal fuzzy number if $\widetilde{P}_{hLR} = (0,0,0,0,0,0,)$. It is denoted as 0_{LR}.

Definition 7: [7] If $\widetilde{R}(\widetilde{P}_{hLR}) = 0$ then $\widetilde{P}_{hLR}$ is called Zero equivalent L-R hexagonal fuzzy number and is denoted as $\widetilde{0}_{LR}$.

Definition 8: [7] An L-R hexagonal fuzzy number $\widetilde{P}_{hLR}$ is called unit L-R hexagonal fuzzy number if $\widetilde{P}_{hLR} = (1,1,0,0,0,0,)$. It is denoted as 1_{LR}.

Definition 9: [7] If $\widetilde{R}(\widetilde{P}_{hLR}) = 1$ then $\widetilde{P}_{hLR}$ is called unit- equivalent L-R hexagonal fuzzy number and it is denoted as $\widetilde{1}_{LR}$.

3. Formulation of Fuzzy Transportation Model

In traditional transportation problem, it is considered that the decision maker is confident about the correct data of shipping cost, availability and demand of the production. In real life utilizations, few constraints in the shipping algorithms may not be known exactly because uncertain elements. For instance, in real world problems are the following positions may appear: Consider a product is to be shipped first time at destination and skilled have no idea about the shipping cost then there exist unpredictability about the shipping cost. For finding transportation algorithms the costs, supply and demands of the commodity are taken as L-R hexagonal fuzzy numbers. The shipping problem, in which a decision maker has a doubt about the exact values of shipping cost from i[th] source to j[th] destination, even so the decision maker confident about the supply and demand of the commodity can be mathematically given as below

Minimize Z = $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \otimes t_{ij}$

Subject to $\sum_{j=1}^{n} t_{ij} \le p_i, \qquad i = 1,2,3,\ldots, m$

$$\sum_{i=1}^{m} t_{ij} \le q_j, \qquad j = 1,2,3,\ldots,n$$

$$t_{ij} \ge 0 \qquad \forall\, i, j$$

where p_i is the total availability of the commodity at i[th] origin, q_j is the total demand of the commodity at j[th] destination, $c_{ij}$is an approximate cost for shipping one unit amount of the commodity from i[th] origin to j[th] destination and $t_{ij}$is the number of units of the commodity that must be shipped from the i[th] origin to j[th] destination or decision variables.

If $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j$ called balance fuzzy transportation problem and if it is not equal then it is unbalanced fuzzy transportation problem.

4. Proposed methodology

There are so many procedures in the several research papers [1, 2, 4, 3, 17] for obtaining initial basic feasible solution[IBFS] and fuzzy optimal solution of balanced fuzzy transportation problem by using various algorithms or different ranking functions with general fuzzy numbers or L-R flat fuzzy numbers. But in the few articles [6, 12, 18] are finding only optimal solution by using several methods and ranking function with general hexagonal fuzzy numbers.

In this article, proposed a new algorithm for solving IBFS and fuzzy optimal solution of balanced and unbalanced transportation problem by using ranking function, in which the transportation costs, supply and demands are represented by L-R hexagonal fuzzy numbers in place of general hexagonal fuzzy numbers. The procedure and the numerical examples are presented below.

The process is given below for obtaining initial basic feasible solution and optimal solution.

Step: 1 General Hexagonal fuzzy numbers transformed to L-R type hexagonal fuzzy numbers.

Step: 2 Check the transportation table is balanced or unbalanced.

Step: 3 If the table is balanced transportation table then continue the following steps. If the table is unbalanced transportation table then it is converted to balanced transportation table after that continue the below steps.

Step: 4 Construct a transportation table using ranking technique with L-R type hexagonal fuzzy numbers.

Step: 5 Obtain initial basic feasible solution using various following algorithms.

4.1 Generalized fuzzy north-west corner method:

The steps to finding initial basic feasible solution using GFNWCM.

Case 1: Start the allotment from the left hand side of the top most corner (North West corner) wing in the transportation matrix and construct a allotment based on availability and demand.

Case 2: After attain the availability or requirements for that row or column respectively, then delete that row or column and prepare a next table.

Case 3: Continue this way until the all allotments of North West corner is completed.

Case 4: Write all allotments of each cell and compute the IBFS.

4.2 Generalized fuzzy Least cost method:

The process is obtaining for IBFS using GFLCM.

Case 1: Select a least cost of the complete transportation table and allot the smallest supply and demand.

Case 2: Delete that row or column whose supply and demands are completed and construct another table.

Case 3: Repeat this process until all allotments are fulfilled.
Case 4: After all allocations are fulfilled then write the allocations and calculate IBFS.
4.3 Generalized fuzzy Vogel's approximation method:
The steps are finding IBFS using GVAM.
Case 1: Check out the each row and column variance of the fuzzy transportation table.
Case 2: Choose the row or column with largest variation in the fuzzy transportation table and allocate a penalty in second smallest cost wing.
Case 3: Delete that row or column whose supply and demands are allocated i.e. all allotment cells with least cost connected with specified largest row or column variance. Next prepare a new transportation table.
Case 4: Maintain this process until all penalties are over in the entire table then take all the allotments in the matrix.
Case5: Obtain the IBFS or least transportation cost.
Step: 4 find the optimal solution of the transportation problem using modified distribution method. This method gives the lowest cost to the fuzzy transportation problem.
4.4 Generalized fuzzy modified distribution method:
The way to find the optimal solution using GFMODI.
Case 1: Determine the IBFS using 4.1, 4.2 or 4.3 methods.
Case 2: Determine the values u_i *and* v_j of dual variables using $c_{ij} = u_i + v_j$.
Case 3: Find the penalty costs using $\Delta_{ij} = c_{ij} - (u_i + v_j)$.
Case 4: Examine the sign of every penalty. If the penalties of all the vacant cells are either positive or zero then the optimum solution of the given problem is obtained. If the penalty has negative then the optimum solution is not gained. So go to further process for shipping costs are possible.
Case 5: Choose the vacant cell with the lowest negative penalty as the cell to be together with immediate solution.
Case 6: Draw a closed path for the vacant cell pick out in the preceding step. Mark that the right angle rotate in this path is allowed only at settled cells at the actual vacant wing.
Case 7: Mark another plus and minus sign at the vacant cells on the corner points of the closed path with a plus sign at the cell being analyzed.
Case 8: Solve the large number of units that must be transported to the vacant cell. The least point with a negative sign on the closed path denoted as the number of units that can be transported to the existing wing.
Case 9: Add this amount to all the cells on the corner points of the closed loop is noted with positive signs and subtract it from those cells marked with negative signs. In this way, a vacancy cell changed to a settled cell.
Case 10: Repeat this way until fuzzy optimum solution is determined for reducing the least fuzzy transportation cost.

UNBALANCED FUZZY TRANSPORTATION PROBLEM CONVERT IN TO BALANCED FUZZY TRANSPORTATION PROBLEM AS GIVEN:
An Unbalanced fuzzy transportation problem transformed in to Basic fuzzy transportation problem by established a temporary source or a temporary destination which will gives for the hugely availability or the prerequisite cost of shipping a unit from this temporary source or destination to any other area is represented by zero. After transforming the unbalanced fuzzy transportation problem to balanced fuzzy transportation problem, take up the regular process for finding the balanced fuzzy transportation problem. A numerical example for the unbalanced fuzzy transportation problem is presented.

5. Numerical Example

In this segment, two examples are given using proposed method for solving fuzzy transportation problem with L-R hexagonal fuzzy numbers using ranking system.

Example 1: The table 1 occupies general hexagonal fuzzy numbers of transportation costs of the product from several origins to several destinations.

Table 1: *Hexagonal fuzzy numbers of fuzzy transportation costs*

	D_1	D_2	D_3	D_4	Supply
O_1	1 4,16,18, 12,16,20	0,1,2, -1,1,3	7,8,9, 6,8,9	11,13,15, 10,13,16	2,4,6, 1,4,7
O_2	8,11,14, 7,11,15	3,4,5, 2,4,6	5,7,9, 4,7,10	8,10,12, 6,10,14	5,6,7, 4,6,8
O_3	6,8,10, 5,8,11	13,15,17, 12,15,18	7,9,11, 6,9,12	1,2,3, 0,2,4	7,8,9, 5,8,11
Demand	3,4,5, 2,4,6	3,5,7, 1,5,9	10,12,14, 8,12,16	6,7,8 5,7,9	

L-R type hexagonal fuzzy numbers for solving fuzzy transportation costs are presented in table-2.

Table 2: *L-R hexagonal fuzzy numbers of transportation costs*

	D_1	D_2	D_3	D_4	Supply
O_1	18,12,2, 2,4,4	2,-1,1, 1,2,2	9,6,1, 1,2,2	15,10,2, 2,3,3	6,1,2, 2,3,3
O_2	14,7,3, 3,4,4	5,2,1, 1,2,2	9,4,2, 2,3,3	12,6,2, 2,4,4	7,4,1, 1,2,2
O_3	10,5,2, 2,3,3	17,12,2, 2,3,3	11,6,2, 2,3,3	3,0,1 1,2,2	9,5,1, 1,3,3
Demand	5,2,1, 1,2,2	7,1,2, 2,4,4	14,8,2, 2,4,4	8,5,1, 1,2,2	

Table 2 represents unbalanced transportation table. So it is changed to balanced transportation problem introducing dummy origin represented in Table – 3.

Table 3: *The balanced Fuzzy Transportation costs with L-R hexagonal fuzzy numbers*

	D_1	D_2	D_3	D_4	Supply
O_1	18,12,2, 2,4,4	2,-1,1, 1,2,2	9,6,1, 1,2,2	15,10,2, 2,3,3	6,1,2, 2,3,3
O_2	14,7,3, 3,4,4	5,2,1, 1,2,2	9,4,2, 2,3,3	12,6,2, 2,4,4	7,4,1, 1,2,2
O_3	10,5,2, 2,3,3	17,12,2, 2,3,3	11,6,2, 2,3,3	3,0,1 1,2,2	9,5,1, 1,3,3
O_4	0,0,0 0,0,0	0,0,0, 0,0,0	0,0,0 0,0,0	0,0,0 0,0,0	12,6,2, 2,4,4
Demand	5,2,1, 1,2,2	7,1,2, 2,4,4	14,8,2, 2,4,4	8,5,1, 1,2,2	

$\Re(\widetilde{P}_{hLR})$ is determined for the fuzzy costs in table - 3using the formula

$$\widetilde{R}(\widetilde{P}_{hLR}) = \left(\frac{3a + 3b - c_1 - c_2 + d_1 + d_2}{6} \right)$$

6.

After implementing the ranking function, the L-R hexagonal Fuzzy transportation problem is shown in Table 4.

Table 4: *After Fuzzy ranking Fuzzy transportation table*

	D_1	D_2	D_3	D_4	Supply
O_1	15.7	0.8	7.8	12.8	3.8
O_2	10.8	3.8	6.8	9.2	5.8
O_3	7.8	14.8	8.8	1.8	7.7
O_4	0	0	0	0	9.7
Demand	3.8	4.7	11.7	6.8	

The various methods used for IBFS and the optimal solution of the fuzzy minimum transportation cost are presented in Table 5.

Table 5: *The optimal solution of the transportation problem*

Methods used for IBFS	IBFS for minimum transportation cost	Number of iterations of fuzzy MODI method for finding the fuzzy optimal solution by using finding IBFS	The total Fuzzy optimal cost
GFNWCM	159	5	61.4
GFLCM	93.4	4	61.4
GFVAM	62.4	2	61.4

Example 2: The table 6 contains general hexagonal fuzzy numbers of transportation costs of the product from various origins to various destinations.

Table 6: *Hexagonal fuzzy numbers of fuzzy transportation costs*

	D_1	D_2	D_3	D_4	Supply
O_1	3,7,11; 15,19,24	13,18,23; 28,33,40	6,13,20; 28,36,45	15,20,25; 31,38,45	7,9,11; 13,16,20
O_2	16,19,24; 29,34,39	3,5,7; 9,10,12	5,7,10; 13,17,21	20,23,26; 30,35,40	6,8,11; 14,19,25
O_3	11,14,17; 21,25,30	7,9,11; 14,18,22	2,3,4; 6,7,9	5,7,8; 11,14,17	9,11,13; 15,18,20
Demand	3,4,5; 6,8,10	3,5,7; 9,12,15	6,7,9; 11,13,16	10,12,14; 16,20,24	

From table 6, the fuzzy transportation costs are changed to LR- type hexagonal fuzzy numbers of fuzzy transportation costs are given in Table 7.

Table 7: *L-R hexagonal fuzzy numbers of transportation costs*

	D_1	D_2	D_3	D_4	Supply
O_1	11,15,4; 4,4,5	23,28,5; 5,5,7	20,28,7; 7,8,9	25,31,5; 5,7,7	11,13,2; 2,3,4
O_2	24,29,5; 3,5,5	7,9,2; 2,1,2	10,13,3; 2,4,4	26,30,3; 3,5,5	11,14,3; 2,5,6
O_3	17,21,3; 3,4,5	11,14,2; 2,4,4	4,6,1; 1,1,2	8,11,1; 2,3,3	13,15,2; 2,3,2

Demand	5,6,1; 1,2,2	7,9,2; 2,3,3	9,11,2; 1,2,3	14,16,2; 2,4,4	

Table 7 represents balanced transportation table. So using the ranking function ($\Re(\widetilde{P}_{hLR})$) is determined for the fuzzy costs in table 7.

$$\Re(\widetilde{P}_{hLR}) = \left(\frac{3a + 3b - c_1 - c_2 + d_1 + d_2}{6} \right)$$

After applying the ranking function, the L-R hexagonal Fuzzy transportation problem is shown in Table 8.

Table 8: *After Fuzzy ranking Fuzzy transportation table*

	D_1	D_2	D_3	D_4	Supply
O_1	13.17	25.83	24.5	28.67	12.5
O_2	26.83	7.83	12	28.67	13.5
O_3	19.5	13.17	5.17	10	14.17
Demand	5.83	8.33	10.33	15.67	

The several methods are used for IBFS and the optimal solution of the fuzzy minimum transportation cost is presented in Table 9.

Table 9: *The optimal solution of the transportation problem*

Methods used for IBFS	IBFS for minimum transportation cost	Number of iterations of fuzzy MODI method for finding the fuzzy optimal solution by using finding IBFS	The total Fuzzy optimal cost
GFNWCM	6 03.67	4	506.36
GFLCM	577.40	2	506.36
GFVAM	509	2	506.36

6. Conclusion

In view of this work, a new method is introduced to gain IBFS and optimum solution of balanced and unbalanced fuzzy transportation model in which the constraints like transportation costs, supply and demand of the product are taken as L-R hexagonal fuzzy numbers. In this, the comparison of UBFTP and BFTP give the optimal transportation cost on LR- hexagonal fuzzy numbers are working to get the fuzzy optimal solutions. We observed that UBFTP get less transportation cost than BFTP. A numerical example shows that the proposed work produce the quality results which are genuine and general in fuzzy environment. It can be also used for other algorithms appearing in real world circumstances.

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Solving Bi-objective Assignment Problem under Neutrosophic Environment

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Abstract

The assignment problem (AP) is a decision-making problem that is used in production planning, industrial organizations, the economy and so on. As the single objective AP is no longer sufficient to handle today's optimization problems, bi-objective AP (BOAP) is considered. This research article introduces BOAP in neutrosophic environment. The neutrosophic BOAP (NBOAP) is formulated by adding the elements of cost matrices with single-valued trapezoidal neutrosophic numbers (SVTrNNs). A new method namely, fixing point approach (FPA) is proposed in this paper. The aim of this study is not only to determine the set of efficient solutions but also to find the optimal compromise solution for NBOAP using FPA. The proposed approach is elucidated with a numerical example and its solutions are plotted in a graph using MATLAB, which demonstrates its efficiency and optimality in practical aspects. This approach is more profitable for decision makers (DMs) and more efficient than other existing approaches because it provides the best optimal compromise solution in a neutrosophic environment.

Keywords: Bi-objective neutrosophic assignment problem, Hungarian method (HM), Fixing point approach, Ideal solution, Efficient solution, Optimal compromise solution.

1. Introduction

AP is one of the most fundamental combinatorial optimization problems which is widely enforced in both mechanized and repair systems and it is one of the most anticipated optimization problems in administration discipline. Many researchers have employed a variety of ways such as HM, linear programming, neural networks, and evolutionary algorithms to solve the AP. As the name suggests, the BOAP consists of two objectives and in solving it, the individual is assigned with a single task in order to optimize the outcomes. Hamou and Mohamed [1] developed the method to construct the set of efficient solutions to MOAP. Sobana and Anuradha [2] primarily focused on determining the set of all solutions to bi-objective interval AP. Przybylski et al. [3] adopted the two-phase technique to solve the BOAP. Bufardi [4] investigated the efficiency of feasible solutions for multi-criteria AP. Adiche et al. [5] developed a hybrid strategy to develop an efficient solution to MOAP. Son et al. [6] developed a compromise programming approach to MOAP. Tilva and Dhodiya [7] modified the algorithm in exponential membership functions for solving MOAP. In some conventional approaches, the parameters are usually defined in an uncertain manner due to the inability of the

DMs to assign accurate values to parameters as they have no idea of the actual value of the parameters. Zadeh [8] introduced the fuzzy sets (FS) which is determined by its membership functions to handle the problems involving imprecise information. Kar et al. [9] developed three different approaches for solving fuzzy BOAP. Pramanik and Biswas [10] analyzed a MOAP with uncertain price, time and ineffectiveness using priority-based fuzzy goal programming technique. Vinoliah et al. [11] proposed a unique approach for the solution of generalised fuzzy AP. Raj et al. [12] investigated an approach involving modified best candidate for solving pentagonal fuzzy AP. In such cases, the results or decisions based on the given data do not seem to be satisfactory. Intuitionistic fuzzy sets (IFS), determined by their membership and non-membership functions and useful in dealing with situations involving uncertainty information were introduced by Atanassov [13]. In such cases, generalisation of the FS eventually failed to deal with difficulties involving imprecise or inconsistent data. To overcome this, Smarandache [14] introduced neutrosophic sets (NS) which is an extension of FS and IFS. The neutrosophic set is determined by the membership, the non-membership and the indeterminacy functions which are independent of one another. The BOAP is examined in a neutrosophical framework to address the truth, indeterminacy and falsity of the data which were caused by issues such as the uncertain magnitude of the problem, imprecise data and inefficient forecasting. Wang et al. [15] introduced the concept of single-valued neutrosophic sets (SVNS) in many real-life situations. A methodology for solving decision-making problems with SVNNs was presented by Deli and Subas [16]. Khalifa [17] proposed a method for solving the MOAP in a neutrosophic environment based on the weighting tchebycheff programme. Khalifa and Pavan kumar [18] developed a neutrosophic AP using the interval-valued trapezoidal neutrosophic number. Bera and Mahapatra [19] proposed a solution methodology for solving AP with neutrosophic costs. Harnpornchai and Wonggattaleekam [20] proposed a neutrosophic set-based relative AP. Risk-Allah et al. [21] developed the neutrosophic compromise programming approach to solve the multi-objective transportation model under neutrosophic environment.

In the literature, many researchers have proposed various methods to solve NBOAP where the solutions are in deterministic form. To fill this gap, we have proposed FPA to determine the set of all efficient solutions and optimal compromise solution for NBOAP in neutrosophic quantities. In traditional BOAP, the DM is supposed to know the precise values of the coefficients of the variables in the objective functions, resources and activities of the product. In real world situations, the precise knowledge of all the parameters of the BOAP may not be possible due to uncontrollable situations. Solution methods based on neutrosophic theory generally have the advantages of not requiring prior prediction of regularities or posterior frequency distributions, as well as that they can handle with unpredictable information based on the subjective judgment of the DM. In general, most of the existing techniques provide only deterministic solution to the optimization problems under neutrosophic environment. Practically, the DM may not have specific, reliable and detailed information regarding these solutions. This motivates us to solve NBOAP under neutrosophic environment. In this paper, the parameters of both objectives for NBOAP are considered as SVTrNNs. The neutrosophic number provides an ideal approach to a decision-making process dealing with the uncertainty of truth, falsity, and an indeterminant state of information. Without converting the given problem into deterministic form, the proposed approach provides the set of all possible solutions for NBOAP. The set of all efficient solutions and a neutrosophic optimal compromise solution can be chosen from the obtained possible solutions of NBOAP. This approach enables the DMs to choose a solution that suits their economic situations and satisfying their goals.

This research article is formulated as follows: In Section 2, basic concepts and preliminaries are presented. In Section 3, assumptions and notations of the proposed NBOAP models are listed. Section 4 proposes solution approach for obtaining neutrosophic optimal compromise solution. Mathematical illustration and graphical interpretations are shown in Section 5. Section 6 illustrates a comparison of the solution method with other existing methods. Section 7 summarizes the conclusions and directions for future research.

2. Preliminaries and Essential Definitions

Some basic definitions related to NS, SVNS and SVTrNNs applied throughout this paper are introduced briefly in this section.

Definition 2.1 Neutrosophic set [14]

Let X be a universe. A NS A over X is defined by $\bar{A}^N = \left\{\left\langle x, P_{\bar{A}^N}(x), Q_{\bar{A}^N}(x), R_{\bar{A}^N}(x)\right\rangle : x \in X\right\}$ where $P_{\bar{A}^N}, Q_{\bar{A}^N}, R_{\bar{A}^N} : X \to \left]0^-, 3^+\right[$ are called the truth, indeterminacy and falsity membership function of the element $x \in X$ to the set $\bar{A}^N$ with $0^- \le P_{\bar{A}^N}(x) + Q_{\bar{A}^N}(x) + R_{\bar{A}^N}(x) \le 3^+$

Definition 2.2 Single-valued neutrosophic sets [15]

A SVNS $\bar{A}^{SVN}$ of a non-empty set X is defined as follows:

$\bar{A}^{SVN} = \left\{\left\langle x, P_{\bar{A}^N}(x), Q_{\bar{A}^N}(x), R_{\bar{A}^N}(x)\right\rangle : x \in X\right\}$ where $P_{\bar{A}^N}(x), Q_{\bar{A}^N}(x)$ *and* $R_{\bar{A}^N}(x) \in [0,1]$ for each $x \in X$ and $0 \le P_{\bar{A}^N}(x) + Q_{\bar{A}^N}(x) + R_{\bar{A}^N}(x) \le 3$

Definition 2.3 Single-valued trapezoidal neutrosophic number [16]

Let $\sigma_{\tilde{d}}, \lambda_{\tilde{d}}, \tau_{\tilde{d}} \in [0,1]$ and $p, q, r, s \in \sim$ such that $p \le q \le r \le s$ Then a SVTrNN, $\tilde{d}^N = \left\langle (p,q,r,s) : \sigma_{\tilde{d}}, \lambda_{\tilde{d}}, \tau_{\tilde{d}} \right\rangle$ is a special NS on $\sim$, whose truth membership, indeterminacy membership and falsity membership functions are given below:

$$\mu_{\tilde{d}^N}(x) = \begin{cases} \sigma_{\tilde{d}^N}\left(\dfrac{x-p}{q-p}\right), & p \le x < q \\ \sigma_{\tilde{d}^N}, & q \le x \le r \\ \sigma_{\tilde{d}^N}\left(\dfrac{s-x}{s-r}\right), & r \le x \le s \\ 0, & otherwise \end{cases}$$

$$\delta_{\tilde{d}^N}(x) = \begin{cases} \dfrac{q-x+\lambda_{\tilde{d}^N}(x-p)}{l-k}, & p \le x < q \\ \lambda_{\tilde{d}^N}, & q \le x \le r \\ \dfrac{x-r+\lambda_{\tilde{d}^N}(s-x)}{s-r}, & r \le x \le s \\ 1, & otherwise \end{cases}$$

$$\rho_{\tilde{d}^N}(x)=\begin{cases}\dfrac{q-x+\tau_{\tilde{d}^N}(x-p)}{q-p}, & p\le x<q\\ \tau_{\tilde{d}^N}, & q\le x\le r\\ \dfrac{x-r+\tau_{\tilde{d}^N}(s-x)}{s-r}, & r\le x\le s\\ 1, & otherwise\end{cases}$$

where $\sigma_{\tilde{d}}, \lambda_{\tilde{d}}$ *and* $\tau_{\tilde{d}}$ denote the maximum truth, minimum indeterminacy, and minimum falsity membership degrees respectively. A SVTrNN $\tilde{d}^N=\langle(p,q,r,s):\sigma_{\tilde{d}},\lambda_{\tilde{d}},\tau_{\tilde{d}}\rangle$ may be expressed as an ill-defined quantity of p, which is approximately equal to $[q,r]$.

Definition 2.4 Arithmetic operations on SVTrNNs [16]

Let $\tilde{d}^N=\langle(p,q,r,s):\sigma_{\tilde{d}^N},\lambda_{\tilde{d}^N},\tau_{\tilde{d}^N}\rangle$ and $\tilde{g}^N=\langle(p',q',r',s');\eta_{\tilde{g}^N},\mu_{\tilde{g}^N},\theta_{\tilde{g}^N}\rangle$ be two SVTrNNs. The arithmetic operations on $\tilde{d}^N$ and $\tilde{g}^N$ are

1. $\tilde{d}^N+\tilde{g}^N=\langle(p+p',q+q',r+r',s+s');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle$
2. $\tilde{d}^N-\tilde{g}^N=\langle(p-p',q-q',r-r',s-s');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle$
3. $$\tilde{d}^N*\tilde{g}^N=\begin{cases}\langle(pp',qq',rr',ss');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle, s,s'>0\\ \langle(ps',qr',qr',ps');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle, s<0,s'>0\\ \langle(ss',qq',rr',pp');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle, s<0,s'<0\end{cases}$$
4. $$\tilde{p}^N/\tilde{q}^N=\begin{cases}\langle(p/s',q/r',r/q',s/p');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle, s,s'>0\\ \langle(s/s',r/r',q/q',p/p');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle, s<0,s'>0\\ \langle(s/p',r/q',q/r',p/s');\sigma_{\tilde{d}^N}\wedge\sigma_{\tilde{g}^N},\lambda_{\tilde{d}^N}\vee\lambda_{\tilde{g}^N},\tau_{\tilde{d}^N}\vee\tau_{\tilde{g}^N}\rangle, s<0,s'<0\end{cases}$$
5. $$d\tilde{g}^N=h(x)=\begin{cases}\langle(dp,dq,dr,ds);\sigma_{\tilde{g}^N},\lambda_{\tilde{g}^N},\tau_{\tilde{g}^N}\rangle, d>0\\ \langle(ds,dr,dq,dp);\sigma_{\tilde{g}^N},\lambda_{\tilde{g}^N},\tau_{\tilde{g}^N}\rangle, d<0\end{cases}$$
6. $\tilde{g}^{N^{-1}}=\langle(1/s',1/r',1/q',1/p');\sigma_{\tilde{g}^N},\lambda_{\tilde{g}^N},\tau_{\tilde{g}^N}\rangle, \tilde{g}^N\neq 0$

Definition 2.5 Efficient solution

A feasible solution U° is said to be an efficient solution to the problem if there exists no other feasible X° such that $Z^{1(N)}(X^\circ)\le Z^{1(N)}(U^\circ)$ and $Z^{2(N)}(X^\circ)\le Z^{2(N)}(U^\circ)$ (or) $Z^{1(N)}(X^\circ)< Z^{1(N)}(U^\circ)$ and $Z^{2(N)}(X^\circ)< Z^{2(N)}(U^\circ)$. Otherwise, it is called non-efficient solution to the problem.

Definition 2.6 Optimal compromise solution

An optimal compromise solution $\left(Z^{1(N)}\left(U^{\circ}\right), Z^{2(N)}\left(V^{\circ}\right)\right)$ is an efficient solution which is closest to the ideal solution $\left(Z^{1(N)}\left(X^{\circ}\right), Z^{2(N)}\left(Y^{\circ}\right)\right)$ where $\tilde{Z}^{1(N)}\left(X^{\circ}\right)$ is an optimal solution to the first objective problem with all constraints and $\tilde{Z}^{2(N)}\left(Y^{\circ}\right)$ is an optimal solution of the second objective problem with all constraints.

3. Description and formulation of NBOAP

For defining a mathematical model, assumption, indices, formulation and related theorem are presented in this section.

3.1. Assumption

Let there be *n* activities to be completed by *n* resources whose costs are determined by their specific task. There must be only one to one relation between the activity and the resource.

3.2. Indices

i: Resources.
j: Activities.

3.3. Formulation

In real life, the goal of every DM is to achieve numerous targets at the same time when the products are assigned under neutrosophic environment. This has motivated the researchers to develop NBOAP. In NBOAP, the quantity $(\tilde{x}_{ij}{}^{N})$ is to be assigned from resources $i\ (i=1,2,...,n)$ to activities $j\ (j=1,2,...,n)$ with cost $(\tilde{c}_{ij}{}^{N})$ where $(\tilde{c}_{ij}{}^{N})$ can be shipping cost, shipping time, deterioration cost, consumption of energy or minimizing the risk while shipping goods, etc. The two objectives $\tilde{Z}^{(1)N}$ and $\tilde{Z}^{(2)N}$ are related to shipping cost and deterioration cost during shipping. The single-valued trapezoidal NBOAP (A) can be represented mathematically as follows:

$$\text{Minimize } \tilde{Z}^{(1)N}\left(x\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_{ij}{}^{(1)N}\tilde{x}_{ij}{}^{N}$$

$$\text{Minimize } \tilde{Z}^{(2)N}\left(x\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_{ij}{}^{(2)N}\tilde{x}_{ij}{}^{N}$$

subject to

$$\sum_{j=1}^{n}\tilde{x}_{ij}{}^{N}=1^{N}, i=1,2,...,n \text{ (only i}^{th}\text{ resource would be assigned to the j}^{th}\text{ activity)} \quad (1)$$

$$\sum_{i=1}^{n}\tilde{x}_{ij}{}^{N}=1^{N}, j=1,2,...,n \text{ (only j}^{th}\text{ activity would be selected by the i}^{th}\text{ resource)} \quad (2)$$

$$\tilde{x}_{ij}^{N}=0^{N} \text{ or } 1^{N} \text{ for all } i \text{ and } j \quad (3)$$

3.4. Parameter

$\tilde{c}_{ij}^{(1)N} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4, c_{ij}'^1, c_{ij}'^2, c_{ij}'^3\right)$ denotes the first objective single valued trapezoidal neutrosophic shipping cost associated with i^{th} resource to j^{th} activity.

$\tilde{c}_{ij}^{(2)N} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4, c_{ij}'^1, c_{ij}'^2, c_{ij}'^3\right)$ denotes the second objective single valued trapezoidal neutrosophic deterioration cost associated with i^{th} resource to j^{th} activity.

$\tilde{x}_{ij}^{N} = \left(x_{ij}^{1}, x_{ij}^{2}, x_{ij}^{3}, x_{ij}^{4}; x_{ij}'^{1}, x_{ij}'^{2}, x_{ij}'^{3}\right)$ denotes the single valued trapezoidal neutrosophic variable assuming 0 or 1 depending upon the entire assignment of j^{th} activity fulfilled from i^{th} resource.

3.5. Theorem

Let $X^{\circ N} = \left\{x_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ be an optimal solution to (A_1) where

$$(A_1)\ Minimize\ \tilde{Z}^{(1)N}(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}^{(1)N} \tilde{x}_{ij}^{N}$$

Subject to (1), (2) and (3)

and $Y^{\circ N} = \left\{y_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ be an optimal solution to (A_2) where

$$(A_2)\ Minimize\ \tilde{Z}^{(2)N}(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{c}_{ij}^{(2)N} \tilde{x}_{ij}^{N}$$

Subject to (1), (2) and (3)

Then $U^{\circ N} = \left\{u_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$, obtained from $X^{\circ N}$ (or) $Y^{\circ N}$ is an efficient solution to the problem (A).

Proof:

Let the problem (A_1) be a square matrix of order 'n'.

Since, $X^{\circ N} = \left\{x_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ is an optimal solution of (A_1), $X^{\circ N} = \left\{x_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ is a feasible solution of (A_2).

Clearly, $X^{\circ N} = \left\{x_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ is an efficient solution to the problem (A) which is trivial.

Let the allocated cell with maximum a_{ij} in (A_2) be chosen. Here a_{ij} is placed where the i^{th} row and the j^{th} column intersect.

Deleting the i^{th} row and j^{th} column of (A_2), we obtain a sub-matrix of order (n-1). Let $U^{\circ N} = \left\{u_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ be the solution to the sub-matrix obtained using the HM.

Repeat the procedure for the remaining allocated cells and obtain the solutions to the problem (A).

The procedure can be repeated for all the remaining cells in decreasing order of their magnitude to obtain all the efficient solutions to the problem (A).

By Definition 5, $U^{\circ N} = \left\{u_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ is an efficient solution to the problem (A).

In the same way, an efficient solution to the problem (A) from the optimal solution $Y^{\circ N} = \left\{y_{ij}^{\circ N}, i = 1,2,..,n; j = 1,2,..,n\right\}$ of (A_2) can be obtained.

Hence the theorem.

4. Solution approach

As to solve bi-objective problems under neutrosophic environment, it is necessary to find the optimal compromise solution. Here we have proposed an approach to find the efficient solutions which lead to optimal compromise solution. The following steps are given to proceed with the proposed approach:

Step 1 Consider the given problem (A) with A_1 as first objective neutrosophic AP (FNAP) and A_2 as second objective neutrosophic AP (SNAP).
Step 2 Determine an optimal solution of A_1 and A_2 by HM.
Step 3 Consider the optimal solution of A_1 as a feasible solution of A_2 which is an efficient solution to the problem (A).
Step 4 Select the allocated cell with the highest cost of problem (A_2) and delete its corresponding row and column. Determine the solution for the resultant sub-matrix using HM.
Step 5 Repeat Step 4 and obtain all the solutions for the remaining cells. The same process can be repeated to all the cells in decreasing order of their magnitude.
Step 6 Consider the optimal solution of A_2 as a feasible solution of A_1 which is an efficient solution to the problem (A).
Step 7 Steps 4 and 5 for A_1 are repeated.
Step 8 Combining all the solutions of A obtained using the optimal solutions of A_1 and A_2, the set of all efficient solutions and optimal compromise solution to the problem (A) can be worked out.

5. Application of MMA in NBOAP

Now we interpret the proposed approach to determine the application for the BOAP under neutrosophic environment. A numerical illustration predicts the shipping cost and deterioration cost of the cargoes in cargo ships. The following subsection discusses the procedure for obtaining the application with FPA.

5.1 Numerical illustration

Let three cargo ships be used for shipping goods from one port to another. Any ship can be chosen at random for each journey. Let us assume that there are two objectives to be considered: (i) the total shipping cost of the cargoes must be minimized; and (ii) the total deterioration cost of the cargoes must be minimized. The shipping cost and deterioration cost of the cargoes are represented as SVTrNNs as shown in Table 1.

Table 1

		Ports					
		P_1		P_2		P_3	
Ships	SH_1	$\tilde{c}_{11}^{(1)N}$	$\tilde{c}_{11}^{(2)N}$	$\tilde{c}_{12}^{(1)N}$	$\tilde{c}_{12}^{(2)N}$	$\tilde{c}_{13}^{(1)N}$	$\tilde{c}_{13}^{(2)N}$
	SH_2	$\tilde{c}_{21}^{(1)N}$	$\tilde{c}_{21}^{(2)N}$	$\tilde{c}_{22}^{(1)N}$	$\tilde{c}_{22}^{(2)N}$	$\tilde{c}_{23}^{(1)N}$	$\tilde{c}_{23}^{(2)N}$
	SH_3	$\tilde{c}_{31}^{(1)N}$	$\tilde{c}_{31}^{(2)N}$	$\tilde{c}_{32}^{(1)N}$	$\tilde{c}_{32}^{(2)N}$	$\tilde{c}_{33}^{(1)N}$	$\tilde{c}_{33}^{(2)N}$

where $\tilde{c}_{11}^{(1)N}$ = (14,17,21,28;0.8,0.2,0.6); $\tilde{c}_{11}^{(2)N}$ = (12,20,25,29;0.9,0.3,0.2);
$\tilde{c}_{12}^{(1)N}$ = (13,18,20,24;0.6,0.4,0.5); $\tilde{c}_{12}^{(2)N}$ = (22,25,30,34;0.8,0.2,0.4);
$\tilde{c}_{13}^{(1)N}$ = (20,25,30,35;0.8,0.4,0.2); $\tilde{c}_{13}^{(2)N}$ = (12,18,21,24;0.7,0.4,0.5);
$\tilde{c}_{21}^{(1)N}$ = (15,18,23,30;0.9,0.2,0.3); $\tilde{c}_{21}^{(2)N}$ = (15,17,19,24;0.7,0.2,0.3);
$\tilde{c}_{22}^{(1)N}$ = (11,16,25,28;0.8,0.3,0.2); $\tilde{c}_{22}^{(2)N}$ = (28,32,35,40;0.9,0.3,0.2);
$\tilde{c}_{23}^{(1)N}$ = (14,15,24,26;0.9,0.1,0.1); $\tilde{c}_{23}^{(2)N}$ = (17,18,22,26;0.8,0.2,0.3);
$\tilde{c}_{31}^{(1)N}$ = (11,17,22,25;0.6,0.5,0.4); $\tilde{c}_{31}^{(2)N}$ = (23,27,30,31;0.9,0.3,0.4);
$\tilde{c}_{32}^{(1)N}$ = (12,14,24,30;0.8,0.6,0.2); $\tilde{c}_{32}^{(2)N}$ = (12,19,24,25;0.8,0.5,0.4);
$\tilde{c}_{33}^{(1)N}$ = (14,16,21,23;0.7,0.5,0.3); $\tilde{c}_{33}^{(2)N}$ = (13,18,23,25;0.9,0.2,0.2)

Now, A_1 and A_2 of the problem A are shown in Table 2.

Table 2

	A_1			A_2		
	P_1	P_2	P_3	P_1	P_2	P_3
SH_1	(14,17,21,28; 0.8,0.2,0.6)	(13,18,20,24; 0.6,0.4,0.5)	(20,25,30,35; 0.8,0.4,0.2)	(12,20,25,29; 0.9,0.3,0.2)	(22,25,30,34; 0.8,0.2,0.4)	(12,18,21,24; 0.7,0.4,0.5)
SH_2	(15,18,23,30; 0.9,0.2,0.3)	(11,16,25,28; 0.8,0.3,0.2)	(14,15,24,26; 0.9,0.1,0.1)	(15,17,19,24; 0.7,0.2,0.3)	(28,32,35,40; 0.9,0.3,0.2)	(17,18,22,26; 0.8,0.2,0.3)
SH_3	(11,17,22,25; 0.6,0.5,0.4)	(12,14,24,30; 0.8,0.6,0.2)	(14,16,21,23; 0.7,0.5,0.3)	(23,27,30,31; 0.9,0.3,0.4)	(12,19,24,25; 0.8,0.5,0.4)	(13,18,23,25; 0.9,0.2,0.2)

Using HM, the optimal allotment of A_1 and A_2 are highlighted in Table 3.

Table 3

	A_1			A_2		
	P_1	P_2	P_3	P_1	P_2	P_3
SH_1	(14,17,21,28; 0.8,0.2,0.6)	**(13,18,20,24; 0.6,0.4,0.5)**	(20,25,30,35; 0.8,0.4,0.2)	(12,20,25,29; 0.9,0.3,0.2)	(22,25,30,34; 0.8,0.2,0.4)	**(12,18,21,24; 0.7,0.4,0.5)**
SH_2	(15,18,23,30; 0.9,0.2,0.3)	(11,16,25,28; 0.8,0.3,0.2)	**(14,15,24,26; 0.9,0.1,0.1)**	**(15,17,19,24; 0.7,0.2,0.3)**	(28,32,35,40; 0.9,0.3,0.2)	(17,18,22,26; 0.8,0.2,0.3)
SH_3	**(11,17,22,25; 0.6,0.5,0.4)**	(12,14,24,30; 0.8,0.6,0.2)	(14,16,21,23; 0.7,0.5,0.3)	(23,27,30,31; 0.9,0.3,0.4)	**(12,19,24,25; 0.8,0.5,0.4)**	(13,18,23,25; 0.9,0.2,0.2)

The optimal allotment and the optimal shipping cost of A_1 are $SH_1 \to P_2$, $SH_2 \to P_3$ and $SH_3 \to P_1$ and (38,50,66,75;0.6,0.5,0.5) respectively. The optimal allotment and the optimal deterioration cost of A_2 are $SH_1 \to P_3$, $SH_2 \to P_1$ and $SH_3 \to P_2$ and (39,54,64,73;0.7,0.5,0.5) respectively.
Now as in Step 3, consider the optimal solution of A_1 as a feasible solution of A_2 as shown in Table 4.

Table 4

	P_1	P_2	P_3
SH_1	(12,20,25,29;0.9,0.3,0.2)	**(22,25,30,34;0.8,0.2,0.4)**	(12,18,21,24;0.7,0.4,0.5)
SH_2	(15,17,19,24;0.7,0.2,0.3)	(28,32,35,40;0.9,0.3,0.2)	**(17,18,22,26;0.8,0.2,0.3)**
SH_3	**(23,27,30,31;0.9,0.3,0.4)**	(12,19,24,25;0.8,0.5,0.4)	(13,18,23,25;0.9,0.2,0.2)

Thus ((38,50,66,75;0.6,0.5,0.5), (62,70,82,91;0.8,0.3.0.4)) is the bi-objective value of NBOAP for the feasible allotment $SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_3$ and $SH_3 \rightarrow P_1$
Using Step 4, the solution for the resultant sub-matrix obtained using HM is shown in Table 5.

Table 5

	P_2	P_3
SH_1	**(22,25,30,34;0.8,0.2,0.4)**	(12,18,21,24;0.7,0.4,0.5)
SH_2	(28,32,35,40;0.9,0.3,0.2)	**(17,18,22,26;0.8,0.2,0.3)**

Thus ((38,50,66,75;0.6,0.5,0.5), (62,70,82,91;0.8,0.3.0.4)) is the bi-objective value of NBOAP for the feasible allotment $SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_3$ and $SH_3 \rightarrow P_1$.
Since all the highest cost cells for A_1 to A_2 are fixed, we terminate the process. Therefore, the set of all possible solutions S_1 from A_1 to A_2 are given in Table 6.

Table 6

S.No	Optimal allotments	Possible solutions (S_1)
1.	$SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_1$, $SH_3 \rightarrow P_3$	((42,52,64,77;0.6,0.5,0.5), (50,60,72,83;0.7,0.2,0.4))
2.	$SH_1 \rightarrow P_1$, $SH_2 \rightarrow P_3$, $SH_3 \rightarrow P_2$	((40,46,69,84;0.8,0.6,0.6), (41,57,71,80;0.8,0.5,0.4))
3.	$SH_1 \rightarrow P_3$, $SH_2 \rightarrow P_1$, $SH_3 \rightarrow P_2$	((47,57,77,95;0.8,0.6,0.3), (39,54,64,73;0.7,0.5,0.5))
4.	$SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_3$, $SH_3 \rightarrow P_1$	((38,50,66,75;0.6,0.5,0.5), (62,70,82,91;0.8,0.3.0.4))

Similarly, by using Steps 6 and 7, we obtain the set of all possible solutions S_2 from A_2 to A_1 as given in Table 7.

Table 7

S.No.	Optimal allotments	Possible solutions (S_2)
1.	$SH_1 \rightarrow P_3$, $SH_2 \rightarrow P_2$, $SH_3 \rightarrow P_1$	((42,58,77,88;0.6,0.5,0.4), (63,77,86,95;0.7,0.4,0.5))
2.	$SH_1 \rightarrow P_1$, $SH_2 \rightarrow P_2$, $SH_3 \rightarrow P_3$	((39,49,67,79;0.7,0.5,0.6), (53,70,83,94;0.9,0.3,0.2))
3.	$SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_3$, $SH_3 \rightarrow P_1$	((38,50,66,75;0.6,0.5,0.5), (62,70,82,91;0.8,0.3,0.4))
4.	$SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_1$, $SH_3 \rightarrow P_3$	((42,52,64,77;0.6,0.5,0.5), (50,60,72,83;0.7,0.2,0.4))
5.	$SH_1 \rightarrow P_1$, $SH_2 \rightarrow P_3$, $SH_3 \rightarrow P_2$	((40,46,69,84;0.8,0.6,0.6), (41,57,71,80;0.8,0.5,0.4))

Now, using Step 8, combine the set of all possible solutions S to the problem (A) obtained from A_1 to A_2 and from A_2 to A_1 as given in Table 8.

Table 8

S.No.	Optimal allotments	Possible solutions ($S=S_1 \cup S_2$)
1.	$SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_1$, $SH_3 \rightarrow P_3$	((42,52,64,77;0.6,0.5,0.5), (50,60,72,83;0.7,0.2,0.4))
2.	$SH_1 \rightarrow P_1$, $SH_2 \rightarrow P_3$, $SH_3 \rightarrow P_2$	((40,46,69,84;0.8,0.6,0.6), (41,57,71,80;0.8,0.5,0.4))
3.	$SH_1 \rightarrow P_3$, $SH_2 \rightarrow P_1$, $SH_3 \rightarrow P_2$	((47,57,77,95;0.8,0.6,0.3), (39,54,64,73;0.7,0.5,0.5))
4.	$SH_1 \rightarrow P_2$, $SH_2 \rightarrow P_3$, $SH_3 \rightarrow P_1$	((38,50,66,75;0.6,0.5,0.5), (62,70,82,91;0.8,0.3.0.4))
5.	$SH_1 \rightarrow P_3$, $SH_2 \rightarrow P_2$, $SH_3 \rightarrow P_1$	((42,58,77,88;0.6,0.5,0.4), (63,77,86,95;0.7,0.4,0.5))
6.	$SH_1 \rightarrow P_1$, $SH_2 \rightarrow P_2$, $SH_3 \rightarrow P_3$	((39,49,67,79;0.7,0.5,0.6), (53,70,83,94;0.9,0.3,0.2))

From Table 8, we obtain the set of all efficient solutions and the optimal compromise solution which is closest to the ideal solution. The obtained ideal, efficient and optimal compromise solutions are plotted in a graph using the MATLAB which are shown in Figure 1.

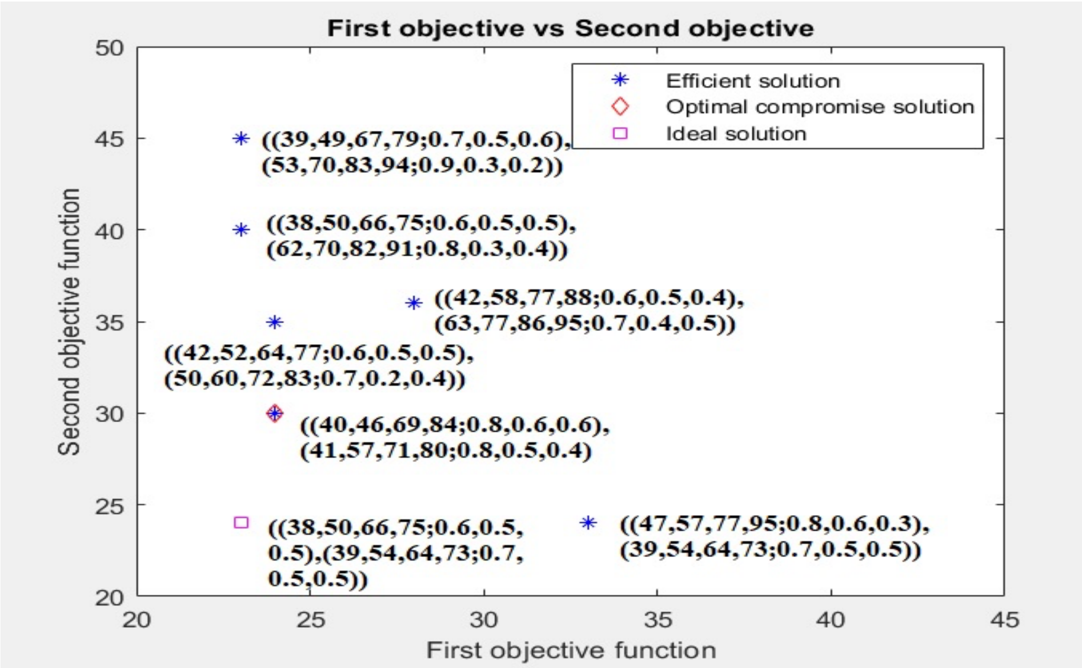

Figure 1: *Graphical representation of all solutions obtained by FPA*

6.Comparative study and discussions

We compare the above example with the existing methods from the literature to prove the efficiency of our approach. Using the method of Risk-Allah et al. [21] we obtain the ideal solution as ((38,50,66,75;0.6,0.5,0.5), (39,54,64,73;0.7,0.5,0.5)) and optimal compromise solution as ((38,50,66,75;0.6,0.5,0.5), (62,70,82,91;0.8,0.3.0.4)) and using the method of Khalifa [17], we obtain the optimal compromise solution as ((47,57,77,95;0.8,0.6,0.3), (39,54,64,73;0.7,0.5,0.5)). Using our proposed approach, we obtain the ideal solution as ((38,50,66,75;0.6,0.5,0.5), (39,54,64,73;0.7,0.5,0.5)), efficient solutions as ((42,52,64,77;0.6,0.5,0.5), (50,60,72,83;0.7,0.2,0.4)); ((40,46,69,84;0.8,0.6,0.6),(41,57,71,80;0.8,0.5,0.4));((47,57,77,95;0.8,0.6,0.3),(39,54,64,73;0.7,0.5,0.5));((38 ,50,66,75;0.6,0.5,0.5),(62,70,82,91;0.8,0.3.0.4));((42,58,77,88;0.6,0.5,0.4),(63,77,86,95;0.7,0.4,0.5));((39,49, 67,79;0.7,0.5,0.6), (53,70,83,94;0.9,0.3,0.2)) and optimal compromise solution as ((40,46,69,84;0.8,0.6,0.6), (41,57,71,80;0.8,0.5,0.4)).

In this comparative study, we find out that our proposed approach provides the set of all efficient solutions and the best optimal compromise solution of the given problem as compared to the other two approaches which are clearly shown in Table 9 and Figure 2.

Table 9: *Comparisons between the proposed approach with other existing approaches*

Methods	Ideal solution	Efficient solutions	Optimal compromise solution
FPA	✓	✓	✓
Rizk-Allah et al. [21]	✓	-	✓
Khalifa [17]	-	-	✓

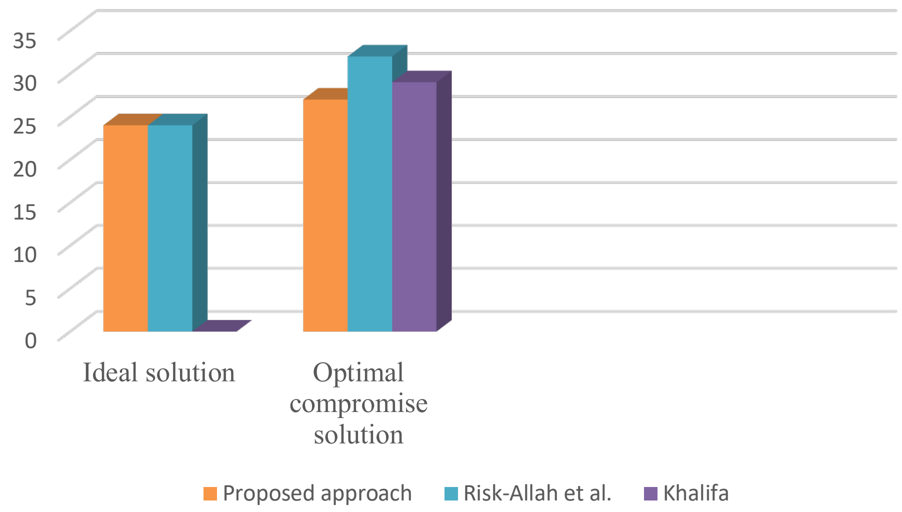

Figure 2: *Comparison between proposed and existing approaches*

7.Conclusions and future scopes

In this research article, the parameters of the model are expressed as SVTrNNs which improve the capacity of DM to make more realistic decisions. The main advantage of our approach is that the efficient solutions and the optimal compromise solution we obtain are neutrosophic quantities rather than deterministic values and they provide greater flexibility to the DM. Our proposed approach provides the best optimal compromise solution to the given problem as compared to the other two approaches which are shown in graph. When the DM deals with a range of logistical issues, the set of all efficient solutions obtained by our proposed approach can serve as a valuable tool. Though our approach analyses the solutions of NBOAP in the best way, there may be some limitations in predicting the solutions of qualitative and complex data due to the computational complexity in handling higher dimensional problems, they can be resolved using evolutionary algorithms. In the future research, one may incorporate this concept in neutrosophic bi-objective fractional assignment problem. The solution approach presented in this article can be aptly used by the DM when dealing with type-2 fuzzy parameters. Furthermore, in areas such as management science, finance, etc. wherever the assignment problems arise in neutrosophic environment, this solution approach will be a great resource.

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Fuzzy Linear Programming Approach for Solving Production Planning Problem

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Abstract

One of the various optimization methods that addresses optimization under uncertainty is fuzzy linear programming. This model can be used when there is ambiguity in the situation because it is not precisely specified or when the problem does not require an exact value. With fuzzy linear programming, there is a range of grey between the two extremes as opposed to binary models, where an event may only be either black or white. As a result, it broadens the range of potential applications because most scenarios involve a spectrum of values rather than a bipolar state. In this article, a new FLP-based method is developed using a single MF, called modified logistics MF. The modified MF logistics and its modifications taking into account the characteristics of the parameter are from the analysis process. This MF was tested for useful performance by modeling using FLP. The developed version of FLP provides confidence in the existing IPPP application. This approach to resolving the IPPP can get feedback from the decision maker, the implementer and the analyst. In this case, this process can be called FLP interaction. FS self-assembly for MPS problems can be developed to find satisfactory solutions. The decision maker, researcher and practitioner can apply their knowledge and experience to get the best results.

Keywords: Fuzzy Linear Programming, Degree of Satisfaction, Production Planning, Fuzzy PF, Vagueness.

Abbreviations

FLP	:	fuzzy linear programming
MF	:	membership function
IPPP	:	industrial production planning problem
FS	:	fuzzy system
NL	:	non linear
NLMF	:	nonlinear membership function
IP	:	industrial Problem

FP	:	Fuzzy parameter
MPS	:	mix product selection
LP	:	Linear Programming

1. Introduction

In previous studies a logistical MF model is developed to overcome the difficulty of using linear MF to solve complex decision making problems. However, it is expected that a new type of logistic MF based on certain NL resources can be obtained and its variability in changing the pattern of real-life problems can be explored. Such patterns of NL logistic MF are reflected in this work with its paradoxical changes in real life problems. The first step in testing such an MF system and its transformation is to apply it to a digital model that illustrates the problem of real decision making. A novel approach for fuzzy linear programming has been created recently employing a particular membership function called the modified logistic membership function. The modified logistic membership function is first developed, then an analytical technique is used to determine its adaptability to unclear parameters. Fuzzy linear programming is used to examine the usefulness of this membership function using an example to provide context. Applying FLP's established technique to actual industrial production planning problems now seems confident. The analyst, the implementer, and the decision-maker may all receive input from this method of solving the industrial production planning problem. This method can therefore be referred to as an IFLP (Interactive FLP). To discover a satisfying solution to the mix product selection problem, it is possible to create a self-organizing fuzzy system. To get the optimum result, the decision-maker, the analyst, and the implementer can pool their expertise and experience. Another study shows, for example, the benefits of MF. Their work is based on exponential LF. His demonstrated example can be accepted to test and compare our newly developed NLMF [1-3], such an attempt to compare this example with the results achieved in this work.

The test based on good intellectual ability should be performed with the newly developed MF to demonstrate that it fits the determination. This IP should be developed by creating multiple products with high FP as well as multiple uncertainties on productivity, product demand, availability and service time. Since it integrates operations and strategies, ties operations with strategies, and is essential to enterprise resource planning and organizational integration, aggregate production planning (APP) is regarded as a crucial stage in production systems. An efficient APP should boost the quality of service offered to the clients while simultaneously minimizing production and inventory expenses. Some cost and demand characteristics can't always be accurately assessed when maintaining an application. Numerous engineering applications use fuzzy logic to manage erroneous data. This gave the problem of aggregate production planning in an environment with uncertain data a mathematical programming foundation. Fuzzy linear programming is used to solve the APP issue when background information about the APP problem is given. An example is shown to illustrate how the model works for various -cut values. A researcher used different types of PI to ensure that its approach used traditional optimization techniques. Complex real-world intelligence tools should be used to test the newly developed MF to ensure it is relevant and decision-making. To test the new MF and the problems shown above, a software platform is required. This platform not only accepts FPs, but also needs to streamline FLP to provide the necessary data for the decision maker. The software, MATLAB, and LP Toolbox are well-suited for resolving such FLP problems, mainly as well as many FPs and unnecessary restrictions. In this study, the author used MATLAB and LP Toolkit to solve the real IP problem of the MPS problem [4-6].

2. Methodology of MF

According to some previous studies, the trapezoidal MF will experience difficulties such as damage when resolving FLP problems. To solve the damage problem, we should use NL LF as a hyperbolic tangent with asymptotes at 1and 0 [7-8]. In this case, we use LF for NLMF as given by:
Minimize

$$g(y) = \frac{C}{1 + De^{\beta y}} \tag{2.1}$$

Where C and D are scalar constants and, $0 < \beta < 1$ is FP considering DOV, where $\beta = 0$ indicates sharp. The difference is higher when you approach the same. Configuration (2.1) will be the same as shown in Figure 1 when $0 < \beta < 1$.

The reason we use this function is that MF logistic is similar to hyperbolic tangent function in previous studies, but more flexible than hyperbolic tangent. It is also known that trapezoidal MF corresponds to LF. Therefore, LF is considered an appropriate function to demonstrate the level of unfounded objective. This work is invaluable in decision-making and implementation by decision-maker and designer. LF, (2.1) is a non-monotonic activity, to be used as fuzzy MF. This is very important because, due to the unpredictable environment, DOV represents the acquisition of change [9-12]. We can show that MF does not increase as:

$$\frac{dg}{dy} = -\frac{CD\beta e^{\beta y}}{\left(1 + De^{\beta y}\right)} \tag{2.2}$$

Where C, D and y are all above zero, $\frac{dg}{dy} \le 0$.Furthermore, it can be shown that (2.1) has asymptotes in $g(y) = 0$ and $g(y) = 1$ with the appropriate values of C, D [13-14]. This means

$$\lim_{y \to \infty} \frac{dg}{dy} = 0 \text{ and } \lim_{y \to 0} \frac{dg}{dy} = 0.$$

This can be expressed as follows:
From (2.2)

$$\lim_{y \to \infty} \frac{dg}{dy} = -\frac{\infty}{\infty}$$

Therefore, using the L-hospital's rule, we obtained:

$$\lim_{y \to \infty} \frac{dg}{dy} = \frac{C\beta}{2\left(1 + De^{\beta y}\right)} = 0 \tag{2.3}$$

As $y \to 0$ the situation is not very vague so $\beta \to 0$.
From (2.2) we have:

$$\lim_{y \to \infty} \frac{dg}{dy} = -\lim_{y \to \infty} \frac{CD\beta}{\left(1 + D\right)^2} = 0, \text{ when } \beta \to 0 \tag{2.4}$$

In addition to the above, LF (2.2) has a vertical tangent at $y = y_0$. Where $g(y_0) = 0.5$. This can be demonstrated by defining tangent as:

$$\lim_{i \to 0} \frac{g\left(y_0 + i\right) - g\left(y_0\right)}{i} = -\infty$$

$$\lim_{i \to 0} \frac{g\left(y_0 + i\right) - g\left(y_0\right)}{i} = \lim_{i \to 0} \frac{\frac{C}{1 + De^{(\beta y_0 + i)}} - \frac{C}{1 + De^{\beta y_0}}}{i} \tag{2.5}$$

$$\lim_{i\to 0}\frac{CDe^{\beta y_0}\left(1-e^{\beta i}\right)}{i\left(1+De^{(\beta y_0+i)}\right)\left(1+De^{\beta y_0}\right)}=\frac{0}{0}$$

So, by using L-hospital's rule:

$$\lim_{i\to 0}\frac{g\left(y_0+i\right)-g\left(y_0\right)}{i}=\lim_{i\to 0}\frac{-\beta CDe^{\beta(y_0+i)}}{i\left(1+De^{\beta y_0}\right)\left(1+De^{\beta(y_0+i)}+i\beta\right.}$$

$$\lim_{i\to 0}\frac{g\left(y_0+i\right)-g\left(y_0\right)}{i}=\lim_{i\to 0}\frac{-\beta CDe^{\beta y_0}}{\left(1+De^{\beta y_0}\right)^2} \tag{2.6}$$

To make $g(y_0)=0.5$ and $g(0)=1$, by (2.1)

$$y_0=\frac{1}{\beta}\ln\left(2+\frac{1}{D}\right) \tag{2.7}$$

$$C=1+D \tag{2.8}$$

Now we use (2.7) as well as (2.8) in (2.6),

$$-\frac{\beta}{4C}(2D-1)\to-\infty \text{ , when } \beta\to\infty, D\,\&\,C\succ 0 \tag{2.9}$$

This shows that the vertical tangent is $y=y_0$.

It can also be shown that the LF has an inflection point at $y=y_0$, such as $g''(y_0)=1$. Where $g''(y)$ is the second derivative of $g(y)$ compared to y. In addition, it can be shown that $g'''(y_0)=0$ at $y=y_0$, where $g'''(y)$ is the third derivative of $g(y)$ compared to y [15-17].

The above argument about vertical, asymptotic, and rotational tangent leads to the conclusion that the recommended LF is variable [18-19]. An MF of this type, unlike linear work, presents real-life problems. From the above description of LF characteristics, the current MF is fully described for FLP problems in the following statistics. NLMF is quickly identified for the FLP problem in the next section.

2.1 Logistic MF

MF logistics for FLP problems are defined as:

$$g(y)=\begin{cases}1; y\prec k\\ \dfrac{C}{1+De^{\beta y}}; y_k\prec y\prec y_v\\ 0; y\succ y_k\end{cases} \tag{2..1.1}$$

Where $g(y)$ is MF value of same parameter y and $0<g(y)<1$.

The size y is considered a member of the fuzzy set associated with it; y_k and y_v respectively are the minimum values as well as the maximum values of FP y. C, D are variable and $b>0$ determines the type of MF. The greater the benefit of this, the greater it's DOV.

2.2 S-Curve MF

S-curve MF is a special case of LF with certain values of C, D and. These principles will be identified. This LF as given by (2.1.1) is expressed as MF in S form by some studies [20-21].
Here, we define the S-curve MF as follows:

$$\theta(y) = \begin{cases} 1; y \prec y^b \\ 0.9999; y = y^b \\ \dfrac{C}{1+De^{\beta y}} \quad ; y^b \prec y \prec y^c \\ 0.001; y = y^c \\ 0; y \succ y^c \end{cases} \tag{2.2.1}$$

Where θ is the MF level. (2.2.1) is similar to (2.1.1) except that MF is adjusted to 0.001 ≤ θ(y) ≤ 0.999. This size is chosen because in the manufacturing process it is not always necessary 100% of the required material. At the same time, the operating capacity will not be below 0%. So there is a gap between y^b and y^c with 0.001 ≤ θ (y) ≤ 0.999. This concept of near (y) is used in this article to solve the output processing problem of the nonlinear MPS problem. We rotate the y-axis as $y^b = 0$ and $y^c = 1$ to find the values of C, D and β. A study has made such an increase in its scientific work [22]. The values of C, D and β are derived from (2.2.1) as:

$$C = 0.999(1+D) \tag{2.2.2}$$

$$\frac{C}{1+De^{\beta}} = 0.001 \tag{2.2.3}$$

By using (2.2.2) into (2.2.3) we have:

$$\frac{0.999(1+D)}{1+De^{\beta}} = 0.001 \tag{2.2.4}$$

Rearranging (2.2.4) we have:

$$\beta = \ln\frac{1}{0.001}\left(\frac{0.998}{D} + 0.999\right) \tag{2.2.5}$$

Since C and β are based on D, we need another condition to obtain the values of C, D, β.
Let,

$$y_0 = \frac{y^b + y^c}{2}, \theta(y_0) = 0.5$$

$$\frac{C}{1+De^{\frac{\beta}{2}}} = 0.5 \tag{2.2.6}$$

and hence:

$$\beta = 2\ln\left(\frac{2C-1}{D}\right) \tag{2.2.7}$$

By using (2.2.2) and (2.2.5) into (2.2.7) we have:

$$2\ln\left(\frac{2(0.999)(1+D)-1}{D}\right) = \ln\frac{1}{0.001}\left(\frac{0.998}{D} + 0.999\right) \tag{2.2.8}$$

By solving (2.2.8):

$$(0.998 + 1.998D)^2 = D(998 + 999D) \tag{2.2.9}$$

By solving (2.2.9):

$$D = \frac{-994.011992 \pm \sqrt{988059.8402 + 3964.127776}}{1990.015992} \tag{2.2.10}$$

Since D has to be positive, (2.2.10) gives $C = 0.001$ and from (2.2.2) and (2.2.5), $C = 1$ and $\beta = 14.120$ respectively.

2.3 MF of the TC of the Matrix $\hat{b}_{kl}$

The MF for the TC is given by:

$$\theta_{bkl} = \begin{cases} 1.000; b_{kl} \prec b_{kl}^{b} \\ 0.9999; b_{kl} = b_{kl}^{b} \\ \dfrac{B}{1 + De^{\beta\left(\frac{b_{kl} - b_{kl}^{b}}{b_{kl}^{c} - b_{kl}^{b}}\right)}}; b_{kl}^{b} \prec b_{kl} \prec b_{kl}^{c} \\ 0.001; b_{kl} = b_{kl}^{c} \\ 0.000; b_{kl} \succ b_{kl}^{c} \end{cases} \tag{2.3.1}$$

Where θ_{bkl} is the degree of adhesion of TC b_{kl}. b_{kl}^{b} and b_{kl}^{c} are individually very low and very high for b_{kl} TCs.

2.4 Fuzzy TC of the Matrix b_{kl}^{*}.

The MF for b_{kl}^{*} is given by

$$\theta_{bkl} = \frac{B}{1 + De^{\beta\left(\frac{b_{kl} - b_{kl}^{b}}{b_{kl}^{c} - b_{kl}^{b}}\right)}}$$

By rearranging exponential term, we have the following:

$$e^{\beta\left(\frac{b_{kl} - b_{kl}^{b}}{b_{kl}^{c} - b_{kl}^{b}}\right)} = \frac{1}{D}\left(\frac{C}{\theta_{bkl}} - 1\right)$$

By taking log on the both sides we have:

$$\beta\left(\frac{b_{kl} - b_{kl}^{b}}{b_{kl}^{c} - b_{kl}^{b}}\right) = \ln\frac{1}{D}\left(\frac{C}{\theta_{bkl}} - 1\right)$$

Hence we have:

$$b_{kl} = b_{kl}^{b} + \left(\frac{b_{kl}^{c} - b_{kl}^{b}}{\beta}\right)\ln\frac{1}{D}\left(\frac{C}{\theta_{bkl}} - 1\right) \tag{2.4.1}$$

Since b_{kl} is the fuzzy TC in (2.4.1), It is denoted by b_{kl}^{*}. Therefore

$$b_{kl}^{*}\Big|_{\theta = \theta_{bkl}} = b_{kl}^{b} + \left(\frac{b_{kl}^{c} - b_{kl}^{b}}{\beta}\right)\ln\frac{1}{D}\left(\frac{C}{\theta_{bkl}} - 1\right) \tag{2.4.2}$$

3. A New Mathematical Model for FLP Problem

Let

$$Max\sum_{l=1}^{8} d_l y_l$$

Subject to:

$$\sum_{k=l}^{29} b_{kl}^{*}\Big|_{\theta=\theta_{bkl}} ; y_l \le c_k \tag{3.1}$$

where $y_l \ge 0; l = 1,2,,4,5,6,7,8$

and (3.1) d_l is a numerical target, $\boldsymbol{b_{kl}^{*}}\,|_{\theta=\theta_{bkl}}$ is TCs, y_l is the decision change and c_k is the default hardware change. (2.4.2) and (3.1) combine to form FLP and (3.2).
Also,

$$Max \sum_{l=1}^{8} d_l y_l$$

Subject to:

$$\sum_{k=1}^{29} b_{kl}^{b} + \left(\frac{b_{kl}^{c} - b_{kl}^{b}}{\beta}\right) \ln \frac{1}{D}\left(\frac{C}{\theta_{bkl}} - 1\right); y_l \le c_k \tag{3.2}$$

where $y_l \ge 0; l = 1,2,,4,5,6,7,8$; $0 \prec \theta_{dl}; \theta_{ck} \prec 1$; $0 \prec \beta \prec \infty$

4. Account on Fuzzy MPS Problem

There are 8 products that can be made by mixing 8 different components and using 9 different configurations. There are also 10 restrictions of the marketing department such as MPS, the requirements of the main product line, as well as the minimum and maximum scope of demand for each product. All the requirements in these circumstances are unclear. It is important to use some DOS to get the maximum benefit from using FLP integration.

4.1 Calculation of w*

Using the LP method, we will be able to address the above-mentioned FLP types as well as the solution of the nonlinear size for the constraints and objective functions that can be achieved. The results obtained in Tables 4.1 and 4.2 are summarized. From Table 4.1 we can see that the values increase the performance. Some previous studies compared this idea as appropriate to represent DOS to describe OF as PF [23-24]. His counsel is becoming more real. PF has a value of 319939 to 0.999. We describe this as 99.9% DOS. As a result, a w * of 207963 has 0.1% DOS. The possible solution is at θ = 0.5 (i.e. 50% DOS) with a value of w * as 247000.

Table 4.1: *OS with S-curve MF for* $\theta = 14.120$.

DOS (θ)	Optimum Values (w*)
0.001000	207963
0.022502	216398
0.121470	224527
0.142474	225177
0.224478	225592
0.283608	230332
0.348042	232317
0.414188	234535
0.456242	245439
0.510467	247826
0.524670	268147
0.542077	273526

0.558422	288537
0.778327	291170
0.783527	292077
0.838137	303324
0.859673	305543
0.914147	307862
0.925365	314989
0.935917	319187
0.999000	319939

Table 4.2: *Distribution of w* against θ and β*

w*	DOV (β)				
DOS (θ)	17	13	9	5	1
0.001	195324	195453	196602	196801	197807
0.250	240136	244800	274791	281523	315592
0.500	267608	287858	289857	295578	316135
0.750	266755	292940	306754	312935	318880
0.999	316185	317208	318635	318733	318931

4.2 Objective Values of Various β

Table 4.1 shows the variability of OV w * compared to DOS θ for the value of DOV $\beta = 14.120$. It will be useful for the decision-maker to see such differences for a number of principles.
Membership value in the analysis above represents DOS and w * is PF. We can conclude that as the DOV increases, the value of the individual increases. This event actually happens with real life problems in an unpredictable environment.
The ideal solution in a nonlinear environment is $\theta = 0.5$. Thus, the results for 50% of the DOS ($\theta = 0.5$) for $3 \le \beta \le 19$ and the corresponding values for w * are shown in Table 4.2. We can see in Table 4.2 that for $\theta = 0.5$ and for that increase, w * decreases. We can conclude that when DOV and TC conversion increase, w * decreases for a single DOS. The data in Table 4.2 are the result of IFLP analysis for (3.1). This information is very useful for the decision maker to make a definite decision about its implementation after the dissertation [25-27].

4.3 Distribution of w* against θ and β

The relationship between w *, θ is provided in Table 4.2. This table is very useful for the decision maker to find out the value and any benefits offered in DOS θ. From Table 4.2 we can see that OV does not depend on DOV and DOS. It cannot be concluded that for the higher value of DOS, the value of value will be higher. This is not true. But at 99.9% DOS, the profit margin will be the highest even at the highest DOV costs. From the diagonal values in Table 4.2, we can conclude that the VO increases at a lower value ($0.001 \le \theta \le 0.250$). Then the w * value is reduced to $0.500 \le \theta \le 0.750$. Finally, the value of w * increases by $0.750 \le \theta \le 1$. These results indicate that the correct resolution (DOS) does not guarantee high value (OV). This means that a person will be satisfied with some DOS when it comes to decision making and the environment.

5. Conclusion and Future Work

The industrial application of FLP interaction is analyzed by modified S-curve MF using real-time data collected from chocolate manufacturers. The problem of non-compliant MPS has been described. Eight cases were identified that could be based on non-FP in the FP system. The required size of each is listed. Value and quality were calculated using the FLP method. Because there are so many decisions to make, the tools to define the solution and the high level of profitability and high DOS are outlined. It should be borne in mind that higher profits will not necessarily lead to higher DOS. FS self-assembly for MPS problems can be developed to find satisfactory solutions. The decision maker, researcher and practitioner can apply their knowledge and experience to get the best results.

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Robust regression algorithms with kernel functions in Support Vector Regression Models

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Abstract

In machine learning, support vector machines (SVM) are supervised learning models with associated learning algorithms that analyze data for classification and regression analysis. SVM is one of the most robust prediction method based on statistical learning frameworks. Regression is a statistical method that attempts to determine the strength and character of the relationship between dependent and independent variables. This paper explores the idea of support vector Regression. The most commonly used classical procedure is Least Squares, which is less efficient and very sensitive when the data contains outliers. To overcome this limitations, alternative robust regression procedures exist such as LMS regression, S-estimator, MM-estimator and Support Vector Regression (SVR). In this study, the comparisons have made for the classical regression procedure and the robust regression procedures. In that, various measures of errors are much efficient when we work with robust regression procedures. In this paper, an attempt has been made to review the existing theory and methods of SVR.

Keywords: Linear regression, Robust regression, kernels, Support Vector Regression

1. Introduction

Support vector regression is a feature of support vector machines. It's worth mentioning that the support vector machine (SVM) is a concept that may be utilized to analyze both regression and classification data [4] and [12]. Support vector classification is the name given to the support vector machine when it is used for classification, while support vector regression is the name given to it when it is used for regression [5]. SVM is a new machine learning technique based on the statistical learning theory proposed by Vapnik and Wolfe dual programming theory. SVM has a robust mathematical theory base, well-generalized ability, and global optimum, as compared to other learning algorithms, and is widely used in pattern recognition and functional regression as a result are seen in [9].

Support vector machine (SVM) has been first introduced by Vapnik. There are two main categories for support vector machines: support vector classification (SVC) and support vector regression (SVR). SVM is a learning system using a high dimensional feature space. It yields prediction functions that are expanded on a subset of support vectors. SVM can generalize complicated gray level structures with only a very few support vectors and thus provides a new mechanism for image compression. A version of a SVM for regression has been proposed in 1997 by Vapnik, Steven Golowich, and Alex Smola [11]. This method is called support vector regression (SVR).

The support vector machine is a more advanced version of the support vector classifier, resulting from the use of kernels to enlarge the feature space in a specified fashion [1] and [6].

The feature space is expanded in this technique to allow a non-linear boundary between the classes. The kernel technique is a computationally efficient way to put such a notion into action [2]. Support vector regression allows us to specify how much error in our model is acceptable, and it will identify an appropriate line (or hyperplane in higher dimensions) to fit the data. See, [3] and [10].

The manuscript of this paper is laid out as follows. The concept of regression procedures is defined in Section 2. This session also covers the LS, LMS, S, MM, and SVR methodologies utilized in this article for examining these ideas. Support vector regression types and various kernel functions are also covered. Section 3 summarizes results of the numerical study of comparative analysis under various kernels along with regression procedures. Section 4 ends with a summary and conclusion.

2. REGRESSION PROCEDURES

The conventional regression procedure, namely, Least Squares Method (LS), the robust procedures, Least Median Squares Method (LMS), S-Estimator (S), and MM-Estimator (MM), and Support Vector Regression (SVR) are briefly discussed in this section.

2.1. Least Squares Method (LS)

A fundamental statistical method for determining a regression line or the best-fit line for a given pattern is the least-squares approach. An equation with specified parameters are described in this procedure. This method is considered a typical strategy in regression analysis for approximating sets of equations with more equations than unknowns. The least squares method determines the best results by minimizing the sum of squares of deviations or errors in each equation's result. Least-square method is the curve that best fits a set of observations with a minimum sum of squared residuals or errors. The exercise of minimizing these residuals would be the trial and error fitting of a line "through" the Cartesian coordinates representing these values. One way to proceed with the Least Squares Method is to solve using matrix multiplication.
he least squares method can more formally be described. Given a dataset of points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) and derive the matrices:

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ . & . \\ . & . \\ 1 & x_n \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}, A = \begin{bmatrix} b \\ m \end{bmatrix}, E \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ e_n \end{bmatrix} \tag{1}$$

Then set up the matrix equation,

$$Y = XA + E \tag{2}$$

where matrix Y contains the y_n values, matrix X contains a row of 1™s and along with the x_n values, matrix A consists of the Y-intercept and slope, and matrix E is the errors. Then solve for A,

$$A = (X^T X)^{-1} X^T Y \tag{3}$$

This is the matrix equation ultimately used for the least squares method of solving a linear system.

2.2. Least Median Squares Method (LMS)

Rousseeuw was the first to introduce the method of Least Median Squares in 1984 [7]. The least median of squares approach estimates the parameters by solving the nonlinear minimization

problem. To put it another way, the estimator must produce the minimum number for the median of squared residuals computed throughout the entire data set. It turns out that this strategy is extremely resistant to false matches and outliers caused by poor localization, implying that it is unaffected by outliers or other violations of the typical normal model's assumptions. The LMS regression was developed to optimise the median of the squares of residuals. The LMS estimate can be obtained as the solution of the following optimization problem.
Let ${x_i}^T = (x_{i1}, x_{i2}, , x_{ip})$, i=1,2,¦.,n and

$$y = (y_1, y_2, , y_n)^T \tag{4}$$

be given real vectors.
It is assumed that $\frac{n}{p} \geq p$ and the (nxp) matrix, X= $[x_{ij}]$ is offull rank to avoid degenerate cases

$$\theta = (\theta_1, \theta_2, , \theta_p)^T \tag{5}$$

be a vector of regression parameters. The optimization problem that arises out of the LMS method is to estimate θ^* providing

$$^{min}{}_{\theta} med(y_i - {x_i}^T\theta)^2 \tag{6}$$

LMS is designed to have a high breakdown point, which is commonly defined as the minimum percentage of "contaminated" data required to change an estimate by a given amount. The LMS breakdown point is 50%, while the comparable LS breakdown point is zero.

2.3. S-Estimator (S)

Rousseeuw and Yohai (1984) proposed the S-estimator, which minimises a scale estimator [8]. S-estimators combine the flexibility and asymptotic features of M-estimators to provide a simple high-breakdown regression estimator. Because they are based on scale estimators, the name S-estimators was chosen. The S-estimator minimizes an estimator of scale, which is given by

$$\hat{\beta_n} = argmin\hat{\alpha_n}(\beta) \tag{7}$$

The estimator of scale may be defined by a function ρ, For any sample $r_1, r_2, , r_n$ of real numbers, we define the scale estimate $s(r_1, r_2, , r_n)$ as the solution of

$$\frac{1}{n}\sum_{i=1}^{n} \rho \frac{r_i}{s} = k \tag{8}$$

where k is the expectation value of ρ for a standard normal distribution. Let $(x_1, y_1), , (x_n, y_n)$ be a sample of regression data with p-dimensional x_i. For each vector θ, obtain the residuals (r$_1(\theta)$,r$_2(\theta)$,¦,r$_n(\theta)$) by solving the above equation and hence the estimator of scale may be defined by a function . Further the function ρ must satisfy the conditions, such as symmetric, continuously differentiable and ρ(0)=0 and also there exists c > 0 such that ρ is strictly increasing on [c, ∞]..
Thus, the S-estimator $\hat{\theta}$ is defined by

$$\hat{\theta} = {}^{min}{}_{\theta} s(r_1(\theta), r_2(\theta), ..r_n(\theta) \tag{9}$$

and the final scale estimator $\hat{\sigma}$ is then

$$\hat{\sigma} = s(r_1(\hat{\theta}), ..., r_n(\hat{\theta})) \tag{10}$$

In least Squares, least absolute deviation estimation, and even generalized M-estimators, outlying observations sometimes strongly influence the estimation result, making an important and interesting relationship existing in the majority of observations. The S-estimators are a class of estimators that overcome this difficulty by smoothly down-weighting outliers in fitting regression functions to data.

2.4. MM-Estimator (MM)

Such estimators are interesting as they combine high efficiency and high breakdown point in a simple and intuitive way. Typically one starts first with a highly-robust regression estimator, typically an S-estimator. Then one can use the scale based upon this preliminary fit along with a better-tuned ρ function to obtain a more efficient M-estimator of the regression parameter. An MM-estimator of α then defined as any solution of an M-type equation where

$$\sum_{i=1}^{n} {\rho_1}'[\frac{y_i - \sum_{j=0}^{k} x_{ij}\beta_j}{S_{MM}}]x_{ij} = 0 \tag{11}$$

Such estimators are interesting as they combine high efficiency and high breakdown point in a simple and intuitive way. Typically one starts first with a highly-robust regression estimator, typically an S-estimator. Then one can use the scale based upon this preliminary fit along with a better-tuned ρ function to obtain a more efficient M-estimator of the regression parameter. An MM-estimator of α then defined as any solution of an M-type equation where

$$\psi_M M(y, x : \alpha) = u_{MM}(X^T \sum_{s}^{-1}(y - x\alpha)) \tag{12}$$

2.5. Support Vector Regression (SVR)

Support Vector Regression is a method for estimating a function that maps from an input item to a real integer. SVR has the same qualities as the classifying SVM, as well as the margin maximization and kernel technique for non-linear mapping.
A dataset for regression is represented as follows,

$$D = (x_1, y_1), (x_2, y_2), ..., (x_m, y_m) \tag{13}$$

where x_i is a n-dimensional vector, y is the real number for each x_i. The SVR function $F(x_i)$ makes a mapping from an input vector x_i to the target y_i and takes the form.

$$F(x) = w.x - b \tag{14}$$

where w is the weight vector and b is the bias. The goal is to estimate the parameters (w and b) of the function that give the best fit of the data. An SVR function F(x) approximates all pairs (x_i, y_i) while maintaining the differences between estimated values and real values under precision.
Unlike LS, SVR's goal is to minimise the coefficients, specifically the L_2 norm of the coefficient vector rather than the squared error. In SVR, we can adjust epsilon to achieve the model's desired accuracy. SVR is an advanced regression technique that makes use of the concept of hyperplane to perform well with large datasets. Simple regression strives to lower error rates, whereas SVR aims to fit the error with a specific threshold.

2.5.1 Kernel function in Support Vector Regression

SVM can be used as a classification machine, as a regression machine, or for novelty detection. Kernel functions, a group of mathematical functions play a significate role for getting better accuracy in SVM. The function of a kernel is to require data as input and transform it into the desired form. Different kernel functions are used by different SVM algorithms. There are several types of these functions, including linear, nonlinear, polynomial, radial basis function (RBF), and sigmoid. The most preferred kind of kernel function is RBF. Because it's localized and has a finite response along the complete x-axis. The kernel functions return the scalar product between two points in an exceedingly suitable feature space. The most widely used kernel functions are briefly furnished as follows.
Linear kernel, which is the most basic sort of kernel and is usually one dimensional. When there are a lot of features, it proves to be the best function. For text-classification tasks, the linear kernel

Table 1: ***Computed error measures under conventional, robust and support vector regression***

Errors	LS	LMS	S	MM	SVR							
					Linear		Polynomial		Radial		Sigmoid	
					γ	ϵ	γ	ϵ	γ	ϵ	γ	ϵ
MDAE	0.47 (0.23)	0.26 (0.22)	0.31 (0.24)	0.47 (0.23)	0.57 (0.24)	0.44 (0.23)	0.40 (0.31)	0.38 (0.29)	0.30 (0.23)	0.23 (0.22)	0.48 (0.49)	0.48 (0.43)
MSE	0.30 (0.09)	2.20 (0.10)	1.60 (0.09)	0.30 (0.09)	0.30 (0.09)	0.31 (0.09)	0.20 (0.12)	0.25 (0.12)	0.10 (0.09)	0.12 (0.09)	1.96 (0.44)	2.10 (0.68)
RMSE	0.55 (0.30)	1.48 (0.32)	1.26 (0.31)	0.55 (0.30)	0.55 (0.30)	0.56 (0.30)	0.50 (0.35)	0.50 (0.35)	0.40 (0.30)	0.35 (0.30)	1.40 (0.66)	1.45 (0.83)
(.)	without outliers											

is usually favoured because most of these problems can be linearly split. Linear kernel functions, denoted by $u^{'}$ v, are faster than other functions.
The polynomial kernel, which is a more generalised form of the linear kernel, is the second kernel. It is not as preferred as other kernel functions as it is less efficient and accurate. Polynomial Kernel function is denoted by $(\gamma u^{'}v + coef0)^{degree}$.
The radial basis function kernel, which is one of the most popular and widely utilized in SVM. It's typically used with non-linear data. When there is no prior knowledge of data, it aids in proper separation. The gamma value ranges from 0 to 1. Radial basis Kernel function is denoted by $e^{((-\gamma|u-v^2))}$.
The sigmoid kernel is mostly preferred for neural networks. This kernel function is similar to a two-layer perceptron model of the neural network, which works as an activation function for neurons. Sigmoid Kernel function is denoted by $tanh(\gamma u^{'}v + coef0)$.
The nu- SVR and eps-SVR has been taken for comparing the performance of various kernel based SVR. In nu-SVR, the parameter γ is used to determine the proportion of the number of support vectors desire to keep in solution with respect to the total number of samples in the dataset. Also the parameter ϵ is introduced into the optimization problem formulation and it is estimated automatically. But in eps-SVR, there is no control on how many data vectors from the dataset become support vectors, it could be a few, it could be many. Nonetheless, the total control of how much error will allow the model to have, and anything beyond the specified ϵ will be penalized in proportion to C, which is the regularization parameter.

3. NUMERICAL STUDY

A real data is used to demonstrate the performance of various approaches by computing various measures of errors values. The dataset used in the numerical analysis is StarsCYG, which is available in a package namely, robustbase in R. The data describes the Hertzsprung-Russell diagram of the star cluster CYG OB1, which contains 47 stars in the direction of Cygnus, the predictor variable, the logarithm of the effective temperature at the star's surface (log.Te), and the response variable, the logarithm of the star's light intensity (log.light).
The experimental study has been carried out to study the performance of various procedures, such as Least Squares (LS), Least Median Squares (LMS), S-Estimator (S), MM-estimator (MM), and Support Vector Regression (SVR) with various kernels by computing error measures for the dataset under with/without outliers and thus obtained results are summarized in the table 1.
(.) without outliers

The result reveals that, robust procedures provide better results when compared with the conventional least square approach. Further, it is observed that $\gamma-$ and $\epsilon-$ type radial kernel based SVR outperforms over other kernels.

4. CONCLUSION

Regression analysis is one of the supervised learning techniques in the context of statistical learning. This paper explores the conventional, robust and support vector based regression procedures. In the context of support vector regression, the study has been carried out under the most widely used kernels. The efficiency of these algorithms have been studied under a real study, with and without outliers. The results indicate that the robust regression procedures more efficient than the traditional regression procedure under with and without outliers. Further, the study reveals that SVR delivers much superior outcomes when compared with the others. In the context of kernels, the γ and ϵ based radial kernel has the maximum efficiency when compared to other kernels, regardless of whether or not the data contains outliers. The study can be extended by incorporating the robust kernel in support vector regression for better accuracy.

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The Seasonal Effect of Working Conditions of an Ice-cream Plant

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Abstract

An ice-cream plant's workings are analyzed in the summer and winter seasons of the paper. The ice-cream unit along with the other three units i.e., flavoring, freezing and combined flavouring and freezing units are always operational in summers, due to the high demand, while in winters the combined flavoring and freezing unit is kept in cold standby as a backup in case there is a demand for ice-cream. In this work, the semi-Markov process and the regenerative point technique have been used to analyze the system. Numerical analysis has been conducted using MATLAB. A variety of measures have been developed to evaluate the effectiveness of a system. The Code Blocks have been used in interpreting the graph in the specific case presented. All evaluation is based on the milk production data collected by the plant. Improvements to the system performance will lead to increased profits. Similar techniques can be applied to other systems.

Keywords: Seasonal functioning, semi-Markov process, Regenerative point technique, profit.

1. Introduction

The primary goal of any industry is to upgrade production through technological interventions and so, is the goal of the dairy industry i.e to improve their production operations to attain competitiveness. Physical models for predicting ice cream's thermal properties were developed by [1]. Analysis of reliability modeling for 2-out-of-3 redundant system was done by [2]. Profit analysis of a two unit standby oil delivering system where priority is given to partially failed unit over the completely failed unit for repair was analyzed by [3]. Reliability models for the fertilzer industry were pioneered by [4], [5]. Availability optimization of ice cream making unit of milk plant was discussed by [6]. Optimized scheduling, production planning and RAM study of an ice-cream plant was given by [7], [12] respectively. Reliability and profit where stand by units functions to accommodate the required demand was analyzed by [8] for system evaluation. Availability analysis of a skim milk powder and profit analysis of a butter-oil production in dairy industry was discussed by [9] respectively. Profit analysis of a system where operation is affected by temperature was discussed by [10]. Description of the four subsystems of the butter-oil production process- the melting vats, the boilers, the clarifier and the settling tanks was given by [11]. The probability of a three-unit induced draft fan system with one standby unit in a working condition was given by [13]. Modeling of two-Unit cold standby system was discussed by [14]. On the basis of progressively censored first-failure data, a problem of estimating parameters for an exponential distribution class and hazard rate functions is studied by [15].

But, none of them have discussed the working of an ice-cream plant. So, in this paper functioning of an ice-cream plant is discussed. The production of functional ice-cream plant consists of one main unit and three units grouped in parallel but in series with the unit 1. The main unit (unit 1) consists of heating, emulsifying, pasteurization, homogenization and ageing. Unit 2 is flavoring unit, unit 3 is freezing unit and unit 4 is combined flavoring and freezing. In summers, due to

high demand the whole system is operative whereas in winters, the system goes to cold standby state and undergoes maintenance. It only operates when the demand occurs. In that case the units 2, 3 along with the unit 1 operates and the unit 4 is in cold standby state and operates on the failure of either of the units 2, 3 or on the failure of both. The system goes to a failed state a) on the failure of the unit 1 or b) on the failure of unit 2 with unit 4 or c) on the failure of unit 3 with unit 4.
Semi-Markov process and regenerative point technique is used to obtain measures of system effectiveness in steady state that include MTSF, availability of the system, busy period of repairman for repair and maintenance, expected number of repair and maintenances, profit of the system.

2. Methods

The stages of methodlogy carried out is given below:

1. In the beginning, industry data on failure rates and maintenance was collected over a five-year period.
2. A comprehensive understanding of how the unit operates is the second step. Through that, reliability models are generated.
3. MATLAB is used to obtain reliability measures using semi-Markov processes and regenerative point techniques that include:
   - Transition probabilities and mean sojourn time in steady state
   - MTSF of the system.
   - Long term availability for the system.
   - Bus period analysis of the repairman.
   - Expected number of repairs.
   - Additionally, the system's profit potential is analyzed graphically.
4. In the following step, graphical analysis is performed by using excel and code blocks on a particular example of exponential distribution.
5. Furthermore, reliability can be improved by identifying key machines and faults, making better decisions, and formulating better strategies.

3. Annotations and Symbols

Table 1

Notations of the model

Notations	Descriptions
λ	Failure rate of unit 1.
λ_1	Failure rate of unit 2.
λ_2	Failure rate of unit 3.
λ_3	Failure rate of unit 4.
λ_4	Maintenance rate of the unit.
α, β	Rate of going to winters and summers respectively.
θ	Repair rate of unit 1.
θ_1	Repair rate of unit 2.
θ_2	Repair rate of unit 3.
θ_3	Repair rate of unit 4.

Notations of the model	
Notations	Descriptions
θ_4	Repair rate after maintenance.
$G(t), g(t)$	c.d.f and p.d.f of repair time of unit 1.
$G_1(t), g_1(t)$	c.d.f and p.d.f of the repair time of unit 2.
$G_2(t), g_2(t)$	c.d.f and p.d.f of the repair time of unit 3.
$G_3(t), g_3(t)$	c.d.f and p.d.f of the repair time of unit .
$G_4(t), g_4(t)$	c.d.f and p.d.f of the maintenance time.
$\star$	Symbol for convolution.
Ⓢ	Laplace Stieltjes Treansform.
$M_l(t)$	Probability that the system is in up state at time t.
$W_l(t)$	Probability that the system is busy for repair at time t.
$A_l(t)$	Availability of the system at time t.
$B_l(t)$	Busy period for repair/maintenance the system at time t.
$V_l(t)$	Expected number of repairs/maintenances.

Descrpition	
States	Description and symbols of the model
S_j	These are the states when the system is operative, j=0,15,17,18.
S_4, S_5, S_6	These are the states when the system works in a reduced capacity.
S_2	This is the cold stanby state.
S_i	These are the failed states, i=3,7,8,9,10,11,12,13,14,16,19,20,21,22.
S, W	Symbols for summer and winter respectively.
o(u1,2,3,4)	The units are in operating state.
o(u1,4)	Units 1 and 4 are in operating state.
o(u1,2,3)	Units 1,2,3 are in operating state.
csu4	Unit 4 is in cold standby state.
fric	Ice-cream unit is under repair due to failure in unit 1.
umic	Ice-cream unit is under repair.
fRu4	Unit 4 is under continuous repair.
fru2,fru3,fru4	Units 2,3,4 are under repair respectively.
fwru2,fwru3	Units 2,3 are waiting for repair.

4. Model Descriptions and Assumptions

Reliability analysis is done of the working of an ice-crream plant w.r.t. seasons. At an initial stage the system is operative, in summers since the demand is high all the units are operative whereas in winters the system is in cold standby state and only operates when there is some demand.The system consists of four units, unit 1 is the main unit of the system where the process starts after that it moves to unit 2 i.e., the flavouring unit and after that unit 3 i.e., the freezing unit. Unit 4 is combined freezing and flavouring unit. In summers, the system operates at reduced capacity when any of the units 2,3,4 fails. The system goes to failed state on the failure of unit 1; unit 2, 4 and unit 3, 4.
Following are the assumptions of the system:

- The system is operating initially.
- A distribution of exponential failure times is assumed for all failure times.
- Unit 1 and unit 4 receive the most priority for repair.

- States always restores the system to its original functionality after every repair.

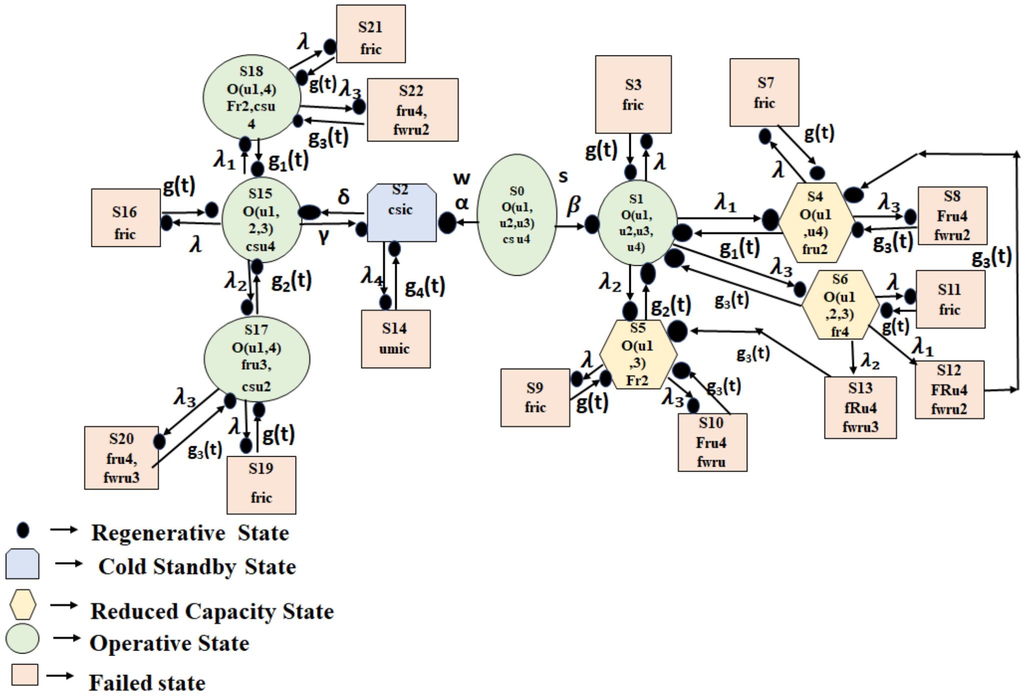

Figure 1: *State Transition Diagram*

5. System Effectiveness Measures

In this model the states $S_0, S_1, S_{15}, S_{17}, s_{18}$ are the operating states. States S_4, S_5, S_6 are the states operating in a reduced capacity. S_2 is a cold standby state, rest are the failed states.

5.1. Mean time to system failure (MTSF)

System effectiveness measures have been achieved using semi-Markov processes and regenerative point techniques. A mean time to failure (MTSF) is determined for the system when considering the failed state as an absorbent state. In terms of probabilistic arguments, we can get the following recursive relation for $\phi_l(t)$:

$$\phi_l(t) = \sum_n Q_{ln}(t) Ⓢ \phi_n(t) + \sum_e Q_{le}(t)$$

where S_n indicates an un-failed regenerative state into which the given regenerative state $S_l$can transit and Se indicates a failed state into which the state S_l can transit directly. By applying the Laplace-Stieltjes Transform (L.S.T.) to the relationships given by the above equation and solving them for $\phi_0^{**}(t)$, we are able to calculate:

$$\phi_0^{**}(t) = \frac{N(s)}{D(s)}$$

The mean time to system failure (MTSF), when the system started at the beginning of state S_0 is: MTSF=$\int_0^\infty R(t)dt = \lim_{s\to 0} R^*(s)$ Using L′ Hospital rule and putting the value of $\phi_0^{**}(s)$ we get

$$\text{MTSF} = T_0 = \lim_{s\to 0} = \frac{1-\phi_0^{**}(s)}{s} = \frac{N}{D}$$

where

N=$\mu_0(p_{13} + p_{14}p_{47} + p_{14}p_{48} + p_{15}p_{59} + p_{15}p_{5,10} + p_{16}p_{6,11} + p_{16}p_{6,12} + p_{16}p_{6,13} - p_{13}p_{15,2}p_{2,15} - p_{13}p_{15,17}p_{17,15} - p_{13}p_{15,18}p_{18,15} - p_{14}p_{47}p_{15,2}p_{2,15} - p_{14}p_{48}p_{15,2}p_{2,15} - p_{15}p_{59}p_{15,2}p_{2,15} - p_{15}p_{15,2}p_{2,15}p_{5,10} - p_{16}p_{15,2}p_{2,15}p_{6,11} - p_{16}p_{15,2}p_{2,15}p_{6,12} - p_{16}p_{15,2}p_{2,15}p_{6,13} - p_{14}p_{47}p_{15,17}p_{17,15} -$

$p_{14}p_{48}p_{15,17}p_{17,15} - p_{15}p_{59}p_{15,17}p_{17,15} - p_{14}p_{47}p_{15,18}p_{18,15} - p_{14}p_{48}p_{15,18}p_{18,15} - p_{15}p_{59}p_{15,18}p_{18,15} - p_{15}p_{5,10}p_{15,17}p_{17,15} - p_{15}p_{5,10}p_{15,18}p_{18,15} - p_{16}p_{6,11}p_{15,17}p_{17,15} - p_{16}p_{6,12}p_{15,17}p_{17,15} - p_{16}p_{6,13}p_{15,17}p_{17,15} - p_{16}p_{6,11}p_{15,18}p_{18,15} - p_{16}p_{6,12}p_{15,18}p_{18,15} - p_{16}p_{6,13}p_{15,18}p_{18,15}) + \mu_1(p_{01} - p_{01}p_{15,2}p_{2,15} - p_{01}p_{15,17}p_{17,15} - p_{01}p_{15,18}p_{18,15}) + \mu_2(p_{02} - p_{02}p_{14}p_{41} - p_{02}p_{15}p_{51} - p_{02}p_{16}p_{61} - p_{02}p_{15,17}p_{17,15} - p_{02}p_{15,18}p_{18,15} + p_{02}p_{14}p_{41}p_{15,17}p_{17,15} + p_{02}p_{15}p_{51}p_{15,17}p_{17,15} + p_{02}p_{16}p_{61}p_{15,17}p_{17,15} + p_{02}p_{14}p_{41}p_{15,18}p_{18,15} + p_{02}p_{15}p_{51}p_{15,18}p_{18,15} + p_{02}p_{16}p_{61}p_{15,18}p_{18,15}) + \mu_4(p_{01}p_{14} - p_{01}p_{14}p_{15,2}p_{2,15} - p_{01}p_{14}p_{15,17}p_{17,15} - p_{01}p_{14}p_{15,18}p_{18,15}) + \mu_5(p_{01}p_{15} - p_{01}p_{15}p_{15,2}p_{2,15} - p_{01}p_{15}p_{15,17}p_{17,15} - p_{01}p_{15}p_{15,18}p_{18,15}) + \mu_6(p_{01}p_{16} - p_{01}p_{16}p_{15,2}p_{2,15} - p_{01}p_{16}p_{15,17}p_{17,15} - p_{01}p_{16}p_{15,18}p_{18,15}) + \mu_{15}(p_{02}p_{2,15} - p_{02}p_{14}p_{41}p_{2,15} - p_{02}p_{15}p_{51}p_{2,15} - p_{02}p_{16}p_{61}p_{2,15}) + \mu_{17}(p_{02}p_{2,15}p_{15,17} - p_{02}p_{14}p_{41}p_{2,15}p_{15,17} - p_{02}p_{15}p_{51}p_{2,15}p_{15,17} - p_{02}p_{16}p_{61}p_{2,15}p_{15,17}) + \mu_{18}(p_{02}p_{2,15}p_{15,18} - p_{02}p_{14}p_{41}p_{2,15}p_{15,18} - p_{02}p_{15}p_{51}p_{2,15}p_{15,18} - p_{02}p_{16}p_{61}p_{2,15}p_{15,18})$

D=$p_{14}p_{41}p_{15,2}p_{2,15} - p_{16}p_{51} - p_{16}p_{61} - p_{15,2}p_{2,15} - p_{15,17}p_{17,15} - p_{15,18}p_{18,15} - p_{14}p_{41} + p_{16}p_{51}p_{15,2}p_{2,15} + p_{16}p_{61}p_{15,2}p_{2,15} + p_{14}p_{41}p_{15,17}p_{17,15} + p_{16}p_{51}p_{15,17}p_{17,15} + p_{16}p_{61}p_{15,17}p_{17,15} + p_{14}p_{41}p_{15,18}p_{18,15} + p_{16}p_{51}p_{15,18}p_{18,15} + p_{16}p_{61}p_{15,18}p_{18,15} + 1$

6. Cost Measures

6.1. Long Term Availability of the System in Summers at Full Capacity

Using the theory of regeneration process, using $A_l(t)$ where l = 0, 1 as the probability that the system will be in upstate at instant t given that it is in state i at t=0, we can find that this value will satisfy the following recursive relations:

$$A_l(t) = M_l(t) + \sum_n q_{ln}(t) \star A_n(t)$$

In this case, Sn can represent any state to which S_l can transit. $M_l(t)$ is the probability that the system will be accessible at time t before visiting any other state.

$M_0 = e^{-(\alpha+\beta)t}, M_1 = e^{-(\lambda+\lambda_1+\lambda_2)t}$

Taking Laplace transform of equations and solving for A_0 we obtain:

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)}$$

Steady state availability is given by:

$$A_0 = \lim_{s\to 0} sA_0^*(s) = \frac{N_1}{D_1}$$

where

$N_1 =$

$\mu_0 + \mu_1 p_{01} - \mu_0 p_{13} - \mu_0 p_{47} - \mu_0 p_{48} - \mu_0 p_{59} - \mu_0 p_{5,10} - \mu_0 p_{6,11} - \mu_1 p_{01} p_{47} - \mu_1 p_{01} p_{48} - \mu_0 p_{14} p_{41} + \mu_0 p_{13} p_{47} + \mu_0 p_{13} p_{48} - \mu_1 p_{01} p_{59} - \mu_0 p_{15} p_{51} + \mu_0 p_{13} p_{59} - \mu_0 p_{16} p_{61} + \mu_0 p_{47} p_{59} + \mu_0 p_{48} p_{59} - \mu_1 p_{01} p_{5,10} + \mu_0 p_{13} p_{5,10} + \mu_0 p_{47} p_{5,10} + \mu_0 p_{48} p_{5,10} - \mu_1 p_{01} p_{6,11} + \mu_0 p_{13} p_{6,11} + \mu_0 p_{47} p_{6,11} + \mu_0 p_{48} p_{6,11} + \mu_0 p_{59} p_{6,11} + \mu_0 p_{5,10} p_{6,11} + \mu_1 p_{01} p_{47} p_{59} + \mu_1 p_{01} p_{48} p_{59} + \mu_0 p_{15} p_{47} p_{51} + \mu_0 p_{14} p_{41} p_{59} + \mu_0 p_{15} p_{48} p_{51} - \mu_0 p_{13} p_{47} p_{59} - \mu_0 p_{13} p_{48} p_{59} - \mu_0 p_{16} p_{41} p_{64} + \mu_0 p_{16} p_{47} p_{61} + \mu_0 p_{16} p_{48} p_{61} - \mu_0 p_{16} p_{51} p_{65} + \mu_0 p_{16} p_{59} p_{61} + \mu_1 p_{01} p_{47} p_{5,10} + \mu_1 p_{01} p_{48} p_{5,10} + \mu_0 p_{14} p_{41} p_{5,10} - \mu_0 p_{13} p_{47} p_{5,10} - \mu_0 p_{13} p_{48} p_{5,10} + \mu_0 p_{16} p_{61} p_{5,10} + \mu_1 p_{01} p_{47} p_{6,11} + \mu_1 p_{01} p_{48} p_{6,11} + \mu_0 p_{14} p_{41} p_{6,11} - \mu_0 p_{13} p_{47} p_{6,11} - \mu_0 p_{13} p_{48} p_{6,11} + \mu_1 p_{01} p_{59} p_{6,11} + \mu_0 p_{15} p_{51} p_{6,11} - \mu_0 p_{13} p_{59} p_{6,11} - \mu_0 p_{47} p_{59} p_{6,11} - \mu_0 p_{48} p_{59} p_{6,11} + \mu_1 p_{01} p_{5,10} p_{6,11} - \mu_0 p_{13} p_{5,10} p_{6,11} - \mu_0 p_{47} p_{5,10} p_{6,11} - \mu_0 p_{48} p_{5,10} p_{6,11} + \mu_0 p_{16} p_{47} p_{51} p_{65} + \mu_0 p_{16} p_{41} p_{59} p_{64} + \mu_0 p_{16} p_{48} p_{51} p_{65} - \mu_0 p_{16} p_{47} p_{59} p_{61} - \mu_0 p_{16} p_{48} p_{59} p_{61} + \mu_0 p_{16} p_{41} p_{64} p_{5,10} - \mu_0 p_{16} p_{47} p_{61} p_{5,10} - \mu_0 p_{16} p_{48} p_{61} p_{5,10} - \mu_1 p_{01} p_{47} p_{59} p_{6,11} - \mu_1 p_{01} p_{48} p_{59} p_{6,11} - \mu_0 p_{15} p_{47} p_{51} p_{6,11} - \mu_0 p_{14} p_{41} p_{59} p_{6,11} - \mu_0 p_{15} p_{48} p_{51} p_{6,11} + \mu_0 p_{13} p_{47} p_{59} p_{6,11} + \mu_0 p_{13} p_{48} p_{59} p_{6,11} - \mu_1 p_{01} p_{47} p_{5,10} p_{6,11} - \mu_1 p_{01} p_{48} p_{5,10} p_{6,11} - \mu_0 p_{14} p_{41} p_{5,10} p_{6,11} + \mu_0 p_{13} p_{47} p_{5,10} p_{6,11} + \mu_0 p_{13} p_{48} p_{5,10} p_{6,11}$

$D_1 = (\mu_3 p_{13} + \mu_1)(p_{51} - p_{47}p_{51} - p_{48}p_{51} - p_{51}p_{6,11} + p_{47}p_{51}p_{6,11} + p_{48}p_{51}p_{6,11}) + (\mu_4 + \mu_7 p_{47} + \mu_8 p_{48})(p_{14} + p_{16}p_{64} - p_{14}p_{59} - p_{14}p_{5,10} - p_{14}p_{6,11} - p_{16}p_{59}p_{64} - p_{16}p_{64}p_{5,10} + p_{14}p_{59}p_{6,11} + p_{14}p_{5,10}p_{6,11}) + (\mu_5 + \mu_9 p_{59} + \mu_{10} p_{5,10})(p_{15} + p_{16}p65 - p_{15}p_{47} - p_{15}p_{48} - p_{15}p_{6,11} - p_{16}p_{47}p65 - p_{16}p_{48}p65 + p_{15}p_{47}p_{6,11} + p_{15}p_{48}p_{6,11}) + (k + \mu_{11} p_{6,11})(p_{16} - p_{16}p_{47} - p_{16}p_{48} - p_{16}p_{59} - p_{16}p_{5,10} + p_{16}p_{47}p_{59} + p_{16}p_{48}p_{59} + p_{16}p_{47}p_{5,10} + p_{16}p_{48}p_{5,10})$......(1)

6.2. Long Term Availability of the System in Summers at Half Capacity

Using the theory of regeneration process, using $A_l(t)$ where l = 4,5,6 as the probability that the system will be in upstate at instant t given that it is in state i at t=0, we can find that this value will satisfy the following recursive relations:

$$A_l(t) = M_l(t) + \sum_n q_{ln}(t) \star A_n(t)$$

In this case, Sn can represent any state to which S_l can transit. $M_l(t)$ is the probability that the system will be accessible at time t before visiting any other state.
$M_4 = e^{-(\lambda+\lambda_3)t}\bar{G_1}(t), M_5 = e^{-(\lambda+\lambda_3)t}\bar{G_2}(t), M_6 = e^{-(\lambda+\lambda_1+\lambda_2)t}\bar{G_3}(t)$
Taking Laplace transform of equations and solving for A_0 we obtain:

$$A_0^*(s) = \frac{N_2(s)}{D_1(s)}$$

Steady state availability is given by:

$$A_0 = \lim_{s\to 0} sA_0^*(s) = \frac{N_2}{D_1}$$

where
$N_2 = -p_{01}(\mu_5 p_{15} p_{47} - \mu_5 p_{15} - kp_{16} - \mu_4 p_{14} + \mu_5 p_{15} p_{48} + kp_{16} p_{47} + kp_{16} p_{48} + \mu_4 p_{14} p_{59} + kp_{16} p_{59} - \mu_4 p_{16} p_{64} - \mu_5 p_{16} p_{65} + \mu_4 p_{14} p_{5,10} + kp_{16} p_{5,10} + \mu_4 p_{14} p_{6,11} + \mu_5 p_{15} p_{6,11} - kp_{16} p_{47} p_{59} - kp_{16} p_{48} p_{59} + \mu_5 p_{16} p_{47} p_{65} + \mu_5 p_{16} p_{48} p_{65} + \mu_4 p_{16} p_{59} p_{64} - kp_{16} p_{47} p_{5,10} - kp_{16} p_{48} p_{5,10} + \mu_4 p_{16} p_{64} p_{5,10} - \mu_5 p_{15} p_{47} p_{6,11} - \mu_5 p_{15} p_{48} p_{6,11} - \mu_4 p_{14} p_{59} p_{6,11} - \mu_4 p_{14} p_{5,10} p_{6,11})$
D_1 is already defined in equation (1).

6.3. Busy Period Analysis for Repair in Summers

Using the theory of regeneration process, using $B_l(t)$ where l = 4,5,6,7,8,9,10,11 as the probability that the system is under repair at an instant t given that it is in state i at t=0, we can find that this value will satisfy the following recursive relations:

$$B_l(t) = W_l(t) + \sum_n q_{ln}(t) \star B_n(t)$$

In this case, Sn can represent any state to which S_l can transit. $W_l(t)$ is the probability that the system will be busy for repair at time t before visiting any other state.
$W_3 = W_7 = W_9 = W_{11} = \bar{G}(t), W_4 = e^{-(\lambda+\lambda_3)}\bar{G_1}(t), W_5 = e^{-(\lambda+\lambda_3)}\bar{G_2}(t), W_6 = e^{-(\lambda+\lambda_1+\lambda_2)}\bar{G_3}(t) + (\lambda_1 e^{-(\lambda+\lambda_1+\lambda_2)} \star 1)\bar{G_3}(t) + (\lambda_2 e^{-(\lambda+\lambda_1+\lambda_2)} \star 1)\bar{G_3}(t), W_8 = W_{10} = W_{12} = W_{13} = \bar{G_3}(t)$
Taking Laplace transform of equations and solving for B_0 we obtain:

$$B_0^*(s) = \frac{N_3(s)}{D_1(s)}$$

Steady state availability is given by:

$$B_0 = \lim_{s\to 0} sB_0^*(s) = \frac{N_3}{D_1}$$

where
$N_3 = -p_{01}(p_{13} p_{47} \mu_3 - p_{14} \mu_4 - p_{15} \mu_5 - p_{16} k - p_{14} p_{47} \mu_7 - p_{14} p_{48} \mu_8 - p_{15} p_{59} \mu_9 - p_{16} p_{64} \mu_4 - p_{16} p_{65} \mu_5 - p_{15} p_{5,10} \mu_{10} - p_{16} p_{6,11} \mu_{11} - p_{16} p_{47} p_{64} \mu_7 - p_{16} p_{48} p_{64} \mu_8 - p_{13} \mu_3 + p_{15} p_{47} \mu_5 + p_{16} p_{47} k + p_{13} p_{48} \mu_3 - p_{16} p_{59} p_{65} \mu_9 + p_{15} p_{48} \mu_5 + p_{16} p_{48} k + p_{13} p_{59} \mu_3 + p_{14} p_{59} \mu_4 + p_{16} p_{59} k - p_{16} p_{65} p_{5,10} \mu_{10} + p_{13} p_{5,10} \mu_3 + p_{14} p_{5,10} \mu_4 + p_{16} p_{5,10} k + p_{13} p_{6,11} \mu_3 + p_{14} p_{6,11} \mu_4 + p_{15} p_{6,11} \mu_5 + p_{15} p_{47} p_{59} \mu_9 + p_{16} p_{47} p_{65} \mu_5 + p_{15} p_{48} p_{59} \mu_9 + p_{16} p_{48} p_{65} \mu_5 + p_{14} p_{47} p_{59} \mu_7 + p_{14} p_{48} p_{59} \mu_8 + p_{16} p_{59} p_{64} \mu_4 + p_{15} p_{47} p_{5,10} \mu_{10} + p_{15} p_{48} p_{5,10} \mu_{10} + p_{14} p_{47} p_{5,10} \mu_7 + p_{14} p_{48} p_{5,10} \mu_8 + p_{16} p_{64} p_{5,10} \mu_4 + p_{16} p_{47} p_{6,11} \mu_{11} + p_{16} p_{48} p_{6,11} \mu_{11} + p_{16} p_{59} p_{6,11} \mu_{11} + p_{14} p_{47} p_{6,11} \mu_7 + p_{14} p_{48} p_{6,11} \mu_8 + p_{15} p_{59} p_{6,11} \mu_9 + p_{16} p_{5,10} p_{6,11} \mu_{11} + p_{15} p_{5,10} p_{6,11} \mu_{10} + p_{16} p_{47} p_{59} p_{65} \mu_9 + p_{16} p_{48} p_{59} p_{65} \mu_9 + p_{16} p_{47} p_{59} p_{64} \mu_7 + p_{16} p_{48} p_{59} p_{64} \mu_8 - p_{13} p_{47} p_{59} \mu_3 - p_{16} p_{47} p_{59} k - p_{13} p_{48} p_{59} \mu_3 - p_{16} p_{48} p_{59} k + p_{16} p_{47} p_{65} p_{5,10} \mu_{10} + p_{16} p_{48} p_{65} p_{5,10} \mu_{10} + p_{16} p_{47} p_{64} p_{5,10} \mu_7 + p_{16} p_{48} p_{64} p_{5,10} \mu_8 - p_{13} p_{47} p_{5,10} \mu_3 - p_{16} p_{47} p_{5,10} k - p_{13} p_{48} p_{5,10} \mu_3 - p_{16} p_{48} p_{5,10} k - p_{13} p_{47} p_{6,11} \mu_3 - p_{15} p_{47} p_{6,11} \mu_5 - p_{13} p_{48} p_{6,11} \mu_3 - p_{15} p_{48} p_{6,11} \mu_5 - p_{13} p_{59} p_{6,11} \mu_3 - p_{14} p_{59} p_{6,11} \mu_4 - p_{13} p_{5,10} p_{6,11} \mu_3 - p_{14} p_{5,10} p_{6,11} \mu_4 - p_{16} p_{47} p_{59} p_{6,11} \mu_{11} - p_{16} p_{48} p_{59} p_{6,11} \mu_{11} - p_{15} p_{47} p_{59} p_{6,11} \mu_9 - p_{15} p_{48} p_{59} p_{6,11} \mu_9 - p_{14} p_{47} p_{59} p_{6,11} \mu_7 - p_{14} p_{48} p_{59} p_{6,11} \mu_8 - p_{16} p_{47} p_{5,10} p_{6,11} \mu_{11} - p_{15} p_{47} p_{5,10} p_{6,11} \mu_{10} - p_{16} p_{48} p_{5,10} p_{6,11} \mu_{11} - p_{15} p_{48} p_{5,10} p_{6,11} \mu_{10} - p_{14} p_{47} p_{5,10} p_{6,11} \mu_7 - p_{14} p_{48} p_{5,10} p_{6,11} \mu_8 + p_{13} p_{47} p_{59} p_{6,11} \mu_3 + p_{13} p_{48} p_{59} p_{6,11} \mu_3 + p_{13} p_{47} p_{5,10} p_{6,11} \mu_3 + p_{13} p_{48} p_{5,10} p_{6,11} \mu_3)$
D_1 is already defined in equation (1).

6.4. Expected Number of Repairs in Summers

Leting $V_l(t)$ be the expected number of repairs in 0 < l $\leq$ t such that it is given the system entered the state S_l at t=0, we get

$$V_l(t) = \sum_n Q_{ln}(t)[h_l + v_l(t)];\ l=4,5,6,7,8,9,10,11$$

$$h_l = \begin{cases} 1, \text{when state } S_l \text{ is the regenerative state} \\ 0, \text{otherwise} \end{cases}$$

Taking LST of equations, we get:

$$V_0^{**}(s) = \frac{N_4}{D_1}$$

The equation describing the number of repairs per unit time in steady state

$$V_0 = \lim_{s\to 0} sv_0^{**}(s) = \frac{N_4}{D_1}$$

where

$N_4 = -p_{01}(p_{13}p_{47} - p_{14}p_{41} - p_{15}p_{51} - p_{16}p_{61} - p_{14}p_{47} - p_{14}p_{48} - p_{15}p_{59} - p_{16}p_{64}p_{41} - p_{16}p_{65}p_{51} - p_{15}p_{5,10} - p_{16}p_{6,11} - p_{16}p_{47}p_{64} - p_{16}p_{48}p_{64} - p_{13} + p_{15}p_{47}p_{51} + p_{16}p_{47}p_{61} + p_{13}p_{48} - p_{16}p_{59}p_{65} + p_{15}p_{48}p_{51} + p_{16}p_{48}p_{61} + p_{13}p_{59} + p_{14}p_{59}p_{41} + p_{16}p_{59}p_{61} - p_{16}p_{65}p_{5,10} + p_{13}p_{5,10} + p_{14}p_{5,10}p_{41} + p_{16}p_{5,10}p_{61} + p_{13}p_{6,11} + p_{14}p_{6,11}p_{41} + p_{15}p_{6,11}p_{51} + p_{15}p_{47}p_{59} + p_{16}p_{47}p_{65}p_{51} + p_{15}p_{48}p_{59} + p_{16}p_{48}p_{65}p_{51} + p_{14}p_{47}p_{59} + p_{14}p_{48}p_{59} + p_{16}p_{59}p_{64}p_{41} + p_{15}p_{47}p_{5,10} + p_{15}p_{48}p_{5,10} + p_{14}p_{47}p_{5,10} + p_{14}p_{48}p_{5,10} + p_{16}p_{64}p_{5,10}p_{41} + p_{16}p_{47}p_{6,11} + p_{16}p_{48}p_{6,11} + p_{16}p_{59}p_{6,11} + p_{14}p_{47}p_{6,11} + p_{14}p_{48}p_{6,11} + p_{15}p_{59}p_{6,11} + p_{16}p_{5,10}p_{6,11} + p_{15}p_{5,10}p_{6,11} + p_{16}p_{47}p_{59}p_{65} + p_{16}p_{48}p_{59}p_{65} + p_{16}p_{47}p_{59}p_{64} + p_{16}p_{48}p_{59}p_{64} - p_{13}p_{47}p_{59} - p_{16}p_{47}p_{59}p_{61} - p_{13}p_{48}p_{59} - p_{16}p_{48}p_{59}p_{61} + p_{16}p_{47}p_{65}p_{5,10} + p_{16}p_{48}p_{65}p_{5,10} + p_{16}p_{47}p_{64}p_{5,10} + p_{16}p_{48}p_{64}p_{5,10} - p_{13}p_{47}p_{5,10} - p_{16}p_{47}p_{5,10}p_{61} - p_{13}p_{48}p_{5,10} - p_{16}p_{48}p_{5,10}p_{61} - p_{13}p_{47}p_{6,11} - p_{15}p_{47}p_{6,11}p_{51} - p_{13}p_{48}p_{6,11} - p_{15}p_{48}p_{6,11}p_{51} - p_{13}p_{59}p_{6,11} - p_{14}p_{59}p_{6,11}p_{41} - p_{13}p_{5,10}p_{6,11} - p_{14}p_{5,10}p_{6,11}p_{41} - p_{16}p_{47}p_{59}p_{6,11} - p_{16}p_{48}p_{59}p_{6,11} - p_{15}p_{47}p_{59}p_{6,11} - p_{15}p_{48}p_{59}p_{6,11} - p_{14}p_{47}p_{59}p_{6,11} - p_{14}p_{48}p_{59}p_{6,11} - p_{16}p_{47}p_{5,10}p_{6,11} - p_{15}p_{47}p_{5,10}p_{6,11} - p_{16}p_{48}p_{5,10}p_{6,11} - p_{15}p_{48}p_{5,10}p_{6,11} - p_{14}p_{47}p_{5,10}p_{6,11} - p_{14}p_{48}p_{5,10}p_{6,11} + p_{13}p_{47}p_{59}p_{6,11} + p_{13}p_{48}p_{59}p_{6,11} + p_{13}p_{47}p_{5,10}p_{6,11} + p_{13}p_{48}p_{5,10}p_{6,11})$

D_1 is already defined in equation (1).

6.5. Availability of the System in Winters

The availability A_0^w of the system in winters is:

$$A_0^w = \lim_{s\to 0}(sA_0^{*w}) = \frac{N_5}{D_2}$$

where

$M_0 = e^{-(\alpha+\beta)}, M_{15} = e^{-(\gamma+\lambda+\lambda_1+\lambda_2)}t, M_{17} = e^{-(\lambda+\lambda_3)t}\bar{G_2}(t), M_{18} = e^{-(\lambda+\lambda_3)t}\bar{G_3}(t)$

$N_5 = \mu_0 + \mu_{15}p_{02}p_{2,15} - \mu_0 p_{2,14} - \mu_0 p_{15,2}p_{2,15} - \mu_0 p_{15,16} - \mu_0 p_{15,17}p_{17,15} - \mu_0 p_{15,18}p_{18,15} - \mu_0 p_{17,19} - \mu_0 p_{17,20} - \mu_0 p_{18,21} - \mu_0 p_{18,22} + \mu_{17}p_{02}p_{2,15}p_{15,17} + \mu_{18}p_{02}p_{2,15}p_{15,18} + \mu_0 p_{2,14}p_{15,16} + \mu_0 p_{2,14}p_{15,17}p_{17,15} + \mu_0 p_{2,14}p_{15,18}p_{18,15} - \mu_{15}p_{02}p_{2,15}p_{17,19} - \mu_{15}p_{02}p_{2,15}p_{17,20} + \mu_0 p_{2,14}p_{17,19} + \mu_0 p_{15,2}p_{2,15}p_{17,19} + \mu_0 p_{2,14}p_{17,20} + \mu_0 p_{15,2}p_{2,15}p_{17,20} - \mu_{15}p_{02}p_{2,15}p_{18,21} - \mu_{15}p_{02}p_{2,15}p_{18,22} + \mu_0 p_{2,14}p_{18,21} + \mu_0 p_{15,2}p_{2,15}p_{18,21} + \mu_0 p_{2,14}p_{18,22} + \mu_0 p_{15,2}p_{2,15}p_{18,22} + \mu_0 p_{15,16}p_{17,19} + \mu_0 p_{15,16}p_{17,20} + \mu_0 p_{15,18}p_{17,19}p_{18,15} + \mu_0 p_{15,16}p_{18,21} + \mu_0 p_{15,18}p_{17,20}p_{18,15} + \mu_0 p_{15,16}p_{18,22} + \mu_0 p_{15,17}p_{17,15}p_{18,21} + \mu_0 p_{15,17}p_{17,15}p_{18,22} + \mu_0 p_{17,19}p_{18,21} + \mu_0 p_{17,19}p_{18,22} + \mu_0 p_{17,20}p_{18,21} + \mu_0 p_{17,20}p_{18,22} - \mu_{18}p_{02}p_{2,15}p_{15,18}p_{17,19} - \mu_{18}p_{02}p_{2,15}p_{15,18}p_{17,20} - \mu_{17}p_{02}p_{2,15}p_{15,17}p_{18,21} - \mu_{17}p_{02}p_{2,15}p_{15,17}p_{18,22} - \mu_0 p_{2,14}p_{15,16}p_{17,19} - \mu_0 p_{2,14}p_{15,16}p_{17,20} - \mu_0 p_{2,14}p_{15,18}p_{17,19}p_{18,15} - \mu_0 p_{2,14}p_{15,16}p_{18,21} - \mu_0 p_{2,14}p_{15,18}p_{17,20}p_{18,15} - \mu_0 p_{2,14}p_{15,16}p_{18,22} - \mu_0 p_{2,14}p_{15,17}p_{17,15}p_{18,21} - \mu_0 p_{2,14}p_{15,17}p_{17,15}p_{18,22} + \mu_{15}p_{02}p_{2,15}p_{17,19}p_{18,21} + \mu_{15}p_{02}p_{2,15}p_{17,19}p_{18,22} + \mu_{15}p_{02}p_{2,15}p_{17,20}p_{18,21} - \mu_0 p_{2,14}p_{17,19}p_{18,21} - \mu_0 p_{15,2}p_{2,15}p_{17,19}p_{18,21} + \mu_{15}p_{02}p_{2,15}p_{17,20}p_{18,22} - \mu_0 p_{2,14}p_{17,19}p_{18,22} - \mu_0 p_{2,14}p_{17,20}p_{18,21} - \mu_0 p_{15,2}p_{2,15}p_{17,19}p_{18,22} - \mu_0 p_{15,2}p_{2,15}p_{17,20}p_{18,21} - \mu_0 p_{2,14}p_{17,20}p_{18,22} - \mu_0 p_{15,2}p_{2,15}p_{17,20}p_{18,22} - \mu_0 p_{15,16}p_{17,19}p_{18,21} - \mu_0 p_{15,16}p_{17,19}p_{18,22} - \mu_0 p_{15,16}p_{17,20}p_{18,21} - \mu_0 p_{15,16}p_{17,20}p_{18,22} + \mu_0 p_{2,14}p_{15,16}p_{17,19}p_{18,21} + \mu_0 p_{2,14}p_{15,16}p_{17,19}p_{18,22} + \mu_0 p_{2,14}p_{15,16}p_{17,20}p_{18,21} + \mu_0 p_{2,14}p_{15,16}p_{17,20}p_{18,22}$

$D_2 = (\mu_{14}p_{2,14} + \mu_2)(p_{15,2} - p_{15,2}p_{17,19} - p_{15,2}p_{17,20} - p_{15,2}p_{18,21} - p_{15,2}p_{18,22} + p_{15,2}p_{17,19}p_{18,21} + p_{15,2}p_{17,19}p_{18,22} + p_{15,2}p_{17,20}p_{18,21} + p_{15,2}p_{17,20}p_{18,22}) + (\mu_{15} + \mu_{16}p_{15,16})(p_{17,15} - p_{2,14}p_{17,15} -$

$p_{17,15}p_{18,21} - p_{17,15}p_{18,22} + p_{2,14}p_{17,15}p_{18,21} + p_{2,14}p_{17,15}p_{18,22}) + (\mu_{17} + \mu_{19}p_{17,19} + \mu_{20}p_{17,20})(p_{15,17} - p_{2,14}p_{15,17} - p_{15,17}p_{18,21} - p_{15,17}p_{18,22} + p_{2,14}p_{15,17}p_{18,21} + p_{2,14}p_{15,17}p_{18,22}) + (\mu_{18} + \mu_{21}p_{18,21} + \mu_{22}p_{18,22})(p_{15,18} - p_{2,14}p_{15,18} - p_{15,18}p_{17,19} - p_{15,18}p_{17,20} + p_{2,14}p_{15,18}p_{17,19} + p_{2,14}p_{15,18}p_{17,20})$.......(2).

6.6. Busy Period for Repair in Winters

The busy period for repair B_0^{wR} of the system in winters is:

$$B_0^{wR} = \lim_{s\to 0}(sB_0^{*wR}) = \frac{N_6}{D_2}$$

where

$W_{17} = e^{-(\lambda+\lambda_3)}\bar{G_2}(t), W_{18} = e^{-(\lambda+\lambda_3)}\bar{G_3}(t), W_{19} = W_{21} = \bar{G}(t), W_{20} = W_{22} = \bar{G_3}(t)$

$N_6 = -p_{02}(p_{2,15}p_{15,16}p_{17,19}\mu_{16} - p_{2,15}p_{15,17}\mu_{17} - p_{2,15}p_{15,18}\mu_{18} - p_{2,15}p_{15,17}p_{17,19}\mu_{19} - p_{2,15}p_{15,17}p_{17,20}\mu_{20} - p_{2,15}p_{15,18}p_{18,21}\mu_{21} - p_{2,15}p_{15,18}p_{18,22}\mu_{22} - p_{2,15}p_{15,16}\mu_{16} + p_{2,15}p_{15,18}p_{17,19}\mu_{18} + p_{2,15}p_{15,16}p_{17,20}\mu_{16} + p_{2,15}p_{15,18}p_{17,20}\mu_{18} + p_{2,15}p_{15,16}p_{18,21}\mu_{16} + p_{2,15}p_{15,17}p_{18,21}\mu_{17} + p_{2,15}p_{15,16}p_{18,22}\mu_{16} + p_{2,15}p_{15,17}p_{18,22}\mu_{17} + p_{2,15}p_{15,18}p_{17,19}p_{18,21}\mu_{21} + p_{2,15}p_{15,18}p_{17,19}p_{18,22}\mu_{22} + p_{2,15}p_{15,18}p_{17,20}p_{18,21}\mu_{21} + p_{2,15}p_{15,18}p_{17,20}p_{18,22}\mu_{22} + p_{2,15}p_{15,17}p_{17,19}p_{18,21}\mu_{19} + p_{2,15}p_{15,17}p_{17,20}p_{18,21}\mu_{20} + p_{2,15}p_{15,17}p_{17,19}p_{18,22}\mu_{19} + p_{2,15}p_{15,17}p_{17,20}p_{18,22}\mu_{20} - p_{2,15}p_{15,16}p_{17,19}p_{18,21}\mu_{16} - p_{2,15}p_{15,16}p_{17,19}p_{18,22}\mu_{16} - p_{2,15}p_{15,16}p_{17,20}p_{18,21}\mu_{16} - p_{2,15}p_{15,16}p_{17,20}p_{18,22}\mu_{16})$

D_2 is already defined in equation (2).

6.7. Busy Period for Maintenance in Winters

The busy period for maintenance B_0^{wM} of the system in winters is:

$$B_0^{wM} = \lim_{s\to 0}(sB_0^{*wM}) = \frac{N_7}{D_2}$$

where

$W_{14} = \bar{G_4}(t)$

$N_7 = p_{02}p_{2,14}\mu_{14}(p_{15,16}p_{17,19} - p_{15,17}p_{17,15} - p_{15,18}p_{18,15} - p_{17,19} - p_{17,20} - p_{18,21} - p_{18,22} - p_{15,16} + p_{15,16}p_{17,20} + p_{15,18}p_{17,19}p_{18,15} + p_{15,16}p_{18,21} + p_{15,18}p_{17,20}p_{18,15} + p_{15,16}p_{18,22} + p_{15,17}p_{17,15}p_{18,21} + p_{15,17}p_{17,15}p_{18,22} + p_{17,19}p_{18,21} + p_{17,19}p_{18,22} + p_{17,20}p_{18,21} + p_{17,20}p_{18,22} - p_{15,16}p_{17,19}p_{18,21} - p_{15,16}p_{17,19}p_{18,22} - p_{15,16}p_{17,20}p_{18,21} - p_{15,16}p_{17,20}p_{18,22} + 1)$

D_2 is already defined in equation (2).

6.7.1. Expected Number of Repairs in Winters

The expected number of repair V_0^{wR} of the system in winters is:

$$V_0^{wR} = \lim_{s\to 0}(sV_0^{*wR}) = \frac{N_8}{D_2}$$

where

$N_8 = -p_{02}(p_{2,15}p_{15,16}p_{17,19} - p_{2,15}p_{15,17}p_{17,15} - p_{2,15}p_{15,18}p_{18,15} - p_{2,15}p_{15,17}p_{17,19} - p_{2,15}p_{15,17}p_{17,20} - p_{2,15}p_{15,18}p_{18,21} - p_{2,15}p_{15,18}p_{18,22} - p_{2,15}p_{15,16} + p_{2,15}p_{15,18}p_{17,19}p_{18,15} + p_{2,15}p_{15,16}p_{17,20} + p_{2,15}p_{15,18}p_{17,20}p_{18,15} + p_{2,15}p_{15,16}p_{18,21} + p_{2,15}p_{15,17}p_{18,21}p_{17,15} + p_{2,15}p_{15,16}p_{18,22} + p_{2,15}p_{15,17}p_{18,22}p_{17,15} + p_{2,15}p_{15,18}p_{17,19}p_{18,21} + p_{2,15}p_{15,18}p_{17,19}p_{18,22} + p_{2,15}p_{15,18}p_{17,20}p_{18,21} + p_{2,15}p_{15,18}p_{17,20}p_{18,22} + p_{2,15}p_{15,17}p_{17,19}p_{18,21} + p_{2,15}p_{15,17}p_{17,20}p_{18,21} + p_{2,15}p_{15,17}p_{17,19}p_{18,22} + p_{2,15}p_{15,17}p_{17,20}p_{18,22} - p_{2,15}p_{15,16}p_{17,19}p_{18,21} - p_{2,15}p_{15,16}p_{17,19}p_{18,22} - p_{2,15}p_{15,16}p_{17,20}p_{18,21} - p_{2,15}p_{15,16}p_{17,20}p_{18,22})$

D_2 is already defined in equation (2).

6.7.2. Expected Number of Maintenances in Winters

The expected number of maintenances V_0^{wM} of the system in winters is:

$$V_0^{wM} = \lim_{s\to 0}(sV_0^{*wM}) = \frac{N_9}{D_2}$$

where

$N_9 = p_{02}p_{2,14}(p_{15,16}p_{17,19} - p_{15,17}p_{17,15} - p_{15,18}p_{18,15} - p_{17,19} - p_{17,20} - p_{18,21} - p_{18,22} - p_{15,16} + p_{15,16}p_{17,20} + p_{15,18}p_{17,19}p_{18,15} + p_{15,16}p_{18,21} + p_{15,18}p_{17,20}p_{18,15} + p_{15,16}p_{18,22} + p_{15,17}p_{17,15}p_{18,21} + p_{15,17}p_{17,15}p_{18,22} + p_{17,19}p_{18,21} + p_{17,19}p_{18,22} + p_{17,20}p_{18,21} + p_{17,20}p_{18,22} - p_{15,16}p_{17,19}p_{18,21} - p_{15,16}p_{17,19}p_{18,22} - p_{15,16}p_{17,20}p_{18,21} - p_{15,16}p_{17,20}p_{18,22} + 1)$
D_2 is already defined in equation (2).

7. Transition Probabilities and Mean Sojourn Time

$p_{01} = \frac{\beta}{\alpha+\beta}, p_{02} = \frac{\alpha}{\alpha+\beta}, p_{13} = \frac{\lambda}{\lambda+\lambda_1+\lambda_2+\lambda_3}, p_{14=\frac{\lambda_1}{\lambda+\lambda_1+\lambda_2+\lambda_3}}, p_{15} = \frac{\lambda_2}{\lambda+\lambda_1+\lambda_2+\lambda_3}, p_{16} = \frac{\lambda_3}{\lambda+\lambda_1+\lambda_2+\lambda_3}, p_{2,14} = \frac{\lambda_4}{\lambda_4+\gamma}, p_{2,15} = \frac{\gamma}{\lambda_4+\gamma}, p_{41} = g_1^*(\lambda+\lambda_3), p_{47} = \frac{\lambda}{\lambda+\lambda_3}(1-g_1^*(\lambda+\lambda_3)), p_{48} = \frac{\lambda_3}{\lambda+\lambda_3}(1-g_1^*(\lambda+\lambda_3)), p_{51} = g_2^*(\lambda+\lambda_3), p_{59} = \frac{\lambda}{\lambda+\lambda_3}(1-g_2^*(\lambda+\lambda_3)), p_{5,10} = \frac{\lambda_3}{\lambda+\lambda_3}(1-g_2^*(\lambda+\lambda_3)), p_{61} = g_3^*(\lambda+\lambda_1+\lambda_2), p_{6,11} = \frac{\lambda}{\lambda+\lambda_1+\lambda_2}(1-g_3^*(\lambda+\lambda_1+\lambda_2)), p_{6,12} = p_{64}^{(12)} = \frac{\lambda_1}{\lambda+\lambda_1+\lambda_2}(1-g_3^*(\lambda+\lambda_1+\lambda_2)), p_{6,13} = p_{65}^{(13)} = \frac{\lambda_2}{\lambda+\lambda_1+\lambda_2}(1-g_3^*(\lambda+\lambda_1+\lambda_2)), p_{15,2} = \frac{\delta}{\delta+\lambda+\lambda_1+\lambda_2}, p_{15,17} = \frac{\lambda_2}{\delta+\lambda+\lambda_1+\lambda_2}, p_{15,18} = \frac{\lambda_1}{\delta+\lambda+\lambda_1+\lambda_2}, p_{17,15} = g_2^*(\lambda+\lambda_3), p_{17,19} = \frac{\lambda}{\lambda+\lambda_3}(1-g_2^*(\lambda+\lambda_3)), p_{17,20} = \frac{\lambda_3}{\lambda+\lambda_3}(1-g_2^*(\lambda+\lambda_3)), p_{18,15} = g_1^*(\lambda+\lambda_3), p_{18,21} = \frac{\lambda}{\lambda+\lambda_3}(1-g_1^*(\lambda+\lambda_3)), p_{18,22} = \frac{\lambda_3}{\lambda+\lambda_3}(1-g_1^*(\lambda+\lambda_3))$
mean sojourn times are as follows:
$\mu_0 = \frac{1}{\alpha+\beta}, \mu_1 = \frac{1}{\lambda+\lambda_1+\lambda_2+\lambda_3}, \mu_2 = \frac{1}{\delta+\lambda_4}, \mu_4 = \frac{1}{\lambda+\lambda_3}(1-g_1^*(\lambda+\lambda_3)), \mu_5 = \frac{1}{\lambda+\lambda_3}(1-g_2^*(\lambda+\lambda_3)), \mu_6 = \frac{1}{\lambda+\lambda_1+\lambda_2}(1-g_3^*(\lambda+\lambda_1+\lambda_2)), \mu_{15} = \frac{1}{\delta+\lambda+\lambda_1+\lambda_2}, \mu_{17} = \frac{1}{\lambda+\lambda_3}(1-g_2^*(\lambda+\lambda_3)), \mu_{18} = \frac{1}{\lambda+\lambda_3}(1-g_1^*(\lambda+\lambda_3))$
If time is measured from the epoch of entry into state S_n, the unconditional mean transit time of the system from any state S_l is:

$$m_{ln}=\int_0^\infty t dQ_{ln}(t) = -q_{ln}^*{}'(0)$$

where,
$m_{01} + m_{02} = \mu_0, m_{13} + m_{14} + m_{15} = \mu_1, m_{2,14} + m_{2,15} = \mu_2, m_{41} + m_{47} + m_{48} = \mu_4, m_{51} + m_{59} + m_{5,10} = \mu_5, m_{61} + m_{6,11} + m_{6,12} + m_{6,13} = \mu_6, m_{61} + m_{6,11} + m_{64}^{(12)} + m_{65}^{(13)} = K, m_{15,2} + m_{15,16} + m_{15,17} + m_{15,18} = \mu_{15}, m_{17,15} + m_{17,19} + m_{17,20} = \mu_{17}, m_{18,15} + m_{18,21} + m_{18,22} = \mu_{18}$
$K = \frac{\lambda}{(\lambda+\lambda_1+\lambda_2)^2} + \frac{\lambda_1+\lambda_2}{\lambda+\lambda_1+\lambda_2}\int_0^\infty t g_3(t)dt + \frac{1}{\lambda+\lambda_1+\lambda_2}\int_0^\infty e^{-(\lambda+\lambda_1+\lambda_2)}g_3(t)dt - \frac{\lambda_1+\lambda_2}{(\lambda+\lambda_1+\lambda_2)^2}\int_0^\infty e^{-(\lambda+\lambda_1+\lambda_2)}g_3(t)dt$
From the above transition probabilites it is verified that:
$p_{01} + p_{02} = 1, p_{13} + p_{14} + p_{15} = 1, p_{2,14} + p_{2,15} = 1, p_{41} + p_{47} + p_{48} = 1, p_{51} + p_{59} + p_{5,10} = 1, p_{61} + p_{6,11} + p_{6,12} + p_{6,13} = 1, p_{61} + p_{6,11} + p_{64}^{(12)} + p_{65}^{(13)} = 1, p_{15,2} + p_{15,16} + p_{15,17} + p_{15,18} = 1, p_{17,15} + p_{17,19} + p_{17,20} = 1, p_{18,15} + p_{18,21} + p_{18,22} = 1$

8. Profit analysis

The profit incurred to the system is:
$P = A_0C_0 + A_0^1C_1 + A_0^wC_2 - (B_0C_3 + B_0^{WR}C_4 + B_0^{WM}C_5 + V_0C_6 + V_0^{WR}C_7 + V_0^{WM}C_8)$
Where
C_0, C_1 are the revenues generated in summers when the system operates at full and half capacity respectively. C_2 is the revenue generated in winters.
C_3, C_4 is the cost per unit time when the repairman in busy for reapir in summers and winters respectively.C_5 is the cost per unit time when the repairman is busy for maintenance.
C_6, C_7 cost per repair in summers and winters respectively.C_8 cost per maintenance.

Rates of the system	Associated Values (per hr)
Failure of unit 1	0.000778752
Failure rate of unit 2	0.0007787652
Failure rate of unit 3	0.00077528
Failure rate of unit 4	0.00053279
Maintenance rate of the unit	0.000594484
Repair rate of unit 1	0.065642055
Repair rate of unit 2	0.0432694
Repair rate of unit 3	0.014205127
Repair rate of unit 4	0.02134
Rate of maintenance	0.00279431
Rate of going to summers	0.00019841
Rate of going to winters	0.0002777
Rate of going to operating state	.0033
Rate of going to standby state	.00027

Figure 2: *Values computed from the data collected*

Costs of the system	Associated Values (Rs.)
Revenue per up time in summers when system works at full capacity	256000
Revenue per up time in summers when system works at half capacity	125000
Revenue per up time in winters	21000
Cost per unit time when repairman is busy for repair in summers	10000
Cost per unit time when repairman is busy for repair in winters	5000
Cost per unit time when repairman is busy for maintenance in winters	12500
Cost per repair in summers	18000
Cost per repair in winters	9000
Cost per maintenance	22000

Figure 3: *Values computed from the data collected*

9. Graphical Analysis and Conclusion using Particular Case

Let us assume an exponential distribution for all the repair rates such that
$g(t) = \theta e^{-\theta(t)}, g_1(t) = \theta_1 e^{-\theta_1(t)}, g_2(t) = \theta_2 e^{-\theta_2(t)}, g_3(t) = \theta_3 e^{-\theta_3(t)}, g_4(t) = \theta_4 e^{-\theta_4(t)}$
$p_{01} = \frac{\beta}{\alpha+\beta}, p_{02} = \frac{\alpha}{\alpha+\beta}, p_{13} = \frac{\lambda}{\lambda+\lambda_1+\lambda_2+\lambda_3}, p_{14} = \frac{\lambda_1}{\lambda+\lambda_1+\lambda_2+\lambda_3}, p_{15} = \frac{\lambda_2}{\lambda+\lambda_1+\lambda_2+\lambda_3}, p_{16} = \frac{\lambda_3}{\lambda+\lambda_1+\lambda_2+\lambda_3}, p_{2,14} = \frac{\lambda_4}{\lambda_4+\gamma}, p_{2,15} = \frac{\gamma}{\lambda_4+\gamma}, p_{41} = \frac{\theta_1}{\theta_1+\lambda+\lambda_3}, p_{47} = \frac{\lambda}{\theta_1+\lambda+\lambda_3}, p_{48} = \frac{\lambda_3}{\theta_1+\lambda+\lambda_3}, p_{51} = \frac{\theta_2}{\theta_2+\lambda+\lambda_3}, p_{59} = \frac{\lambda}{\theta_2+\lambda+\lambda_3}, p_{5,10} = \frac{\lambda_3}{\theta_2+\lambda+\lambda_3}, p_{61} = \frac{\theta_3}{\theta_3+\lambda+\lambda_1+\lambda_2}, p_{6,11} = \frac{\lambda}{\theta_3+\lambda+\lambda_1+\lambda_2}, p_{6,12} = p_{64}^{(12)} = \frac{\lambda_1}{\theta_3+\lambda+\lambda_1+\lambda_2}, p_{6,13} = p_{65}^{(13)} = \frac{\lambda_2}{\theta_3+\lambda+\lambda_1+\lambda_2}, p_{15,2} = \frac{\delta}{\delta+\lambda+\lambda_1+\lambda_2}, p_{15,17} = \frac{\lambda_2}{\delta+\lambda+\lambda_1+\lambda_2}, p_{15,18} = \frac{\lambda_1}{\delta+\lambda+\lambda_1+\lambda_2}, p_{17,15} = \frac{\theta_2}{\theta_2+\lambda+\lambda_3}, p_{17,19} = \frac{\lambda}{\theta_2+\lambda+\lambda_3}, p_{17,20} = \frac{\lambda_3}{\theta_2+\lambda+\lambda_3}, p_{18,15} = \frac{\theta_1}{\theta_1+\lambda+\lambda_3}, p_{18,21} = \frac{\lambda}{\theta_2+\lambda+\lambda_3}, p_{18,22} = \frac{\lambda_3}{\theta_2+\lambda+\lambda_3}, \mu_0 = \frac{1}{\alpha+\beta}, \mu_1 = \frac{1}{\lambda+\lambda_1+\lambda_2+\lambda_3}, \mu_2 = \frac{1}{\delta+\lambda_4}, \mu_4 = \frac{1}{\lambda+\lambda_3}(1 - g_1^*(\lambda + \lambda_3)), \mu_5 = \frac{1}{\lambda+\lambda_3+\theta_2}, \mu_6 = \frac{1}{\lambda+\lambda_1+\lambda_2+\theta_3} \mu_{15} = \frac{1}{\delta+\lambda+\lambda_1+\lambda_2}, \mu_{17} = \frac{1}{\lambda+\lambda_3+\theta_2}, \mu_{18} = \frac{1}{\lambda+\lambda_3+\theta_1}$

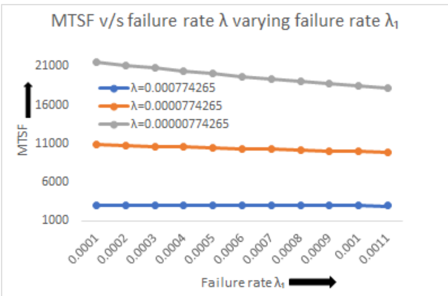

Figure 4: *MTSF v/s Failure rate*

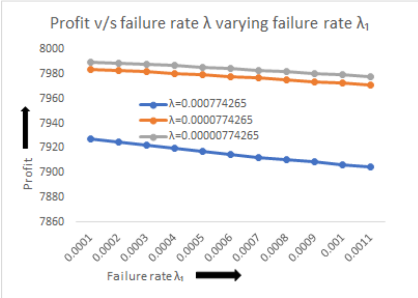

Figure 5: *Profit v/s Failure rate*

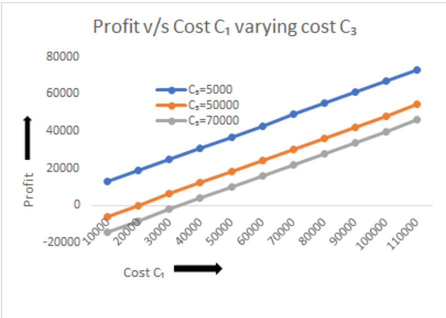

Figure 6: *Profit v/s Cost*

Figures 4, 5 are the MTSF and profit graphs showing a similar trend against the failure rate λ_1 varying failure rate λ.
It shows that as the failure rate λ or λ_1 increases the MTSF and profit decreases.
cut off points for figure 6 are as follows:

Table 2: *Cut-off Points*

Cost C_3Rs.	Revenue per up timeRs.
50000	19736.6955
70000	33564.2752

In table 2 the cut-off points have been shown and from figure 6 it is also clear that with increase in the cost C_1 the profit of the system increases. A number of research papers have been reviewed in this review that have contributed greatly to the field of reliability engineering over the years. The authors have conducted a substantial literature review with an aim of providing reliability engineers and industry leaders with recommendations on improving system reliability. As a final point, we see a wide range of potential uses for the new methods, techniques, and models. The system analysis was performed using a semi-Markov process and regenerative point technique. Code Blocks and Excel are used for graphical analysis, while MATLAB is used for calculation.In conclusion, researchers should consider cost factors as well as reliability factors in order to attain maximum reliability at a minimal cost in the future.

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An Upgraded Approach to Solve Fuzzy Transportation Problems

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Abstract

TP has many applications and applications and applications to reduce costs. A good algorithm has been developed to adjust the TP in the context of all given parameters, namely the supply, demand and TC team one, well. However, in real applications, there are many different situations due to uncertainty. It is therefore important to study PT in an uncertain environment. In this paper, an updated procedure is proposed to fix FTP where all parameters represents the non-triangular FN. The first is to use a non-trivial assembly to convert FTP to an LP with FC and net resistance. The second is to use a new vending system to turn the problem-solving lab into a three-wire lab. The value of a well-updated system is assessed compared to existing systems from an application model. The results obtained show that the updated method proposed in this study is simpler and more efficient than some existing methods commonly used in literature.

Keywords: Fuzzy Transportation Problem, Fuzzy Numbers, Solid Transportation Problems, Linear Programming Problem.

Abbreviations

Transportation Problem	:	TP
Transportation Cost	:	TC
Fuzzy Transportation Problem	:	FTP
Fuzzy Number	:	FN
Linear Programming	:	LP
Fuzzy Cost	:	FC
Linear Programming Problem	:	LPP
Fuzzy Linear Programming	:	FLP

Membership Function	:	MF
Objective Function	:	OF
Solid Transportation Problem	:	STP
Fuzzy Variables	:	FV
Goal Programming	:	GP
Optimal Solution	:	OS
Initial Basic Feasible Solution	:	IBFS
Fuzzy Supply	:	FS
Fuzzy Demand	:	FD
Fuzzy Quantity	:	FQ
Objective Value	:	OV

1. Introduction

TP is an important LPP installed network that appears in many situations and should receive a lot of attention in documentation. The main idea of this problem is to find the minimum TC of the product to meet the requirements in the destination by using the resources at the beginning. TP can be used for a variety of situations such as planning, production, investment, plant location, product management, project management and many more [26]. In general, TP is handled assuming TC and the price and rejection values are directly related, i.e. around the network. However, in most cases, the decision maker has no information about the TP rate. As [35] explained, the following factors may affect the state of fuzziness in TPs: (a) the decision-maker lacks sufficient information about the TC unit of the transport function and therefore the TC does not n ', (b) there may be some kind of misconception about the demand for a new product on the market, (c) there may be uncertainty about the availability of the product from the source or importer due to time constraints. Also, in the fast, there are many different conditions due to uncertainty, such as changing weather conditions, oil prices, and traffic conditions. It is therefore important to study TP in an uncertain environment. Since TP is actually LPP, the easiest way is to add the current FLP format to FTP [4, 19, 22, 25, 33, 39, 44, 49]. However, some of the existing systems provide only a clever solution that represents agreement in the case of nonlinear data [3, 28-29, 32, 36].

FTP studies are also available, [27] demonstrated that the solution obtained by FLP always worked and developed in the previous studies, FLP in several of the best ways to fix TP. [13] developed an algorithm to adjust TPs where supply and demand are solid foundations with linear or triangular MF. [46] used FSs to fix the TP and the required by the parametric program. Their approach provides a solution that meets the highest objectives and objectives at the same time. [34] discussed the TP type and FC rate and turned the problem into a TP bicriterion on OF net. Their system provides only a good solution based on a good solution of the changed problems. [5] proposed a free STP of trapezoidal FN representing transportation cost, requirement and authority. They put together a parametric system to find a non-trivial solution. [21] devised a method based on the extension principle to obtain unnecessary OV via FTP as well as FC and FS rates as well as the required number. The LP system requires several objectives to solve the FTP problem. Liu incorporated a similar mechanism to regulate nonlinear STP [12]. [42] showed two ways to reduce the FTP cost of supply chain and the required are trapezoidal FNs. They used a parametric method to find a non-invasive solution with the aim of reducing the TC concentration in both systems. [50] studied STP bicriterion in the stochastic parameter and built three mathematical models for the problem, including the expected GP value, the unpredictable GP block, and the GP-based space. [37] proposed a STP of a specific load in an unfamiliar environment, where immediate and unpaid prices, supply standards and requirements, as well as transportation capacity were FV. [18] demonstrated a non-

invasive GP approach to solving the generation / transfer configuration problem embedded in a number of unimportant objectives in an uncertain environment. [14] devised a new GP-based system to streamline FTP and FC. [23] proposed several ways to fix TP in a non-invasive network. [17] used algorithm cells to solve TPs in random numbers. [43] produced FTP, using trapezoidal FNs and developed a modified fuzzy distribution system to achieve the OS in FN format. [24] introduced a new algorithm, a zero-sum algorithm for finding seamless FTP operating systems with critical parameters. [30] devised a strategy for reducing TC as well as travel time when demand, supply and TC per minute are available in FN. [40] developed a new FLPP-based system to find the FTP OS. They developed a new system based on the quality function to fix FTP on the TC, product supply and demand are fully represented by trapezoidal [38]. After that, [11] introduced a similar algorithm to fix the same type of FTP assuming that the decision maker was unsure of the correct TC values but there was nothing wrong with that involved in products and demand. [20] introduced a simpler algorithm to solve an FTP problem that was simpler and easier to understand than the method suggested in [45]. In addition, a new mechanism was developed to locate a non-invasive OS that had no transport problems on the new trapezoidal FN representation of [48]. [1] devised a systematic approach to fix all types of FTP, both increasing or decreasing OF. [15] proposed a new complex strategy for the consideration and choice of investing in intelligent travel systems. A unique set of many STPs with nonsense penalties, sources, requirements and liability was developed and implemented by [31]. FLP-based messaging systems for FS, request, and travel capabilities have been included in this report. [6] developed a fuzzy version of the Vogel method and MODI to achieve a non-invasive IBFS and a viable solution that could be done, one by one, without translating them into classic TP. Furthermore, [52] developed algorithms to find the OS of FTP, where supply, demand and price are all FN. Their algorithm provides the decision maker a better solution compared to current systems. [16] used examples to show that their system will not always lead to a useless OS. [41] discussed the psychological analysis of FTP. [7] demonstrated an ingenious GP approach to resolving TP with multiple objectives and intermediate costs. Furthermore, [10] compared two working TPs with 2 fuzzy parameters where unit TCs, the specific charge and the first problem in TC units, charge , supply and demand and the second problem are type FV 2. [9] focused on the generation of PTS OS in a nonlinear environment, which assumes that all parameters are type 2 FV due to lack of transparency.
In this study, the following contributions were made by developing a new destruction method to mimic and modify FTPs: (a) Unlike some current methods, it is assumed that the arrival, desire and group TC values are negative triangular FNs. (b) The shortcomings of the existing FTP method are briefly discussed. (c) How to manage FTP is divided into two main parts. The first is to convert FTP to LPP with FC blocking on the net. The second is to develop a new sales strategy to convert the resulting LPP into three TP net. (d) The computational compression of the required method is greatly reduced compared to some conventional methods commonly used in literature.

2. Preliminaries

In this section, we examine some important concepts related to the fuzzy set concept, which will be used in other scripts [2].
First Definition: A FN is a convex normalized fuzzy set v * of real row R; the MF who goes on and on.
Second Definition: FN $v^* = (v_1, v_2, v_3)$ is said to be FN triangular if its MF is given as follows:

$$\theta_{v^*}(y) = \frac{y - v_1}{v_2 - v_1}, v_1 \le y \le v_2$$

$$\theta_{v^*}(y) = \frac{v_3 - y}{v_3 - v_2}, v_2 \le y \le v_3 \tag{A1}$$

$$\theta_{v^*}(y) = 0, \textit{ elsewhere}$$

FN $v^* = (v_1, v_2, v_3)$ can also mean $v^* = [v(m), v(d), v(v)]$. In this case, the point *(v) d* of the triangle FN v * gives the highest point $\boldsymbol{\theta_{v^*}(y)}$, i.e., $\boldsymbol{\theta_{v^*}(v^d)} = 1$, this is also called v * triangular FN core. Furthermore, *(v)l* and *(v)v* are individually spaced supporting the bottom and top of the triangular FN v *. All FN settings are defined as triangular TF (R).

Third Definition: It is said that FN triangular $v^* = (v_1, v_2, v_3)$ is the positive FN triangular if $v_1 \ge 0$. All these FN triangular curves are defined as all TF (R) *.

Forth Definition: It is said that the two triangular FNs $v^* = (v_1, v_2, v_3)$ and $u^* = (u_1, u_2, u_3)$ are equal, $v^* = u^*$, $v_1 = u_1$, $v_2 = u_2$ if only and $v_3 = u_3$.

Fifth Definition: Let $v^* = (v_1, v_2, v_3)$ and $u^* = (u_1, u_2, u_3)$ be two negative triangular FNs in $l \in R$. Then the mathematical function is given in v * and u * by:

i) $l \ge 0,\ lv^* = (lv_1, lv_2, lv_3)$

ii) $l < 0,\ lv^* = (lv_3, lv_2, lv_1)$

iii) $v^* \times u^* = (v_1 + u_1, v_2 + u_2, v_3 + u_3)$

iv) $v^* \times u^* = (v_1 u_1, v_2 u_2, v_3 u_3)$

3. FTP

In a typical TP, the decision maker is thought to have accurate information about the magnitude of the problem. In real-world applications, TC standards, product supply and demand may not be fully understood due to uncontrolled events. To deal with such situations, a fuzzy program is put on paper to fix TP. FTP, by a decision maker who is unsure of the exact principles of TC, supply and demand, can be developed [51] as follows:

$$min \sum_{j=1}^{n} \sum_{k=1}^{o} d^*_{jk} \times y^*_{jk}$$

Subject to:

$$\sum_{k=1}^{o} y^*_{jk} = t_j \qquad j = 1, 2, 3, ..., n$$

$$\sum_{k=1}^{o} y^*_{jk} = e_k \qquad k = 1, 2, 3, ..., o$$

$$y^*_{jk} \in TF(R), \quad j = 1, 2, ..., n, \quad k = 1, 2, ..., o. \tag{A2}$$

where, n = total number of points, o = total number of places, t_j = FS of product and j^{th} origin, e_k = FD of product and place kth where, d_{jk} = fuzzy TC per minute product from j^{th} from k^{th} point and y^*_{jk} = FQ of the product to be transferred from j^{th} point from k^{th} to reduce the total TC concentration.

Sixth Definition: Eq.A2 is considered a balanced FTP, if:

$$\sum_{j=1}^{n} t_j = \sum_{k=1}^{o} e_k$$

Elsewhere, it is known as unbalanced FTP.

First Remark: Without a general hiccup, we think the Eq.A2 is balanced one.

Second Remark: Since the negative number of products in the negative TC is insignificant, it is assumed that all parts of the FTP are not negative triangular FNs. About 2 words, take it as d_{jk}, t_j, e_k and y_{jk} and all represent it $d_{jk} = (d_{jk})\,m, (d_{jk})\,d, (d_{jk})\,v,\ t_j = (t_j)\,m, (t_j)\,d, (t_j)\,v.,\ e_k = (e_k)\,m, (e_k)\,d, (e_k)\,v$ and $y^*_{jk} = (y^*_{jk})\,m, (y^*_{jk})\,d, (y^*_{jk})\,v$, one. So from description 5, Eq.A2 can be interpreted as follows:

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{m}\left(y_{jk}\right)^{m},\left(d_{jk}\right)^{d}\left(y_{jk}\right)^{d},\left(d_{jk}\right)^{v}\left(y_{jk}\right)^{v}\right)$$

Subject to Constraints:

$$\begin{gathered}
\sum_{k=1}^{o}\left(\left(y_{jk}\right)^{m},\left(y_{jk}\right)^{d},\left(y_{jk}\right)^{v}\right)=\left(\left(e_{k}\right)^{m},\left(e_{k}\right)^{d},\left(e_{k}\right)^{v}\right),\ k=1,2,3,\ldots,o\\
\sum_{j=1}^{n}\left(\left(y_{jk}\right)^{m},\left(y_{jk}\right)^{d},\left(y_{jk}\right)^{v}\right)=\left(\left(t_{j}\right)^{m},\left(t_{j}\right)^{d},\left(t_{j}\right)^{v}\right),\ j=1,2,3,\ldots,n\\
\left(y_{jk}\right)^{v}-\left(y_{jk}\right)^{d}\geq o,\ j=1,2,3,\ldots,n;\ k=1,2,3,\ldots,o\\
\left(y_{jk}\right)^{d}-\left(y_{jk}\right)^{m}\geq o,\ j=1,2,3,\ldots,n;\ k=1,2,3,\ldots,o\\
\left(y_{jk}\right)^{m}\geq 0, j=1,2,3,\ldots,n; k=1,2,3,\ldots,o
\end{gathered}\tag{A3}$$

Similarly, with respect to definition 4, Eq.A3 can be rewritten as follows:

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{m}\left(y_{jk}\right)^{m}\right),\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{d}\left(y_{jk}\right)^{d}\right),\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{v}\left(y_{jk}\right)^{v}\right)\tag{A3.0}$$

Subject to Constraints:

$$\sum_{j=1}^{n}\left(\left(y_{jk}\right)^{m}\right)=\left(t_{j}\right)^{m},\ j=1,2,3,\ldots,n\tag{A3.1}$$

$$\sum_{j=1}^{n}\left(\left(y_{jk}\right)^{d}\right)=\left(t_{j}\right)^{d},\ j=1,2,3,\ldots,n\tag{A3.2}$$

$$\sum_{j=1}^{n}\left(\left(y_{jk}\right)^{v}\right)=\left(t_{j}\right)^{v},\ j=1,2,3,\ldots,n\tag{A3.3}$$

$$\sum_{k=1}^{o}\left(\left(y_{jk}\right)^{m}\right)=\left(t_{j}\right)^{m},\ k=1,2,3,\ldots,o\tag{A3.4}$$

$$\sum_{k=1}^{o}\left(\left(y_{jk}\right)^{d}\right)=\left(t_{j}\right)^{d},\ k=1,2,3,\ldots,o\tag{A3.5}$$

$$\sum_{k=1}^{o}\left(\left(y_{jk}\right)^{v}\right)=\left(t_{j}\right)^{v},\ k=1,2,3,\ldots,o\tag{A3.6}$$

$$\left(y_{jk}\right)^{v}-\left(y_{jk}\right)^{d}\geq o,\ j=1,2,3,\ldots,n;\ k=1,2,3,\ldots,o\tag{A3.7}$$

$$\left(y_{jk}\right)^{d} - \left(y_{jk}\right)^{m} \geq o,\ j = 1,2,3,...,n;\ k = 1,2,3,...,o \tag{A3.8}$$

$$\left(y_{jk}\right)^{m} \geq 0, j = 1,2,3,...,n; k = 1,2,3,...,o \tag{A3.9}$$

Also, for the Eq.A2 equilibrium inhibitors, they use Forth Definition. In the next section, new ways to overcome these weaknesses are suggested.

4. Newly Developed Approach

It is useful to point out that we can think of Eq.A3.0 as LPP has several objectives having OF:

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{m}\left(y_{jk}\right)^{m}\right), \sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{d}\left(y_{jk}\right)^{d}\right), \sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{v}\left(y_{jk}\right)^{v}\right)$$

based on the operating space of Eq.A.3.0. Also, blockchain (Eq.A3.7), (Eq.A3.8) and (Eq.A3.9) in Eq.A3.0 (and Eq.A2 and Eq.A3) are guaranteed only in the system d exploitation must have been a negative triangular FN. This means that without these constraints, the potential gap of Eq.A3.0 is separated by a series of changes (y_k) *m*, (y_k) *d* and (y_k) *v*. Therefore, we remove these barriers from the scope of the A3.0 scale and solve the underlying problem that makes these barriers satisfactory. This confirms that the OS was obtained as a negative FN triangular. Regarding the above discussion, we first find out the operating system of this net problem:

$$min\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{m}\left(y_{jk}\right)^{m}\right) \tag{A4}$$

Subject to Constraints:

$$\sum_{j=1}^{n}\left(\left(y_{jk}\right)^{m}\right) = \left(t_{j}\right)^{m},\ j = 1,2,3,...,n \tag{A4.1}$$

$$\sum_{k=1}^{o}\left(\left(y_{jk}\right)^{m}\right) = \left(t_{j}\right)^{m},\ k = 1,2,3,...,o \tag{A4.2}$$

$$\left(y_{jk}\right)^{m} \geq 0, j = 1,2,3,...,n; k = 1,2,3,...,o \tag{A4.3}$$

As we have seen, the problem (Eq.A4) is pure TP that can be solved using standard travel simplex algorithm. The OS of this issue is a left-hander of the non-essential OS of Eq.A4.2. Note that the Eq.A4.3 bandwidth guarantees an unobtrusive OS of the Eq.A2 as a negative FN. Now assuming that (*y*) *m* = (y_{jk}) on × 1 is the OS of the problem (Eq.A4), we solve the following problem to enter the OS of Eq. A2:

$$min\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{d}\left(y_{jk}\right)^{d}\right) \tag{A5}$$

Subject to Constraints:

$$\sum_{j=1}^{n}\left(y_{jk}\right)^{d} = \left(t_{j}\right)^{d},\ j = 1,2,3,...,n \tag{A5.1}$$

$$\sum_{k=1}^{o}\left(y_{jk}\right)^{d} = \left(t_{k}\right)^{d},\ k = 1,2,3,...,o \tag{A5.2}$$

$$\left(y_{jk}\right)^{d} \geq \left(y_{jk}\right)^{m},\ j = 1,2,3,...,n;\ k = 1,2,3,...,o \tag{A5.3}$$

Note that the barrier (Eq.A5.3) ensures that the fuzzy OS of the Eq.A2 is not lower than its left. Furthermore, it is clear that Eq.A5 is a limited TP that can be corrected using the method provided by [24]. Finally, assuming that $(y)d = \left(y_{jk}\right)d$ on × 1 are the OS of the problem (Eq.A5), we solve the following problem to get the exact point of the fuzzy OS of the Eq .A2:

$$min\sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{v}\left(y_{jk}\right)^{v}\right) \tag{A6}$$

Subject to Constraints:

$$\sum_{j=1}^{n}\left(y_{jk}\right)^{v}=\left(t_{j}\right)^{v},\ j=1,2,3,\ldots,n \tag{A6.1}$$

$$\sum_{k=1}^{o}\left(y_{jk}\right)^{v}=\left(t_{k}\right)^{v},\ k=1,2,3,\ldots,o \tag{A6.2}$$

$$\left(y_{jk}\right)^{v}\geq\left(y_{jk}\right)^{d},\ j=1,2,3,\ldots,n;\ k=1,2,3,\ldots,o \tag{A6.3}$$

As we have seen, this problem is also limited TP can be solved by using the method given in [8]. Moreover, the barrier (Eq.A6.3) supports that the correct position of the fuzzy OS of Eq.A2 is greater than or equal to its center. Therefore, the OS of the problem of Eq.A4, Eq.A5 and Eq.A6 ensure that the base OS of Eq.A2 is a triangular FN that is not negative. In summary, if *(y) m*, *(y) d* and *(y) v* are SO of network problems Eq.A4, Eq.A5 and Eq.A6, respectively, then *y = (y) m*, *(y) d*, *(y) v* would be the base OS of Eq.A2. Finally, the optimal value of the Eq.A2 problem is obtained by adding y and $d \times y$ as follows:

$$\begin{aligned} d\times y &= \sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{m},\left(d_{jk}\right)^{d},\left(d_{jk}\right)^{v}\right)\times\left(\left(y_{jk}\right)^{m},\left(y_{jk}\right)^{d},\left(y_{jk}\right)^{v}\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{o}\left(\left(d_{jk}\right)^{m}\left(y_{jk}\right)^{m},\left(d_{jk}\right)^{d}\left(y_{jk}\right)^{d},\left(d_{jk}\right)^{v}\left(y_{jk}\right)^{v}\right) \end{aligned}$$

5. Merits of Newly Proposed Method

In this section, the benefits of an expected FTP processing method are defined.

- Not only will the required system to implement FTP and TC representing FN be implemented in the principle of donations and requests as existing numbers, but it can also be used to configure FTP on all FN-enabled devices.
- The OS looks like a negative FN, that is, there are no negative components either in the FQ product or in the non-core TC.
- Using different methods such as northwest corner system, minimum price system and Vogel fuzzy approximation system to find the IBFS of the problem (Eq.A4), (Eq.A5) and (Eq.A6) lead to the same total CT.
- The main advantage of the proposed method is that solving the Eq.A2 problem (Eq.A4), (Eq.A5) and (Eq.A6) is relatively large compared to the problem (Eq.A3. 0) by a. mathematical considerations, regarding the number of barriers to change. There is a direct relationship between the conventional complexity of LPP and the number of barriers to change. In particular, since the memory size required to place the backup in the simplex algorithm is square of the number of constraints, reducing the number of constraints of the LP type is very important from a mathematical point of view. Thus, the reduction in the number of inhibitions and the variability of the LP type leads to a reduction in the complexity of the LP type modified by

simplex algorithms and key methods such as Khachian's ellipsoid algorithm and projective algorithms.

Compare the resistance and variability of the problem (Eq.A4), (Eq.A5) and (Eq.A6) the problem (Eq.A3.0). Problem (Eq.A4) has no barrier *(n + o)* and no change, while problem (Eq.A3.0) has *3 (n + o) +2* no barrier and 3 no change. This indicates that the problem (Eq.A4) has *2 (n + o + no)* barriers and 2 less change than the problem (Eq.A3.0), so the use of the problem (Eq.A4) is strong the economy in terms of the problem (Eq. .A3.0) is based on mathematical concepts. There is a similar comparison between the resistance and exchange rate (Eq.A5) and (Eq.A6) and (Eq.A3.0). It is noteworthy that the constraints (Eq.A5.3) and (Eq.A6.3) agree that the variance (y_k)*d* and (y_k)*v* are limited. The simplex bounded system also uses these blockchains in the same way as the simplex bounded block (y_k)*m≥0*. This means that these barriers do not increase the number of barriers immediately. Therefore, regarding the above discussion, we recommend using problem (Eq.A4), (Eq.A5), and (Eq.A6) instead of problem (Eq.A3.0) to solve Eq.A2 from a mathematical point of view.

6. Applications of Newly Proposed Method

In this section, the model application is analyzed using the recommended method and evaluating the obtained results.

First Example: One company has two sources P1 and P2, as well as three E1, E2 and E3 sources; TC fuzzy for quantity of products from j^{th} source to where k^{th} is d_{jk} where,

$$d_{jk} = \left[(20,30,40)(60,70,90)(90,100,110)(70,80,90)(85,95,115)(35,45,55)\right]$$

The FS of the product in the first and second stages are (80, 100, 130) and (50, 70, 100), respectively. FD of the product in the first place, second and third are (40, 50, 60), (30, 40, 50) and (70, 90, 120), respectively. The company wants to determine the FQ of the product that needs to be shipped from any start to anywhere so that the total TC value is minimal. This problem can be solved with the following FTP:

$$min\ \left(20,30,40\right)y_{11} \times \left(60,70,90\right)y_{12} \times \left(90,100,110\right)y_{13} \times \left(70,80,90\right)y_{21} \times \left(90,100,120\right)y_{22} \times \left(40,50,60\right)y_{23}$$

Subject to Constraints:

$$\begin{aligned} &y_{11} \times y_{12} \times y_{13} = \left(80,100,130\right), y_{21} \times y_{22} \times y_{23} = \left(50,70,100\right), y_{11} \times y_{21} = \left(40,50,70\right), \\ &y_{12} \times y_{22} = \left(30,40,50\right), y_{13} \times y_{23} = \left(65,85,115\right), y_{11},\ y_{12},\ y_{13},\ y_{21},\ y_{22},\ y_{23} \in TF\left(R\right). \end{aligned} \tag{A7}$$

Total FS = (125, 165, 225) = total FD so FTP is appropriate. Regarding the problem (Eq.A6), this problem can be translated into the following FTP:

$$\begin{aligned} &Min\ 20\left(y_{11}\right)m + 60\left(y_{12}\right)m + 90\left(y_{13}\right)m + 70\left(y_{21}\right)m + 85\left(y_{22}\right)m + 35\left(y_{23}\right)m, \\ &30\left(y_{11}\right)d + 70\left(y_{12}\right)d + 100\left(y_{13}\right)d + 80\left(y_{21}\right)d + 95\left(y_{22}\right)d + 45\left(y_{23}\right)d, \\ &40\left(y_{11}\right)v + 90\left(y_{12}\right)v + 110\left(y_{13}\right)v + 90\left(y_{21}\right)v\ + 115\left(y_{22}\right)v + 55\left(y_{23}\right)v \end{aligned}$$

Subject to Constraints:

$$\left(y_{11}\right)m + \left(y_{12}\right)m + \left(y_{13}\right)m = 80, \left(y_{11}\right)d + \left(y_{12}\right)d + \left(y_{13}\right)d = 100, \left(y_{11}\right)v + \left(y_{12}\right)v + \left(y_{13}\right)v = 130,$$

$$\begin{aligned}&(y_{21})m+(y_{22})m+(y_{23})m=50,(y_{21})d+(y_{22})d+(y_{23})d=70,(y_{21})v+(y_{22})v+(y_{23})v=100,\\&(y_{11})m+(y_{21})m=40,(y_{11})d+(y_{21})d=50,(y_{11})v+(y_{21})v=70,(y_{12})m+(y_{22})m=30,\\&(y_{12})d+(y_{22})d=40,(y_{12})v+(y_{22})v=50,(y_{13})m+(y_{23})m=65,(y_{13})d+(y_{23})d=85,\\&(y_{13})v+(y_{23})v=115,\\&(y_{jk})d-(y_{jk})m\ge 0,(y_{jk})v-(y_{jk})d\ge 0,(y_{jk})m\ge 0,\ j=1,\ 2;k=1,\ 2,\ 3\end{aligned} \tag{A8}$$

In this case, regarding the problem (Eq.A4), we will first fix the following TP net to get the left side of Eq.A7:

$$Min\ 20(y_{11})m+60(y_{12})m+90(y_{13})m+70(y_{21})m+85(y_{22})m+35(y_{23})m,$$

Subject to Constraints:

$$\begin{aligned}&(y_{11})m+(y_{12})m+(y_{13})m=80,(y_{21})m+(y_{22})m+(y_{23})m=50,(y_{11})m+(y_{21})m=40,\\&(y_{12})m+(y_{22})m=30,(y_{13})m+(y_{23})m=65,(y_{11})m,(y_{12})m,(y_{13})m,(y_{21})m,(y_{22})m,(y_{23})m\ge 0\end{aligned} \tag{A9}$$

Fix TP (Eq.A9) using the standard transportation simplex algorithm that delivers the best OS and OV:

$$(y_{11})m=40,(y_{12})m=30,(y_{13})m=20,(y_{21})m=5,(y_{22})m=5,(y_{23})m=50,(d\times y)m=5030. \tag{A10}$$

Now based on the OS (Eq.A10) and related to the problem (Eq.A5), we are setting up a limited LPP to access the free OS space of FTP (Eq.A7):

$$Min\ 30(y_{11})d+70(y_{12})d+100(y_{13})d+80(y_{21})d+95(y_{22})d+45(y_{23})d$$

Subject to Constraints:

$$\begin{aligned}&(y_{11})d+(y_{12})d+(y_{13})d=100,(y_{21})d+(y_{22})d+(y_{23})d=70,(y_{11})d+(y_{21})d=50,\\&(y_{12})d+(y_{22})d=40,(y_{13})d+(y_{23})d=85,\\&(y_{11})d\ge 40,(y_{12})d\ge 30,(y_{13})d\ge 20,(y_{21})d\ge 0,(y_{22})d\ge 0,(y_{23})d\ge 50.\end{aligned} \tag{A11}$$

Classical TP can be corrected using the method provided above. OS and OV (Eq.A11) problems are found as follows:

$$(y_{11})d=50,(y_{12})d=40,(y_{13})d=20,(y_{21})d=5,(y_{22})d=5,(y_{23})d=70,(d\times y)d=7930. \tag{A12}$$

Also, regarding OS (Eq.A12) and Troubleshooting (Eq.A6), we solve the following bounded LPP to find the true fuzzy OS core of Eq.A7:

$$Min\ 40(y_{11})v+90(y_{12})v+130(y_{13})v+90(y_{21})v+115(y_{22})v+55(y_{23})v$$

Subject to Constraints:

$$(y_{11})v+(y_{12})v+(y_{13})v=130,(y_{21})v+(y_{22})v+(y_{23})v=100,$$

$$(y_{11})v+(y_{21})v=70,(y_{12})v+(y_{22})v=50,(y_{13})v+(y_{23})v=115, \tag{A13}$$
$$(y_{11})v\geq 50,(y_{12})v\geq 40,(y_{13})v\geq 20,(y_{21})v\geq 5,(y_{22})v\geq 5,(y_{23})v\geq 70.$$

OS and OV (Eq.A11) problems are found as follows:

$$(y_{11})v=70,(y_{12})v=50,(y_{13})v=20,(y_{21})v=5,(y_{22})v=5,(y_{23})v=100,(d\times y)v=12930. \tag{A14}$$

Finally, based on OS (Eq.A10), (Eq.A12) and (Eq.A14), the OS and OV of Eq.A7 are offered as follows:

$$\begin{aligned}
&y=y_{11}=(y_{11})m,(y_{11})d,(y_{11})v=(40,\ 50,\ 70);y=y_{12}=(y_{12})m,(y_{12})d,(y_{12})v=(30,\ 40,\ 50)\\
&y=y_{13}=(y_{13})m,(y_{13})d,(y_{13})v=(20,\ 20,\ 20);y=y_{21}=(y_{21})m,(y_{21})d,(y_{21})v=(5,\ 5,\ 5)\\
&y=y_{22}=(y_{22})m,(y_{22})d,(y_{22})v=(5,\ 5,\ 5);y=y_{23}=(y_{23})m,(y_{23})d,(y_{23})v=(50,\ 70,\ 100)
\end{aligned}$$

$$d\times y=\sum_{j=1}^{n}\sum_{k=1}^{o}d_{jk}\times y_{jk}=(5030,7930,12930) \tag{A15}$$

The minimum TC minimum can be converted to the default OS. Using the required method, the minimum nonlinear TC is w = (5030,7930,12930) which can be translated as follows:

- The minimum TC total is 5030 minutes.
- The maximum possible TC is 7930 minutes.
- The minimum travel time is 12930 minutes.

This result indicates that the minimum TC minimum will be more than 5030 minutes and less than 12930 minutes and the maximum chance is that the minimum TC total is 7930 minutes. It can be seen that there is no negative aspect of the OS that fuzzy got in the fuzzy TC collection, as you consider the existing system, if we apply the general distribution system fuzzy switch to find the OS without issue. An Eq.A6 with the help of IFBFS, obtained based on generalized fuzzy northwest corner system, then we get the following fuzzy OS and there is a negative part in the FQ of the product y_{13} which should be changed by originally from third place and therefore no physical meaning.

$$\begin{aligned}
&y=y_{11}=(y_{11})m,(y_{11})d,(y_{11})v=(40,\ 50,\ 70);y=y_{12}=(y_{12})m,(y_{12})d,(y_{12})v=(30,\ 40,\ 50)\\
&y=y_{13}=(y_{13})m,(y_{13})d,(y_{13})v=(20,\ 20,\ 20);y=y_{21}=(y_{21})m,(y_{21})d,(y_{21})v=(5,\ 5,\ 5)\\
&y=y_{22}=(y_{22})m,(y_{22})d,(y_{22})v=(5,\ 5,\ 5)
\end{aligned} \tag{A16}$$

Also, the levels y_{13} and fuzzy OS (Eq.A15) and (Eq.A16) are equal, i.e. R (-40, 20, 70) = R (20, 20, 20) = 20, where obviously. (-40, 20, 70) ≠ (20, 20, 20) and this indicates another setback of the method, which used the degree function to adjust the FTP. Finally, to get the base OS of Eq.A7 based on the problem (Eq.A3.0), we will fix the following LPPs as follows:

$$\begin{aligned}
&Min\ \left(20(y_{11})m+55(y_{11})d+40(y_{11})v\right)/4+\left(60(y_{12})m+135(y_{12})d+90(y_{12})v\right)/4\\
&+\left(90(y_{13})m+195(y_{13})d+110(y_{13})v\right)/4+\left(70(y_{21})m+155(y_{21})d+90(y_{21})v\right)/4\\
&+\left(85(y_{22})m+185(y_{22})d+115(y_{22})v\right)/4+\left(35(y_{23})m+85(y_{23})d+55(y_{23})v\right)/4
\end{aligned} \tag{A17}$$

Subject to Constraints of Problem (Eq.A8).

The classic LPP provides an unparalleled OS of Eq.A7 that is compatible with the rare OS (Eq.A15) obtained on the basis of the system we recommended. However, there are two main reasons why we consider the proposed method as follows:

- The classical LPP (Eq.A17) applied to fix FTP (Eq.A7) is not a modified LPP, while the problem (Eq.A9), (Eq.A11) and (Eq.A13) fix FTP (Eq. .A7) as classic TPs. To fix TPs, a tabular method is preferred over the LP method [11] so it may be recommended to use the expected method instead of the current method to fix FTPs.
- The primitive LPP (Eq.A17) applied to fix FTP (Eq.A7) has 27 obstacles and 18 variables, while the problem (Eq.A9) has 5 obstacles and 6 variables. There is a similar comparison between rates of prevention and change problems (Eq.A11) and (Eq.A13) and (Eq.A17). Therefore, using the problem (Eq.A9), (Eq.A11) and (Eq.A13) to fix FTP (Eq.A7) is a big deal compared to the problem (Eq.A17) based on the assumption of math. , about the number of restrictions and changes. In summary, it is better to use the methods we have recommended than the existing methods to resolve FTP from a conventional view.

7. Conclusion

A large number of TPs at different levels of sophistication have been documented in the paper. However, some of these models have a small global application because common TPs take net data for TC, requesting and query values. Unlike conventional TPs, we analyzed inaccurate data on real TP and developed a simpler method and corrected these weaknesses in the form described. In the FTP discussed in this study, non-negative triangular FNs represent all aspects of the problem. In this article, we are trying to create some important LPPs to fix FTP, which have fewer restrictions and changes compared to other existing LPPs. In particular, the proposed method can be useful in large-scale applications. Since the proposed system is based on the ancient travel simplex algorithm, it is easy to learn and apply the required system to get a real-time FTP operating system in the application world. One of the advantages of the recommended methods is that the fuzzy OS acquired and the best advantage are the negative triangular FNs. Finally, we believe that there are many more studies that need to be further explored. Some of these points are discussed below:

- The recommended method works well in fixing FTP because all invalid parameters are represented as triangular FNs. The consolidation of this system to find the free OS of TP and trapezoidal FN will be an interesting research project in the future.
- STP assesses supply, demand and transportation to meet transportation needs effectively. Therefore, research on the topic to create the process required to obtain non-trivial OV from nonlinear STP when price, supply and required quantities in non-triangular FNs travel capacity are negative, as leave it at another checkpoint.

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Minimax Estimation of the Scale Parameter of Inverse Rayleigh Distribution under Symmetric and Asymmetric Loss Functions

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Abstract

In this article, minimax estimation of the scale parameter λ of the inverse Rayleigh distribution is performed under symmetric (QLF) and asymmetric (SLELF and GELF) loss functions by applying the Lehmann's theorem (1950). An extended Jeffrey's prior and gamma prior are assumed to derive the minimax estimators under each of the considered loss functions. An extensive simulation study is carried out to compare the performance of the minimax estimators with the maximum likelihood (MLE), which is traditionally used as a classical estimator, on the basis of biases and mean squared errors (MSE). The obtained results suggest that under the assumption of extended Jeffrey's prior, minimax estimators with positive c values are superior as compared to the MLE. Moreover, it is found that in most of the cases, minimax estimator under quadratic loss function (QLF) performs satisfactory on the assumption of gamma prior.

Keywords: Minimax estimator, squared log error loss function, quadratic loss function, general entropy loss function, extended Jeffrey's prior, risk function

1. Introduction

Minimax estimation is a Bayesian estimation approach in statistical inference, which was introduced by Wald [1] relating to the concept of Game theory. It brings different dimensions to statistical estimation and improves the point estimation process. In recent years, a vast amount of research works have been devoted to study the minimax estimators of some well known distributions. Roy et al. [4] developed the minimax estimation of the scale parameter of the Weibull distribution using Quadratic and MLINEX loss functions. The minimax estimator of the scale parameter of Rayleigh distribution under Quadratic loss function was investigated in [5]. Li [3] discussed the Minimax estimation of the parameter of Maxwell distribution under different loss functions considering non-informative quasi-prior density. The problem of finding the minimax estimator of the scale parameter in a class of lifetime distributions under different loss functions are discussed in [2].

The fundamental differences between the classical and minimax estimation approach is that in classical estimation the parameter is assumed to be a fixed point, whereas in minimax estimation the parameter of interest is considered to be a random variable. The most important element in the minimax approach is the specification of a distribution function on the parameter space, which is called prior distribution. In addition to the prior distribution the assumed loss functions also have a significant impact on the minimax estimator for a given model. Recently, in literature the inverted version of a standard probability distribution got a lot of attention by many researchers including [6], [7], [8]. In this study, our concern is to derive the minimax estimator of the unknown scale parameter λ of the inverse Rayleigh distribution having the following probability density function

$$f(x;\lambda) = \frac{2\lambda}{x^3} exp\left[-\left(\frac{\lambda}{x^2}\right)\right]; \quad x > 0,\ \lambda > 0. \tag{1}$$

For modeling lifetime data, inverse Rayleigh (IR) distribution, which is a special case of inverse Weibull (IW) distribution has many applications in reliability studies. Voda [9] discussed some statistical properties of IR distribution like maximum likelihood estimator, confidence intervals etc. Bayes estimators for the parameter of inverse Rayleigh distribution under squared error and zero one loss functions based on lower record values are studied by Soliman et al. [10]. Bayesian estimation of the parameter and reliability function of an inverse Rayleigh distribution under symmetric and asymmetric linear exponential loss functions using a non-informative prior has been done in [11].

The aim of this article is to make a comparison between the maximum likelihood estimator (MLE) and minimax estimators of the scale parameter of inverse Rayleigh distribution under three different loss functions. These are quadratic loss function, which is symmetric in nature and another two are asymmetric loss functions, namely, squared log error and general entropy loss functions. As a prior knowledge of the unknown scale parameter λ, we consider both non-informative and informative prior. In case of non-informative prior, our choice is the extended Jeffrey's prior which is also a generalization of the Jeffrey's prior and for informative prior, gamma prior is chosen which is also conjugate in structure. The Bayes estimates of λ as well as the risk functions are derived under the mentioned loss functions and further by applying *Lehmann's* theorem, it is shown that the obtained estimators are also the minimax estimators.

The rest of the article is structured in following manner. In section 2, maximum likelihood estimator for the scale parameter λ is derived. In section 3, we discuss about the prior and posterior distributions of λ by considering both the non informative and informative prior respectively. Bayes estimators under quadratic loss (QLF), squared log error loss (SLELF) and general entropy loss (GELF) functions for the scale parameter of the inverse Rayleigh distribution are developed in section 4. In section 5, minimax estimators under different loss functions are discussed. Extensive simulation study for different parameter choices are performed and results are presented in section 6. Finally in section 7, the conclusion of the paper is provided.

2. Maximum Likelihood Estimation

Several desirable properties for a good estimator such as consistency, asymptotic efficiency, invariance property etc. are satisfied by the Maximum likelihood estimator. This makes the MLE one of the most frequently used techniques for parameter estimation. Let $x_1, x_2, \cdots, x_n$ be a random sample of size n from the density function (1). Then the likelihood function is given by

$$L\left(x_i;\lambda\right) = (2\lambda)^n \prod_{i=1}^{n} \frac{1}{x_i^3} e^{-\lambda \sum_{i=1}^{n} \frac{1}{x_i^2}}. \tag{2}$$

Taking logarithm, the log-likelihood function becomes

$$ln\, L\left(x_i;\lambda\right) = n\, ln\, 2 + n\, ln\, \lambda + \sum_{i=1}^{n} ln\left(\frac{1}{x_i^3}\right) - \lambda \sum_{i=1}^{n} \frac{1}{x_i^2}.$$

Now, by differentiating the above equation with respect to λ and equating it with zero, we obtain the MLE of λ as,

$$\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i^2}}. \tag{3}$$

3. Prior and Posterior density function of Scale parameter λ

Specification of a prior distribution over the parameter space is a substantial part for deriving the posterior probability distribution under the Bayesian paradigm. The posterior distribution is defined as proportional to the likelihood function for the data multiplied by the prior information for the parameter(s), which is useful for future inferences and prediction. In literature, there is no specification about the choice of prior from which one can conclude the superiority of one prior over the others. Generally, selection of prior(s) is based on ones subjective knowledge and beliefs. However, it is preferable to use informative prior when sufficient information about the parameter(s) of interest is available, otherwise it is better to use non-informative prior [12]. Here we consider both type of prior distributions for estimating the unknown scale parameter λ.

3.1. Posterior distribution under the assumption of extended Jeffrey's prior

The extended Jeffrey's prior was proposed by Al-Kutobi [13] and given as

$$\Pi(\lambda) \propto [I(\lambda)]^c; \;\; c \in \mathbb{R}^+$$

where, $I(\lambda) = -nE\left(\frac{\partial^2 lnf(x;\lambda)}{\partial\lambda^2}\right)$ is the Fisher's information matrix. From the probability model (1) we found $I(\lambda) = \frac{n}{\lambda^2}$ and therefore, the extended Jeffrey's prior becomes

$$\Pi_1(\lambda) \propto \left(\frac{n}{\lambda^2}\right)^c. \tag{4}$$

The prior distribution (4) and the likelihood function (2) are combined to get the posterior distribution of λ and it is given by

$$\Pi_1\left(\lambda|\underset{\sim}{X}\right) = \frac{\left(\sum_{i=1}^n \frac{1}{x_i^2}\right)^{n-2c+1}}{\Gamma(n-2c+1)}\lambda^{n-2c}e^{-\lambda\sum_{i=1}^n \frac{1}{x_i^2}}.$$

Therefore, the distribution of $\lambda|\underset{\sim}{X}$ can be written as $G\left(n-2c+1, \sum_{i=1}^n \frac{1}{x_i^2}\right)$.

Remark 1. Extended Jeffrey's prior is the generalized version of many non informative priors. We get Jeffrey's prior if we replace c with $\frac{1}{2}$. Also it reduces to Hartigan's prior when c = $\frac{3}{2}$.

3.2. Posterior distribution under the assumption of Gamma prior

The gamma distribution with known hyperparameters α and p, is considered here as an informative prior for the parameter λ. For the inverse Rayleigh distribution, gamma prior also becomes the conjugate prior as the posterior distribution belongs to the gamma family.

For, $\lambda \sim Gamma(\alpha, p)$ the prior density becomes

$$\Pi_2(\lambda) = \frac{p^\alpha}{\Gamma\alpha}\lambda^{\alpha-1}e^{-p\lambda}; \;\; \lambda > 0, \; \alpha > 0, \; p > 0. \tag{5}$$

Now, combining the prior distribution (5) and the likelihood function (2) the posterior distribution of λ takes the form

$$\begin{aligned}\Pi_2\left(\lambda|\underset{\sim}{X}\right) &= \frac{\lambda^{n+\alpha-1}e^{-\lambda\left[\sum_{i=1}^n \frac{1}{x_i^2}+p\right]}}{\int_0^\infty \lambda^{n+\alpha-1}e^{-\lambda\left[\sum_{i=1}^n \frac{1}{x_i^2}+p\right]}d\lambda} \\ &= \frac{\left(\sum_{i=1}^n \frac{1}{x_i^2}+p\right)^{n+\alpha}}{\Gamma(n+\alpha)}\lambda^{n+\alpha-1}e^{-\lambda\left[\sum_{i=1}^n \frac{1}{x_i^2}+p\right]}.\end{aligned}$$

Therefore, the distribution of $\lambda|\underset{\sim}{X}$ can be written as $G\left(n+\alpha, \sum_{i=1}^n \frac{1}{x_i^2}+p\right)$.

4. Bayes Estimation of Scale parameter λ under different Loss functions

The selection of an appropriate loss function $L(\lambda, \hat{\lambda})$ in Bayesian Inference is a major aspect for the estimation of unknown parameter. Most of the research works on point estimation and prediction considered the underlying loss function as squared error due to its elegant statistical properties and mathematical simplicity. The reason being that it is symmetric in nature and assigns equal importance to the overestimation and underestimation of the parameter. In many practical situations when the loss is not symmetric, use of squared error loss function (SELF) is inappropriate. Basu and Ebrahimi [14] pointed out that overestimation and underestimation have different consequences. Thus, in order to make the statistical inference more practical and applicable we often use asymmetric loss function. In this present study, we consider both the symmetric and asymmetric loss functions to derive the Bayes estimate of λ.

4.1. Estimation under Quadratic loss function

Here we consider the quadratic loss function (QLF) for obtaining the Bayes estimate under the assumption of both non informative and informative prior simultaneously. It is well known that, SELF is useful for estimation of location parameter but in case of scale parameter a modified form of this loss, known as QLF is preferable and it is defined as follows [15]

$$L_1(\lambda, \hat{\lambda}) = \left(\frac{\hat{\lambda} - \lambda}{\lambda}\right)^2,$$

which is a non-negative, symmetric and continuous loss function.
The risk function under QLF is denoted by $R_{QLF}\left(\lambda, \hat{\lambda}\right)$ and is defined as

$$\begin{aligned} R_{QLF}\left(\lambda, \hat{\lambda}\right) &= E\left[L_1(\lambda, \hat{\lambda})\right] \\ &= 1 - 2\hat{\lambda} E\left(\lambda^{-1} | \underset{\sim}{X}\right) + \hat{\lambda}^2 E\left(\lambda^{-2} | \underset{\sim}{X}\right). \end{aligned} \tag{6}$$

By differentiating the above risk function with respect to $\hat{\lambda}$ and equating it to zero, we will get the Bayes estimate for which the risk would be minimized. Hence under QLF the Bayes estimate of λ takes the form as

$$\hat{\lambda}_{QLF} = \frac{E\left(\lambda^{-1} | \underset{\sim}{X}\right)}{E\left(\lambda^{-2} | \underset{\sim}{X}\right)}. \tag{7}$$

Now, based on the extended Jeffrey's prior we have

$$E\left(\lambda^{-1} | \underset{\sim}{X}\right) = \frac{1}{n - 2c}\left(\sum_{i=1}^{n} \frac{1}{x_i^2}\right) \text{ and}$$
$$E\left(\lambda^{-2} | \underset{\sim}{X}\right) = \frac{\Gamma\left(n - 2c - 1\right)}{\Gamma\left(n - 2c + 1\right)}\left(\sum_{i=1}^{n} \frac{1}{x_i^2}\right)^2.$$

Therefore, by putting these values in (7), we obtain the Bayes estimate of λ under QLF based on Extended Jeffrey's prior as

$$\hat{\lambda}_{QLF_1} = \frac{(n-2c)\,\Gamma\left(n-2c\right)}{(n-2c)\,\Gamma\left(n-2c-1\right)} \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{x_i^2}\right)} = \frac{\Gamma\left(n-2c\right)}{\Gamma\left(n-2c-1\right)} \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{x_i^2}\right)}. \tag{8}$$

Similarly, based on the assumption of gamma prior we have

$$E\left(\lambda^{-1} | \underset{\sim}{X}\right) = \frac{\Gamma\left(n + \alpha - 1\right)}{\Gamma\left(n + \alpha\right)}\left(\sum_{i=1}^{n} \frac{1}{x_i^2} + p\right) \text{ and}$$

$$E\left(\lambda^{-2}|\underline{X}\right) = \frac{\Gamma\left(n+\alpha-2\right)}{\Gamma\left(n+\alpha\right)}\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)^2.$$

After putting these values in (7), we find the Bayes estimate of λ under QLF based on gamma prior as

$$\hat{\lambda}_{QLF_2} = \frac{\Gamma\left(n+\alpha-2+1\right)}{\Gamma\left(n+\alpha-2\right)}\frac{1}{\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)} = \frac{\left(n+\alpha-2\right)}{\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)}. \tag{9}$$

4.2. Estimation under Squared log error loss function

In order to obtain the Bayes estimate of λ, we consider the squared log error loss function (SLELF) which is proposed by Brown [16] and defined as

$$L_2\left(\lambda,\hat{\lambda}\right) = \left(ln\hat{\lambda}-ln\lambda\right)^2 = \left(ln\frac{\hat{\lambda}}{\lambda}\right)^2,$$

where both $\hat{\lambda}$ and λ are positive. This is a balanced loss function with $lim\, L_2\left(\lambda,\hat{\lambda}\right) \to \infty$ as $\hat{\lambda} \to 0$ or ∞. A balanced loss function considers both estimation error and goodness of fit, while an unbalanced loss function only considers estimation error [18]. This loss is asymmetric and convex [17]. It is convex when $\frac{\hat{\lambda}}{\lambda} \leq e$, and concave otherwise, but its risk function is minimum with respect to $\hat{\lambda}_{SLELF}$.

The risk function under SLELF is denoted by $R_{SLELF}\left(\lambda,\hat{\lambda}\right)$ and expressed as

$$\begin{aligned} R_{SLELF}\left(\lambda,\hat{\lambda}\right) &= E\left[L_2\left(\lambda,\hat{\lambda}\right)\right] \\ &= \left(ln\hat{\lambda}\right)^2 - 2ln\hat{\lambda}E\left[ln\lambda|\underline{X}\right] + E\left[\left(ln\lambda\right)^2|\underline{X}\right]. \end{aligned} \tag{10}$$

Now, by differentiating the risk function with respect to $\hat{\lambda}$ and equating it to zero, we will be able to find the Bayes estimate for which the above risk is minimized. Hence under SLELF, we obtain the Bayes estimate of λ which have the following expression

$$\hat{\lambda}_{SLELF} = exp\left[E\left(ln\lambda|\underline{X}\right)\right]. \tag{11}$$

So, we calculate $\left[E\left(ln\lambda|\underline{X}\right)\right]$ by using the posterior density derived under both the extended Jeffrey's prior and gamma prior respectively.

Hence, under the assumption of the extended Jeffrey's prior

$$E\left(ln\lambda|\underline{X}\right) = \Psi\left(n-2c+1\right) - ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right), \tag{12}$$

$$E\left((ln\lambda)^2|\underline{X}\right) = \frac{\Gamma''\left(n-2c+1\right)}{\Gamma\left(n-2c+1\right)} - 2\Psi\left(n-2c+1\right)ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right) + \left\{ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)\right\}^2 \tag{13}$$

where, $\Psi\left(n-2c+1\right) = \frac{\Gamma'(n-2c+1)}{\Gamma(n-2c+1)}$, is a digamma function.

Similarly, under the gamma prior, expressions are

$$E\left(ln\lambda|\underline{X}\right) = \Psi\left(n+\alpha\right) - ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right), \tag{14}$$

$$E\left((ln\lambda)^2|\underline{X}\right) = \frac{\Gamma''\left(n+\alpha\right)}{\Gamma\left(n+\alpha\right)} - 2\Psi\left(n+\alpha\right)ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right) + \left\{ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)\right\}^2 \tag{15}$$

where, $\Psi\left(n+\alpha\right)=\frac{\Gamma'(n+\alpha)}{\Gamma(n+\alpha)}$, is a digamma function.

Therefore, to obtain the Bayes estimate of the parameter λ under SLELF based on both prior assumptions, we substitute the expressions (12) and (14) respectively in (11). After simplification, we get,

$$\hat{\lambda}_{SLELF_1} = \frac{e^{\Psi(n-2c+1)}}{\sum_{i=1}^{n}\frac{1}{x_i^2}} \text{ and} \tag{16}$$

$$\hat{\lambda}_{SLELF_2} = \frac{e^{\Psi(n+\alpha)}}{\sum_{i=1}^{n}\frac{1}{x_i^2}+p}. \tag{17}$$

4.3. Estimation under General entropy loss function

Another well known asymmetric loss function is general entropy loss function (GELF) proposed by Calabria and Pulcini [19]. Many authors like [20], [21] referred this loss as the modified linear exponential (MLINEX) loss function and defined as

$$L_3\left(\lambda,\hat{\lambda}\right)=\omega\left[\left(\frac{\hat{\lambda}}{\lambda}\right)^{\gamma}-\gamma\, ln\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right];\;\; \omega>0, \gamma\neq 0.$$

The constant γ, involved in the loss function is the shape parameter and indicates the deviation from symmetry. It is clear that if the value of the shape parameter $\gamma=1$, this loss reduces to the entropy loss function which is also used by several authors like [22], [23] etc. Dey [11] mentioned that if we replace $\left(\hat{\lambda}-\lambda\right)$ in place of $ln\left(\frac{\hat{\lambda}}{\lambda}\right)$ i.e. $ln\hat{\lambda}-ln\lambda$, linear exponential (LINEX) loss function has been obtained, which is proposed by Zellner [24].

Now by considering GELF, the expression of the risk function denoted as $R_{GELF}\left(\lambda,\hat{\lambda}\right)$ is given below

$$\begin{aligned} R_{GELF}\left(\lambda,\hat{\lambda}\right) &= E\left[L_3\left(\lambda,\hat{\lambda}\right)\right] \\ &= \omega\hat{\lambda}^{\gamma}E\left(\lambda^{-\gamma}|\underline{X}\right)-\omega\,\gamma\, ln\hat{\lambda}+\omega\,\gamma\, E\left(ln\lambda|\underline{X}\right)-\omega. \end{aligned} \tag{18}$$

So, for minimizing the risk function we differentiate the above equation with respect to $\hat{\lambda}$ and equate it to zero. After simplification, we have

$$\hat{\lambda}_{GELF}=\left[E\left(\lambda^{-\gamma}|\underline{X}\right)\right]^{-\frac{1}{\gamma}}. \tag{19}$$

Now, we solve the above expression by considering extended Jeffrey's prior and gamma prior simultaneously. Therefore, under the extended Jeffrey's prior

$$E\left(\lambda^{-\gamma}|\underline{X}\right)=\frac{\Gamma\left(n-2c-\gamma+1\right)}{\Gamma\left(n-2c+1\right)}\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{\gamma}$$

and under the gamma prior

$$E\left(\lambda^{-\gamma}|\underline{X}\right)=\frac{\Gamma\left(n+\alpha-\gamma\right)}{\Gamma\left(n+\alpha\right)}\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)^{\gamma}.$$

After substituting the values of $E\left(\lambda^{-\gamma}|\underline{X}\right)$ in (19), we have the following Bayes estimators under both non-informative and informative prior respectively,

$$\hat{\lambda}_{GELF_1}=\left[\frac{\Gamma(n-2c-\gamma+1)}{\Gamma(n-2c+1)}\right]^{-\frac{1}{\gamma}}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{x_i^2}}\right), \tag{20}$$

$$\hat{\lambda}_{GELF_2} = \left[\frac{\Gamma(n+\alpha-\gamma)}{\Gamma(n+\alpha)}\right]^{-\frac{1}{\gamma}} \left(\frac{1}{\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)}\right). \tag{21}$$

5. Minimax Estimators

In this section, we derive the minimax estimators of the scale parameter λ under symmetric (QLF) and asymmetric (SLELF and GELF) loss functions. Bayes estimators are derived primarily and then minimax etimators are obtained by applying the *Lehmann's* theorem, which can be described as follows.

Theorem 1. Suppose, $\tau = \{F_\theta; \theta \in \Theta\}$ be a family of distribution functions and D is a class of estimators of θ. Let, $d^* \in$ D is a Bayes estimator against a prior distribution $\xi^*(\theta)$ on the parameter space Θ and the risk function $R(d^*, \theta)$ = constant on Θ; then d^* is a minimax estimator of θ.

The motivation behind this study is to check whether the risk functions developed in 4 are constant or not for the corresponding Bayes estimators. If the risk functions are constant then according to the *Lehmann's* theorem, the respective Bayes estimators are minimax estimators.

First of all, to verify the above *Lehmann's* theorem we consider the quadratic loss function. The risk function (6) is derived after considering the Bayes estimators (8) and (9) for both the non-informative and informative prior respectively. So, the risk function $R_{QLF}(\lambda, \hat{\lambda})$ for the estimators $\hat{\lambda}_{QLF_1}$ and $\hat{\lambda}_{QLF_2}$ becomes

$$\begin{aligned}
&R_{QLF}(\lambda, \hat{\lambda}_{QLF_1})\\
=&1-2\hat{\lambda}_{QLF_1}E\left(\lambda^{-1}|\underset{\sim}{X}\right)+\hat{\lambda}^2_{QLF_1}E\left(\lambda^{-2}|\underset{\sim}{X}\right)\\
=&1-2\left\{\frac{\Gamma(n-2c)}{\Gamma(n-2c-1)}\frac{1}{\sum_{i=1}^{n}\frac{1}{x_i^2}}\frac{1}{n-2c}\sum_{i=1}^{n}\frac{1}{x_i^2}\right\}+\left(\frac{\Gamma(n-2c)}{\Gamma(n-2c-1)}\right)^2\frac{1}{\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^2}\frac{\Gamma(n-2c-1)}{\Gamma(n-2c+1)}\\
&\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^2\\
=&1-2\left(\frac{n-2c-1}{n-2c}\right)+\frac{n-2c-1}{n-2c}\\
=&\frac{1}{n-2c}; \quad \text{which is a constant and}
\end{aligned}$$

$$\begin{aligned}
&R_{QLF}(\lambda, \hat{\lambda}_{QLF_2})\\
=&1-2\hat{\lambda}_{QLF_2}E\left(\lambda^{-1}|\underset{\sim}{X}\right)+\hat{\lambda}^2_{QLF_2}E\left(\lambda^{-2}|\underset{\sim}{X}\right)\\
=&1-2\left\{\frac{n+\alpha-2}{\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)}\frac{\Gamma(n+\alpha-1)}{\Gamma(n+\alpha)}\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)\right\}+\frac{(n+\alpha-2)^2}{\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)^2}\frac{\Gamma(n+\alpha-2)}{\Gamma(n+\alpha)}\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)^2\\
=&1-2\left(\frac{n+\alpha-2}{n+\alpha-1}\right)+\frac{n+\alpha-2}{n+\alpha-1}\\
=&\frac{1}{n+\alpha-1}; \quad \text{which is also constant.}
\end{aligned}$$

Therefore, as per the Lehmann's theorem, $\hat{\lambda}_{QLF_1}$ and $\hat{\lambda}_{QLF_2}$ are the minimax estimators of the scale parameter λ under the quadratic loss function for extended Jeffrey's prior and gamma prior respectively.

Next for SLELF, we use the Bayes estimators (16) and (17) in (10) to obtain the risk functions corresponding to the Bayes estimators $\hat{\lambda}_{SLELF_1}$ and $\hat{\lambda}_{SLELF_2}$ under the non-informative and informative priors respectively.

$$
\begin{aligned}
&R_{SLELF}\left(\lambda, \hat{\lambda}_{SLELF_1}\right)\\
&= \left(ln\,\hat{\lambda}_{SLELF_1}\right)^2 - 2\,ln\,\hat{\lambda}_{SLELF_1} E\left[ln\lambda|\underset{\sim}{X}\right] + E\left[(ln\lambda)^2\,|\underset{\sim}{X}\right]\\
&= \left(\Psi(n-2c+1) - ln\sum_{1=1}^{n}\frac{1}{x_i^2}\right)^2 - 2\left(\Psi(n-2c+1) - ln\sum_{i=1}^{n}\frac{1}{x_i^2}\right)\left(\Psi(n-2c+1) - ln\sum_{i=1}^{n}\frac{1}{x_i^2}\right)\\
&\quad + \frac{\Gamma''(n-2c+1)}{\Gamma(n-2c+1)} - 2\Psi(n-2c+1)\,ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right) + \left(ln\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^2\\
&= \frac{\Gamma''(n-2c+1)}{\Gamma(n-2c+1)} - \left(\Psi(n-2c+1)\right)^2; \quad \text{which is a constant and}
\end{aligned}
$$

$$
\begin{aligned}
&R_{SLELF}\left(\lambda, \hat{\lambda}_{SLELF_2}\right)\\
&= \left(ln\,\hat{\lambda}_{SLELF_2}\right)^2 - 2\,ln\,\hat{\lambda}_{SLELF_2} E\left[ln\lambda|\underset{\sim}{X}\right] + E\left[(ln\lambda)^2\,|\underset{\sim}{X}\right]\\
&= -\left(\frac{\Gamma'(n+\alpha)}{\Gamma(n+\alpha)}\right)^2 + 2\Psi(n+\alpha)\,ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right) - \left[ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)\right]^2 + \frac{\Gamma''(n+\alpha)}{\Gamma(n+\alpha)}\\
&\quad - 2\Psi(n+\alpha)ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right) + \left[ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)\right]^2\\
&= \frac{\Gamma''(n+\alpha)}{\Gamma(n+\alpha)} - \left(\Psi(n+\alpha)\right)^2; \quad \text{which is also constant.}
\end{aligned}
$$

Therefore, according to the *Lehmann's* theorem the Bayes estimators $\hat{\lambda}_{SLELF_1}$ and $\hat{\lambda}_{SLELF_2}$ are the minimax estimators under SLELF.

Finally, we calculate the risk functions $R_{GELF}\left(\lambda, \hat{\lambda}\right)$ for the Bayes estimators $\hat{\lambda}_{GELF_1}$ and $\hat{\lambda}_{GELF_2}$ respectively, as

$$
\begin{aligned}
&R_{GELF}\left(\lambda, \hat{\lambda}_{GELF_1}\right)\\
&= \omega\hat{\lambda}_{GELF_1}^{\gamma} E\left(\lambda^{-\gamma}|\underset{\sim}{X}\right) - \omega\,\gamma\,ln\hat{\lambda}_{GELF_1} + \omega\,\gamma\,E\left(ln\lambda|\underset{\sim}{X}\right) - \omega\\
&= \omega\frac{\Gamma(n-2c+1)}{\Gamma(n-2c-\gamma+1)\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{\gamma}}\,\frac{\Gamma(n-2c-\gamma+1)\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{\gamma}}{\Gamma(n-2c+1)} - \omega\gamma\left[-\frac{1}{\gamma}ln\frac{\Gamma(n-2c-\gamma+1)}{\Gamma(n-2c+1)}\right.\\
&\quad \left. - ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)\right] + \omega\gamma\left[\Psi(n-2c+1) - ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)\right] - \omega\\
&= \omega\,ln\frac{\Gamma(n-2c-\gamma+1)}{\Gamma(n-2c+1)} + \omega\,\gamma\Psi(n-2c+1); \quad \text{which is a constant and}
\end{aligned}
$$

$$
\begin{aligned}
&R_{GELF}\left(\lambda, \hat{\lambda}_{GELF_2}\right)\\
&= \omega\hat{\lambda}_{GELF_2}^{\gamma} E\left(\lambda^{-\gamma}|\underset{\sim}{X}\right) - \omega\,\gamma\,ln\hat{\lambda}_{GELF_2} + \omega\,\gamma\,E\left(ln\lambda|\underset{\sim}{X}\right) - \omega\\
&= \omega\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha-\gamma)}\left(\frac{1}{\sum_{i=1}^{n}\frac{1}{x_i^2}+p}\right)^{\gamma}\frac{\Gamma(n+\alpha-\gamma)}{\Gamma(n+\alpha)}\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)^{\gamma} - \omega\gamma\left[-\frac{1}{\gamma}ln\frac{\Gamma(n+\alpha-\gamma)}{\Gamma(n+\alpha)}\right.\\
&\quad \left. - ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)\right] + \omega\gamma\left[\Psi(n+\alpha) - ln\left(\sum_{i=1}^{n}\frac{1}{x_i^2}+p\right)\right] - \omega
\end{aligned}
$$

$$=\omega - \omega\gamma \left[-\frac{1}{\gamma} ln \frac{\Gamma(n+\alpha-\gamma)}{\Gamma(n+\alpha)} - ln \left(\sum_{i=1}^{n} \frac{1}{x_i^2} + p \right) \right] + \omega\gamma \left[\Psi(n+\alpha) - ln \left(\sum_{i=1}^{n} \frac{1}{x_i^2} + p \right) \right] - \omega$$

$$=\omega\, ln \frac{\Gamma(n+\alpha-\gamma)}{\Gamma(n+\alpha)} + \omega\gamma\Psi(n+\alpha); \quad \text{which is also constant.}$$

Therefore, according to the *Lehmann's* theorem, both the Bayes estimators $\hat{\lambda}_{GELF_1}$ and $\hat{\lambda}_{GELF_2}$ are the minimax estimators of λ under the extended Jeffrey's prior and gamma prior respectively. So, the minimax estimators under various loss functions are derived and we compare their performances numerically in the next section.

6. Simulation Study

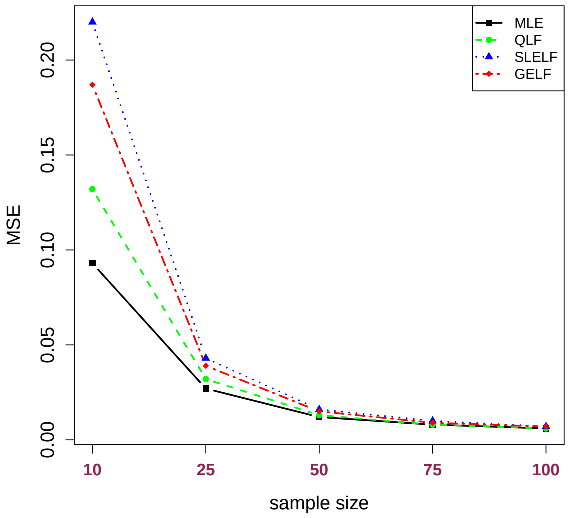
In this section, the numerical comparisons between the minimax estimators and the maximum likelihood estimator have been conducted through an extensive Monte Carlo simulation study. The performance of the estimators is evaluated on the basis of biases and mean squared errors (MSE) criteria. The initial choices of the scale parameter are taken as λ = 0.75 and 1. We generate random samples of sizes n = 10, 25, 50, 75, 100 from (1) by using inverse transformation method and replicate the process for $K = 10,000$ times. Based on these replicated samples, the bias and MSE of the estimators will be calculated by using the following formula,

Table 1: *Estimated values, Bias and MSE of different estimators under extended Jeffrey's when* $\lambda = 0.75$.

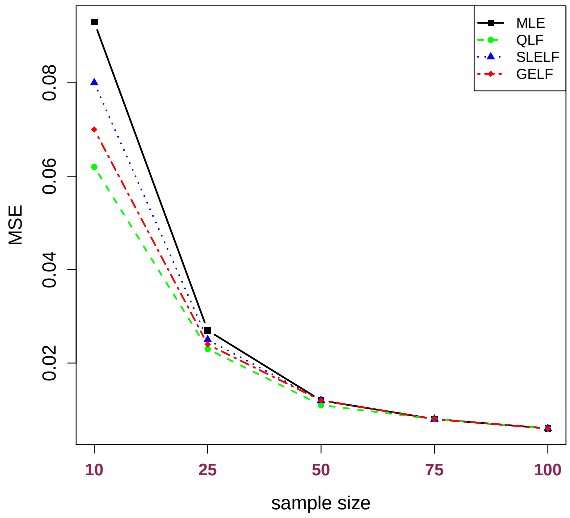
sample sizes(n)	criteria	$\hat{\lambda}_{MLE}$	c=-1 $\hat{\lambda}_{QLF}$	c=-1 $\hat{\lambda}_{SLELF}$	c=-1 $\hat{\lambda}_{GELF}$	c=0.5 $\hat{\lambda}_{QLF}$	c=0.5 $\hat{\lambda}_{SLELF}$	c=0.5 $\hat{\lambda}_{GELF}$	c=1 $\hat{\lambda}_{QLF}$	c=1 $\hat{\lambda}_{SLELF}$	c=1 $\hat{\lambda}_{GELF}$	c=1.5 $\hat{\lambda}_{QLF}$	c=1.5 $\hat{\lambda}_{SLELF}$	c=1.5 $\hat{\lambda}_{GELF}$
10	Estimate	0.833	0.916	1.042	1.000	0.667	0.792	0.75	0.583	0.709	0.667	0.500	0.625	0.583
	Bias	0.083	0.166	0.292	0.250	-0.083	0.042	0.00	-0.167	-0.041	-0.083	-0.250	-0.125	-0.167
	MSE	0.093	0.132	0.220	0.187	0.062	0.080	0.07	0.070	0.064	0.062	0.094	0.064	0.070
25	Estimate	0.779	0.810	0.857	0.841	0.717	0.764	0.748	0.686	0.732	0.717	0.654	0.701	0.686
	Bias	0.029	0.060	0.107	0.091	-0.033	0.014	-0.002	-0.064	-0.018	-0.033	-0.096	-0.049	-0.064
	MSE	0.027	0.032	0.043	0.039	0.023	0.025	0.024	0.024	0.023	0.023	0.027	0.023	0.024
50	Estimate	0.763	0.778	0.801	0.794	0.733	0.756	0.748	0.717	0.740	0.733	0.702	0.725	0.717
	Bias	0.013	0.028	0.051	0.044	-0.017	0.006	-0.002	-0.033	-0.010	-0.017	-0.048	-0.025	-0.033
	MSE	0.012	0.013	0.016	0.015	0.011	0.012	0.012	0.012	0.011	0.011	0.013	0.012	0.012
75	Estimate	0.758	0.768	0.783	0.778	0.738	0.753	0.748	0.727	0.743	0.738	0.717	0.733	0.727
	Bias	0.008	0.018	0.033	0.028	-0.012	0.003	-0.002	-0.023	-0.007	-0.012	-0.033	-0.017	-0.023
	MSE	0.008	0.008	0.010	0.009	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.008	0.008
100	Estimate	0.756	0.764	0.775	0.771	0.741	0.752	0.748	0.733	0.745	0.741	0.726	0.737	0.733
	Bias	0.006	0.014	0.025	0.021	-0.009	0.002	-0.002	-0.017	-0.005	-0.009	-0.024	-0.013	-0.017
	MSE	0.006	0.006	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006

$$\text{Bias}\,(\hat{\lambda}) = \frac{1}{K}\sum_{i=1}^{K}\left(\hat{\lambda}_i - \lambda\right) \quad \text{and} \quad \text{MSE}\,(\hat{\lambda}) = \frac{1}{K}\sum_{i=1}^{K}\left(\hat{\lambda}_i - \lambda\right)^2.$$

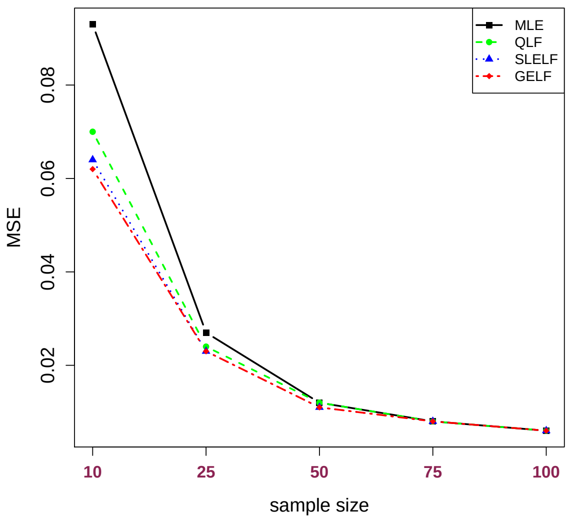
In case of classical estimation, $\hat{\lambda}_{MLE}$ can be easily obtained for K times from the expression (3) for each of the chosen λ with different sample sizes. In Bayesian setup, to obtain the minimax estimators of λ, we consider three different loss functions QLF, SLELF and GELF respectively. For GELF, the value of the shape parameter is fixed at $\gamma = 1$. Now, under the assumption of extended Jeffrey's prior, we choose different values of c, such as, $c = \pm 1, 0.5, 1.5$. It is to be noted that, when $c = 0.5$, then the extended Jeffrey's prior is simplified as Jeffrey's prior and for $c = 1.5$, it reduces to Hartigan's prior. Also, in this empirical study, the choices of hyper parameters are taken as $(\alpha, p) = (0.5, 0.5), (0.5, 5.0), (1.0, 0.25)$ and $(5.0, 5.0)$ under the assumption of gamma prior. For every combinations of (α, p), we calculate the minimax estimators of λ under the three various loss functions. Finally, the average minimax and MLE estimators with their corresponding biases



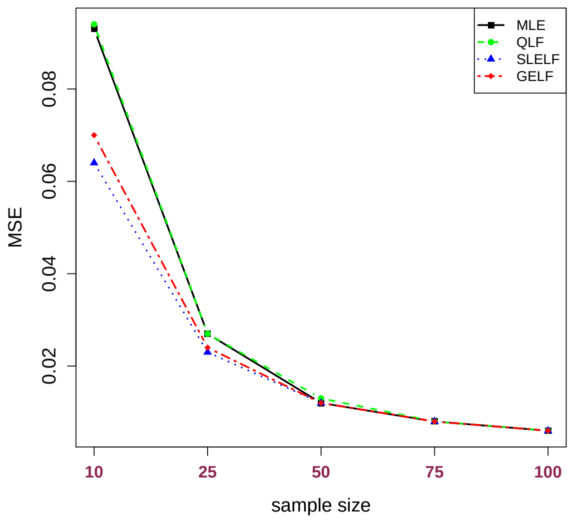
(a) *when* $c = -1$



(b) *when* $c = 0.5$



(c) *when* $c = 1$



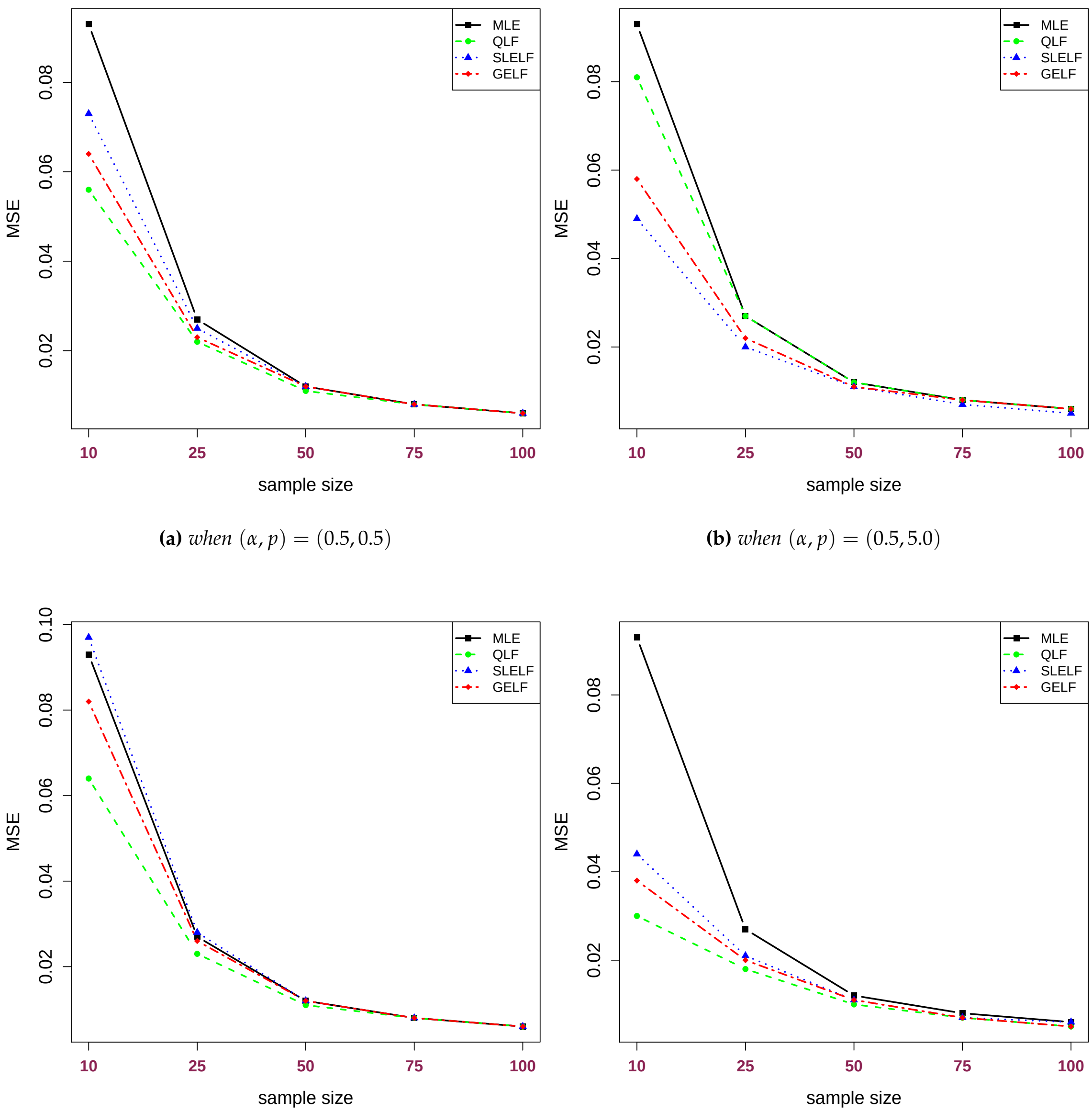
(d) *when* $c = 1.5$

Figure 1: *MSEs of MLE and minimax estimators under extended Jeffrey's prior with different values of c when* $\lambda = 0.75$

and MSE values are summarized in Tables 1 - 2 and 3 - 4 under the extended Jefferey's prior and gamma prior respectively.

In certain cases, a graphical representation of data is a superior representation of information. The aim is to graphically display comparable findings in order to provide a comprehensive evaluation of the estimators based on their biases and MSEs obtained in subsequent tables. The MSE values are plotted in vertical axis against the increasing order of sample sizes in horizontal axis. Here, for instances we only provide the graph for $\lambda = 0.75$ under different conditions both for the extended Jeffrey's and gamma prior. The observations obtained from the simulation results are listed below.

1. When $c = -1$, then it is clearly seen that the MLE is appeared to be better than all the minimax estimators under three loss functions.



(a) *when* $(\alpha, p) = (0.5, 0.5)$

(b) *when* $(\alpha, p) = (0.5, 5.0)$

(c) *when* $(\alpha, p) = (1.0, 0.25)$

(d) *when* $(\alpha, p) = (5.0, 5.0)$

Figure 2: *MSEs of MLE and minimax estimators under gamma prior with different values of hyperparameters when* $\lambda = 0.75$

2. Under Jeffrey's prior $(c = 0.5)$, minimax estimator under QLF has the smallest MSE value.

3. When $c = 1$, minimax estimator under GELF performs better than the other estimators.

4. Under Hartigan's prior $(c = 1.5)$, minimax estimator under the SLELF has the smallest MSE compared to the others estimators. Also, both the MSE of MLE and the minimax estimator under QLF are coincided.

5. It is found from Tables 1 and 2 that Hartigan's prior and Jeffrey's prior are identical when sample size $n > 50$.

6. Under gamma prior, it is observe that in most of the cases the minimax estimator under QLF performs better than the other estimators.

Table 2: *Estimated values, Bias and MSE of different estimators under extended Jeffrey's prior when* $\lambda = 1$.

sample sizes(n)	criteria	$\hat{\lambda}_{MLE}$	c=-1 $\hat{\lambda}_{QLF}$	c=-1 $\hat{\lambda}_{SLELF}$	c=-1 $\hat{\lambda}_{GELF}$	c=0.5 $\hat{\lambda}_{QLF}$	c=0.5 $\hat{\lambda}_{SLELF}$	c=0.5 $\hat{\lambda}_{GELF}$	c=1 $\hat{\lambda}_{QLF}$	c=1 $\hat{\lambda}_{SLELF}$	c=1 $\hat{\lambda}_{GELF}$	c=1.5 $\hat{\lambda}_{QLF}$	c=1.5 $\hat{\lambda}_{SLELF}$	c=1.5 $\hat{\lambda}_{GELF}$
	Estimate	1.111	1.222	1.389	1.333	0.889	1.056	1.000	0.778	0.945	0.889	0.667	0.834	0.778
10	Bias	0.111	0.222	0.389	0.333	-0.111	0.056	0.000	-0.222	-0.055	-0.111	-0.333	-0.166	-0.222
	MSE	0.166	0.236	0.392	0.333	0.111	0.142	0.125	0.125	0.114	0.111	0.167	0.114	0.125
	Estimate	1.039	1.080	1.143	1.122	0.956	1.018	0.997	0.914	0.977	0.956	0.873	0.935	0.914
25	Bias	0.039	0.080	0.143	0.122	-0.044	0.018	-0.003	-0.086	-0.023	-0.044	-0.127	-0.065	-0.086
	MSE	0.048	0.056	0.076	0.069	0.041	0.045	0.042	0.043	0.041	0.041	0.049	0.042	0.043
	Estimate	1.018	1.038	1.068	1.058	0.977	1.007	0.997	0.956	0.987	0.977	0.936	0.967	0.956
50	Bias	0.018	0.038	0.068	0.058	-0.023	0.007	-0.003	-0.044	-0.013	-0.023	-0.064	-0.033	-0.044
	MSE	0.022	0.024	0.028	0.027	0.020	0.021	0.021	0.021	0.020	0.020	0.022	0.021	0.021
	Estimate	1.010	1.024	1.044	1.037	0.983	1.004	0.997	0.970	0.990	0.983	0.956	0.977	0.970
75	Bias	0.010	0.024	0.044	0.037	-0.017	0.004	-0.003	-0.030	-0.010	-0.017	-0.044	-0.023	-0.030
	MSE	0.014	0.015	0.017	0.016	0.014	0.014	0.014	0.014	0.014	0.014	0.015	0.014	0.014
	Estimate	1.008	1.018	1.033	1.028	0.988	1.003	0.998	0.978	0.993	0.988	0.968	0.983	0.978
100	Bias	0.008	0.018	0.033	0.028	-0.012	0.003	-0.002	-0.022	-0.007	-0.012	-0.032	-0.017	-0.022
	MSE	0.010	0.011	0.012	0.012	0.010	0.010	0.010	0.010	0.010	0.010	0.011	0.010	0.010

Table 3: *Estimated values, Bias and MSE of different estimators under gamma prior when* $\lambda = 0.75$.

sample sizes(n)	criteria	$\hat{\lambda}_{MLE}$	(0.5, 0.5) $\hat{\lambda}_{QLF}$	(0.5, 0.5) $\hat{\lambda}_{SLELF}$	(0.5, 0.5) $\hat{\lambda}_{GELF}$	(0.5, 5.0) $\hat{\lambda}_{QLF}$	(0.5, 5.0) $\hat{\lambda}_{SLELF}$	(0.5, 5.0) $\hat{\lambda}_{GELF}$	(1.0, 0.25) $\hat{\lambda}_{QLF}$	(1.0, 0.25) $\hat{\lambda}_{SLELF}$	(1.0, 0.25) $\hat{\lambda}_{GELF}$	(5.0, 5.0) $\hat{\lambda}_{QLF}$	(5.0, 5.0) $\hat{\lambda}_{SLELF}$	(5.0, 5.0) $\hat{\lambda}_{GELF}$
	Estimate	0.833	0.677	0.796	0.756	0.488	0.575	0.546	0.733	0.855	0.814	0.747	0.833	0.804
10	Bias	0.083	-0.073	0.046	0.006	-0.262	-0.175	-0.204	-0.017	0.105	0.064	-0.003	0.083	0.054
	MSE	0.093	0.056	0.073	0.064	0.081	0.049	0.058	0.064	0.097	0.082	0.030	0.044	0.038
	Estimate	0.779	0.721	0.767	0.751	0.631	0.671	0.657	0.742	0.788	0.773	0.751	0.792	0.778
25	Bias	0.029	-0.029	0.017	0.001	-0.119	-0.079	-0.093	-0.008	0.038	0.023	0.001	0.042	0.028
	MSE	0.027	0.022	0.025	0.023	0.027	0.020	0.022	0.023	0.028	0.026	0.018	0.021	0.020
	Estimate	0.763	0.735	0.757	0.750	0.687	0.708	0.701	0.745	0.768	0.760	0.751	0.772	0.765
50	Bias	0.013	-0.015	0.007	0.000	-0.063	-0.042	-0.049	-0.005	0.018	0.010	0.001	0.022	0.015
	MSE	0.012	0.011	0.012	0.012	0.012	0.011	0.011	0.011	0.012	0.012	0.010	0.011	0.011
	Estimate	0.758	0.739	0.754	0.749	0.706	0.721	0.716	0.746	0.761	0.756	0.750	0.764	0.759
75	Bias	0.008	-0.011	0.004	-0.001	-0.044	-0.029	-0.034	-0.004	0.011	0.006	0.000	0.014	0.009
	MSE	0.008	0.008	0.008	0.008	0.008	0.007	0.008	0.008	0.008	0.008	0.007	0.007	0.007
	Estimate	0.756	0.742	0.753	0.749	0.717	0.728	0.725	0.747	0.758	0.755	0.750	0.761	0.757
100	Bias	0.006	-0.008	0.003	-0.001	-0.033	-0.022	-0.025	-0.003	0.008	0.005	0.000	0.011	0.007
	MSE	0.006	0.006	0.006	0.006	0.006	0.005	0.006	0.006	0.006	0.006	0.005	0.006	0.005

7. The minimax estimator under gamma prior has less MSE value as compared with the extended Jeffrey's prior.

8. Bias of $\hat{\lambda}$ decreases with an increasing sample sizes for all the estimators.

9. Bias and MSE of all the estimators of λ increases with the value of true scale parameter increases.

10. In all the cases MSE of the estimators reduced with the increase in sample size which verifies the consistency of all the estimators. Further, for large size of sample, they all converge to an almost same MSE value.

Table 4: *Estimated values, Bias and MSE of different estimators under gamma prior when $\lambda = 1$.*

sample sizes(n)	criteria	$\hat{\lambda}_{MLE}$	(0.5, 0.5) $\hat{\lambda}_{QLF}$	$\hat{\lambda}_{SLELF}$	$\hat{\lambda}_{GELF}$	(0.5, 5.0) $\hat{\lambda}_{QLF}$	$\hat{\lambda}_{SLELF}$	$\hat{\lambda}_{GELF}$	(1.0, 0.25) $\hat{\lambda}_{QLF}$	$\hat{\lambda}_{SLELF}$	$\hat{\lambda}_{GELF}$	(5.0, 5.0) $\hat{\lambda}_{QLF}$	$\hat{\lambda}_{SLELF}$	$\hat{\lambda}_{GELF}$
10	Estimate	1.111	0.889	1.046	0.994	0.592	0.696	0.661	0.97	1.132	1.077	0.905	1.010	0.975
	Bias	0.111	-0.111	0.046	-0.006	-0.408	-0.304	-0.339	-0.03	0.132	0.077	-0.095	0.010	-0.025
	MSE	0.166	0.097	0.120	0.106	0.182	0.114	0.134	0.110	0.165	0.140	0.045	0.045	0.043
25	Estimate	1.039	0.956	1.017	0.996	0.804	0.855	0.838	0.987	1.048	1.028	0.958	1.009	0.992
	Bias	0.039	-0.044	0.017	-0.004	-0.196	-0.145	-0.162	-0.013	0.048	0.028	-0.042	0.009	-0.008
	MSE	0.048	0.039	0.042	0.041	0.057	0.042	0.046	0.041	0.048	0.045	0.028	0.029	0.028
50	Estimate	1.018	0.977	1.007	0.997	0.894	0.922	0.913	0.992	1.022	1.012	0.977	1.005	0.996
	Bias	0.018	-0.023	0.007	-0.003	-0.106	-0.078	-0.087	-0.008	0.022	0.012	-0.023	0.005	-0.004
	MSE	0.022	0.020	0.021	0.020	0.025	0.020	0.022	0.020	0.022	0.021	0.017	0.017	0.017
75	Estimate	1.010	0.983	1.004	0.997	0.927	0.946	0.940	0.993	1.014	1.007 7	0.984	1.003	0.996
	Bias	0.010	-0.017	0.004	-0.003	-0.073	-0.054	-0.060	-0.007	0.014	0.007	-0.016	0.003	-0.004
	MSE	0.014	0.013	0.014	0.014	0.016	0.014	0.014	0.014	0.014	0.014	0.012	0.012	0.012
100	Estimate	1.008	0.988	1.003	0.998	0.945	0.959	0.954	0.995	1.010	1.005	0.988	1.002	0.998
	Bias	0.008	-0.012	0.003	-0.002	-0.055	-0.041	-0.046	-0.005	0.010	0.005	-0.012	0.002	-0.002
	MSE	0.010	0.010	0.010	0.010	0.011	0.010	0.010	0.010	0.010	0.010	0.009	0.009	0.009

7. Conclusion

In this article, an attempt has been made towards a comparison between the minimax estimators and the maximum likelihood estimator of the scale parameter λ of the inverse Rayleigh distribution. In order to obtain the minimax estimator of λ, we consider extended Jeffrey's prior and gamma prior under the symmetric (QLF) and asymmetric (SLELF and GELF) loss functions. An extensive simulation process is performed to investigate the performance of the MLE as well as minimax estimators in terms of bias and MSE values. From the simulation results it can be observed that in large sample cases the MLE and minimax estimators under different loss functions have approximately same MSE values.

In case of extended Jeffrey's prior, when the value of c is negative (i.e. $c = -1$), the maximum likelihood estimator (MLE) appears to be better than minimax estimators under all the considered loss functions. However, when c has positive values, then the minimax estimators are more efficient than the classical estimator MLE.

While comparing the MLE with the minimax estimators under gamma prior, it has been observe that the minimax estimators are appeared to be better for all the choices of hyperparameters. It is also remarked that the minimax estimators under gamma prior have less MSE as compared to the extended Jeffrey's prior. Therefore, choosing an informative prior is always superior to that of the non-informative prior. Finally, an increasing order of sample size results in a noticeable decrease in MSEs for all choices of parameter values which established that all the estimators are consistent.

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The power continuous Bernoulli distribution: Theory and applications

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Abstract

The continuous Bernoulli distribution is a recently introduced one-parameter distribution with support $[0,1]$, finding numerous applications in applied statistics. The idea of this article is to propose a natural extension of this distribution by adding a shape parameter through a power transformation. We introduce the power continuous Bernoulli distribution, aiming to extend the modeling scope of the continuous Bernoulli distribution. Basics of its mathematical properties are derived, such as the shapes of the related functions, the determination of various moment measures, and an evaluation of the overall amount of its randomness via the Rényi entropy. A statistical analysis of the distribution is then performed, showing how it can be applied when dealing with data. Estimates of the parameters are discussed through the maximum likelihood method. A Monte Carlo simulation study investigates the asymptotic behavior of these estimates. The flexibility of the power continuous Bernoulli distribution in real-life data fitting is analyzed using two data sets. Also, fair competitors are considered to highlight the accuracy of this distribution. At all stages, numerous graphics and tables illustrate the findings.

Keywords: Continuous Bernoulli distribution; moments; quantiles; entropy; data fitting.

1. Introduction

In order to understand the mathematical foundation of the study, let us first present the so-called continuous Bernoulli distribution as introduced in [19].

Definition 1. The continuous Bernoulli distribution with parameter $\lambda \in [0,1]$, also denoted as $\mathcal{CB}(\lambda)$, is defined by the following probability density function (pdf):

$$f(x;\lambda) = \begin{cases} 1, & \lambda = \dfrac{1}{2} \text{ and } x \in [0,1], \\ c_\lambda \lambda^x (1-\lambda)^{1-x}, & \lambda \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \in [0,1], \\ 0, & x \notin [0,1], \end{cases} \tag{1}$$

where c_λ is the following constant:

$$c_\lambda = \frac{2\operatorname{arctanh}(1-2\lambda)}{1-2\lambda}. \tag{2}$$

Here, $\operatorname{arctanh}(x)$ denotes the inverse hyperbolic tangent defined by $\operatorname{arctanh}(x) = (1/2)\ln[(1+x)/(1-x)]$ (as a minor remark, the following expressions are equivalent: $2\operatorname{arctanh}(1-2\lambda) = \ln(1-\lambda) - \ln(\lambda) = \ln(1/\lambda - 1)$).

Alternatively, the $\mathcal{CB}(\lambda)$ distribution can be defined by its cumulative distribution function (cdf), which is given by

$$F(x;\lambda) = \begin{cases} 0, & x < 0, \\ x, & \lambda = \dfrac{1}{2} \text{ and } x \in [0,1], \\ \dfrac{\lambda^x(1-\lambda)^{1-x}+\lambda-1}{2\lambda-1}, & \lambda \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \in [0,1], \\ 1, & x > 1. \end{cases} \tag{3}$$

Thus, the $\mathcal{CB}(\lambda)$ distribution, like the power distribution, is a one-parameter continuous distribution with support of $[0,1]$. It is useful in a variety of fields, including probability theory, statistics, with an emphasis on machine learning. In particular, it is good at simulating the pixel intensities of natural images in deep learning and computer vision, especially when putting up variational autoencoders. We advise the reader to [19] and [13] for more information on these topics.

More broadly, bounded support distributions have proven useful in modeling real-world data, particularly in scenarios where the data are measured in percentages and proportions. So when the observations take on value within the unit interval $[0,1]$. In recent decades, the beta and Kumaraswamy distributions have gained more popularity in this regard. However, there are situations where these classical distributions provide poor fit in data analysis. This has become a quest for many researchers to develop alternative bounded distributions with better flexibility in real-life data fitting. With this in mind, [12] introduced the log-Lindley distribution, [21] developed the unit-logistic distribution, [1] created the log-Xgamma distribution, [20] proposed the unit-Gompertz distribution, [23] developed the Kumaraswamy unit-Gompertz distribution, [16] examined the unit-Burr XII distribution, [15] introduced the unit-Chen distribution, [26] developed the transmuted Marshall-Olkin extended Topp-Leone distribution, [6] proposed the log-XLindley distribution, [2] studied the unit-Rayleigh distribution, etc. The $\mathcal{CB}(\lambda)$ distribution belongs to the list.

In this paper, by including a shape parameter, we hope to increase the flexibility of the $\mathcal{CB}(\lambda)$ distribution for a variety of applications. In other words, for a random variable X following the $\mathcal{CB}(\lambda)$ distribution, we consider the distribution of the power random variable $Y = X^{1/\alpha}$, where $\alpha > 0$. In this way, we introduce the power continuous Bernoulli distribution with parameters α and λ, $\mathcal{PCB}(\alpha,\lambda)$ distribution for short. The used power scheme is somewhat classic in statistics, and allows to flexibilize various "rigid distributions". We may mention the Weibull distribution, which is the power version of the exponential distribution, the power Lindley distribution by [9], which is the power version of the Lindley distribution (see [18]), etc. Recent examples include the power beta distribution by [5], the power Lomax distribution by [28], the power Ailamujia distribution by [14], etc.

In fact, at the time of writing, no extensions of the $\mathcal{CB}(\lambda)$ distribution exist, and the $\mathcal{PCB}(\alpha,\lambda)$ distribution is a strong contender for being useful from both theoretical and applied perspectives. After a detailed presentation, we investigate its main features, such as the related probability functions, moments of various kinds, and entropy (Rényi entropy). Then, we examine the practice on the statistical side. We estimate the $\mathcal{PCB}(\alpha,\lambda)$ distribution parameters, i.e., α and λ, by the maximum likelihood (ML) method. A Monte Carlo simulation study is then conducted to validate the asymptotic behavior of these estimates. We present significant applications of the $\mathcal{PCB}(\alpha,\lambda)$ distribution in a data fitting context, with the use of two real-life data sets: one containing trade share data, and the other containing tensile strength of polyester fibers. In addition, several distributions are considered for fair comparison in terms of efficiency in fitting. Illustrations, via tables and graphics, are given to support the findings. We have thus laid the foundation for the use of the $\mathcal{PCB}(\alpha,\lambda)$ distribution for statistical purposes.

The organization of the paper is as follows: Section 2 describes the $\mathcal{PCB}(\alpha,\lambda)$ distribution, including its underlying functions of interest. A moment analysis is performed in Section 3. The entropy is studied in Section 4. Parameter estimation, simulation study and real-life data fitting are developed in Section 5. A conclusion is given in Section 6.

2. Power continuous Bernoulli distribution

The $\mathcal{PCB}(\alpha, \lambda)$ distribution is defined below through its related probabilistic functions.

Definition 2. Based on its stochastic definition and the functions (1) and (3), the $\mathcal{PCB}(\alpha, \lambda)$ distribution with $\alpha > 0$ and $\lambda \in [0, 1]$ is defined by the following cdf:

$$\begin{aligned} F(x; \alpha, \lambda) &= F(x^\alpha; \lambda) \\ &= \begin{cases} 0, & x < 0 \\ x^\alpha, & \lambda = \dfrac{1}{2} \text{ and } x \in [0, 1], \\ \dfrac{\lambda^{x^\alpha}(1-\lambda)^{1-x^\alpha} + \lambda - 1}{2\lambda - 1}, & \lambda \in (0, 1) / \left\{\dfrac{1}{2}\right\} \text{ and } x \in [0, 1], \\ 1, & x > 1, \end{cases} \end{aligned} \tag{4}$$

or, equivalently, by the following pdf:

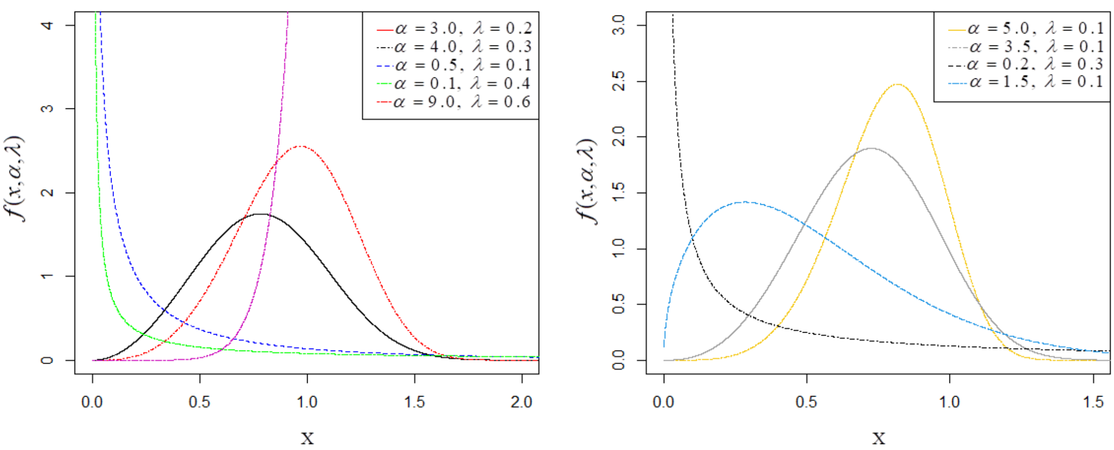
$$\begin{aligned} f(x; \alpha, \lambda) &= \alpha x^{\alpha-1} f(x^\alpha; \lambda) \\ &= \begin{cases} \alpha x^{\alpha-1}, & \lambda = \dfrac{1}{2} \text{ and } x \in [0, 1], \\ c_\lambda \alpha x^{\alpha-1} \lambda^{x^\alpha}(1-\lambda)^{1-x^\alpha}, & \lambda \in (0, 1) / \left\{\dfrac{1}{2}\right\} \text{ and } x \in [0, 1], \\ 0, & x \notin [0, 1], \end{cases} \end{aligned} \tag{5}$$

where c_λ is the constant defined in (2).

Basically, by taking $\alpha = 1$ into (4) and (5), we obtain the cdf and pdf of the $\mathcal{CB}(\lambda)$ distribution, as described in (3) and (1), respectively.

It is important to note that the $\mathcal{PCB}(\alpha, \lambda)$ distribution has one mode in the case $\lambda \in (1/2, 1)$, and it is given by the following mathematical formula: $x = [(\alpha - 1)/(2\alpha \operatorname{arctanh}(1 - 2\lambda))]^{1/\alpha}$. In this case, the $\mathcal{PCB}(\alpha, \lambda)$ distribution is unimodal, and the mode differs from 0 if, and only if, $\alpha \in [0, 1)$, i.e., $\alpha \neq 1$. So by considering the power version of the $\mathcal{CB}(\lambda)$ distribution, we introduce a unimodality property that can be used quite efficiently for statistical aims, including data fitting purposes.

Figure 1 presents the plots for $f(x; \alpha, \lambda)$ in order to illustrate the effect of the parameter α on its possible shapes.



(a) (b)

Figure 1: *Pdf plots of the $\mathcal{PCB}(\alpha, \lambda)$ distribution at different choices of the parameter settings: (a) $(\alpha, \lambda) \in \{(3, 0.2), (4, 0.3), (0.5, 0.1), (0.1, 0.4), (9, 0.6)\}$ and (b) $(\alpha, \lambda) \in \{(5, 0.1), (3.5, 0.1), (0.2, 0.3), (1.5, 0.1)\}$*

Clearly, we observe that the pdf of the $\mathcal{PCB}(\alpha,\lambda)$ distribution accommodates decreasing (reversed-J) or increasing, left-skewed, right-skewed and symmetric shapes.

As a complementary function to the pdf, the hazard rate function (hrf) of the $\mathcal{PCB}(\alpha,\lambda)$ distribution is given as

$$\begin{aligned} h(x;\alpha,\lambda) &= \frac{f(x;\alpha,\lambda)}{1-F(x;\alpha,\lambda)} \\ &= \begin{cases} \dfrac{\alpha x^{\alpha-1}}{1-x^{\alpha}}, & \lambda=\dfrac{1}{2} \text{ and } x\in[0,1], \\ \dfrac{c_\lambda^{*}\alpha x^{\alpha-1}\lambda^{x^\alpha}(1-\lambda)^{1-x^\alpha}}{\lambda-\lambda^{x^\alpha}(1-\lambda)^{1-x^\alpha}}, & \lambda\in(0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x\in[0,1], \\ 0, & x\notin[0,1], \end{cases} \end{aligned}$$

where $c_\lambda^{*}=(2\lambda-1)c_\lambda=-2\,\mathrm{arctanh}(1-2\lambda)$. The graphical representation of this function is displayed in Figure 2.

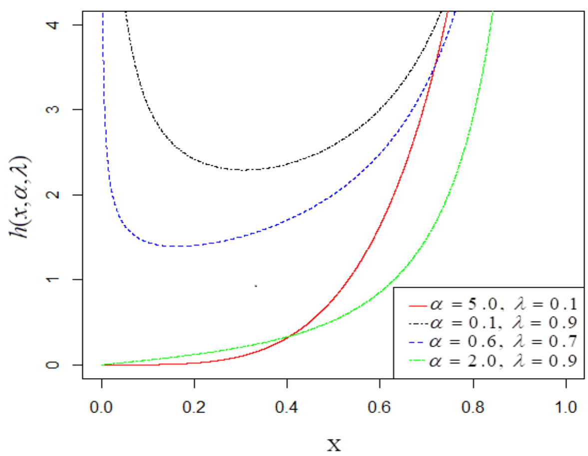


Figure 2: *Hrf plots of the $\mathcal{PCB}(\alpha,\lambda)$ distribution at different choices of the parameter settings: $(\alpha,\lambda)\in\{(5,0.1),(0.1,0.9),(0.6,0.7),(2,0.9)\}$*

Figure 2 indicates that the $\mathcal{PCB}(\alpha,\lambda)$ distribution exhibits increasing and bathtub-shaped hazard properties. These are demanded properties for data analysis purposes with values in $[0,1]$.

As the inverse function of the cdf, the quantile function (qf) of the $\mathcal{PCB}(\alpha,\lambda)$ distribution is given as

$$\begin{aligned} Q(x;\alpha,\lambda) &= F^{-1}(x;\alpha,\lambda) \\ &= \begin{cases} x^{1/\alpha}, & \lambda=\dfrac{1}{2} \text{ and } x\in[0,1], \\ \left\{\dfrac{\ln[(2\lambda-1)x+1-\lambda]-\ln(1-\lambda)}{\ln(\lambda)-\ln(1-\lambda)}\right\}^{1/\alpha}, & \lambda\in(0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x\in[0,1]. \end{cases} \end{aligned} \tag{6}$$

By inserting $x=1/2$ in (6), we obtain the median of the $\mathcal{PCB}(\alpha,\lambda)$ distribution, which is given by

$$M=\begin{cases} 2^{-1/\alpha}, & \lambda=\dfrac{1}{2}, \\ \left\{\dfrac{\ln(2)+\ln(1-\lambda)}{2\,\mathrm{arctanh}(1-2\lambda)}\right\}^{1/\alpha}, & \lambda\in(0,1)/\left\{\dfrac{1}{2}\right\}. \end{cases}$$

Traditionally, the qf and random values from the uniform distribution over $[0,1]$ can be used to generate random values from a random variable Y following the $\mathcal{PCB}(\alpha,\lambda)$ distribution. Table 1 shows some quantiles from the $\mathcal{PCB}(\alpha, \lambda)$ distribution using the expression in (6) as illustrative numerical examples.

Table 1: *Some values of the qf of the $\mathcal{PCB}(\alpha, \lambda)$ distribution.*

x	$\alpha = 0.5, \lambda = 0.3$	$\alpha = 0.5, \lambda = 0.8$	$\alpha = 1, \lambda = 0.3$	$\alpha = 1, \lambda = 0.8$
0.05	0.0012	0.0102	0.0342	0.1008
0.1	0.0048	0.0358	0.0694	0.1893
0.2	0.0205	0.1149	0.1432	0.3390
0.3	0.0493	0.2144	0.2219	0.4630
0.4	0.0938	0.3235	0.3063	0.5688
0.5	0.1577	0.4369	0.3971	0.6610
0.6	0.2455	0.5516	0.4955	0.7427
0.7	0.3635	0.6661	0.6029	0.8161
0.8	0.5199	0.7793	0.7210	0.8828
0.9	0.7264	0.8907	0.8523	0.9438

The quantile values of the $\mathcal{PCB}(\alpha, \lambda)$ distribution fall into $[0, 1]$ for different parameter values.
On the other hand, based on the qf, advanced quantile modeling can be performed. For more information, see [10].

3. Moments

The moment measures of the $\mathcal{PCB}(\alpha, \lambda)$ distribution are of interest to describe it in an in-depth manner in terms of central, dispersion, and form parameters, and reveal some statistical features.
The following proposition is about the mathematical expressions of the moments of a random variable following the $\mathcal{PCB}(\alpha, \lambda)$ distribution.

Proposition 1. Let Y be a random variable following the $\mathcal{PCB}(\alpha, \lambda)$ distribution and m be an integer. Then the m-th moment (or raw moment) of Y is given by

$$\begin{aligned}\mathfrak{M}_m &= E(Y^m)\\ &= \begin{cases} \dfrac{\alpha}{\alpha+m}, & \lambda = \dfrac{1}{2},\\ \dfrac{(1-\lambda)c_\lambda}{[2\,\mathrm{arctanh}(1-2\lambda)]^{m/\alpha+1}}\gamma_-\left[\dfrac{m}{\alpha}+1, 2\,\mathrm{arctanh}(1-2\lambda)\right], & \lambda \in \left(0, \dfrac{1}{2}\right),\\ \dfrac{(1-\lambda)c_\lambda}{[-2\,\mathrm{arctanh}(1-2\lambda)]^{m/\alpha+1}}\gamma_+\left[\dfrac{m}{\alpha}+1, -2\,\mathrm{arctanh}(1-2\lambda)\right], & \lambda \in \left(\dfrac{1}{2}, 1\right),\end{cases}\end{aligned}$$

where

$$\gamma_-(x,u) = \int_0^u t^{x-1}e^{-t}dt, \quad \gamma_+(x,u) = \int_0^u t^{x-1}e^{t}dt. \tag{7}$$

Proof. For the case $\lambda = 1/2$, we have

$$\mathfrak{M}_m = \int_{-\infty}^{+\infty} x^m f(x;\alpha,\lambda)dx = \alpha\int_0^1 x^m x^{\alpha-1}dx = \frac{\alpha}{\alpha+m}.$$

For the case $\lambda \in (0,1)/\{1/2\}$, by introducing a random variable X with the $\mathcal{CB}(\lambda)$ distribution, we have

$$\mathfrak{M}_m = E(X^{m/\alpha}) = \int_{-\infty}^{+\infty} x^{m/\alpha} f(x;\lambda)dx = c_\lambda \int_0^1 x^{m/\alpha}\lambda^x(1-\lambda)^{1-x}dx.$$

Since m/α is not necessarily an integer, let us distinguish the case $\lambda \in (0, 1/2)$ and the case $\lambda \in (1/2, 1)$.

- In the case $\lambda \in (0, 1/2)$, by applying the change of variable $y = 2\,\mathrm{arctanh}(1-2\lambda)x \geq 0$, we obtain

$$\begin{aligned}\mathfrak{M}_m &= c_\lambda \int_0^1 x^{m/\alpha} e^{x\ln(\lambda)+(1-x)\ln(1-\lambda)} dx = (1-\lambda)c_\lambda \int_0^1 x^{m/\alpha} e^{-x[2\,\mathrm{arctanh}(1-2\lambda)]} dx \\ &= \frac{(1-\lambda)c_\lambda}{[2\,\mathrm{arctanh}(1-2\lambda)]^{m/\alpha+1}} \int_0^{2\,\mathrm{arctanh}(1-2\lambda)} y^{m/\alpha} e^{-y} dy \\ &= \frac{(1-\lambda)c_\lambda}{[2\,\mathrm{arctanh}(1-2\lambda)]^{m/\alpha+1}} \gamma_- \left[\frac{m}{\alpha}+1, 2\,\mathrm{arctanh}(1-2\lambda)\right].\end{aligned}$$

- In the case $\lambda \in (1/2, 1)$, we must take into account a sign detail; by applying the change of variable $y = -2\,\mathrm{arctanh}(1-2\lambda)x \geq 0$, we obtain

$$\begin{aligned}\mathfrak{M}_m &= c_\lambda \int_0^1 x^{m/\alpha} e^{x\ln(\lambda)+(1-x)\ln(1-\lambda)} dx = (1-\lambda)c_\lambda \int_0^1 x^{m/\alpha} e^{-x[2\,\mathrm{arctanh}(1-2\lambda)]} dx \\ &= \frac{(1-\lambda)c_\lambda}{[-2\,\mathrm{arctanh}(1-2\lambda)]^{m/\alpha+1}} \int_0^{-2\,\mathrm{arctanh}(1-2\lambda)} y^{m/\alpha} e^{y} dy \\ &= \frac{(1-\lambda)c_\lambda}{[-2\,\mathrm{arctanh}(1-2\lambda)]^{m/\alpha+1}} \gamma_+ \left[\frac{m}{\alpha}+1, -2\,\mathrm{arctanh}(1-2\lambda)\right].\end{aligned}$$

The desired expressions are obtained, ending the proof. □

It is worth noting that the integral function $\gamma_-(x,u)$ corresponds to the lower incomplete gamma function, which is implemented in most of the mathematical software.

In the case $\alpha = 1$, m/α is an integer, and we have

$$\mathfrak{M}_m = \begin{cases} \dfrac{1}{m+1}, & \lambda = \dfrac{1}{2}, \\ \dfrac{(1-\lambda)c_\lambda}{[2\,\mathrm{arctanh}(1-2\lambda)]^{m+1}} \gamma_- [m+1, 2\,\mathrm{arctanh}(1-2\lambda)], & \lambda \in (0,1) \,/\, \left\{\dfrac{1}{2}\right\}, \end{cases}$$

giving the m-th moment related to the $\mathcal{CB}(\alpha, \lambda)$ distribution, which missing in the list of properties in [19]. In this particular case, by using the expression $\gamma_-(2,u) = 1-(1+u)e^{-u}$, we refind the mean of Y as precised in [19, Equation (8)]:

$$\mathfrak{M}_1 = \begin{cases} \dfrac{1}{2}, & \lambda = \dfrac{1}{2}, \\ \dfrac{(1-\lambda)c_\lambda}{[2\,\mathrm{arctanh}(1-2\lambda)]^{2}} \gamma_- [2, 2\,\mathrm{arctanh}(1-2\lambda)] & \\ = \dfrac{\lambda}{2\lambda-1} + \dfrac{1}{2\,\mathrm{arctanh}(1-2\lambda)}, & \lambda \in (0,1) \,/\, \left\{\dfrac{1}{2}\right\}. \end{cases}$$

More generally, based on the expression of the moments established in Proposition 1, we can easily derive the mean of a random variable Y following the $\mathcal{PCB}(\alpha, \lambda)$ distribution; it is given as

$$\mathfrak{M}_1 = \begin{cases} \dfrac{\alpha}{\alpha+1}, & \lambda = \dfrac{1}{2}, \\ \dfrac{(1-\lambda)c_\lambda}{[2\,\mathrm{arctanh}(1-2\lambda)]^{1/\alpha+1}} \gamma_- \left[\dfrac{1}{\alpha}+1, 2\,\mathrm{arctanh}(1-2\lambda)\right], & \lambda \in \left(0, \dfrac{1}{2}\right), \\ \dfrac{(1-\lambda)c_\lambda}{[-2\,\mathrm{arctanh}(1-2\lambda)]^{1/\alpha+1}} \gamma_+ \left[\dfrac{1}{\alpha}+1, -2\,\mathrm{arctanh}(1-2\lambda)\right], & \lambda \in \left(\dfrac{1}{2}, 1\right), \end{cases}$$

as well the moment of order 2 of Y:

$$\mathfrak{M}_2 = \begin{cases} \dfrac{\alpha}{\alpha+2}, & \lambda = \dfrac{1}{2}, \\ \dfrac{(1-\lambda)c_\lambda}{[2\,\mathrm{arctanh}(1-2\lambda)]^{2/\alpha+1}} \gamma_- \left[\dfrac{2}{\alpha}+1, 2\,\mathrm{arctanh}(1-2\lambda)\right], & \lambda \in \left(0, \dfrac{1}{2}\right), \\ \dfrac{(1-\lambda)c_\lambda}{[-2\,\mathrm{arctanh}(1-2\lambda)]^{2/\alpha+1}} \gamma_+ \left[\dfrac{2}{\alpha}+1, -2\,\mathrm{arctanh}(1-2\lambda)\right], & \lambda \in \left(\dfrac{1}{2}, 1\right). \end{cases}$$

The variance of Y follows from the standard formula: $\sigma^2 = \mathfrak{M}_2 - \mathfrak{M}_1^2$. The m-th central moment of Y is given by

$$\mathfrak{M}_m^* = E[(Y - \mathfrak{M}_1)^m] = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \mathfrak{M}_k \mathfrak{M}_1^{m-k}.$$

Based on these central moments, the skewness and kurtosis coefficients of Y are, respectively, given by

$$S = \frac{\mathfrak{M}_3^*}{\sigma^3}, \quad K = \frac{\mathfrak{M}_4^*}{\sigma^4}.$$

Numerical computation of the mean, variance, measures of skewness and kurtosis for the $\mathcal{PCB}(\alpha, \lambda)$ distribution are shown in Table 2.

Table 2: *Theoretical moment measures of the $\mathcal{PCB}(\alpha, \lambda)$ distribution*

α	λ	$\mathfrak{M}_1$	σ^2	S	K
0.5	0.1	0.1755	0.0528	1.6646	5.0278
	0.4	0.3001	0.0837	0.8087	2.4314
	0.9	0.5152	0.0923	-0.1377	1.7674
1.0	0.1	0.3301	0.0665	0.7430	2.5785
	0.4	0.4663	0.0827	0.1417	1.8116
	0.9	0.6699	0.0664	-0.7388	2.5633
2.0	0.1	0.5234	0.0562	0.0559	2.0882
	0.4	0.6394	0.0575	-0.4382	2.2506
	0.9	0.7962	0.0360	-1.3390	4.3644

From Table 2, we conclude that the $\mathcal{PCB}(\alpha, \lambda)$ distribution can be left- and right-skewed, as negative and positive values for S are observed, and it has all kurtosis states, as K varies around the limit value of 3.

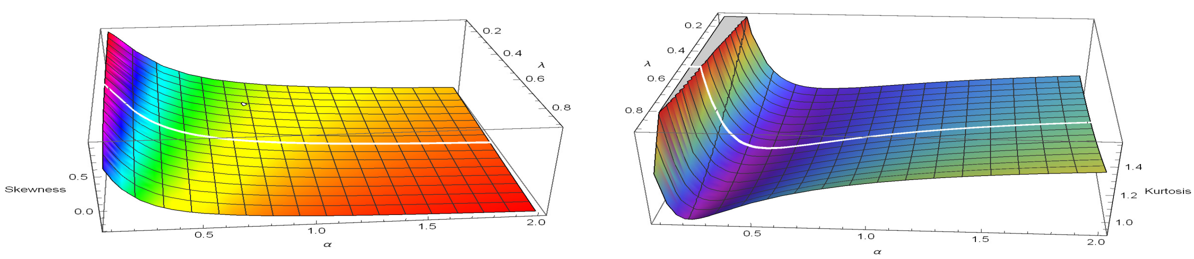
Complement: Alternative measures of skewness and kurtosis are the ones based on the qf of the distribution as proposed by [7] and [22], respectively. The Galton skewness and the Moors kurtosis are, respectively, defined as

$$S_G = \frac{Q(6/8; \alpha, \lambda) - 2Q(4/8; \alpha, \lambda) + Q(2/8; \alpha, \lambda)}{Q(6/8; \alpha, \lambda) - Q(2/8; \alpha, \lambda)}$$

and

$$K_M = \frac{Q(7/8; \alpha, \lambda) - Q(5/8; \alpha, \lambda) + Q(3/8; \alpha, \lambda) - Q(1/8; \alpha, \lambda)}{Q(6/8; \alpha, \lambda) - Q(2/8; \alpha, \lambda)}.$$

In order to complete the previous numerical work, Figure 3 presents the nature of the Galton skewness and Moors kurtosis of the $\mathcal{PCB}(\alpha, \lambda)$ distribution.



(a) (b)

Figure 3: *Plots of (a) the Galton skewness and (b) Moors kurtosis for the $\mathcal{PCB}(\alpha, \lambda)$ distribution with $\alpha \in [0, 2]$ and $\lambda \in [0, 1]$*

From Figure 3, we can observe that the Galton skewness seems monotonic according to α and λ, with possible negative and positive values. On the other hand, the Moors kurtosis is more complex, being non-monotonic in α. This illustrates the versatility of the $\mathcal{PCB}(\alpha,\lambda)$ distribution on these form aspects.

4. Entropy

The amount of randomness in the $\mathcal{PCB}(\alpha,\lambda)$ distribution is now the object of all the attention. In order to accomplish this, we recall that the Rényi entropy of a random variable X with pdf $f(x)$ is given by

$$\mathfrak{R}_\theta = \frac{1}{1-\theta}\ln\left[\int_{-\infty}^{+\infty} f(x)^\theta dx\right],$$

with $\theta > 0$ and $\theta \neq 1$. The following proposition is about the mathematical expression of the Rényi entropy of a random variable Y following the $\mathcal{PCB}(\alpha,\lambda)$ distribution.

Proposition 2. Let $\theta > 0$ with $\theta \neq 1$, and Y be a random variable following the $\mathcal{PCB}(\alpha,\lambda)$ distribution. Then the Rényi entropy of Y is given by

$$\mathfrak{R}_\theta = \begin{cases} \frac{1}{1-\theta}\ln\left(\frac{\alpha^\theta}{\theta(\alpha-1)+1}\right), & \lambda = \frac{1}{2}, \\ \frac{1}{1-\theta}\ln\left\{\frac{(1-\lambda)^\theta c_\lambda^\theta \alpha^{\theta-1}}{[2\theta\,\text{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\gamma_-\left[\frac{(\theta-1)(\alpha-1)}{\alpha}+1, 2\theta\,\text{arctanh}(1-2\lambda)\right]\right\}, & \lambda \in \left(0,\frac{1}{2}\right), \\ \frac{1}{1-\theta}\ln\left\{\frac{(1-\lambda)^\theta c_\lambda^\theta \alpha^{\theta-1}}{[-2\theta\,\text{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\gamma_+\left[\frac{(\theta-1)(\alpha-1)}{\alpha}+1, -2\theta\,\text{arctanh}(1-2\lambda)\right]\right\}, & \lambda \in \left(\frac{1}{2},1\right), \end{cases}$$

where $\gamma_-(x,u)$ and $\gamma_+(x,u)$ are defined as in (7).

Proof. For the case $\lambda = 1/2$, we have

$$\int_{-\infty}^{+\infty} f(x;\alpha,\lambda)^\theta dx = \alpha^\theta \int_0^1 x^{\theta(\alpha-1)}dx = \frac{\alpha^\theta}{\theta(\alpha-1)+1}.$$

Hence,

$$\mathfrak{R}_\theta = \frac{1}{1-\theta}\ln\left(\frac{\alpha^\theta}{\theta(\alpha-1)+1}\right).$$

For the case $\lambda \in (0,1)/\{1/2\}$, by applying the change of variable $y = x^\alpha$, we have

$$\int_{-\infty}^{+\infty} f(x;\alpha,\lambda)^\theta dx = c_\lambda^\theta \alpha^\theta \int_0^1 x^{\theta(\alpha-1)}\lambda^{\theta x^\alpha}(1-\lambda)^{\theta(1-x^\alpha)}dx$$
$$= c_\lambda^\theta \alpha^{\theta-1}\int_0^1 y^{(\theta-1)(\alpha-1)/\alpha}\lambda^{\theta y}(1-\lambda)^{\theta(1-y)}dy.$$

Let us now distinguish the case $\lambda \in (0,1/2)$ and the case $\lambda \in (1/2,1)$.

- In the case $\lambda \in (0,1/2)$, by applying the change of variable $t = 2\theta\,\text{arctanh}(1-2\lambda)y \geq 0$, we obtain

$$\int_{-\infty}^{+\infty} f(x;\alpha,\lambda)^\theta dx = c_\lambda^\theta \alpha^{\theta-1}\int_0^1 y^{(\theta-1)(\alpha-1)/\alpha}e^{\theta y\ln(\lambda)+\theta(1-y)\ln(1-\lambda)}dy$$
$$= (1-\lambda)^\theta c_\lambda^\theta \alpha^{\theta-1}\int_0^1 y^{(\theta-1)(\alpha-1)/\alpha}e^{-y[2\theta\,\text{arctanh}(1-2\lambda)]}dy$$
$$= \frac{(1-\lambda)^\theta c_\lambda^\theta \alpha^{\theta-1}}{[2\theta\,\text{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\int_0^{2\theta\,\text{arctanh}(1-2\lambda)} t^{(\theta-1)(\alpha-1)/\alpha}e^{-t}dt$$
$$= \frac{(1-\lambda)^\theta c_\lambda^\theta \alpha^{\theta-1}}{[2\theta\,\text{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\gamma_-\left[\frac{(\theta-1)(\alpha-1)}{\alpha}+1, 2\theta\,\text{arctanh}(1-2\lambda)\right].$$

Hence,

$$\mathfrak{R}_\theta = \frac{1}{1-\theta}\ln\left\{\frac{(1-\lambda)^\theta c_\lambda^\theta \alpha^{\theta-1}}{[2\theta\,\text{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\gamma_-\left[\frac{(\theta-1)(\alpha-1)}{\alpha}+1, 2\theta\,\text{arctanh}(1-2\lambda)\right]\right\}.$$

- In the case $\lambda \in (1/2, 1)$, by applying the change of variable $t = -2\theta \operatorname{arctanh}(1-2\lambda)y \geq 0$, we obtain

$$\int_{-\infty}^{+\infty} f(x;\alpha,\lambda)^{\theta} dx = c_{\lambda}^{\theta}\alpha^{\theta-1}\int_{0}^{1} y^{(\theta-1)(\alpha-1)/\alpha} e^{\theta y \ln(\lambda)+\theta(1-y)\ln(1-\lambda)} dy$$
$$= (1-\lambda)^{\theta} c_{\lambda}^{\theta}\alpha^{\theta-1}\int_{0}^{1} y^{(\theta-1)(\alpha-1)/\alpha} e^{-y[2\theta \operatorname{arctanh}(1-2\lambda)]} dy$$
$$= \frac{(1-\lambda)^{\theta} c_{\lambda}^{\theta}\alpha^{\theta-1}}{[-2\theta \operatorname{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\int_{0}^{-2\theta \operatorname{arctanh}(1-2\lambda)} t^{(\theta-1)(\alpha-1)/\alpha} e^{t} dt$$
$$= \frac{(1-\lambda)^{\theta} c_{\lambda}^{\theta}\alpha^{\theta-1}}{[-2\theta \operatorname{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\gamma_{+}\left[\frac{(\theta-1)(\alpha-1)}{\alpha}+1, -2\theta \operatorname{arctanh}(1-2\lambda)\right].$$

Hence,

$$\mathfrak{R}_{\theta} = \frac{1}{1-\theta}\ln\left\{\frac{(1-\lambda)^{\theta} c_{\lambda}^{\theta}\alpha^{\theta-1}}{[-2\theta \operatorname{arctanh}(1-2\lambda)]^{(\theta-1)(\alpha-1)/\alpha+1}}\gamma_{+}\left[\frac{(\theta-1)(\alpha-1)}{\alpha}+1, -2\theta \operatorname{arctanh}(1-2\lambda)\right]\right\}.$$

We end the proof by compiling the above expressions. □

Table 3 shows some numerical values of the Rényi entropy of the $\mathcal{PCB}(\alpha,\lambda)$ distribution.

Table 3: *Numerical results of the Rényi entropy of the $\mathcal{PCB}(\alpha,\lambda)$ distribution*

γ	$\alpha = 1, \lambda = 0.3$	$\alpha = 1, \lambda = 0.8$	$\alpha = 2, \lambda = 0.3$	$\alpha = 2, \lambda = 0.8$
0.01	-0.0003	-0.0008	-0.0019	-0.0061
0.03	-0.0009	-0.0024	-0.0057	-0.0813
0.5	-0.0148	-0.0390	-0.0712	-0.2523
0.7	-0.0207	-0.0542	-0.0901	-0.3290
2.0	-0.0574	-0.1443	-0.1604	-0.6414
4.0	-0.1069	-0.2468	-0.2056	-0.8486
7.0	-0.1624	-0.3379	-0.2360	-0.9827
9.0	-0.1892	-0.3756	-0.2474	-1.0320

Consequently, from Table 3, some useful properties of the Rényi entropy provided in [11] are applicable here. In particular, (i) for any $\theta_1 < \theta_2$, we have $\mathfrak{R}_{\theta_2} \leq \mathfrak{R}_{\theta_1}$, (ii) the Rényi entropy can be negative.

5. Statistical applications

This section is devoted to the applicability of the $\mathcal{PCB}(\alpha,\lambda)$ distribution.

5.1. Estimation

In the setting of the $\mathcal{PCB}(\alpha,\lambda)$ distribution, we aim to estimate the unknown parameters α and λ based on data that can be conceptually fitted with this distribution. To accomplish this, we can use the ML method, described as follows: Let $y_1, \ldots, y_n$ represent n independent observations from a random variable Y following the $\mathcal{PCB}(\alpha,\lambda)$ distribution. Then, based on the pdf indicated in (5), the likelihood function is specified by

$$L(\alpha,\lambda;y_1,\ldots,y_n)=\prod_{i=1}^{n} f(y_i;\alpha,\lambda)$$
$$=\begin{cases}\alpha^n\left[\prod_{i=1}^{n} y_i\right]^{\alpha-1}, & \lambda=\frac{1}{2},\\ c_\lambda^n\alpha^n\left[\prod_{i=1}^{n} y_i\right]^{\alpha-1}\lambda^{\sum_{i=1}^{n} y_i^\alpha}(1-\lambda)^{n-\sum_{i=1}^{n} y_i^\alpha}, & \lambda\in(0,1)/\left\{\frac{1}{2}\right\}.\end{cases}$$

The ML estimates (MLEs) of α and λ are given by

$$(\widehat{\alpha},\widehat{\lambda})=\mathrm{argmax}_{(\alpha,\lambda)}\, L(\alpha,\lambda;y_1,\ldots,y_n).$$

Alternatively, they are defined by

$$(\widehat{\alpha},\widehat{\lambda})=\mathrm{argmax}_{(\alpha,\lambda)}\, \ell(\alpha,\lambda;y_1,\ldots,y_n),$$

where $\ell(\alpha,\lambda;y_1,\ldots,y_n)$ refers to the log-likelihood function given by

$$\ell(\alpha,\lambda;y_1,\ldots,y_n)=\ln\left[L(\alpha,\lambda;y_1,\ldots,y_n)\right]$$
$$=\begin{cases}n\ln(\alpha)+(\alpha-1)\sum_{i=1}^{n}\ln(y_i), & \lambda=\frac{1}{2},\\ n\ln(c_\lambda)+n\ln(\alpha)+(\alpha-1)\sum_{i=1}^{n}\ln(y_i)+\ln(\lambda)\sum_{i=1}^{n} y_i^\alpha+\ln(1-\lambda)\left[n-\sum_{i=1}^{n} y_i^\alpha\right], & \lambda\in(0,1)/\left\{\frac{1}{2}\right\}.\end{cases}$$

They can be obtained by solving the following non-linear equations with respect to α and λ:

$$\frac{\partial\ell(\alpha,\lambda;y_1,\ldots,y_n)}{\partial\alpha}=0,\quad \frac{\partial\ell(\alpha,\lambda;y_1,\ldots,y_n)}{\partial\lambda}=0.$$

The standard errors of $\widehat{\alpha}$ and $\widehat{\lambda}$ can be approximated, and they are denoted in the next as $\mathrm{se}(\widehat{\alpha})$ and $\mathrm{se}(\widehat{\lambda})$, respectively. The advantage of the ML method is that it guarantees interesting properties for the MLEs, such as asymptotic unbiasedness and normality. More information on these properties can be found in [4]. Based on the MLEs, we can estimate all the underlying functions of the $\mathcal{PCB}(\alpha,\lambda)$ distribution. In particular, an estimate of the cdf $F(x;\alpha,\lambda)$ is given by $\widehat{F}(x)=F(x;\widehat{\alpha},\widehat{\lambda})$ and an estimate of the pdf $f(x;\alpha,\lambda)$ is given by $\widehat{f}(x)=f(x;\widehat{\alpha},\widehat{\lambda})$.

5.2. Simulation study

In this portion, we investigate the asymptotic behavior of the MLEs of α and λ using Monte Carlo simulation. Random samples from the $\mathcal{PCB}(\alpha,\lambda)$ distribution were generated using (6). The simulation is repeated $N=2000$ times for different sample sizes $n\in\{20,50,100,200,500\}$ and different choices of the parameter values $(\alpha=0.3,\lambda=0.1)$, $(\alpha=0.5,\lambda=0.3)$ and $(\alpha=0.8,\lambda=0.6)$. The performance of the MLEs is examined using various statistical criteria presented below. For $\phi\in\{\alpha,\lambda\}$, we consider

1. the average bias (Bias) defined by $\frac{1}{N}\sum_{i=1}^{N}(\widehat{\phi}_i-\phi)$, where the index i refers to the i-th experiment among the N,
2. the root mean square error (RMSE) defined by $\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\phi}_i-\phi)^2}$,
3. the coverage probability (CP) of the 95% confidence interval defined by
$\frac{1}{N}\sum_{i=1}^{N} I\left(\widehat{\phi}_i-u_*\,\mathrm{se}(\widehat{\phi}_i)<\phi<\widehat{\phi}_i+u_*\,\mathrm{se}(\widehat{\phi}_i)\right)$,

where $I(.)$ is the indicator function, $\mathrm{se}(\phi_i)$ is the standard error related to $\widehat{\phi}_i$ and $u_* = 1.959964$.
Table 4 presents the simulation results based on these criteria.

Table 4: *Simulation results for the unknown parameters estimates of $\mathcal{PCB}(\alpha, \lambda)$ distribution*

		Bias		RMSE		CP	
Parameters	n	α	λ	α	λ	α	λ
	25	0.0189	0.0234	0.0807	0.1314	0.9420	0.8065
$\alpha = 0.3$	50	0.0056	0.0156	0.0520	0.0934	0.9455	0.8425
$\lambda = 0.1$	100	0.0042	0.0062	0.0358	0.0565	0.9570	0.9000
	200	0.0028	0.0028	0.0255	0.0406	0.9450	0.9060
	500	0.0006	0.0014	0.0155	0.0250	0.9520	0.9395
	25	0.0528	-0.0061	0.1736	0.2073	0.9440	0.8120
$\alpha = 0.5$	50	0.0228	-0.0040	0.1139	0.1608	0.9545	0.8675
$\lambda = 0.3$	100	0.0095	0.0075	0.0819	0.1251	0.9470	0.9150
	200	0.0037	0.0040	0.0554	0.0902	0.9515	0.9300
	500	0.0029	-0.0002	0.0341	0.0569	0.9575	0.9405
	25	0.1847	-0.1243	0.3970	0.2635	0.9485	0.8380
$\alpha = 0.8$	50	0.0959	-0.0711	0.2636	0.2073	0.9420	0.8835
$\lambda = 0.6$	100	0.0460	-0.0380	0.1832	0.1628	0.9445	0.9090
	200	0.0189	-0.0155	0.1276	0.1221	0.9620	0.9285
	500	0.0065	-0.0051	0.0827	0.0818	0.9540	0.9365

From Table 4, we notice that the RMSE of both MLEs decreases as the sample size n increases. While $\widehat{\alpha}$ is a positively biased parameter estimate, $\widehat{\lambda}$ can be both positively and negatively biased. Furthermore, the CP of both MLEs approaches 0.95, and the CP of λ increases as the sample size n increases.

5.3. Real-life data fitting

Among other purposes, the $\mathcal{PCB}(\alpha, \lambda)$ distribution can be used for fitting data with values into $[0, 1]$. We thus illustrate this application by considering two real-life data sets, and compare their fit with the ones obtained from some existing distributions with support of $[0, 1]$. More specifically, we consider the following recently developed unit distributions, including the beta and Kumaraswamy distributions.

1. Marshall-Olkin extended Kumaraswamy distribution (MOEKD) introduced by [8], and defined with the following pdf:
$$f(x; \alpha, a, b) = \frac{\alpha a b x^{a-1}\left(1 - x^a\right)^{b-1}}{\left[1 - \bar{\alpha}\left(1 - x^a\right)^b\right]^2}, \quad x \in [0, 1], \text{ where } \bar{\alpha} = 1 - \alpha, \text{ with } \alpha, a, b > 0.$$
2. Marshall-Olkin extended Topp-Leone distribution (MOETLD) introduced by [25], and specified by the following pdf:
$$f(x; \alpha, \lambda) = \frac{2\alpha\lambda(1 - x)\left[1 - (1 - x)^2\right]^{\lambda - 1}}{\left[1 - \bar{\alpha}\left(1 - [1 - (1 - x)^2]^\lambda\right)\right]^2}, \quad x \in [0, 1], \text{ with } \alpha, \lambda > 0.$$
3. Unit-Gompertz distribution (UGD) introduced by [20], and defined with the following pdf: $f(x; a, b) = abx^{-a-1}e^{-b(x^{-a}-1)}, \quad x \in (0, 1]$, with $a, b > 0$.
4. Unit-Burr XII distribution (UBXIID) introduced by [16], and defined with the following pdf: $f(x; \alpha, \beta) = \alpha\beta x^{-1}(-\ln(x))^{\beta-1}\left(1 + (-\ln(x))^\beta\right)^{-\alpha-1}, \quad x \in (0, 1]$, with $\alpha, \beta > 0$.

5. Kumaraswamy distribution introduced by [17], and characterized by the following pdf: $f(x;a,b) = abx^{a-1}(1-x^a)^b, \quad x \in (0,1]$, with $a,b > 0$.

6. Beta distribution reported in [24], and defined with the following pdf: $f(x;\alpha,\beta) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1)$, where $\Gamma(x)$ refers to the standard gamma function, with $\alpha, \beta > 0$.

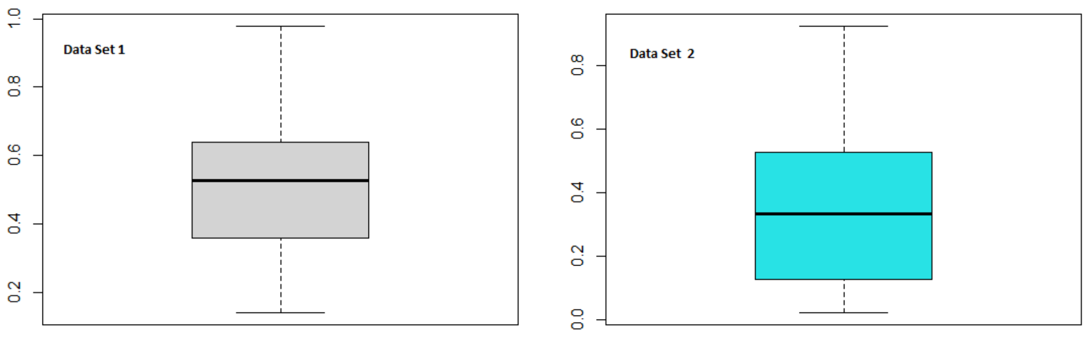
7. Continuous Bernoulli distribution (CBD) reported in [30], and defined with the pdf given in (1).

For clarity in exposition, we finally mention that the $\mathcal{PCB}(\alpha,\lambda)$ distribution will sometimes be denoted as PCBD in the figures and tables to come.

Data set 1: The first data set consists of trade share data from [3]. The trade share data are as follows: 0.140501976, 0.156622976, 0.157703221, 0.160405084, 0.160815045, 0.22145839, 0.299405932, 0.31307286, 0.324612707, 0.324745566, 0.329479247, 0.330021679, 0.337879002, 0.339706242, 0.352317631, 0.358856708, 0.393250912, 0.41760394, 0.425837249, 0.43557933, 0.442142904, 0.444374621, 0.450546652, 0.4557693, 0.46834656, 0.473254889, 0.484600782, 0.488949597, 0.509590268, 0.517664552, 0.527773321, 0.534684658, 0.543337107, 0.544243515, 0.550812602, 0.552722335, 0.56064254, 0.56074965, 0.567130983, 0.575274825, 0.582814276, 0.603035331, 0.605031252, 0.613616884, 0.626079738, 0.639484167, 0.646913528, 0.651203632, 0.681555152, 0.699432909, 0.704819918, 0.729232311, 0.742971599, 0.745497823, 0.779847085, 0.798375845, 0.814710021, 0.822956383, 0.830238342, 0.834204197, and 0.979355395. The data set is approximately symmetric with a skewness value of 0.0059. Details of this data set can be accessed in [29].

Data set 2: The second data set relates to 30 measurements of the tensile strength of polyester fibers reported in [20]. It was first reported in [27]. The data are as follows: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, and 0.926. The data set is right-skewed with a skewness value of 0.5193.

Figure 4 presents the boxplot for the two data sets, showing some of their quantile characteristics.



(a) (b)

Figure 4: *Boxplot for (a) data set 1 and (b) data set 2*

Figure 4 further supports the claim that data set 1 is approximately symmetric while data set 2 is right-skewed. Observe also that there are no outliers in the two data sets.

The distribution comparison will be based on the distribution parameter estimates, log-likelihood (LogL), Akaike information criterion (AIC), and Kolmogorov-Smirnov test statistic (K-S), along with the corresponding p-value. Tables 5 and 6 present the summary statistics for data sets 1 and 2, respectively.

Table 5: *Summary statistics for data set 1*

Models	Estimates	LogL	AIC	K-S	p-value
PCBD	$\alpha = 2.8491$	15.1002	-26.2005	0.0565	0.9837
	$\lambda = 0.0094$				
	$\alpha = 0.3011$				
MOEKD	$a = 3.0586$	14.3183	-22.6367	0.0582	0.9783
	$b = 1.9513$				
MOETLD	$\alpha = 0.6630$	14.3606	-24.7211	0.0568	0.9831
	$\lambda = 3.3521$				
Beta	$\alpha = 2.7940$	13.9561	-23.9121	0.1162	0.3546
	$\beta = 2.6038$				
UGD	$a = 0.6162$	10.8759	-17.7518	0.1098	0.4235
	$b = 1.0921$				
UBXIID	$\alpha = 2.1247$	14.1186	-24.2371	0.0578	0.9804
	$\beta = 2.2237$				
Kumaraswamy	$a = 2.3297$	13.6251	-23.2503	0.0689	0.9142
	$b = 2.7630$				
CBD	$\lambda = 0.5424$	0.0734	1.8532	0.1834	0.0287

Table 6: *Summary statistics for data set 2*

Models	Estimates	LogL	AIC	K-S	p-value
PCBD	$\alpha = 1.1240$	3.4469	-2.8938	0.0578	0.9998
	$\lambda = 0.1069$				
	$\alpha = 0.4363$				
MOEKD	$a = 1.1874$	3.6043	-1.2088	0.0627	0.9992
	$b = 1.2582$				
MOETLD	$\alpha = 1.0929$	2.9136	-1.8272	0.0672	0.9978
	$\lambda = 1.0628$				
Beta	$\alpha = 0.9666$	3.3051	-2.6101	0.1646	0.3515
	$\beta = 1.6203$				
UGD	$a = 1.0373$	3.9488	-3.8976	0.0734	0.9932
	$b = 0.4213$				
UBXIID	$\alpha = 1.0331$	1.0390	1.9220	0.0993	0.9007
	$\beta = 1.8465$				
Kumaraswamy	$a = 0.9627$	3.3110	-2.6221	0.0649	0.9987
	$b = 1.6084$				
CBD	$\lambda = 0.1565$	3.3118	-4.6236	0.0594	0.9997

5.4. Discussion of the results

In the model selection concept, the model that best fits the data set is traceable to the one having the maximized LogL, least value in terms of AIC and K-S with the highest p-value. A close look at Tables 5 and 6 reveals that the $\mathcal{PCB}(\alpha, \lambda)$ distribution outperforms the competitors in fitting the two data sets under study. In particular, all the comparison criteria in Table 5 are in favor of the $\mathcal{PCB}(\alpha, \lambda)$ distribution. Whereas, in Table 6, we observe that LogL as a criterion supports the Marshall-Olkin extended Kumaraswamy and unit-Gompertz distributions over the $\mathcal{PCB}(\alpha, \lambda)$ distribution, while the AIC supports the unit-Gompertz and continuous Bernoulli distributions over the $\mathcal{PCB}(\alpha, \lambda)$ distribution. However, the $\mathcal{PCB}(\alpha, \lambda)$ distribution outperforms all of the competitors in terms of the K-S statistic and its corresponding p-value. Figures 5,6,7 and 8 show additional evidence of its flexibility over the competitors.

Especially, Figure 5 displays the estimated pdf and cdf fits of the distributions for data set 1.

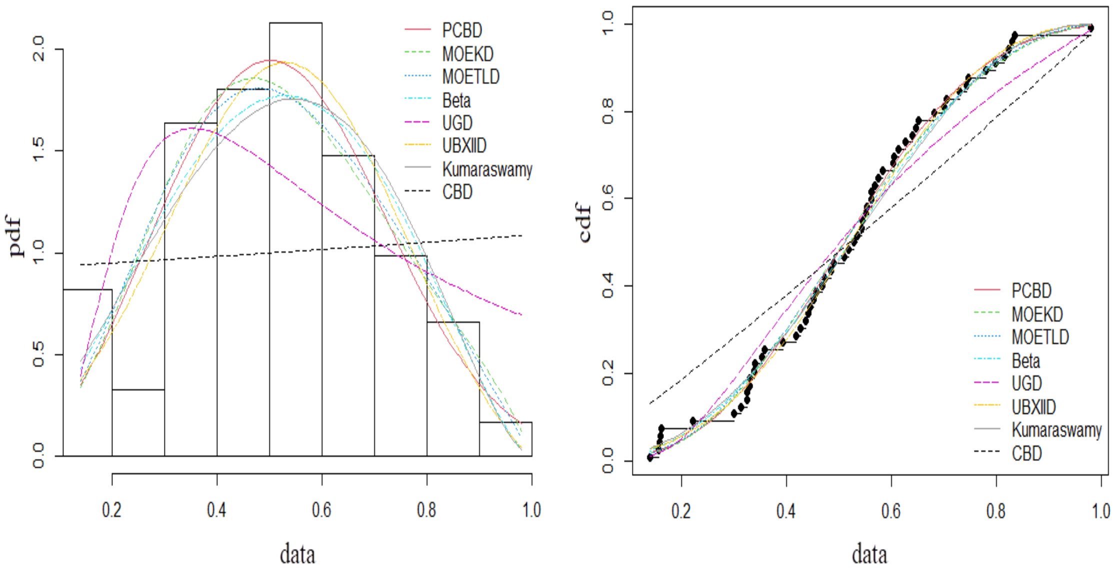


Figure 5: *Estimated pdf and cdf fits of the distributions for data set 1*

We observe that the estimated pdf fit of the $\mathcal{PCB}(\alpha, \lambda)$ distribution perfectly captures the shape of the unimodal histogram, and the estimated cdf fit approaches well the curvature of the empirical cdf.

In Figure 6 the probability-probability (P-P) plots of the distributions for data set 1 are shown.

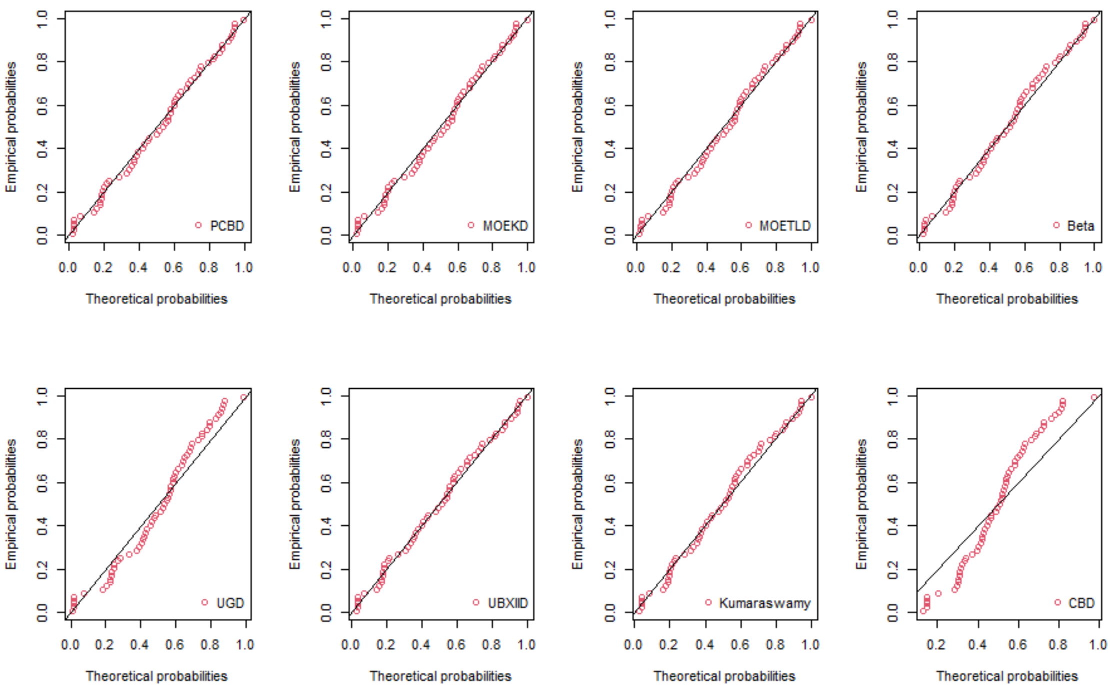


Figure 6: *P-P plost of the distributions for data set 1*

Visually, the P-P line of the $\mathcal{PCB}(\alpha, \lambda)$ distribution better adjusts the associated scatter plot than the others.

Figure 7 is analogous to Figure 5 but for data set 2.

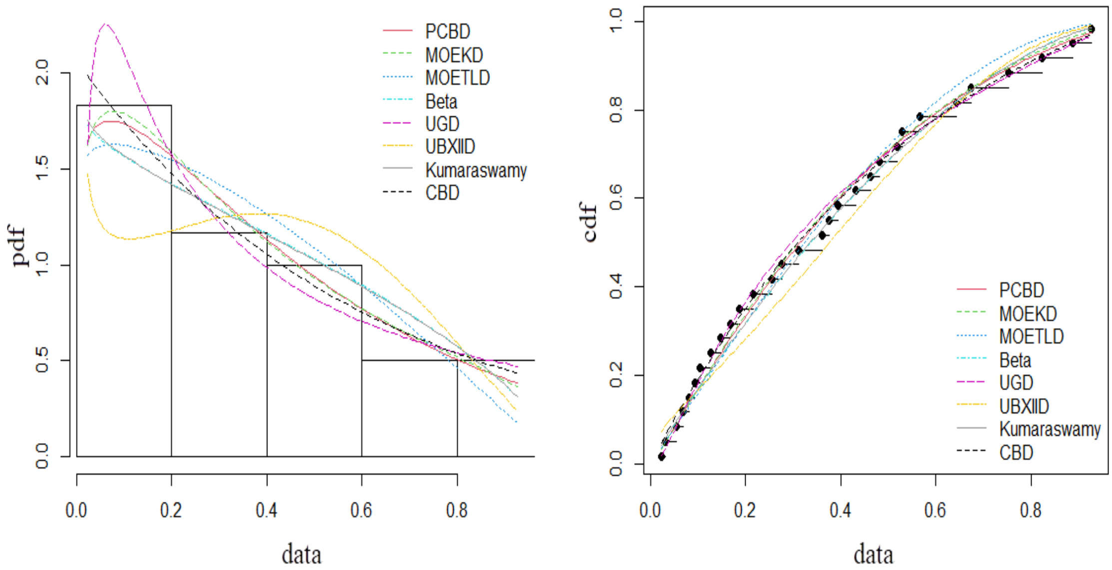


Figure 7: *Estimated pdf and cdf fits of the distributions for data set 2*

In Figure 7, the estimated pdf fit of the $\mathcal{PCB}(\alpha, \lambda)$ distribution captures well the decreasing shape of the histogram, and the estimated cdf fit approaches correctly the concave trend of the empirical cdf.

Figure 8 is analogous to Figure 6 but for data set 2.

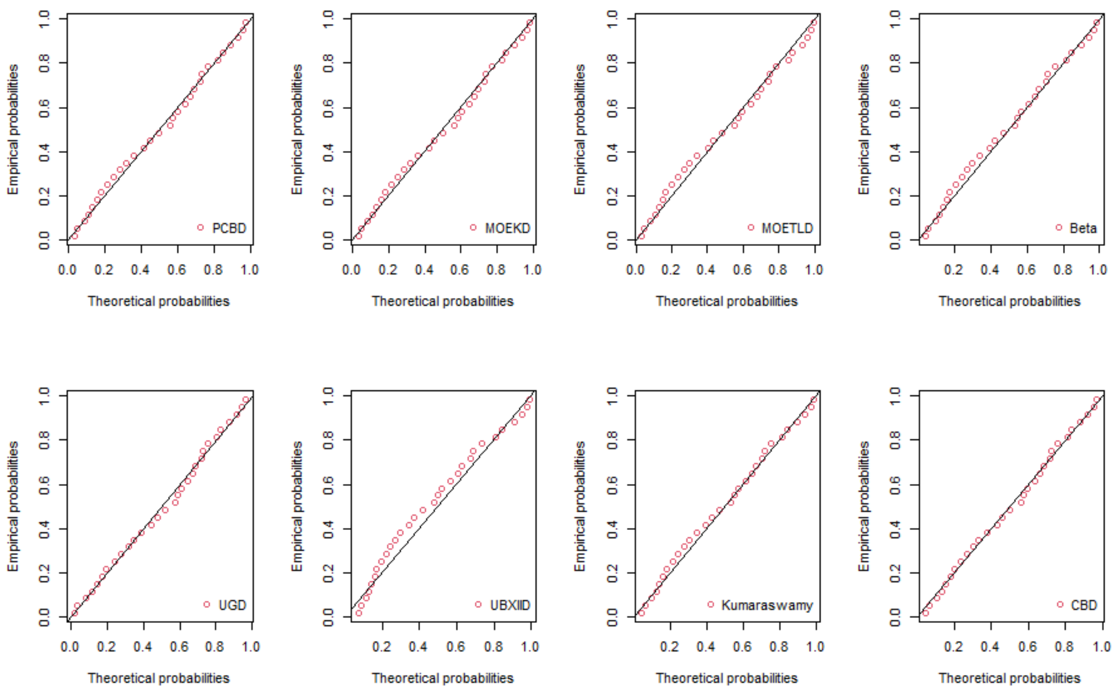


Figure 8: *P-P plots of the distributions for data set 2*

In Figure 8, the P-P line of the $\mathcal{PCB}(\alpha, \lambda)$ distribution is quite acceptable in terms of fitting, as for some other competitors.

In summary, from these figures, it is clear that the fit accuracy of the $\mathcal{PCB}(\alpha, \lambda)$ distribution is excellent, making it a golden distribution to analyze the considered data sets.

6. Conclusion

We proposed a natural extension of the novel continuous Bernoulli distribution by adding a shape parameter through power transformation. The so-called power continuous Bernoulli distribution is aimed at extending the modeling scope of the continuous Bernoulli distribution. Some of its mathematical properties were derived (moments, quantiles, entropy, etc.). A parametric estimation exercise has been given through the maximum likelihood method, and the asymptotic behavior of the parameter estimates was investigated through a Monte Carlo simulation study. Finally, we illustrate the flexibility of the power continuous Bernoulli distribution in real-life data fitting using two real data sets. The potential for probability and statistics applications, such as regression modeling and machine learning applications, is enormous, and this study provides the first steps in that direction.

Conflict of Interests

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Stress-strength Reliability for Equi-correlated Multivariate Normal and its estimation

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Abstract

In this article it is mainly focused on discussion about estimation of stress-strength reliability under equi-correlated multivariate setup. It is seen in some situations that the components of a system are equi-correlated. Generally, the form of the equi-correlation structure within the components of a system is known for a given situation, however parameters that are involved in the equi-correlation structure always unknown. In this article, we propose a procedure to compute and estimate the stress-strength reliability R= Pr($\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}$) when $\boldsymbol{x}$ and $\boldsymbol{y}$ are distributed non-independently equi-correlated multivariate normal distribution, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two known vectors. Here we have proposed the method of moments estimator to estimate these unknown parameters. Actually, we want to find out overall strength is larger than overall stress. In order to do that we take $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\boldsymbol{y}$ as their representatives e.g. principal components of the respective vectors do the job approximately. An asymptotic distribution used to obtain confidence intervals for the stress-strength reliability. The performance of these intervals checked through the simulation study. Finally, we provide a real data analysis.

Keywords: Equi-correaled; Principal Component, Method of Moments Estimator (MOM); Asymptotic.

1. Introduction

The strength-stress model measured by R=Pr(X>Y), the lifetime of a component has a random strength X and it's subjected to random stress Y. In stress-strength model, at any time, the system fails if and only if, the stress is greater than its strength. First introduced to this model by Birnbaum [1] and then developed by Birnbaum and McCarty [2]. There has been a huge number of works as regards estimation of the reliability R= P(X>Y) in the field of stress-strength models. It has several applications particularly in engineering ideas, like structures, deterioration of rocket motors, static fatigue of ceramic parts, fatigue failure of craft structures, and also in mechanical, civil engineering. The R=Pr(X>Y) has been formulated for the huge majority of the popular statistical distributions when X and Y are independent random variables belonging to the same univariate family and also (X,Y) follows the bivariate distribution with dependence between X and Y. This form of R has been established for the bulk of popular statistical distributions, including Normal, uniform, exponential, gamma, beta, extreme value, Weibull, Laplace, logistic and the Pareto distributions…etc [3-7]. This model may be applied in clinical trial to comparing two treatment effects, it may be more useful to draw conclusions regarding the unit's free measure, rather than comparing the means [8]. Simonoff, Hochberg and Reiser [9] also used this model to find the effect of the treatment, if Y is the response for a control group, and X refers to a treatment group.

A numerical procedure obtained by Birnbaum and McCarty [2] based on the asymptotic distribution to find the sample size needed for setting up an upper confidence bound with the defined width and confidence coefficient. Sen [10] obtained the non-parametric confidence bounds for P(X<Y) based on independent samples. Govindarazulu [11] obtained two-sided confidence limits for R when X and Y are independent and also dependent normal variates. Church and Harris [12] obtained confidence intervals for R in case of independent normal varieties.

All these above existing works were done under the univariate or bivariate setup, Gupta and Gupta [13] first introduced the concept of estimating stress-strength reliability under multivariate normal setup. They considered the forms of R= $\Pr(\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y})$, when $(\boldsymbol{x}_{p_1\text{x}1}, \boldsymbol{y}_{p_2\text{x}1})$ follows multivariate normal distribution with non-independent vector between $\boldsymbol{x}_{p_1\text{x}1}$ and $\boldsymbol{y}_{p_2\text{x}1}$, $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are two known vectors. This problem arises when a system in the energy is supplied to the system by p_1 sources and is consumed through p_2 sources and the sources of energy supplied and consumed are linearly related with known vector $\boldsymbol{a}'$ and $\boldsymbol{b}'$. Under this set up Gupta and Gupta [13] considered only special cases of $\boldsymbol{a}'$ and $\boldsymbol{b}'$ and compared the MVUE and MLE estimates of R using given mean vector and dispersion matrix. Reiser and Farragi [14] derived the lower confidence bounds for R=P($\boldsymbol{a}'\boldsymbol{x}^* > \boldsymbol{b}'\boldsymbol{y}^*$) and solved it iteratively and also derived an approximate lower confidence bounds for R. Enis and Geisser [15] have demonstrated that, how to obtain the exact confidence bounds for R.

In many instances, it is seen that the components of a system are equi-correlated. Generally, the form of the equi-correlation structure is known for a given situation within the components of a system, however parameters that are involved in the equi-correlation structure are always unknown. Thus, we compute the stress strength reliability analytically for the special case of equi-correlated multivariate normal setup. We consider the principal component analysis to estimate the $\boldsymbol{a}'$ and $\boldsymbol{b}'$ where as Gupta and Gupta [13] considered only spatial cases of $\boldsymbol{a}'$ and $\boldsymbol{b}'$ and we present estimation of R using method of moment (MOM) estimates of the parameters for equi-correlated multivariate normal setup in Section 2.1. Determine the asymptotic distribution of $\hat{\delta}$ in Section 2.2. Finally, Simulation studies and data analysis are carried out in Section 3 for performance of MOM of R in teams of mean squared errors (MSE), relative bias (RB) and mean absolute error (MAE).

2. Estimation of stress-strength reliability (R)

Let, $\boldsymbol{x}_{p_1\text{x}1}$ and $\boldsymbol{y}_{p_2\text{x}1}$ be two random vector such that the distribution of $\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \sim N_{p1+p2}(\mu, \Sigma)$

$$\text{where, } \mu = \begin{pmatrix}\mu_1 \mathbf{1}_{\mathbf{p_1}\text{x}1}\\ \mu_2 \mathbf{1}_{\mathbf{p_2}\text{x}1}\end{pmatrix}_{(p_1+p_2)\text{x}1} \text{ and } \Sigma = \sigma^2 \left(\begin{matrix} \begin{pmatrix} 1\ \rho_1 \ldots .\rho_1 \\ \rho_1 1 \ldots .\rho_1 \\ \ldots\ldots\ldots\ldots \\ \rho_1\rho_1 \ldots .1 \end{pmatrix} & \begin{pmatrix} \rho_3\ \rho_3 \ldots ..\rho_3 \\ \rho_3\ \rho_3 \ldots ..\rho_3 \\ \ldots\ldots\ldots\ldots . \\ \rho_3\ \rho_3 \ldots ..\rho_3 \end{pmatrix} \\ \begin{pmatrix} \rho_3\ \rho_3 \ldots ..\rho_3 \\ \rho_3\ \rho_3 \ldots ..\rho_3 \\ \ldots\ldots\ldots\ldots . \\ \rho_3\ \rho_3 \ldots ..\rho_3 \end{pmatrix} & \begin{pmatrix} 1\ \rho_2 \ldots .\rho_2 \\ \rho_2 1 \ldots .\rho_1 \\ \ldots\ldots\ldots\ldots \\ \rho_2\rho_2 \ldots .1 \end{pmatrix} \end{matrix} \right)_{(p_1+p_2)\text{x}(p_1+p_2)};$$

$$\Sigma_{\mathbf{11}} = \sigma^2 \begin{pmatrix} 1\rho_1 \ldots \rho_1 \\ \rho_1 1 \ldots \rho_1 \\ \ldots\ldots\ldots\ldots \\ \rho_1\rho_1 \ldots 1 \end{pmatrix}_{(p_1 x p_1)}, \Sigma_{\mathbf{22}} = \sigma^2 \begin{pmatrix} 1\rho_2 \ldots \rho_2 \\ \rho_2 1 \ldots \rho_2 \\ \ldots\ldots\ldots\ldots \\ \rho_2\rho_2 \ldots 1 \end{pmatrix}_{(p_2 x p_2)} \ \&\ \Sigma_{12} = \Sigma_{21}' = \sigma^2 \begin{pmatrix} \rho_3\rho_3 \ldots \rho_3 \\ \rho_3\rho_3 \ldots \rho_3 \\ \ldots\ldots\ldots\ldots \\ \rho_3\rho_3 \ldots \rho_3 \end{pmatrix}_{(p_1 x p_2)};$$

$$-\frac{1}{p_1-1} < \rho_1 < 1, -\frac{1}{p_2-1} < \rho_2 < 1 \ \& -1 \le \rho_3 \le \frac{1}{\sqrt{p_1 p_2}}\left\{1 + \frac{(p_1-1)\rho_1 + (p_2-1)\rho_2}{2}\right\}$$

Now, we are interested to find out the overall strain vector is more than overall stress vector and a gross idea of doing this is to find that in terms of their principal components $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\boldsymbol{y}$.

Then, we want to find the approximate reliability in terms of $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\mathbf{y}$. Then, R= $\Pr(\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y})$ = $\Pr(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y} > 0)$.

Now, the distribution of $u = \boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y}$ follows $N(\mu_{u,}\sigma_u^2)$,

where, μ_u=$E\ (\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y}) = \mu_1 \boldsymbol{a}'\mathbf{1}_{\mathbf{p_1}\mathbf{x1}} - \mu_2 \boldsymbol{b}'\mathbf{1}_{\boldsymbol{p_2x1}}$

and $\sigma_u^2 = Var(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y}) = \boldsymbol{a}'\sum_{\mathbf{11}}\boldsymbol{a} - \mathbf{2}\boldsymbol{a}'\sum_{\mathbf{12}}\boldsymbol{b} + \boldsymbol{b}'\sum_{\mathbf{22}}\boldsymbol{b}$

So, R= $\Pr(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\boldsymbol{y} > 0) = \Pr(u > 0)$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_u}\exp\left\{-\frac{1}{2}\left(\frac{u-\mu_u}{\sigma_u}\right)^2\right\}du = \int_{-\frac{\mu_u}{\sigma_u}}^\infty \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}dz = \Phi\left(\frac{\mu_u}{\sigma_u}\right) \tag{1}$$

The overall representation of the two sets or vectors are related to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that they are approximated by $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\boldsymbol{y}$ as in principal component analysis. Principal component analysis explaining the variance-Covariance structure $\sum_{\mathbf{11}}$ & $\sum_{\mathbf{22}}$ of a set of variables $\boldsymbol{x}$ and $\boldsymbol{y}$ through a linear combination ($\boldsymbol{a}'$ & $\boldsymbol{b}'$) of these variables, i.e, explain maximum variability. It is noted that, the first principal component has the largest possible variance (that is, accounts for as much of the variability in the data as possible), and each succeeding component in turn has the highest variance possible under the constraint that it is orthogonal to the preceding components.

Let, the estimate of $\boldsymbol{a}'$ by $\boldsymbol{e}_1'$ normalized eigenvector of $\sum_{\mathbf{11}}$ corresponding to eigen value $\boldsymbol{\lambda}_1$ and $\boldsymbol{b}'$ by $\boldsymbol{l}_1'$ normalized eigenvector of $\sum_{\mathbf{22}}$ corresponding to eigen value $\boldsymbol{\lambda}_1$.
Thus, we have $\boldsymbol{e}_1' = \frac{1}{\sqrt{p_1}}\mathbf{1}'_{\mathbf{p_1x1}}$ and $\boldsymbol{l}_1' = \frac{1}{\sqrt{p_2}}\mathbf{1}'_{\mathbf{p_2x1}}$ [16]
Then, $\mu_u = \frac{1}{\sqrt{p_1}}\mu_1\,\mathbf{1}'_{\mathbf{p_1x1}}\mathbf{1}_{\mathbf{p_1x1}} - \frac{1}{\sqrt{p_2}}\mu_2\,\mathbf{1}'_{\mathbf{p_2x1}}\mathbf{1}_{\mathbf{p_2x1}} = \sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2$
and,

$$\sigma_u^2 = \frac{1}{p_1}\mathbf{1}'_{\mathbf{p_1x1}}\sum\nolimits_{\mathbf{11}}\mathbf{1}_{\mathbf{p_1x1}} - 2\frac{1}{\sqrt{p_1p_2}}\mathbf{1}'_{\mathbf{p_1x1}}\sum\nolimits_{\mathbf{12}}\mathbf{1}_{\mathbf{p_2x1}} + \frac{1}{p_2}\mathbf{1}'_{\mathbf{p_2x1}}\sum\nolimits_{\mathbf{22}}\mathbf{1}_{\mathbf{p_2x1}}$$
$$= \sigma^2\left[\frac{p_1(1+(p_1-1)\rho_1)}{p_1} - 2\frac{p_1p_2}{\sqrt{p_1p_2}}\rho_3 + \frac{p_2(1+(p_2-1)\rho_2)}{p_2}\right]$$
$$= \sigma^2\left(2 + (p_1-1)\rho_1 + (p_2-1)\rho_2 - 2\sqrt{p_1p_2}\rho_3\right)$$

Then, from equation (1)

$$\text{R}= \Pr(\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}) = \Phi\left[\frac{\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2}{\sigma\sqrt{(2+(p_1-1)\rho_1+(p_2-1)\rho_2-2\sqrt{p_1p_2}\rho_3)}}\right] = \Phi(\delta) \tag{2}$$

where Φ = Distribution function of univariate standard normal distribution.

Now to estimate the R, we need to estimate the parameters of $\mu_1, \mu_2, \rho_1, \rho_2, \rho_3$ and σ^2 in equation (2). Thus, we obtain the method of moments estimator (MOM) of these unknow parameters to estimate δ, denoted by $\hat{\delta}$ and obtain its asymptotic distribution.

2.1.Method of Moments Estimation

Suppose, $m_1, m_2, \ldots, m_6$ are the sample moments of the random sample of $\binom{\boldsymbol{x}_\alpha}{\boldsymbol{y}_\alpha}$, $\alpha = 1,2,\ldots,n$, where $\binom{\boldsymbol{x}_\alpha}{\boldsymbol{y}_\alpha} \sim N_{p1+p2}(\mu, \sum)$. The sample moments are defined as

$$m_1 = \frac{1}{p_1}\sum\nolimits_{i=1}^{p_1}\left(\frac{1}{n}\sum\nolimits_{r=1}^{n} x_{\text{ir}}\right)$$
$$m_2 = \frac{1}{p_2}\sum\nolimits_{j=1}^{p_2}\left(\frac{1}{n}\sum\nolimits_{r=1}^{n} y_{\text{jr}}\right)$$
$$m_3 = \frac{1}{p_1}\sum\nolimits_{i=1}^{p_1}\left(\frac{1}{n}\sum\nolimits_{r=1}^{n} x_{\text{ir}}^2\right)$$
$$m_4 = \frac{1}{p_2}\sum\nolimits_{j=1}^{p_2}\left(\frac{1}{n}\sum\nolimits_{r=1}^{n} y_{\text{jr}}^2\right)$$

$$m_5 = \frac{2}{p_1(p_1-1)} \sum_{\substack{i=1 \\ i<j}}^{p_1} \sum_{j=1}^{p_1} \left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}} x_{\mathrm{jr}}\right)$$

$$m_6 = \frac{2}{p_2(p_2-1)} \sum_{\substack{i=1 \\ i<j}}^{p_2} \sum_{j=1}^{p_2} \left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}} y_{\mathrm{jr}}\right)$$

$$m_7 = \frac{1}{p_1 p_2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}} y_{\mathrm{jr}}\right)$$

Then the moment estimators (MOM) of $\mu_1, \mu_2, \rho_1, \rho_2, \rho_3$ and σ^2 are define as

$$\hat{\mu}_1 = m_1 = \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right)$$

$$\hat{\mu}_2 = m_2 = \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}\right)$$

$$\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{2}\left[\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right) + \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right) - \hat{\mu}_1{}^2 - \hat{\mu}_2{}^2\right] \\
&= \frac{1}{2}\left[\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right) + \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right) - \right. \\
&\qquad \left. \left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right)\right)^2 - \left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}\right)\right)^2\right] \\
&= \frac{1}{2}(m_3 + m_4 - m_1^2 - m_2^2)
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_1 &= \frac{1}{\hat{\sigma}^2}\left(\frac{2}{p_1(p_1-1)}\sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{\mathrm{n}}\sum_{r=1}^{n} x_{\mathrm{ir}} x_{\mathrm{jr}}\right) - \hat{\mu}_1{}^2\right) \\
&= \frac{2}{(m_3 + m_4 - m_1^2 - m_2^2)}(m_5 - m_1^2)
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_2 &= \frac{1}{\hat{\sigma}^2}\left(\frac{2}{p_2(p_2-1)}\sum_{\substack{i=1 \\ i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}} y_{\mathrm{jr}}\right) - \hat{\mu}_2{}^2\right) \\
&= \frac{2}{(m_3 + m_4 - m_1^2 - m_2^2)}(m_6 - m_2^2)
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_3 &= \frac{1}{\hat{\sigma}^2}\left(\frac{1}{p_1 p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}} y_{\mathrm{jr}}\right) - \hat{\mu}_1\hat{\mu}_2\right) \\
&= \frac{2}{(m_3 + m_4 - m_1^2 - m_2^2)}(m_7 - m_1 m_2)
\end{aligned}$$

Thus, the method of moment estimator of R by $\hat{R} = \Phi(\hat{\delta})$

where, $\hat{\delta} = \frac{\sqrt{p_1}\hat{\mu}_1 - \sqrt{p_2}\hat{\mu}_2}{\hat{\sigma}\sqrt{\{2+(p_1-1)\hat{\rho}_1+(p_2-1)\hat{\rho}_2-2\sqrt{p_1p_2}\hat{\rho}_3\}}}$

$$= \frac{m_1\sqrt{p_1} - m_2\sqrt{p_2}}{\sqrt{-m_1^2-m_2^2+m_3+m_4}\sqrt{\frac{-m_3-m_4+m_5+m_6+m_1^2p_1-m_5p_1+m_2^2p_2-m_6p_2-2m_1m_2\sqrt{p_1p_2}+2m_7\sqrt{p_1p_2}}{m_1^2+m_2^2-m_3-m_4}}}$$

2.2. Asymptotic distribution of $\hat{\delta}$

In this Section, we obtain the asymptotic distribution of $\hat{\delta}$ using delta method [17]. Using this we may determine the confidence intervals.

Let us define as

$$\hat{\delta} = g\left(\underset{\sim}{m}\right), \underset{\sim}{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} \text{ and } \delta = h\left(\underset{\sim}{\mu}\right), \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ {\mu_1}^2 + \sigma^2 \\ {\mu_2}^2 + \sigma^2 \\ {\mu_1}^2 + \sigma^2\rho_1 \\ {\mu_2}^2 + \sigma^2\rho_2 \\ \mu_1\mu_2 + \sigma^2\rho_3 \end{pmatrix}$$

By using the central limit theorem,

$$\left(\sqrt{n}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \\ {\mu_1}^2 + \sigma^2 \\ {\mu_2}^2 + \sigma^2 \\ {\mu_1}^2 + \sigma^2\rho_1 \\ {\mu_2}^2 + \sigma^2\rho_2 \\ \mu_1\mu_2 + \sigma^2\rho_3 \end{pmatrix}\right) \xrightarrow{d} N_7\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma^*\right)$$

where, $\Sigma^* = D\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} = \begin{pmatrix} Var(m_1) & Cov(m_1,m_2) & Cov(m_1,m_3) & \ldots\ldots & Cov(m_1,m_7) \\ Cov(m_1,m_2) & Var(m_2) & Cov(m_2,m_3) & \ldots\ldots & Cov(m_2,m_7) \\ Cov(m_1,m_3) & Cov(m_2,m_3) & Var(m_3) & \ldots\ldots & Cov(m_3,m_7) \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ Cov(m_1,m_7) & Cov(m_2,m_7) & Cov(m_3,m_7) & \ldots\ldots & Var(m_7) \end{pmatrix}$

$$Var(m_1) = Var\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right)\right)$$

$$= \frac{1}{n^2{p_1}^2}\left[\sum_{i=1}^{p_1} Var\left(\sum_{r=1}^{n} x_{\mathrm{ir}}\right) + 2\sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{r=1}^{n} x_{\mathrm{jr}}\right)\right]$$

$$= \frac{\sigma^2}{p_1}[1 + (p_1 - 1)\rho_1]$$

$$Var(m_2) = Var\left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}\right)\right)$$

$$= \frac{\sigma^2}{p_2}[1 + (p_2 - 1)\rho_2]$$

$$Var(m_3) = Var\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right)\right)$$

$$= \frac{1}{n^2{p_1}^2}\left[\sum_{i=1}^{p_1} \mathrm{Var}\left(\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right) + 2\sum_{\substack{i=1 \\ i<j}}^{p_1}\sum_{j=1}^{p_1} \mathrm{Cov}\left(\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{r=1}^{n} x_{\mathrm{jr}}^2\right)\right]$$

$$= \frac{1}{p_1}[4\sigma^2{\mu_1}^2 + 2\sigma^4 + (p_1 - 1)(4{\mu_1}^2\rho_1\sigma^2 + 2{\rho_1}^2\sigma^4)]$$

$$Var(m_4) = Var\left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right)\right)$$

$$= \frac{1}{p_2}[4\sigma^2{\mu_2}^2 + 2\sigma^4 + (p_2 - 1)(4{\mu_2}^2\rho_2\sigma^2 + 2{\rho_2}^2\sigma^4)]$$

$$Var(m_5) = Var\left(\frac{2}{p_1(p_1-1)}\sum_{i=1}^{p_1}\sum_{j=1 i<j}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right)$$
$$= \frac{4}{n^2{p_1}^2(p_1-1)^2}\left[\sum_{i=1}^{p_1}\sum_{j=1 i<j}^{p_1}\left(Var\left(\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right) + \right.$$
$$\left.\sum_{i=1}^{p_1}\sum_{\substack{j=1 \\ i<j,k<l(i,j)\neq(k,l)}}^{p_1}\sum_{k=1}^{p_1}\sum_{l=1}^{p_1}\left(Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{kr}}x_{\mathrm{lr}}\right)\right)\right]$$
$$= \frac{4}{({p_1}^2(p_1-1)^2)}\left[\frac{p_1(p_1-1)}{2}\{2{\mu_1}^2\sigma^2 + 2{\mu_1}^2\rho_1\sigma^2 + \sigma^4 + {\rho_1}^2\sigma^4\}\right.$$
$$+\frac{1}{4}\{((p_1-1)(p_1-2))((p_1-1)(p_1-2)-2)(4{\mu_1}^2\rho_1\sigma^2 + 2{\rho_1}^2\sigma^4)\}$$
$$+\frac{1}{4}\Big\{\{(p_1(p_1-1))(p_1(p_1-1)-2) - ((p_1-1)(p_1-2))((p_1-1)(p_1-2)-2)\}$$
$$(3{\mu_1}^2\rho_1\sigma^2 + {\mu_1}^2\sigma^2 + {\rho_1}^2\sigma^4 + \rho_1\sigma^4)\}]$$

$$Var(m_6) = Var\left(\frac{2}{p_2(p_2-1)}\sum_{i=1}^{p_2}\sum_{j=1 i<j}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{4}{n^2{p_2}^2(p_2-1)^2}\left[\sum_{i=1}^{p_2}\sum_{j=1 i<j}^{p_2}\left(\mathrm{Var}\left(\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right) + \right.$$
$$\left.\sum_{i=1}^{p_2}\sum_{\substack{j=1 \\ i<j,k<l(i,j)\neq(k,l)}}^{p_2}\sum_{k=1}^{p_2}\sum_{l=1}^{p_2}\left(Cov\left(\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} y_{\mathrm{kr}}y_{\mathrm{lr}}\right)\right)\right]$$
$$= \frac{4}{({p_2}^2(p_2-1)^2)}\left[\frac{p_2(p_2-1)}{2}\{2{\mu_2}^2\sigma^2 + 2{\mu_2}^2\rho_2\sigma^2 + \sigma^4 + {\rho_2}^2\sigma^4\} + \right.$$
$$\frac{1}{4}\{((p_2-1)(p_2-2))((p_2-1)(p_2-2)-2)(4{\mu_2}^2\rho_2\sigma^2 + 2{\rho_2}^2\sigma^4)\} +$$
$$\frac{1}{4}\Big\{\{(p_2(p_2-1))(p_2(p_2-1)-2) - ((p_2-1)(p_2-2))((p_2-1)(p_2-2)-2)\}$$
$$(3{\mu_2}^2\rho_2\sigma^2 + {\mu_2}^2\sigma^2 + {\rho_2}^2\sigma^4 + \rho_2\sigma^4)\}]$$

$$Var(m_7) = Var\left(\frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{1}{n^2{p_1}^2{p_2}^2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\mathrm{Var}\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right) + \right.$$
$$2\sum_{i}\sum_{\substack{j \\ i\neq j\neq k}}\sum_{k} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{kr}}\right) +$$
$$2\sum_{i}\sum_{\substack{j \\ i\neq j\neq k}}\sum_{k} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{kr}}y_{\mathrm{jr}}\right) +$$
$$\left.2\sum_{i}\sum_{\substack{j \\ i\neq j\neq k\neq l}}\sum_{k}\sum_{l} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{r=1}^{n} x_{\mathrm{kr}}y_{\mathrm{lr}}\right)\right]$$
$$= \frac{1}{p_1p_2}[({\mu_1}^2\sigma^2 + {\mu_2}^2\sigma^2 + 2\mu_1\mu_2\sigma^2\rho_3 + \sigma^4 + {\rho_3}^2\sigma^4) +$$
$$(p_1-1)({\mu_1}^2\sigma^2 + 2\mu_1\mu_2\sigma^2\rho_3 + {\mu_2}^2\rho_1\sigma^2 + \sigma^4\rho_1 + \sigma^4{\rho_3}^2) +$$
$$(p_2-1)({\mu_2}^2\sigma^2 + 2\mu_1\mu_2\sigma^2\rho_3 + {\mu_1}^2\rho_2\sigma^2 + \sigma^4\rho_2 + \sigma^4{\rho_3}^2) +$$
$$(p_1-1)(p_2-1)({\mu_1}^2\sigma^2\rho_2 + 2\mu_1\mu_2\sigma^2\rho_3 + {\mu_2}^2\sigma^2\rho_1 + \sigma^4\rho_1\rho_2 + \sigma^4{\rho_3}^2)]$$

$$Cov(m_1, m_2) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right), \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}\right)\right)$$
$$= \frac{1}{n^2 p_1 p_2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} \mathrm{Cov}\left(\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{r=1}^{n} y_{\mathrm{jr}}\right)\right]$$
$$= \sigma^2\rho_3$$

$$Cov(m_1, m_3) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right), \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right)\right)$$
$$= \frac{1}{n^2 {p_1}^2}\left[\sum_{i=1}^{p_1} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{r=1}^{n} x_{\mathrm{ir}}^2\right) + \ 2\sum_{\substack{i=1\\ i<j}}^{p_1}\sum_{j=1}^{p_1} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{r=1}^{n} x_{\mathrm{jr}}^2\right)\right]$$
$$= \frac{2}{p_1}[\mu_1\sigma^2 + (p_1 - 1)\sigma^2\mu_1\rho_1]$$

$$Cov(m_1, m_4) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right), \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right)\right)$$
$$= \frac{1}{n^2 p_1 p_2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{r=1}^{n} y_{\mathrm{jr}}^2\right)\right]$$
$$= 2\sigma^2\rho_3\mu_2$$

$$Cov(m_1, m_5) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right), \frac{2}{p_1(p_1 - 1)}\sum_{\substack{i=1\\ i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2 p_1^2(p_1 - 1)} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{\substack{i=1\\ i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)$$
$$= \frac{2}{p_1}\mu_1\sigma^2[\rho_1(p_1 - 1) + 1]$$
$$Cov(m_1, m_6) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right), \frac{2}{p_2(p_2 - 1)}\sum_{\substack{i=1\\ i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2 p_1 p_2(p_2 - 1)}\left(\sum_{i}\sum_{\substack{j\\ j<k}}\sum_{k} Cov\left(\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{r=1}^{n} y_{\mathrm{jr}}y_{\mathrm{kr}}\right)\right)$$
$$= 2\mu_2\sigma^2\rho_3$$
$$Cov(m_1, m_7) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}\right), \frac{1}{p_1 p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{1}{n^2 p_1^2 p_2} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= \frac{1}{p_1}[\mu_1\sigma^2\rho_3 + \mu_2\sigma^2 + (p_1 - 1)(\mu_1\sigma^2\rho_3 + \mu_2\sigma^2\rho_1)]$$

$$Cov(m_2, m_3) = Cov\left(\frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}\right), \frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right)\right)$$
$$= \frac{1}{n^2 p_1 p_2} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}\right)$$
$$= 2\sigma^2\rho_3\mu_1$$

$$Cov(m_2, m_4) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}\right), \frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}^2\right)\right)$$

$$= \frac{1}{n^2 p_2^2}\left[\sum_{i=1}^{p_2} \mathrm{Cov}\left(\sum_{r=1}^{n} y_{\mathrm{ir}}, \sum_{r=1}^{n} y_{\mathrm{ir}}^2\right) + 2\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2} \mathrm{Cov}\left(\sum_{r=1}^{n} y_{\mathrm{ir}}, \sum_{r=1}^{n} y_{\mathrm{jr}}^2\right)\right]$$

$$= \frac{2}{p_2}[\mu_2\sigma^2 + (p_2 - 1)\sigma^2\mu_2\rho_2]$$

$$Cov(m_2, m_5) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}\right), \frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right)$$

$$= \frac{2}{n^2 p_1 p_2 (p_1 - 1)} Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}, \sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)$$

$$= 2\mu_1\sigma^2\rho_3$$

$$Cov(m_2, m_6) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}\right), \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$

$$= \frac{2}{n^2 p_2^2 (p_2 - 1)} Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$

$$= \frac{2\mu_2\sigma^2}{p_2}[\rho_2(p_2 - 1) + 1]$$

$$Cov(m_2, m_7) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}\right), \frac{1}{p_1 p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$

$$= \frac{1}{n^2 p_1 p_2^2} Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$

$$= \frac{1}{p_2}[\mu_2\sigma^2\rho_3 + \mu_2\sigma^2 + (p_2 - 1)(\mu_2\sigma^2\rho_3 + \mu_2\sigma^2\rho_2)]$$

$$Cov(m_3, m_4) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right), \frac{1}{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right)\right)$$

$$= \frac{1}{n^2 p_1 p_2} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right)$$

$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_3, m_5) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right), \frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right)$$

$$= \frac{2}{n^2 p_1^2 (p_1 - 1)} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)$$

$$= \frac{1}{p_1}[4({\mu_1}^2\sigma\rho_1 + {\mu_1}^2\sigma^2 + \sigma^4\rho_1) + 4{\mu_1}^2\sigma^2\rho_1 + 2\sigma^4{\rho_1}^2]$$

$$Cov(m_3, m_6) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right), \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2 p_1 p_2 (p_2-1)} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_3, m_7) = Cov\left(\frac{1}{p_1}\sum_{i=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}^2\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{1}{n^2 p_1^2 p_2} Cov\left(\sum_{i=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}^2, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= \frac{2}{p_1}[({\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3) + (p_1-1)({\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_1 + \sigma^4\rho_1\rho_3)]$$

$$Cov(m_4, m_5) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right), \frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2 p_1 p_2 (p_1-1)} \mathrm{Cov}\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}^2, \sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right)$$
$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_4, m_6) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right), \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{2}{n^2 p_2^2 (p_2-1)} \mathrm{Cov}\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}^2, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= \frac{1}{p_2}[4({\mu_2}^2\sigma\rho_2 + {\mu_2}^2\sigma^2 + \sigma^4\rho_2) + 4{\mu_2}^2\sigma^2\rho_2 + 2\sigma^4{\rho_2}^2]$$

$$Cov(m_4, m_7) = Cov\left(\frac{1}{p_2}\sum_{i=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{jr}}^2\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{1}{n^2 p_1 p_2^2} Cov\left(\sum_{i=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{jr}}^2, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= \frac{2}{p_2}[({\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3) + (p_2-1)({\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_2 + \sigma^4\rho_2\rho_3)]$$

$$Cov(m_5, m_6) =$$
$$Cov\left(\frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right), \frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$
$$= \frac{4}{n^2 p_1 p_2 (p_1-1)(p_2-1)} Cov\left(\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}, \sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$
$$= 4\mu_1\mu_2\sigma^2\rho_3 + 2\sigma^4{\rho_3}^2$$

$$Cov(m_5, m_7) = Cov\left(\frac{2}{p_1(p_1-1)}\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$

$$= \frac{2}{n^2p_1^2(p_1-1)p_2}Cov\left(\sum_{\substack{i=1\\i<j}}^{p_1}\sum_{j=1}^{p_1}\sum_{r=1}^{n} x_{\mathrm{ir}}x_{\mathrm{jr}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$

$$= \frac{2}{p_1}[(2{\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_1 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3 + \sigma^4\rho_1\rho_3) +$$
$$(p_1-2)({\mu_1}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_1 + \sigma^4\rho_1\rho_3)]$$

$$Cov(m_6, m_7) = Cov\left(\frac{2}{p_2(p_2-1)}\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}\right), \frac{1}{p_1p_2}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\left(\frac{1}{n}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)\right)$$

$$= \frac{2}{n^2p_1p_2^2(p_2-1)}Cov\left(\sum_{\substack{i=1\\i<j}}^{p_2}\sum_{j=1}^{p_2}\sum_{r=1}^{n} y_{\mathrm{ir}}y_{\mathrm{jr}}, \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\sum_{r=1}^{n} x_{\mathrm{ir}}y_{\mathrm{jr}}\right)$$

$$= \frac{2}{p_2}[(2{\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_2 + \mu_1\mu_2\sigma^2 + \sigma^4\rho_3 + \sigma^4\rho_2\rho_3) +$$
$$(p_2-2)({\mu_2}^2\sigma^2\rho_3 + \mu_1\mu_2\sigma^2\rho_2 + \sigma^4\rho_2\rho_3)]$$

Thus we find,

$$\underset{\sim}{g}'(\hat{\delta}) = \left(\frac{\partial g}{\partial m_1}, \frac{\partial g}{\partial m_2}, \frac{\partial g}{\partial m_3}, \frac{\partial g}{\partial m_4}, \frac{\partial g}{\partial m_5}, \frac{\partial g}{\partial m_6}, \frac{\partial g}{\partial m_7}\right)_{\underset{\sim}{m}=\underset{\sim}{\mu}}$$

where,

$$\left(\frac{\partial g}{\partial m_1}\right)_{m_1=\mu_1} = \frac{p_1\mu_1(\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2)}{\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_2}\right)_{m_2=\mu_2} = \frac{p_2\mu_2(\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2)}{\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_3}\right)_{m_3=({\mu_1}^2+\sigma^2)} = \frac{-\sqrt{p_1}\mu_1 + \sqrt{p_2}\mu_2}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_4}\right)_{m_4=({\mu_2}^2+\sigma^2)} = \frac{-\sqrt{p_1}\mu_1 + \sqrt{p_2}\mu_2}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_5}\right)_{m_5=({\mu_1}^2+\sigma^2\rho_1)} = \frac{-(-1+p_1)(\sqrt{p_1}\mu_1 - \sqrt{p_2}\mu_2)}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

$$\left(\frac{\partial g}{\partial m_6}\right)_{m_6=({\mu_2}^2+\sigma^2\rho_2)} = \frac{(-1+p_2)(-\sqrt{p_1}\mu_1 + \sqrt{p_2}\mu_2)}{2\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{\frac{3}{2}}}$$

$$\left(\frac{\partial g}{\partial m_7}\right)_{m_7=(\mu_1\mu_2+\sigma^2\rho_3)} = \frac{\sqrt{p_1p_2}\left(\sqrt{p_1}\mu_1-\sqrt{p_2}\mu_2\right)}{\left(\sigma^2\left(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\right)\right)^{3/2}}$$

Using the delta method, we have

$$\sqrt{n}\left(\hat{\delta}-\delta\right)\xrightarrow{d}N(0,\sigma_\delta^2)$$

or,

$$\hat{\delta}\xrightarrow{d}N\left(\delta,\frac{\sigma_\delta^2}{n}\right)$$

where,

$$\begin{gathered}
\sigma_\delta^2 = \underset{\sim}{g}'\left(\hat{\delta}\right)\Sigma^*\underset{\sim}{g}\left(\hat{\delta}\right) \\
= ((\sigma^3p_1^4p_2\rho_1^2(\mu_1^2+2\sigma^2\rho_1)-2p_2^2\mu_2^2\rho_1((2+4\sigma)\mu_1^2+3\sigma^3\rho_1)+ \\
\sigma^3\sqrt{p_1p_2}\mu_2(-1+\rho_1)(-2+\rho_1-(-1+p_2)\rho_2))- \\
2\sigma^3p_1^{7/2}p_2^{3/2}\rho_1^2(\mu_1\mu_2+2\sigma^2\rho_3)+p_1p_2(-4(1+2\sigma)\mu_1^4\rho_1+ \\
2\mu_1^2\rho_1(2(1+3\sigma)p_2\mu_2^2-3\sigma^3\rho_1)+\sigma^3p_2^3\rho_2^2(\mu_2^2+2\sigma^2\rho_2)- \\
4\sigma\sqrt{p_1p_2}\mu_1\mu_2(p_2^2\mu_2^2\rho_2-\sigma^2(-1+\rho_1)(-2+\rho_1+\rho_2)+ \\
p_2(-\mu_2^2(-1+\rho_2)+\sigma^2(-1+\rho_1)\rho_2))- \\
2(2(1+2\sigma)\mu_2^4\rho_2+\sigma^3\mu_2^2(-4\sqrt{p_1p_2}-2\sqrt{p_1p_2}\rho_1 \\
(-1+\rho_2)+6\sqrt{p_1p_2}\rho_2+(3-2\sqrt{p_1p_2})\rho_2^2)+ \\
\sigma^5(-2+\rho_1+\rho_2)^2(-2+\rho_1+\rho_2+4\sqrt{p_1p_2}\rho_3))- \\
2\sigma p_2^2(\sigma^2\mu_2^2\rho_2(-2+\rho_1-2(-1+\sqrt{p_1p_2})\rho_2)+\mu_2^4(-1+\rho_1-2 \\
(-1+\sqrt{p_1p_2})\rho_2)+\sigma^4\rho_2^2(-6+3\rho_1+3\rho_2+4\sqrt{p_1p_2}\rho_3))+ \\
p_2(4\mu_2^4(\sigma\sqrt{p_1p_2}+(1-\sigma(-3+\sqrt{p_1p_2}))\rho_2)+ \\
\sigma^3\mu_2^2(4+9\rho_1^2+4(-1+3\sqrt{p_1p_2})\rho_2+(9-8\sqrt{p_1p_2}) \\
\rho_2^2+\rho_1(-4+(2-4\sqrt{p_1p_2})\rho_2))+ \\
2\sigma^5\rho_2(-2+\rho_1+\rho_2)(-6+3\rho_1+3\rho_2+8\sqrt{p_1p_2}\rho_3)))- \\
2p_1^{3/2}\sqrt{p_2}(-2\mu_1\mu_2(-2\sigma^3\sqrt{p_1p_2}-\sigma^3\sqrt{p_1p_2}\rho_1(-1+\rho_2)+ \\
(3\sigma^3\sqrt{p_1p_2}+(2+4\sigma)\mu_2^2)\rho_2-\sigma^3(-3+\sqrt{p_1p_2})\rho_2^2)+ \\
\sigma^3p_2^3\rho_2^2(\mu_1\mu_2+2\sigma^2\rho_3)+p_2(4(1+3\sigma)\mu_1^3\mu_2\rho_1+ \\
4\sigma\sqrt{p_1p_2}\mu_1^2\mu_2^2(-1+\rho_2)-2\sigma^3\sqrt{p_1p_2}\mu_2^2\rho_1(-3+2\rho_1+\rho_2)+ \\
\mu_1\mu_2(4\mu_2^2(\sigma\sqrt{p_1p_2}+(1-\sigma(-3+\sqrt{p_1p_2}))\rho_2)+\sigma^3 \\
(4+9\rho_1^2+(-4+6\sqrt{p_1p_2})\rho_2+(9-4\sqrt{p_1p_2})\rho_2^2- \\
2\rho_1(2+(-1+\sqrt{p_1p_2})\rho_2)))+ \\
2\sigma^5(-2+\rho_1+\rho_2)\rho_3(-2+\rho_1+\rho_2+4\sqrt{p_1p_2}\rho_3))- \\
2\sigma p_2^2(2\sqrt{p_1p_2}\mu_1^2\mu_2^2\rho_2+\mu_1\mu_2(\sigma^2\rho_2(-2+\rho_1- \\
(-2+\sqrt{p_1p_2})\rho_2)+\mu_2^2(-1+\rho_1-2(-1+\sqrt{p_1p_2})\rho_2))+ \\
2\sigma^4\rho_2\rho_3\left(-2+\rho_1+\rho_2+2\sqrt{p_1p_2}\rho_3\right)-\sigma^2\mu_2^2 \\
(-2\rho_3+\rho_1(\sqrt{p_1p_2}\rho_2+2\rho_3))))+ \\
\sigma p_1^3p_2(-2\mu_1^4(-1+2\rho_1+\rho_2-p_2\rho_2)+4p_2\mu_1^3\mu_2\rho_3+ \\
4\sigma^2\mu_1\mu_2\rho_1\left(\sqrt{p_1p_2}\rho_1-2p_2\rho_3\right)+ \\
\mu_1^2(-4\sigma^2\rho_1^2+2\rho_1(-\sigma^2(-2+\rho_2)+p_2(\mu_2^2+\sigma^2\rho_2))- \\
4\sigma^2p_2\rho_3^2)+\sigma^2\rho_1(-6\sigma^2\rho_1^2+8\sigma^2p_2\rho_3^2+ \\
\rho_1(p_2(\mu_2^2+6\sigma^2\rho_2)-2\sigma^2(-6+3\rho_2+4\sqrt{p_1p_2}\rho_3))))- \\
4\sigma p_1^{5/2}\sqrt{p_2}(-\sigma^2\sqrt{p_1p_2}\mu_1\mu_2\rho_1(-1+\rho_2)+ \\
p_2(-\mu_1^3\mu_2(-1+2\rho_1+\rho_2)+ \\
\sigma^2\mu_1\mu_2\left(-2\rho_1^2+\rho_1\left(2+\left(-1+\sqrt{p_1p_2}\right)\rho_2\right)+2(-1+\rho_2)\rho_3\right)+ \\
\sigma^2\rho_1(\sqrt{p_1p_2}\mu_2^2\rho_1-2\sigma^2\rho_3(-2+\rho_1+\rho_2+2\sqrt{p_1p_2}\rho_3)))+ \\
p_2^2(\mu_1^3\mu_2\rho_2+2\mu_1^2\mu_2^2\rho_3+2\sigma^2\rho_3(-\mu_2^2\rho_1+\sigma^2(\rho_1\rho_2+2\rho_3^2))+ \\
\mu_1\mu_2(\mu_2^2\rho_1+\sigma^2(\rho_1\rho_2-2\rho_3(\rho_2+\rho_3)))))+ \\
p_1^2(-2\mu_1^2\rho_2((2+4\sigma)\mu_2^2+3\sigma^3\rho_2)+p_2(4(1+3\sigma)\mu_1^4\rho_1+ \\
4\sigma\sqrt{p_1p_2}\mu_1^3\mu_2(-1+\rho_2)-4\sigma^3\sqrt{p_1p_2}\mu_1\mu_2\rho_1
\end{gathered}$$

$$(-3+2\rho_1+\rho_2)+\mu_1^2(4\mu_2^2(\sigma\sqrt{p_1p_2}+(1+3\sigma-\sigma\sqrt{p_1p_2})$$
$$\rho_2)+\sigma^3(4+9\rho_1^2+2\rho_1(-2+\rho_2)-4\rho_2+9\rho_2^2))+$$
$$2\sigma^3\rho_1(-2\sqrt{p_1p_2}\mu_2^2(-1+\rho_2)+\sigma^2(-2+\rho_1+\rho_2)$$
$$(-6+3\rho_1+3\rho_2+8\sqrt{p_1p_2}\rho_3)))+\sigma p_2^3(2\mu_2^4\rho_1+$$
$$4\mu_1\mu_2^3\rho_3+\sigma^2\rho_2(\mu_1^2\rho_2+2\sigma^2(3\rho_1\rho_2+4\rho_3^2))+$$
$$2\mu_2^2(\mu_1^2\rho_2+\sigma^2(\rho_1\rho_2-2\rho_3(2\rho_2+\rho_3))))-$$
$$2\sigma p_2^2(2\sqrt{p_1p_2}\mu_1^3\mu_2\rho_2+\mu_1^2(\sigma^2\rho_2(-2+\rho_1+2\rho_2)+$$
$$\mu_2^2(-2+3\rho_1+(3-2\sqrt{p_1p_2})\rho_2))+$$
$$\sigma^2\mu_2^2\big(2\rho_1^2+\rho_1\big(-2+\big(1-2\sqrt{p_1p_2}\big)\rho_2\big)-4(-1+\rho_2)\rho_3\big)-$$
$$2\sigma^2\mu_1\mu_2\left(-2\rho_3+\rho_1\big(\sqrt{p_1p_2}\rho_2+2\rho_3\big)\right)+$$
$$2\sigma^4(3\rho_1^2\rho_2+2(-2+\rho_2)\rho_3^2+\rho_1$$
$$(3\rho_2^2+2\rho_3^2+\rho_2(-6+4\sqrt{p_1p_2}\rho_3))))))/$$
$$(2n\sigma^5p_1p_2\big(2+(-1+p_1)\rho_1+(-1+p_2)\rho_2-2\sqrt{p_1p_2}\rho_3\big)^3))$$

2.3. Asymptotic Confidence Intervals for R

Based on the asymptotic distribution of $\hat{\delta}$, we obtain the asymptotic confidence interval of R. Here, the estimate of R by $\hat{R}=\Phi(\hat{\delta})$, i.e. $\hat{\delta}=\Phi^{-1}(R)$ and we have $\hat{\delta}\xrightarrow{d}N\left(\delta,\frac{\sigma_\delta^2}{n}\right)$ as $n\to\infty$. In order to determine the two sided confidence Intervals, we find out the two numbers $L_1$and L_2 $(L_1<L_2)$, such that, for a given α, we have

$$P(L_1\leqslant\Phi(\delta)\leqslant L_2)=1-\alpha$$

or, $$P(\Phi^{-1}(L_1)\leqslant\delta\leqslant\Phi^{-1}(L_2))=1-\alpha \tag{3}$$

Then, an asymptotic (1-α) level confidence Intervals for δ is given by

$$P\left(-z_{\alpha/2}\leqslant\frac{\sqrt{n}(\hat{\delta}-\delta)}{\sigma_\delta}\leqslant z_{\alpha/2}\right)=1-\alpha$$

or, $$P\left(-\frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}}\leqslant(\hat{\delta}-\delta)\leqslant\frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}}\right)=1-\alpha$$

or, $$P\left(\hat{\delta}-\frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}}\leqslant\delta\leqslant\hat{\delta}+\frac{z_{\alpha/2}\sigma_\delta}{\sqrt{n}}\right)=1-\alpha$$

We can replace $\sigma_\delta$by $\hat{\sigma}_\delta$ to obtain asymptotic confidence Intervals for δ. Thus, we can write

$$P\left(\hat{\delta}-\frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\leqslant\delta\leqslant\hat{\delta}+\frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right)=1-\alpha \tag{4}$$

Comparing (3) and (4), we have $L_1$and L_2respectively as

$$\Phi^{-1}(L_1)=\hat{\delta}-\frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}$$

or, $$L_1=\Phi\left(\hat{\delta}-\frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right)$$

and, $$L_2=\Phi\left(\hat{\delta}+\frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right)$$

Then, an asymptotic (1-α) level confidence Intervals for R is represented by

$$(L_1, L_2) = \left\{\Phi\left(\hat{\delta} - \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right), \Phi\left(\hat{\delta} + \frac{z_{\alpha/2}\hat{\sigma}_\delta}{\sqrt{n}}\right)\right\}$$

where, $z_{\alpha/2}$ upper critical value for the standard normal distribution.

Thus, an asymptotic (1-α) confidence lower bound for R as

$$L_B = \Phi\left(\hat{\delta} - \frac{z_\alpha\hat{\sigma}_\delta}{\sqrt{n}}\right)$$

3. Simulation study and Data analysis

3.1. Simulation study

Now, we compute convergence and performance of MOM estimator, we considered different scenarios, each corresponding to a different combination of distributional parameters with different reliabilities for p_1=10 and p_2=10, reported in Table 1. We set the six parameters in order to get a high value (>0.5) for the reliability, since one typically looks for high reliability for the study component or system in real practice. Through these scenarios we cover the large range of reliability, since the range of R from 0.5825 to 0.9736.

For this above purpose we compute the following measures:

(i) Sample mean of $\hat{R}$ using MOM

(ii) Mean square error (MSE) of $\hat{R}$: $E(\hat{R} - R)^2$

(iii) Mean Relative Bias (RB) of $\hat{R}$: $\frac{E(\hat{R})-R}{R}$

(iv) Mean absolute error (MAE) of $\hat{R}$: $E(|\hat{R} - R|)$

It is difficult to obtain the analytical form of the equation (1) for various 'R'. So, we figure out these by using simulation study. Hence, we generate the random samples of size n from $\binom{\boldsymbol{x}}{\boldsymbol{y}} \sim N_{p1+p2}(\mu, \Sigma)$ for different scenarios. For each of sample drown of size n, we compute the R using MOM by taking 500 replications each time and also compute the above measures. Here we consider the different sample sizes (n) as 10, 30, 50 and 100. For this purpose, here, R programming language is used. The simulation results are reported in Table 2. It is noted that, the MSE, RB and MAE of $\hat{R}$ are reduces as the sample size increases decrease as expected and when n=100, $\hat{R}$ achieved the true value of R under each scenario. Thus, the result seems to be supportive for R in larger sample. Also, the performance of the R using MOM is quite satisfactory in terms of MSE, RB and MAE for small sample sizes. Hence, it is satisfying the consistency property of the MOM of R.

We take the components as p_1=14, p_2=12 and set the parameters are μ_1=4, μ_2=3.5, ρ_1=0.8, ρ_2=0.6, ρ_3=0.7, σ^2=4 in order to verify the asymptotic distribution of $\hat{\delta}$ as follows normal distribution, described in section 2.2. Then, we have $\hat{\delta} \sim N(1.538, 0.0196)$ and generate n=500 samples using this as theoretical quantiles. For each of sample drown of size n=500, we compute the $\hat{\delta}$ using MOM by taking 500 replications each time is treated as sample quantiles. Figure 1, Q-Q plot [18] and Shapiro-Wilk normality test [19] (result as W = 0.99686, p-value = 0.4469) provided satisfactory result that the $\hat{\delta}$ follows asymptotic normal distribution.

The results of the simulation study for the confidence intervals as lower (L_1) limit, upper (L_2) limit and lower bound (L_B) are recorded in Table 3. Table 1 and 2, represent the asymptotic and bootstrap confidence belt at 90%, 95% and 99% levels. It has been observed that for a small sample size, the estimate of R is getting high and also confidence intervals in case of asymptotic. The results get better as the sample sizes increase and the reliability R gets closer to true value. The overall band of asymptotic confidence is going to sink as the sample sizes increase and it has consistent variation.

Table 1: *Parameters for the equi-correlated set-up*

Scenario	Parameters and values						
	μ_1	μ_2	ρ_1	ρ_2	ρ_3	σ^2	R
1	1.5	1	0.2	0.2	0.1	16	0.5825
2	2.5	1.5	0.3	0.2	0.1	16	0.6453
3	3	2	0.3	0.2	0.2	16	0.6915
4	4	2.5	0.5	0.4	0.3	16	0.7209
5	4	2.5	0.5	0.4	0.3	8	0.7962
6	4	2.5	0.6	0.4	0.4	8	0.8335
7	4	2.5	0.6	0.5	0.5	8	0.8881
8	4	2.5	0.7	0.6	0.5	4	0.8912
9	4	2.5	0.8	0.6	0.6	4	0.9293
10	4	2.5	0.8	0.7	0.7	4	0.9736

Table 2: *Simulation results: Coverage Probability, MSE, RB and MAE*

Scenario	Sample Size	Sample Mean ($\hat{R}$)	MSE	RB	MAE
1	10	0.5840	0.0177	0.0025	0.1064
	30	0.5833	0.0054	0.0013	0.0576
	50	0.5831	0.0034	0.0010	0.0467
	100	0.5828	0.0017	0.0006	0.0328
2	10	0.6491	0.0154	0.0060	0.1002
	30	0.6456	0.0048	0.0005	0.0537
	50	0.6455	0.0029	0.0004	0.0431
	100	0.6454	0.0015	0.0001	0.0315
3	10	0.6918	0.0171	0.0006	0.1034
	30	0.6918	0.0052	0.0005	0.0582
	50	0.6917	0.0027	0.0004	0.0423
	100	0.6917	0.0015	0.0003	0.0309
4	10	0.7238	0.0161	0.0040	0.1024
	30	0.7222	0.0046	0.0017	0.0538
	50	0.7209	0.0026	-0.0001	0.0409
	100	0.7206	0.0013	-0.0004	0.0295
5	10	0.7977	0.0110	0.0018	0.0856
	30	0.7973	0.0033	0.0014	0.0457
	50	0.7969	0.0022	0.0008	0.0385
	100	0.7965	0.0012	0.0004	0.0271
6	10	0.8376	0.0091	0.0048	0.0776
	30	0.8347	0.0033	0.0014	0.0457
	50	0.8341	0.0020	0.0007	0.0356
	100	0.8336	0.0009	0.0001	0.0245
7	10	0.8903	0.0057	0.0024	0.0603
	30	0.8895	0.0019	0.0016	0.0354
	50	0.8888	0.0014	0.0008	0.0298
	100	0.8884	0.0006	0.0003	0.0192
8	10	0.8934	0.0057	0.0025	0.0622
	30	0.8926	0.0020	0.0016	0.0355
	50	0.8918	0.0013	0.0007	0.0284
	100	0.8917	0.0006	0.0005	0.0205

Scenario	Sample Size	Sample Mean ($\widehat{R}$)	MSE	RB	MAE
9	10	0.9310	0.0031	0.0018	0.0434
	30	0.9304	0.0013	0.0012	0.0278
	50	0.9298	0.0008	0.0005	0.0224
	100	0.9296	0.0004	0.0003	0.0159
10	10	0.9740	0.0008	0.0004	0.0213
	30	0.9737	0.0003	0.0001	0.0139
	50	0.9736	0.0002	0.0000	0.0119
	100	0.9735	0.0001	-0.0001	0.0085

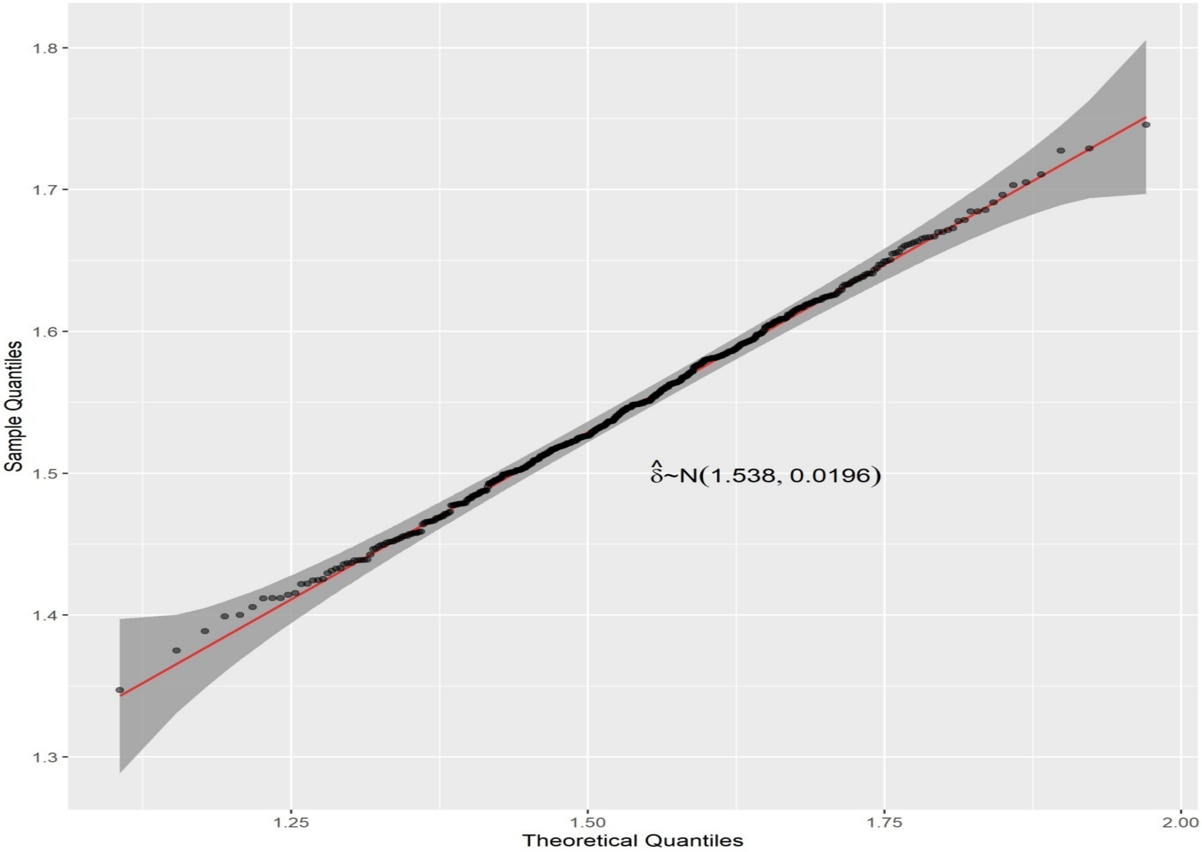

Figure 1: *Normal Q-Q Plot*

Table 3: *Asymptotic Confidence Intervals*

Scen-ario	Sample Size	$\widehat{R}$	90%			95%			99%		
			L	U	LB	L	U	LB	L	U	LB
1	10	0.5884	0.5167	0.6572	0.5326	0.5028	0.6699	0.5167	0.4757	0.6942	0.4867
	30	0.5842	0.5624	0.6058	0.5672	0.5582	0.6099	0.5624	0.5499	0.6179	0.5533
	50	0.5838	0.5706	0.5969	0.5735	0.5681	0.5994	0.5706	0.5631	0.6042	0.5651
	100	0.5830	0.5762	0.5897	0.5777	0.5749	0.5910	0.5762	0.5724	0.5935	0.5734
2	10	0.7109	0.6465	0.7692	0.6612	0.6336	0.7796	0.6465	0.6079	0.7991	0.6184
	30	0.6978	0.6712	0.7235	0.6771	0.6659	0.7283	0.6712	0.6557	0.7375	0.6598
	50	0.6876	0.6744	0.7005	0.6774	0.6719	0.7029	0.6744	0.6669	0.7077	0.6689
	100	0.6444	0.6371	0.6516	0.6387	0.6357	0.6530	0.6371	0.6330	0.6557	0.6341
3	10	0.7583	0.6551	0.8420	0.6793	0.6335	0.8555	0.6551	0.5902	0.8797	0.6079
	30	0.7235	0.6925	0.7530	0.6995	0.6863	0.7584	0.6925	0.6742	0.7688	0.6792
	50	0.7135	0.6859	0.7400	0.6921	0.6805	0.7449	0.6859	0.6698	0.7543	0.6741

Scen-ario	Sample Size	$\hat{R}$	90%			95%			99%		
			L	U	LB	L	U	LB	L	U	LB
	100	0.6954	0.6853	0.7054	0.6875	0.6833	0.7073	0.6853	0.6795	0.7110	0.6810
4	10	0.8041	0.6534	0.9063	0.6903	0.6202	0.9202	0.6534	0.5530	0.9429	0.5805
	30	0.7571	0.7004	0.8074	0.7134	0.6889	0.8162	0.7004	0.6659	0.8328	0.6753
	50	0.7420	0.7226	0.7606	0.7269	0.7188	0.7641	0.7226	0.7113	0.7709	0.7143
	100	0.7243	0.7140	0.7344	0.7163	0.7121	0.7363	0.7140	0.7081	0.7400	0.7097
5	10	0.8613	0.7002	0.9502	0.7418	0.6619	0.9603	0.7002	0.5823	0.9753	0.6151
	30	0.8570	0.7952	0.9048	0.8101	0.7818	0.9124	0.7952	0.7542	0.9260	0.7656
	50	0.8352	0.8057	0.8616	0.8125	0.7997	0.8663	0.8057	0.7877	0.8751	0.7926
	100	0.7996	0.7854	0.8133	0.7886	0.7826	0.8158	0.7854	0.7771	0.8208	0.7793
6	10	0.8800	0.7306	0.9586	0.7699	0.6940	0.9673	0.7306	0.6170	0.9799	0.6490
	30	0.8613	0.7941	0.9117	0.8104	0.7793	0.9195	0.7941	0.7488	0.9334	0.7614
	50	0.8506	0.8098	0.8852	0.8193	0.8012	0.8911	0.8098	0.7839	0.9020	0.7910
	100	0.8340	0.8111	0.8551	0.8163	0.8064	0.8589	0.8111	0.7972	0.8662	0.8010
7	10	0.9310	0.5747	0.9973	0.6824	0.4762	0.9988	0.5747	0.2930	0.9998	0.3639
	30	0.9253	0.7169	0.9896	0.7780	0.6581	0.9934	0.7169	0.5327	0.9975	0.5846
	50	0.9079	0.8218	0.9586	0.8442	0.8008	0.9650	0.8218	0.7557	0.9752	0.7746
	100	0.8839	0.8527	0.9101	0.8601	0.8462	0.9145	0.8527	0.8328	0.9228	0.8383
8	10	0.9146	0.6030	0.9934	0.6935	0.5194	0.9964	0.6030	0.3570	0.9991	0.4214
	30	0.8957	0.7818	0.9587	0.8117	0.7538	0.9662	0.7818	0.6940	0.9776	0.7190
	50	0.8939	0.8218	0.9421	0.8399	0.8051	0.9490	0.8218	0.7699	0.9605	0.7846
	100	0.8937	0.8723	0.9124	0.8772	0.8678	0.9157	0.8723	0.8589	0.9218	0.8626
9	10	0.9602	0.7672	0.9973	0.8304	0.7032	0.9985	0.7672	0.5597	0.9996	0.6200
	30	0.9486	0.7554	0.9949	0.8157	0.6954	0.9970	0.7554	0.5633	0.9990	0.6186
	50	0.9451	0.8894	0.9759	0.9043	0.8751	0.9797	0.8894	0.8438	0.9857	0.8571
	100	0.9306	0.9004	0.9532	0.9078	0.8936	0.9568	0.9004	0.8795	0.9631	0.8853
10	10	0.9893	0.0092	1.0000	0.0922	0.0006	1.0000	0.0092	0.0000	1.0000	0.0000
	30	0.9821	0.6637	0.9999	0.7862	0.5403	1.0000	0.6637	0.2991	1.0000	0.3926
	50	0.9770	0.8131	0.9990	0.8715	0.7509	0.9995	0.8131	0.6038	0.9999	0.6668
	100	0.9768	0.9252	0.9945	0.9409	0.9091	0.9960	0.9252	0.8705	0.9979	0.8873

3.2. Data Analysis

In this section, we apply the above methods to find out values of the estimators as $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, $\hat{\sigma}^2$and $\hat{R}$ from a given data set. The secondary data set of "Wave Energy Converters Data Set" is taken from the UCI Machine Learning site. The data set can be downloaded at https://archive.ics.uci.edu/ml/datasets/Wave+Energy+Converters. This data set consists of positions and absorbed power outputs of wave energy converters (WECs) in four real wave scenarios from the southern coast of Australia (Sydney, Adelaide, Perth and Tasmania). From this date set we take only two place of data set as Adelaide and Perth. We consider the eleven variables names as WECs absorbed power from each data set, which are consistent with the positive correlation among the variables. Then, we find out the stress strength reliability of absorbed power between the Adelaide and Perth respectively. Here we select the number of variables as p_1=11, p_2=11 and the MOM estimates as $\hat{\mu}_1$ =88175.2, $\hat{\mu}_2$=87244.27, $\hat{\rho}_1$=0.06567, $\hat{\rho}_2$=0.05251, $\hat{\rho}_3$=-0.04049, $\hat{\sigma}^2$ =107224128 and $\hat{R}$=0.55872. Jennrich test [20] used to examine the differences between correlation matrices of elevens variables of Adelaide and Perth data sets. The result shows that, the sample and estimated correlations by MOM are equal, reported in Table 4 and 5. This means that there is an equi-correlation between variables of the above data sets. The mean vectors of each data set reported in Table 6 and all are mostly equal. The performance of MOM quite good for sample size. The confidence intervals result on "Wave Energy Converters Data Set" shows in Table 7. The asymptotic confidence intervals in terms of lower limit, upper limits and lower bound are almost same and also band at different levels, but confidence interval and band of bootstrap is lesser then asymptotic confidence intervals values.

Table 4. *Correlation Matrix and Estimated Correlation Matrix of Adelaide data set*

Correlation Matrix											
Variable	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
V1	1	0.022	0.009	0.044	0.051	0.022	0.018	0.12	0.07	0.115	0.07
V2	0.022	1	0.017	0.032	0.019	0.069	0.013	0.049	0.0004	0.07	0.039
V3	0.009	0.017	1	0.048	0.047	0.047	0.082	0.07	0.062	0.091	0.056
V4	0.044	0.032	0.048	1	0.052	0.067	0.06	0.041	0.059	0.085	0.098
V5	0.051	0.019	0.047	0.052	1	0.05	0.101	0.101	0.118	0.092	0.045
V6	0.022	0.069	0.047	0.067	0.05	1	0.063	0.068	0.078	0.098	0.026
V7	0.018	0.013	0.082	0.06	0.101	0.063	1	0.093	0.14	0.125	0.065
V8	0.12	0.049	0.07	0.041	0.101	0.068	0.093	1	0.071	0.145	0.052
V9	0.07	0.0004	0.062	0.059	0.118	0.078	0.14	0.071	1	0.071	0.092
V10	0.115	0.07	0.091	0.085	0.092	0.098	0.125	0.145	0.071	1	0.12
V11	0.07	0.039	0.056	0.098	0.045	0.026	0.065	0.052	0.092	0.12	1
Estimated Correlation Matrix using MOM											
V1	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V2	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V3	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V4	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066	0.066
V5	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066	0.066
V6	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066	0.066
V7	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066	0.066
V8	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066	0.066
V9	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066	0.066
V10	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1	0.066
V11	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	0.066	1
Jennrich test: χ^2= 30.1314, p-value= 0.9974615 (H_0: all the correlations are equal)											

Table 5. *Correlation Matrix and Estimated Correlation Matrix of Perth data set*

Correlation Matrix											
Variable	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
V1	1	0.052	0.063	0.121	0.113	0.03	0.023	0.117	0.003	0.075	0.109
V2	0.052	1	0.048	0.037	0.031	0.05	0.018	0.065	0	0.018	0.052
V3	0.063	0.048	1	0.11	0.05	0.032	0.04	0.034	0.048	0.029	0.044
V4	0.121	0.037	0.11	1	0.086	0.048	0.029	0.096	0.019	0.068	0.038
V5	0.113	0.031	0.05	0.086	1	0.022	0.002	0.118	0.003	0.128	0.049
V6	0.03	0.05	0.032	0.048	0.022	1	0.024	0.045	0.029	0.041	0.061
V7	0.023	0.018	0.04	0.029	0.002	0.024	1	0.026	0.088	0.068	0.09
V8	0.117	0.065	0.034	0.096	0.118	0.045	0.026	1	0.008	0.069	0.114
V9	0.003	0	0.048	0.019	0.003	0.029	0.088	0.008	1	0.011	0.029
V10	0.075	0.018	0.029	0.068	0.128	0.041	0.068	0.069	0.011	1	0.077
V11	0.109	0.052	0.044	0.038	0.049	0.061	0.09	0.114	0.029	0.077	1
Estimated Correlation Matrix using MOM											

V1	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V2	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V3	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V4	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053	0.053
V5	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053	0.053
V6	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053	0.053
V7	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053	0.053
V8	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053	0.053
V9	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053	0.053
V10	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1	0.053
V11	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	1
Jennrich test: χ^2= 30.1976, p-value= 0.9973857 (H_0: all the correlations are equal)											

Table 6. *Mean Vector of data sets*

Data Set	Mean Vector										
	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11
Adelaide	87821.85	87785.72	88185.84	87680.8	88436.53	87564.24	88660.64	88424.98	87703.94	89191.15	88471.47
Perth	88115.64	86299.28	87054.98	87490.71	87172.86	87227.25	87479.91	87259.51	86110.12	88026.43	87450.23

Table 7: *Confidence Intervals for tests of the "Wave Energy Converters Data Set"*

Confidence Intervals	90%			95%			99%		
	L	**U**	**LB**	**L**	**U**	**LB**	**L**	**U**	**LB**
Asymptotic	0.5567	0.5587	0.5577	0.5566	0.5588	0.5587	0.5565	0.5589	0.5588
Bootstrap	0.5563	0.5612	0.5568	0.5558	0.5617	0.5563	0.5549	0.5626	0.5553

4. Conclusions

In this article, we proposed a method to estimate the stress–strength reliability and all unknown parameters under the equi-corelated multivariate normal setup. We provide MOM method to estimate these unknown parameters and use them to estimate of δ and R. We also obtain the asymptotic distribution of estimated δ. The simulation results indicate that performance than MOM in terms of MSE, RB and MAE for different choices of the parameters. Simulation studies illustrate that, the MSE, RB and MAE of this estimator reduce as the sample size increases and they almost achieved the true value of R. Also, the simulation studies illustrate that the proposed method has the best coverage probability and also produces the shortest band of confidence intervals. The stress-strength reliability of the given data set is $\hat{R}$=0.55872. The performance of method of moments estimator (MOM) of R is consistent for different sample size and quite good for small sample size.

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RELIABILITY ANALYSIS FOR GDC SYSTEM USING REPAIR AND REPLACEMENT FACILITY IN PISTON FOUNDRY PLANT

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Abstract

The system in industries is greatly impacted by failure. Eliminating these defects is therefore essential for enhancing system performance. This study aims to assess the range of repair/replacement facilities in the GDC (Gravity Die Casting) system at the Piston Foundry Plant. Two sub-units are connected to one main unit, which makes up the GDC system. Any component failure results in system failure. In this situation, the system will first attempt to be repaired, and if that is unsuccessful, it will be replaced. To operate effectively, the primary unit needs to be built of aluminium alloy (Al). Lack of raw materials is what leads to a system failing.Using semi-Markov processes and the regenerating point method, the aforementioned measurements were computed numerically and graphically. The results of this study are unusual since no prior research has concentrated on the GDC system repair/replacement facilities at piston foundries. The conclusions, according to the discussion, are very helpful for businesses who manufacture pistons and utilise the GDC system.

Keywords: GDC, repair, replacement, semi-Markov process, regenerating point technique.

1. Introduction

A system is made up of a variety of parts that function in concert to create a whole. Finally, the system's operation is influenced by how well each component performs. A component-based system can be in both an operational and a failing state. Failures have an impact on the system's use and dependability. As a result, many systems, such as those that control hydraulic, computer, and electric power supplies, nuclear power plants, aviation engines, vehicle engines, and so on, now demand reliability as a component. For systems that function in the dependability domain, researchers have significantly contributed to the creation of reliability models. There have been several research on backup systems, including: Srinivasan [10] gave an examination of warm standby system dependability for a repair facility. The stochastic standby system behaviour with repair time was handled by Kumar et al. [4]. Sharma and Kaur [8] conducted a cost-benefit analysis of a compressor standby system. A power plant system's cold standby unit was stochastically modelled by Sharma and Sharma[9].

Some authors provided an overview of the different reliability modelling methodologies used in die casting systems such as: High Pressure Grain structure and segregation in die casting of magnesium and aluminium alloys Characteristics mentioned by Laukli [5]. High pressure die cast AlSi9Cu3 (Fe) alloys are provided by Timelli [11] using constitutive and stochastic models to anticipate the impact of casting flaws on the mechanical properties. Die Casting Process Modeling and Optimization for ZAMAK Alloy given by Sharma [7]. Existing epistemic uncertainty in die-casting is modelled for reliability and optimised by Yourui et al.[12]. Sensitivity study for the casting method provided by Kumar [3]. An Early Investigation of a Lightweight provided by Muller et al. [6] Die Casting Die Using a Modular Design Approach. High pressure die casting machine reliability analysis of two unit standby system offered by Bhatia and Sharma [1]. The Casting Process Optimization Case Study: A Review of the Reliability Techniques used by Chaudhari and Vasudevan [2]. According to the discussion above, every researcher has addressed reliability analysis of the die casting method used in piston foundries. Research findings pertaining to the GDC system in piston foundries have not been discovered. There are a variety of systems in piston foundry operations that must be analysed using real data at various rates and costs. Our efforts are closing this gap by gathering genuine data from a company called Federal-Mogul Powertrain, India Limited, which is based in Bahadurgarh, Punjab, near Patiala. Federal-Mogul is the world's leading maker of world-class pistons, piston rings and cylinder linears, with products for two-and three -wheelers, vehicles and tractors, among other applications.

We create a Reliability model for the Gravity Die Casting (GDC) system at the piston foundry using the ideas presented above as our inspiration. This model includes the ability for repair and replacement. The goal of this study is to evaluate the range of repair and replacement capabilities offered by the GDC (Gravity Die Casting) system at the Piston Foundry Plant. One primary unit and two supporting units make up the GDC system. The system as a whole fails if even one of the constituent components fails. In this instance, the system will be fixed, and if repair is not possible, it will be replaced. The primary component needs to be made of aluminum alloy for it to work effectively (Al). Lack of raw materials might cause a system to fail.

There are a few assumptions that must be made for the model:

- The system works initially at state S0.
- All failures/repairs /replacement times are exponential distribution.
- In the states, the system is restored to working order after each repair/replacement.
- The unit is brought online as quickly as possible.
- Visit of repairman is immediate upon failure.

2. Methods

The following tools and procedures were utilised to accomplish this study:

In order to overcome the difficulties, semi-Markov processes and regenerating point techniques are used. System availability, mean time to system failure, busy period for repairs/replacements, and expected number of repairs/replacements are only a few of the data that have been collected about system efficiency. Also made are the profits. For a particular case, graphical assessments are produced using the programming languages C++, Python, and MS Excel.

3. Notations and States for the Model

$\lambda \rightarrow$ Failure rate of the main unit i.e. DC.

$\lambda_1 \rightarrow$ Failure rate of the sub-unit one.

$\lambda_2 \rightarrow$ Failure rate of the sub-unit two.

$p \rightarrow$ Probability that raw material is Available.

$q \rightarrow$ Probability that raw material is Non-Available.

$a \rightarrow$ Probability that repair is feasible.

$b \rightarrow$ Probability that replacement is feasible.

$\beta \rightarrow$ Rate of metal treatment.

$O \rightarrow$ Operative unit.

$DC \rightarrow$ Main unit of the system i.e. DC.

$O(DC) \rightarrow$ Main unit of the system is in operating state.
$SU_1, SU_2 \rightarrow$ Sub-unit one and sub-unit two of the system.
$O(SU_1) \rightarrow$ Sub unit one is in operating state.
$O(SU_2) \rightarrow$ Sub unit two is in operating state.
$Al \rightarrow$ Aluminum alloy.
$Av(Al) \rightarrow$ Aluminum alloy is Available.
$NA(Al) \rightarrow$ Aluminum alloy is Non- Available .
$CS(DC) \rightarrow$ Main unit is in cold standby state.
$CS(SU_1) \rightarrow$ Sub unit one is in cold standby state.
$CS(SU_2) \rightarrow$ Sub unit two is in cold standby state.
$F(t), f(t) \rightarrow$ c.d.f. and p.d.f. of availablility of the raw material.
$G(t), g(t) \rightarrow$ c.d.f. and p.d.f of time to repair/replacement of the main unit.
$G_1(t), g_1(t) \rightarrow$ c.d.f. and p.d.f of time to repair/replacement of the sub-unit one.
$G_2(t), g_2(t) \rightarrow$ c.d.f. and p.d.f of time to repair/replacement of the sub-unit two.
$Fr(DC) \rightarrow$ Main unit is under repair.
$Fr(SU_1), Fr(SU_2) \rightarrow$ Sub-unit one and Sub-unit two are under repair.
$Frp(DC) \rightarrow$ Main unit is under replacement.
$Frp(SU_1), Frp(SU_2) \rightarrow$ Sub-unit one and Sub-unit two are under replacement. (1)

4. The System's Reliability Measures

4.1. Transition Probabilities

The transition diagram in **Fig.1** depicts the system's many states.

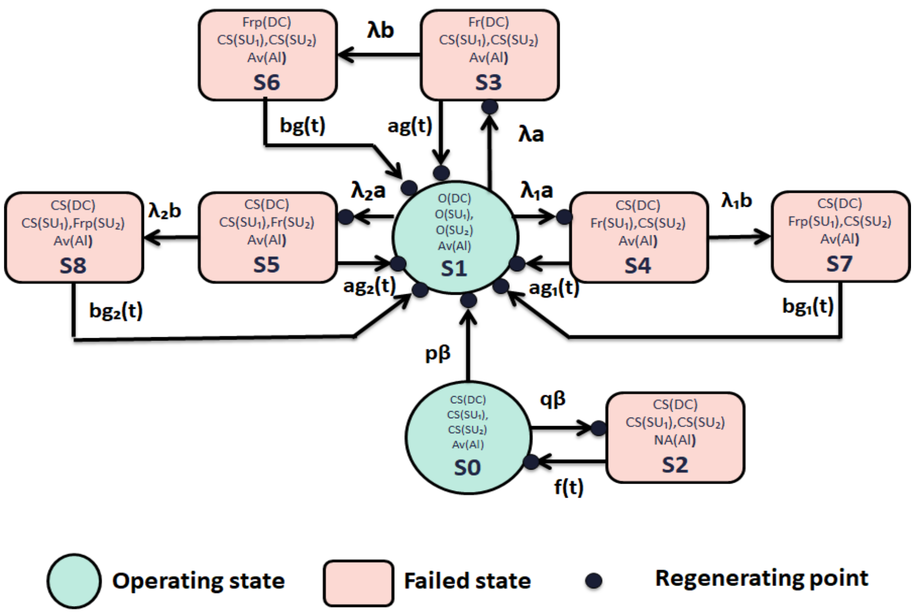

Figure 1: *State Transition Diagram*

The epochs of entry into states S0, S1, S2, S3, S4, and S5 are regenerative states; the remaining states are non-regenerative stages. States S0 and S1 are operating states, while states S2, S3, S4, S5, S6, S7, and S8 are failed states. The following sources provide the transition probabilities:

$$
\begin{array}{ll}
dQ_{01}(t) = p\beta e^{-\beta t}dt & dQ_{02}(t) = q\beta e^{-\beta t}dt \\
dQ_{13}(t) = \lambda a e^{-(\lambda+\lambda_1+\lambda_2)at}dt & dQ_{14}(t) = \lambda_1 a e^{-(\lambda+\lambda_1+\lambda_2)at}dt \\
dQ_{15}(t) = \lambda_2 a e^{-(\lambda+\lambda_1+\lambda_2)at}dt & dQ_{36}(t) = \lambda b e^{-\lambda bt}\bar{G(t)}dt \\
dQ_{31}^{(6)}(t) = [\lambda b e^{-\lambda bt} © 1]g(t)dt & dQ_{31}(t) = g(t)e^{-\lambda bt}dt \\
dQ_{41}(t) = g_1(t)e^{-\lambda_1 bt}dt & dQ_{47}(t) = \lambda_1 b e^{-\lambda_1 bt}\bar{G_1(t)}dt \\
dQ_{41}^{(7)}(t) = [\lambda_1 b e^{-\lambda_1 bt} © 1]g_1(t)dt & dQ_{51}(t) = g_2(t)e^{-\lambda_2 bt}dt \\
dQ_{58}(t) = \lambda_2 b e^{-\lambda_2 bt}\bar{G_2(t)}dt & dQ_{51}^{(8)}(t) = [\lambda_2 b e^{-\lambda_2 bt} © 1]g_2(t)dt \\
dQ_{20}(t) = f(t)dt & dQ_{61}(t) = g(t)dt \\
dQ_{71}(t) = g_1(t)dt & dQ_{81}(t) = g_2(t)dt
\end{array} \quad (2)
$$

The non-zero elements p_{ij} can be represented as below:

$$p_{ij} = Q_{ij}(\infty) = \int_0^\infty q_{ij}dt \quad (3)$$

As we get

$$
\begin{array}{ll}
p_{01} = p & p_{02} = q \\
p_{13} = \dfrac{\lambda a}{(\lambda+\lambda_1+\lambda_2)a} & p_{14} = \dfrac{\lambda_1 a}{(\lambda+\lambda_1+\lambda_2)a} \\
p_{15} = \dfrac{\lambda_2 a}{(\lambda+\lambda_1+\lambda_2)a} & p_{31} = g^*(\lambda b) \\
p_{36} = p_{31}^{(6)} = \dfrac{\lambda b[1-g^*(\lambda b)]}{\lambda b} & p_{41} = g_1^*(\lambda_1 b) \\
p_{47} = p_{41}^{(7)} = \dfrac{\lambda_1 b[1-g_1^*(\lambda_1 b)]}{\lambda_1 b} & p_{51} = g_2^*(\lambda_2 b) \\
p_{58} = p_{51}^{(8)} = \dfrac{\lambda_2 b[1-g_2^*(\lambda_2 b)]}{\lambda_2 b} & p_{20} = f^*(0) \\
p_{61} = g^*(0) & p_{71} = g_1^*(0) \\
p_{81} = g_2^*(0) &
\end{array} \quad (4)
$$

It is also verified that:

$$
\begin{array}{ll}
p_{01} + p_{02} = 1 & p_{13} + p_{14} + p_{15} = 1 \\
p_{31} + p_{36} = 1 & p_{31} + p_{31}^{(6)} = 1 \\
p_{41} + p_{47} = 1 & p_{41} + p_{41}^{(7)} = 1 \\
p_{51} + p_{58} = 1 & p_{51} + p_{51}^{(8)} = 1 \\
p_{20} = p_{61} = p_{71} = p_{81} = 1 &
\end{array} \quad (5)
$$

The unconditional mean time taken by the system to transit for any regenerative state 'j'when it (time) is counted from the epoch of entrance into state 'i'is mathematically state as:

$$m_{ij} = \int_0^\infty t\,dQ_{ij}(t) = -q_{ij}^{*'}(0) \quad (6)$$

it is also verified that

$$\begin{aligned} &m_{01} + m_{02} = \mu_0 && m_{13} + m_{14} + m_{15} = \mu_1 \\ &m_{31} + m_{36} = \mu_3 && m_{31} + m_{31}^{(6)} = K \\ &m_{41} + m_{47} = \mu_4 && m_{41} + m_{41}^{(7)} = K_1 \\ &m_{51} + m_{58} = \mu_5 && m_{51} + m_{51}^{(8)} = K_2 \\ &m_{20} = \mu_2 && m_{61} = \mu_6 \\ &m_{71} = \mu_7 && m_{81} = \mu_8 \end{aligned} \tag{7}$$

where

$$\begin{aligned} m_{01} &= \int_0^\infty tp\beta e^{-(\beta)t}dt & m_{02} &= \int_0^\infty tq\beta e^{-(\beta)t}dt \\ m_{13} &= \int_0^\infty t\lambda a e^{-(\lambda+\lambda_1+\lambda_2)at}dt & m_{14} &= \int_0^\infty \lambda_1 a t e^{-(\lambda+\lambda_1+\lambda_2)at}dt \\ m_{15} &= \int_0^\infty \lambda_2 a t e^{-(\lambda+\lambda_1+\lambda_2)at}dt & m_{31} &= \int_0^\infty g(t) t e^{-(\lambda b)t}dt \\ m_{36} &= \int_0^\infty \lambda b t e^{-(\lambda b)t}\bar{G}(t)dt & m_{31}^{(6)} &= \int_0^\infty t[\lambda b e^{-(\lambda b)t} © 1]g(t)dt \\ m_{41} &= \int_0^\infty g_1(t) t e^{-(\lambda_1 b)t}dt & m_{47} &= \int_0^\infty \lambda_1 b t e^{-(\lambda_1 b)t}\bar{G_1}(t)dt \\ m_{41}^{(7)} &= \int_0^\infty t[\lambda_1 b e^{-(\lambda_1 b)t} © 1]g_1(t)dt & m_{51} &= \int_0^\infty g_2(t) t e^{-(\lambda_2 b)t}dt \\ m_{58} &= \int_0^\infty \lambda_2 b t e^{-(\lambda_2 b)t}\bar{G_2}(t)dt & m_{51}^{(8)} &= \int_0^\infty t[\lambda_2 b e^{-(\lambda_2 b)t} © 1]g_2(t)dt \\ m_{20} &= \int_0^\infty t f(t)dt & m_{61} &= \int_0^\infty t g(t)dt \\ m_{71} &= \int_0^\infty t g_1(t)dt & m_{81} &= \int_0^\infty t g_2(t)dt \\ K &= \int_0^\infty \bar{G}(t)dt & K_1 &= \int_0^\infty \bar{G_1}(t)dt \\ K_2 &= \int_0^\infty \bar{G_2}(t)dt \end{aligned} \tag{8}$$

The mean sojourn time (μ_i) in the regenerative state 'i'is defined as time of stay in that state before transition to any other state :

$$\begin{aligned} &\mu_0 = \frac{1}{\beta} && \mu_1 = \frac{1}{\lambda + \lambda_1 + \lambda_2} \\ &\mu_3 = \frac{1 - g^*(\lambda b)}{\lambda b} && \mu_4 = \frac{1 - g_1^*(\lambda_1 b)}{\lambda_1 b} \\ &\mu_5 = \frac{1 - g_2^*(\lambda_2 b)}{\lambda_2 b} && \mu_2 = -f^*(0) \\ &\mu_6 = -g^*(0) && \mu_7 = -g_1^*(0) \\ &\mu_8 = -g_2^*(0) \end{aligned} \tag{9}$$

5. Reliability Analysis

5.1. Mean Time To System Failure

To determine the mean time to system failure (MTSF) of the system, we regard the failed states of system absorbing. By probabilities arguments; we obtain the following recursive relation for

$\phi_i(t)$:

$$\begin{aligned}\phi_0(t) &= Q_{01}(t)Ⓢ\phi_1(t) + Q_{02}(t)\\ \phi_1(t) &= Q_{13}(t) + Q_{14}(t) + Q_{15}(t)\end{aligned} \tag{10}$$

Taking Laplace Stieltje Transforms(L.S.T) of these relations in equations(10) and solving for $\phi_o^{**}(s)$ we obtain

$$\phi_o^{**}(s) = \frac{N(s)}{D(s)} \tag{11}$$

where

$$\begin{aligned}N(s) &= Q_{01}^{**}(s)[Q_{13}^{**}(s) + Q_{14}^{**}(s) + Q_{15}^{**}(s)] + Q_{02}^{**}(s)\\ D(s) &= 1\end{aligned} \tag{12}$$

Now the mean time to system failure (MTSF) , when the system started at the beginning of state S0 is

$$T = \lim_{s \longrightarrow 0} \frac{1 - \phi_o^{**}(s)}{s} \tag{13}$$

Using L' Hospital rule and putting the value of $\phi_o^{**}(s)$ from equation(13), we have

$$T_0 = \frac{N}{D} \tag{14}$$

where

$$\begin{aligned}N &= \mu_0 + \mu_1[p_{01}]\\ D &= 1\end{aligned} \tag{15}$$

5.2. Availability Analysis

Let $A_i(t)$ be the probability that the system is in up state at instant t given that the system entered regenerative state i at t=0. The availability $A_i(t)$ is to satisfy the following recursive relations:

$$\begin{aligned}A_0(t) &= M_0(t) + q_{01}(t)©A_1(t) + q_{02}(t)©A_2(t)\\ A_1(t) &= M_1(t) + q_{13}(t)©A_3(t) + q_{14}(t)©A_4(t) + q_{15}(t)©A_5(t)\\ A_2(t) &= q_{20}(t)©A_0(t)\\ A_3(t) &= q_{31}(t)©A_1(t) + q_{31}^{(6)}(t)©A_1(t)\\ A_4(t) &= q_{41}(t)©A_1(t) + q_{41}^{(7)}(t)©A_1(t)\\ A_5(t) &= q_{51}(t)©A_1(t) + q_{51}^{(8)}(t)©A_1(t)\end{aligned} \tag{16}$$

where

$$M_0(t) = e^{-\beta t} \qquad M_1(t) = e^{-(\lambda+\lambda_1+\lambda_2)t} \tag{17}$$

Taking Laplace Transformation of the above equation(17) and letting s$\longrightarrow$ 0, we get

$$M_0^*(0) = \mu_0 \qquad M_1^*(0) = \mu_1 \tag{18}$$

Taking Laplace transform of the above equations(16) and solving them for

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{19}$$

where

$$\begin{aligned}N_1(s) = M_0^*(s)[1 - q_{13}^*(s)(q_{31}^*(s) + q_{31}^{(6)*}(s)) - q_{14}^*(s)(q_{41}^*(s) + q_{41}^{(7)*}(s)) - q_{15}^*(s)\\ (q_{51}^*(s) + q_{51}^{(8)*}(s))] + M_1^*(s)q_{01}^*(s)\end{aligned} \tag{20}$$

$$\begin{aligned}D_1(s) = 1 - q_{13}^*(s)(q_{31}^*(s) + q_{31}^{(6)*}(s)) - q_{14}^*(s)(q_{41}^*(s) + q_{41}^{(7)*}(s)) - q_{15}^*(s)(q_{51}^*(s) + q_{51}^{(8)*}(s))\\ - q_{02}^*(s)q_{20}^*(s)[1 - q_{13}^*(s)(q_{31}^*(s) + q_{31}^{(6)*}(s)) - q_{14}^*(s)(q_{41}^*(s) + q_{41}^{(7)*}(s))\\ - q_{15}^*(s)(q_{51}^*(s) + q_{51}^{(8)*}(s))]\end{aligned} \tag{21}$$

In steady state, system availability is given as

$$A_0 = \lim_{s \longrightarrow 0} sA_0^*(s) = \frac{N_1}{D_1} \tag{22}$$

where

$$N_1 = \mu_1[p_{01}] \tag{23}$$

$$D_1 = \mu_1[p_{01}] + K[p_{01}p_{13}] + K_1[p_{01}p_{14}] + K_2[p_{01}p_{15}] \tag{24}$$

5.3. Busy Period Analysis for the Repairman

Let $BR_i(t)$ be the probability that the repairman is busy at time t given that the system entered regenerative state i at i=0. The recursive relation for $BR_i(t)$ are as follows:

$$\begin{aligned}BR_0(t) &= q_{01}(t)©BR_1(t) + q_{02}(t)©BR_2(t)\\ BR_1(t) &= q_{13}(t)©BR_3(t) + q_{14}(t)©BR_4(t) + q_{15}(t)©BR_5(t)\\ BR_2(t) &= W_2(t) + q_{20}(t)©BR_0(t)\\ BR_3(t) &= W_3(t) + q_{31}(t)©BR_1(t)\\ BR_4(t) &= W_4(t) + q_{41}(t)©BR_1(t)\\ BR_5(t) &= W_5(t) + q_{51}(t)©BR_1(t)\end{aligned} \tag{25}$$

where

$$\begin{aligned}W_2(t) &= \bar{F(t)} & W_3(t) &= \bar{G(t)}\\ W_4(t) &= \bar{G_1(t)} & W_5(t) &= \bar{G_2(t)}\end{aligned} \tag{26}$$

Taking Laplace Transformation of the above equation(26) and letting s$\longrightarrow$ 0, we get

$$W_2^*(0) = \mu_2 \qquad W_3^*(0) = K \qquad W_4^*(0) = K_1$$
$$W_5^*(0) = K_2$$

Taking Laplace transform of the above equations(25) and solving them for

$$BR_0^*(s) = \frac{N_2(s)}{D_1(s)}$$

where

$$\begin{aligned}N_2(s) = W_3^*(s)q_{01}^*(s)q_{13}^*(s) + W_4^*(s)q_{01}^*(s)q_{14}^*(s) + W_5^*(s)q_{01}^*(s)q_{15}^*(s) + W_2^*(s)q_{02}^*(s)\\ [1 - q_{13}^*(s)q_{31}^*(s) - q_{14}^*(s)q_{41}^*(s) - q_{15}^*(s)q_{51}^*(s)]\end{aligned}$$

The value of $D_1(s)$ is already defined in equation(21).

System total fraction of the time when it is under repair in steady state is given by

$$BR_0 = \lim_{s \longrightarrow 0} sBR_0^*(s) = \frac{N_2}{D_1} \tag{27}$$

where

$$N_2 = \mu_2 p_{02}[1 - p_{13}p_{31} - p_{14}p_{41} - p_{15}p_{51}] + K[p_{01}p_{13}] + K_1[p_{01}p_{14}] + K_2[p_{01}p_{15}]$$

The value of D_1 is already defined in equation(24).

5.4. Busy Period Analysis for the Replacement

Let $BRP_i(t)$ be the probability that the repairman is busy at time t given that the system entered regenerative state i at i=0. The recursive relation for $BRP_i(t)$ are as follows:

$$\begin{aligned}
BRP_0(t) &= q_{01}(t)©BRP_1(t) + q_{02}(t)©BRP_2(t)\\
BRP_1(t) &= q_{13}(t)©BRP_3(t) + q_{14}(t)©BRP_4(t) + q_{15}(t)©BRP_5(t)\\
BRP_2(t) &= q_{20}(t)©BRP_0(t)\\
BRP_3(t) &= W_3(t) + q_{31}^{(6)}(t)©BRP_1(t)\\
BRP_4(t) &= W_4(t) + q_{41}^{(7)}(t)©BRP_1(t)\\
BRP_5(t) &= W_5(t) + q_{51}^{(8)}(t)©BRP_1(t)
\end{aligned} \tag{28}$$

where

$$W_3(t) = \bar{G(t)} \qquad W_4(t) = \bar{G_1(t)} \qquad W_5(t) = \bar{G_2(t)} \tag{29}$$

Taking Laplace Transformation of the above equation(29) and letting s$\longrightarrow$ 0, we get

$$W_3^*(0) = K \qquad W_4^*(0) = K_1 \qquad W_5^*(0) = K_2 \tag{30}$$

Taking Laplace transform of the above equations(28) and solving them for

$$BRP_0^*(s) = \frac{N_3(s)}{D_1(s)} \tag{31}$$

where

$$\begin{aligned}
N_3(s) = {} & W_3^*(s)q_{01}^*(s)q_{13}^*(s) + W_4^*(s)q_{01}^*(s)q_{14}^*(s) + W_5^*(s)q_{01}^*(s)q_{15}^*(s) + q_{02}^*(s)\\
& [1 - q_{13}^*(s)q_{31}^{(6)*}(s) - q_{14}^*(s)q_{41}^{(7)*}(s) - q_{15}^*(s)q_{51}^{(8)*}(s)]
\end{aligned} \tag{32}$$

The value of $D_1(s)$ is already defined in equation(21).

System total fraction of the time when it is under repair in steady state is given by

$$BRP_0 = \lim_{s \longrightarrow 0} sBRP_0^*(s) = \frac{N_3}{D_1} \tag{33}$$

where

$$N_3 = p_{02}[1 - p_{13}p_{31}^{(6)} - p_{14}p_{41}^{(7)} - p_{15}p_{51}^{(8)}] + K[p_{01}p_{13}] + K_1[p_{01}p_{14}] + K_2[p_{01}p_{15}] \tag{34}$$

The value of D_1 is already defined in equation(24).

5.5. Expected Number of Repairs

Let $ER_i(t)$ be the expected no. of repairs in (0,t] given that the system entered regenerative state i at i=0. The recursive relations for $ER_i(t)$ are as follows:

$$\begin{aligned}
ER_0(t) &= Q_{01}(t)Ⓢ ER_1(t) + Q_{02}(t)Ⓢ ER_2(t)\\
ER_1(t) &= Q_{13}(t)Ⓢ ER_3(t) + Q_{14}(t)Ⓢ ER_4(t) + Q_{15}(t)Ⓢ ER_5(t)\\
ER_2(t) &= Q_{20}(t)Ⓢ[1 + ER_0(t)]\\
ER_3(t) &= Q_{31}(t)Ⓢ[1 + ER_1(t)]\\
ER_4(t) &= Q_{41}(t)Ⓢ[1 + ER_1(t)]\\
ER_5(t) &= Q_{51}(t)Ⓢ[1 + ER_1(t)]
\end{aligned} \tag{35}$$

Taking L.S.T.of above relations and obtain the value of $ER_0^{**}(s)$, we get

$$ER_0^{**}(s) = \frac{N_4(s)}{D_1(s)}$$

where

$$N_4(s) = Q_{01}^{**}(s)[Q_{13}^{**}(s)Q_{31}^{**}(s) + Q_{14}^{**}(s)Q_{41}^{**}(s) + Q_{15}^{**}(s)Q_{51}^{**}(s)] + Q_{02}^{**}(s)Q_{20}^{**}(s)[1 - Q_{13}^{**}(s)Q_{31}^{**}(s) - Q_{14}^{**}(s)Q_{41}^{**}(s) - Q_{15}^{**}(s)Q_{51}^{**}(s)]$$

The value of $D_1(s)$ is already defined in equation(21).

For system steady state, the number of repairs per unit time is given by

$$ER_0 = \lim_{s \longrightarrow 0} sER_0^{**}(s) = \frac{N_4}{D_1} \tag{36}$$

where

$$N_4 = p_{01}[p_{13}p_{31} + p_[14]p_{41} + p_{15}p_{51}] + p_{02}[1 - p_{13}p_{31} - p_{14}p_{41} - p_{15}p_{51}]$$

The value of D_1 is already defined in equation(24).

5.6. Expected Number of Replacements

Let $ERP_i(t)$ be the expected no. of replacements in (0,t] given that the system entered regenerative state i at i=0. The recursive relations for $ERP_i(t)$ are as follows:

$$\begin{aligned}
ERP_0(t) &= Q_{01}(t)Ⓢ ERP_1(t) + Q_{02}(t)Ⓢ ERP_2(t)\\
ERP_1(t) &= Q_{13}(t)Ⓢ ERP_3(t) + Q_{14}(t)Ⓢ ERP_4(t) + Q_{15}(t)Ⓢ ERP_5(t)\\
ERP_2(t) &= Q_{20}(t)Ⓢ ERP_0(t)\\
ERP_3(t) &= Q_{31}^{(6)}(t)Ⓢ[1 + ERP_1(t)]\\
ERP_4(t) &= Q_{41}^{(7)}(t)Ⓢ[1 + ERP_1(t)]\\
ERP_5(t) &= Q_{51}^{(8)}(t)Ⓢ[1 + ERP_1(t)]
\end{aligned} \tag{37}$$

Taking L.S.T.of above relations and obtain the value of $ERP_0^{**}(s)$, we get

$$ERP_0^{**}(s) = \frac{N_5(s)}{D_1(s)}$$

where

$$N_5(s) = Q_{01}^{**}(s)[Q_{13}^{**}(s)Q_{31}^{(6)**}(s) + Q_{14}^{**}(s)Q_{41}^{(7)**}(s) + Q_{15}^{**}(s)Q_{51}^{(8)**}(s)]$$

The value of $D_1(s)$ is already defined in equation(21).

For system steady state, the number of replacements per unit time is given by

$$ERP_0 = \lim_{s \longrightarrow 0} sERP_0^{**}(s) = \frac{N_5}{D_1} \tag{38}$$

where

$$N_5 = p_{01}[p_{13}p_{31}^{(6)} + p_{14}p_{41}^{(7)} + p_{15}p_{51}^{(8)}]$$

The value of D_1 is already defined in equation(24).

6. Profit Analysis

Profit incurred to the system model in steady state is given by

$$P = Z_0A_0 - Z_1BR_0 - Z_2BRP_0 - Z_3ER_0 - Z_4ERP_0 \tag{39}$$

where

P = Profit Analysis.
Z_0 = Revenue per unit up time.
Z_1 = Cost per unit up time for which the repairman is busy for repair.
Z_2 = Cost per unit up time for which the repairman is busy for replacement.
Z_3 = Cost per repair.
Z_4 = Cost per replacement.

7. Particular Cases

For the particular case, the failure rates and repair rates are exponentially distributed as follows:

$$f(t) = \gamma e^{-\gamma t} \qquad g(t) = \alpha e^{-\alpha t}$$
$$g_1(t) = \alpha_1 e^{-\alpha_1 t} \qquad g_2(t) = \alpha_2 e^{-\alpha_2 t}$$

As we get,

$$p_{01} = p \qquad p_{02} = q$$
$$p_{13} = \frac{\lambda}{(\lambda + \lambda_1 + \lambda_2)} \qquad p_{14} = \frac{\lambda_1}{(\lambda + \lambda_1 + \lambda_2)}$$
$$p_{15} = \frac{\lambda_2}{(\lambda + \lambda_1 + \lambda_2)} \qquad p_{31} = \frac{\alpha}{\lambda + \alpha}$$
$$p_{36} = p_{31}^{(6)} = \frac{\lambda}{\lambda + \alpha} \qquad p_{41} = \frac{\alpha_1}{\lambda_1 + \alpha_1}$$
$$p_{47} = p_{41}^{(7)} = \frac{\lambda_1}{\lambda_1 + \alpha_1} \qquad p_{51} = \frac{\alpha_2}{\lambda_2 + \alpha_2}$$
$$p_{58} = p_{51}^{(8)} = \frac{\lambda_2}{\lambda_2 + \alpha_2} \qquad p_{20} = p_{61} = p_{71} = p_{81} = 1$$
$$\mu_0 = \frac{1}{\beta} \qquad \mu_1 = \frac{1}{(\lambda + \lambda_1 + \lambda_2)}$$
$$\mu_2 = \frac{1}{\gamma} \qquad \mu_3 = \frac{1}{\lambda + \alpha}$$

$$\mu_4 = \frac{1}{\lambda_1 + \alpha_1} \qquad \mu_5 = \frac{1}{\lambda_2 + \alpha_5}$$
$$\mu_6 = K = \frac{1}{\alpha} \qquad \mu_7 = K_1 = \frac{1}{\alpha_1}$$
$$\mu_8 = K_2 = \frac{1}{\alpha_2} \tag{40}$$

To study the reliability and profit analysis of the GDC systems, We have visited the piston foundry of a firm named **Federal-Mogul Powertrain** and contacted the concerned persons and obtain the information regarding the failures/repairs and replacements. Based on the facts received i.e.,

Table 1: *Information Gathered*

Description	Notation	Rate(/hr)
Failure Rate of Main unit	λ	$0.001391281/hr$
Failure Rate of Sub-unit one	λ_1	$0.001390573/hr$
Failure Rate of Sub-unit two	λ_2	$0.001420852/hr$
Repair/Replacement Rate of Main unit	α	$0.193686798/hr$
Repair/Replacement Rate of Sub-unit one	α_1	$0.206397204/hr$
Repair/Replacement Rate of Sub-unit two	α_2	$0.201849607/hr$

The remaining values are assumed and are listed in Table 2:

Table 2: *Assumed Values*

Description	Notation	Rate(/hr)
Rate of Metal Treatment	β	$1.1762493/hr$
Rate of raw material is available	γ	$0.1158713/hr$
Probability that raw material is available	p	0.75
Probability that raw material is non-available	q	0.25
Probability that repair is feasible	a	0.75
Probability that replacement is feasible	b	0.25
Revenue per unit uptime of the system(per month)	Z_0	$Rs.8,85,000$
Cost per unit uptime, when repairman is busy for repair(per month)	Z_1	$Rs.12,466$
Cost per unit uptime, when repairman is busy for replacement(per month)	Z_2	$Rs.19,480$
Cost per repair(per month)	Z_3	$Rs.18,350$
Cost per replacement(per month)	Z_4	$Rs.25,650$

Various measures of system effectiveness are shown in Table 3:

Table 3: *Results*

Description	Notation	Rate(/hr)
Mean Time to System Failure	T_0	$179.0661/hrs$
Availability of the system	A_0	0.994787
Busy period of Repairman	BR_0	0.020706
Busy period of Replacement	BRP_0	0.002816
Expected no. of Repairs	ER_0	0.004181
Expected no. of Replacements	ERP_0	0.000056
Profit	P	$Rs.8,80,328.59$

8. Graphical Representation

Graphical study has been made for the MTSF, Profit with respect to failure rate of sub-unit one(λ_1), revenue per unit uptime of the system(Z_0) for different values of cost of repairman for busy in doing repair(Z_1).

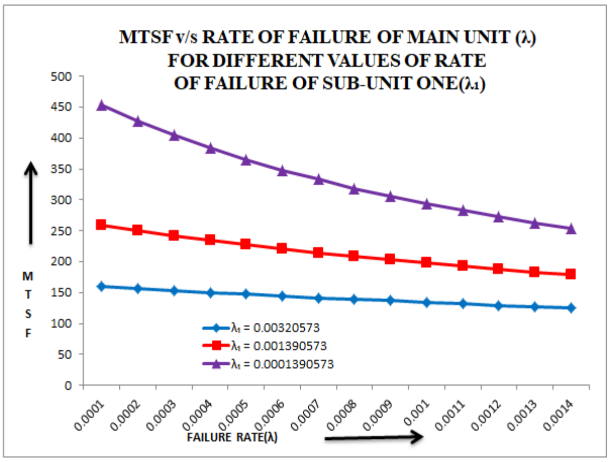

Figure 2: *MTSF v/s Failure Rate*

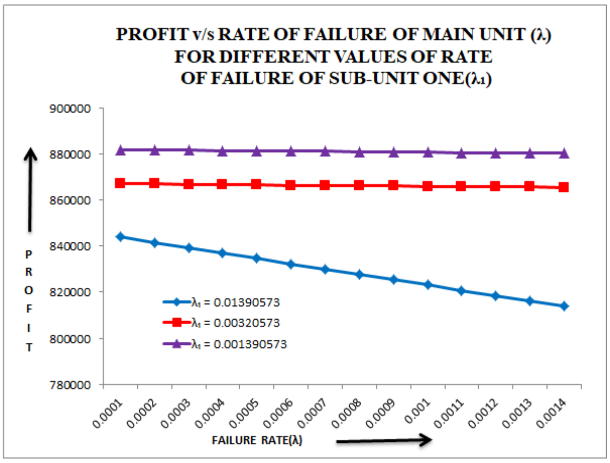

Figure 3: *Profit v/s Failure Rate*

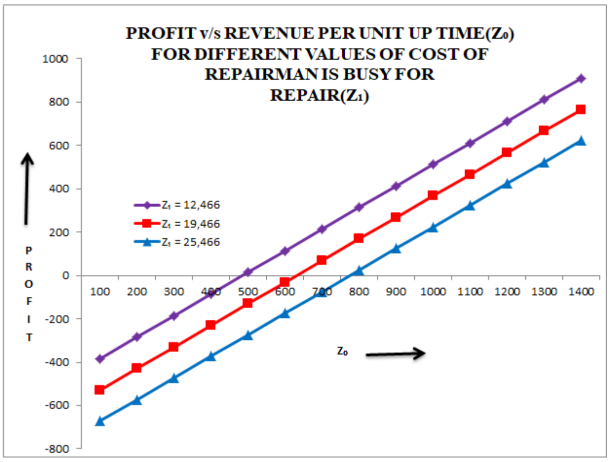

Figure 4: *Profit v/s Revenue*

9. Discussion

Discussion for the FAILURE RATE v/s MTSF and PROFIT v/s FAILURE RATE in the **(Table 4)**

Table 4: *Results*

Variation	Effect
λ/λ_1 increasing ($\uparrow$)	MTSF decreases ($\downarrow$)
λ/λ_1 increasing ($\uparrow$)	Profit decreases ($\downarrow$)

As shown in above table, the behaviour of MTSF and Profit w.r.t. rate of failure of Main unit for the different values of the rate of failure of sub-unit one. It clear from the table that MTSF and Profit gets decreased with increase in values of rate of failure of Main unit i.e. λ. Also MTSF and Profit decreases as failure rate of sub-unit one i.e. λ_1 increases.

Discussion for the PROFIT v/s REVENUE in the **(Table 5)** as below:

Table 5: *Results*

Variation	Effect
Z_0 increasing ($\uparrow$)	Profit increases ($\uparrow$)
$Z_1 = 12,466$; Profit $>=<$ according as z_0	when Z_0 is $>=<$ INR 428
$Z_1 = 19,466$; Profit $>=<$ according as z_0	when it Z_0 $>=<$ INR 582
$Z_1 = 25,466$; Profit $>=<$ according as z_0	when it Z_0 $>=<$ INR 722

Above table depicts the behaviour of the profit w.r.t. revenue per unit uptime of the system (Z_0) for different values of cost of repairman is busy under repair (Z_1). The graph exhibits that there is inclination in the trend of profit increases with increases in the values of Z_0. Also, following conclusion can be drawn from the discussion for Profit v/s Revenue :

For $Z_1 = 12,466$, the profit is positive or zero or negative according as Z_0 is $>=<$ INR 428. Hence, for this case the revenue per unit up time should be fixed, equal or greater than INR 428.

Similarly, discussion for other values of Z_1.

10. Conclusion

It's vital to use the outcomes of the mathematical metrics to enhance the reliability model (Table 3). To better comprehend the significant genuine influencing factors, these results must be employed. This study's findings are ground-breaking since no prior research has emphasised the critical function that GDC system repair and replacement facilities play at piston plants. The analysis's conclusions are very intriguing, and employing the GDC system by piston manufacturing companies is advantageous, according to the argument. Similar to how it is used in other domains, system designers can apply the proposed strategy in their own. Utilize the acquired equations to assess the applicability of various mechanism-type systems.

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Comparison of Bridge Systems with Multiple Types of Components

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Abstract

This paper aims to compare some bridge systems with multiple types of components in stochastic, hazard rate, and likelihood ratio order. Such systems are generally used in the designing and production industries. These systems are supported by a buffer store that balances the fluctuation in two production lines during the production process. The survival signature tool and distortion function technique are employed to compare the performance of four different bridge systems. Survival signature and henceforth survival function is computed for each considered system. The findings of comparisons are facilitated with the help of tables and figures. The comparison of large size coherent systems based on the structure-function approach is quite challenging. As this study is based on survival signature, so it is not so complex and has future scope.

Keywords: survival signature; bridge system; survival function; distortion function.

1. Introduction

In today's competitive and technology-driven world, it has become consequential to develop safe, reliable and long-lasting systems. The accurate reliability assessment of components and systems is crucial, and hence the branch of reliability engineering is in very much demand. In reliability theory, the stochastic comparison of systems is an imperative concept and has been explored by many researchers. It is quite challenging to compare complex systems, and most realistic cases generally have complex structures. Birnbaum *et al.* [2] and Barlow and Proschan [1] compared the same order coherent systems based on component lifetime using the structure-function approach. But these methods involve analytical complexities while comparing complex manufacturing systems. Recently, system signature and survival signature have emerged as advanced and promising tools in reliability analysis. These tools have suitable applications in studying system reliability and comparing various coherent systems.

A system having monotonic structure function with each of its components being relevant is known as coherent system. Samaniego [12] introduced the concept of system signature for the systems having independent and identically distributed (*iid*) components, with common distribution function F. For such coherent systems, Samaniego [12] derived an explicit expression of the failure rate in terms of components' failure rate and *F*. The IFR closure theorem for k-out-of-n system is also discussed by researcher. Kochar *et al.* [6] further derived the expression of system signature for k-out-of-n systems with component-wise and system-wise redundancy. Samaniego

[11] extended the concept of signature for preservation, characterisation, and system reliability. The applications of network reliability and economical reliability to systems having shared components are also presented. Navarro *et al.* [9] defined a joint signature for coherent systems with shared components. They discussed the sufficient condition for bivariate stochastic ordering between the joint lifetimes of two pairs of the systems.

Coolen and Coolen-Maturi [3] extended the concept of system signature to systems with multiple types of components, and they coined the new term 'survival signature'. The survival function of the coherent systems having *iid* and exchangeable components is evaluated using the survival signature tool. Coolen *et al.* [4] further adopted this technique and developed non parametric predictive inference for studying the reliability of systems. Krpelık *et al.* [8] introduced the formula for computing system survival signatures by means of merging survival signatures of multiple subsystems. They also introduced a decomposition method that allows decoupling the dependencies among subsystems. Huang *et al.* [5] analysed the reliability of the phased mission systems having identical components in each phase using survival signature.

Several authors have worked on the stochastic comparison of coherent systems. Kochar *et al.* [6] compared various systems on the basis of stochastic, hazard rate, and likelihood ordering using the notion of system signature. Authors derived an important theorem on hazard rate ordering of the system based on its components' hazard rate ordering. Coolen and Coolen-Maturi [3] compared some coherent systems with *iid* and non-*iid* components based on a novel technique of survival signature. Koutras *et al.* [7] stochastically compared two systems having exchangeable components. They further provided a necessary and sufficient condition for examining hazard rate ordering and reverse hazard rate ordering. Samaniego and Navarro [13] presented the methodology to compare some systems having heterogeneous components in different modes (stochastic, hazard rate, and likelihood ratio ordering) using survival signature and distortion function.

The bridge systems are broadly used in system designing in addition to the series and the parallel systems. Such systems are found in the production process in various industries. The production system having two parallel production lines connected by a buffer store to balance their productivity variation is investigated as a bridge structure system [10]. The analytical evaluation of the lifetime of the bridge system is too dense. Therefore, the comparison among such systems becomes more complicated. The present study compares the lifetimes of the bridge systems having multiple types of components at different positions. The survival signature technique is used to compare these complex systems. This paper investigates some bridge systems having two/three types of components shown in Figure 1, Figure 2, and Figure 3. The comparative analysis of considered systems is done using the survival signature approach [13].

2. Definitions and Notations

The present section includes prevalent concepts, definitions, and theorems. For 'm' components system, the state vector $x = (x_1, x_2, \dots. x_m) \in \{0,1\}^m$, where

$$x_i = \begin{cases} 1, & when\ i^{th}\ component\ of\ system\ is\ working \\ 0, & when\ i^{th} component\ of\ system\ is\ not\ working \end{cases}$$

for all $i = 1, 2, 3, \dots, m$. Thus, the set $\{0,1\}^m$ represents all the possible state vectors of m-order binary coherent system. Barlow and Proschan [1] defined the structure function ϕ mapped from the set $\{0,1\}^m$ to $\{0,1\}$ as follows

$$\phi(x_1, x_2, .., x_m) = \begin{cases} 1, if\ systems\ works \\ 0, if\ systems\ fails. \end{cases}$$

As compared to structure function, system signature [12] is less general but more significant. For the coherent system of order 'm', the system signature is a probability vector such that some i^{th} component causes system failure. Mathematically, the i^{th} element 's_i', of the system signature $s = (s_1, s_2, .., s_m)$ is expressed as

$$s_i = P\ (T = X_{i:m}) = \frac{m_i}{m!}$$

where T denotes the lifetime of the system, $X_{i:m}$ represents the i^{th} order statistic of the failure time of the m-components and m_i is number of those orderings corresponding to which system fails on failure of i^{th} component. It is evident that $\forall\ i,\ s_i \geq 0$ and $\sum_{i=1}^{m} s_i = 1$.

For a coherent system with m *iid* components having a continuous lifetime distribution, the survival signature $\Phi(l)$ for $l = 0,1,2, \dots, m$ is defined as the probability of functioning of system, provided that its exactly l components are working [3]. Mathematically, the survival signature of coherent system is given by

$$\Phi(l) = \frac{\sum_{x \epsilon s_l} \phi\ (x)}{|s_l|} = \binom{m}{l}^{-1} \sum_{x \epsilon s_l} \phi(x)$$

where s_l is the set of all such state vectors whose exactly l components (x_i) are 1 and remaining are 0. The system reliability $\bar{F}_T(t)$ in terms of survival signature for iid components is

$$\bar{F}_T(t) = P\ (T > t) = \sum_{l=0}^{m} \Phi(l) \binom{m}{l} [F(t)]^{m-l}\ [\bar{F}(t)]^l$$

where $F(t)$, $\bar{F}(t)$ be the distribution and survival function respectively of components.

Coolen and Coolen-Maturi [3] considered the coherent system of order m, with $K > 1$ types of independent components. All the components of certain type are assumed to be identically distributed. Considering m_k components of type k, the survival signature $\Phi(l_1, l_2, \dots, l_K)$ is given by

$$\Phi(l_1, l_2, \dots, l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right] \sum_{x \epsilon s_{l_1, l_2, \dots, l_k}} \phi(x)$$

where l_k $(k = 1,2, \dots, K)$ is the number of functioning units of type k. In the above expression, x is a state vector given by $x = (x^1, x^2, \dots, x^K)$, where $x^k = (x_1^k, x_2^k, \dots, x_{m_k}^k)$. In case l_k $(k = 1,2, \dots, K)$ units of type k are working, then the vector x^k has precisely its l_k components (x_i^k) as 1 and remaining are 0. The set of all such state vectors is denoted by $s_{l_1, l_2, \dots, l_k}$. The reliability function $\bar{F}_T(t)$ of such systems in terms of survival signature as given by Coolen and Coolen-Maturi [3] is

$$\bar{F}_T(t) = P(T > t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_K=0}^{m_k} \left[\Phi(l_1, l_2, \dots, l_K) \prod_{i=1}^{K} \binom{m_i}{l_i}\ F_i(t)^{m_i - l_i} \bar{F}_i(t)^{l_i}\right]$$

where $F_i(t)$, $\bar{F}_i(t)$ be the distribution and survival function of the i^{th} component.

Some results on stochastic order properties which appeared in [14] are discussed below. Let T_1, T_2 be the random variables with the distribution functions $F_1(t), F_2(t)$ and reliability functions $\overline{F_1}(t), \overline{F_2}(t)$ respectively, then

- T_1 is smaller than T_2 in usual stochastic order, i.e. $T_1 \leq_{ST} T_2$ if $\overline{F_1}(t) \leq \overline{F_2}(t)$for all t;
- T_1 is smaller than T_2 in the hazard rate order, i.e. $T_1 \leq_{HR} T_2$ if $\overline{F_2}(t)/\overline{F_1}(t)$ is increasing in t;
- T_1 is smaller than T_2 in the likelihood ratio order, i.e. $T_1 \leq_{LR} T_2$ if $f_2(t)/f_1(t)$ is increasing in t; where $f_1(t)$ and $f_2(t)$ are probability density functions (pdfs) of T_1 and T_2 respectively.

Samaniego and Navarro [13] also derived a result for the comparison of two systems having m_k independent type k components with distribution function F_k for $k \in \{1,2,..,r\}$. The following theorem appeared as Theorem 2.1. in Samaniego and Navarro [13] .

Theorem 1. If T_1, T_2 be the lifetimes and Φ_1, Φ_2 be survival signatures of two systems A and B respectively and if for all vectors $(l_1, \dots, l_r)$, with $l_k = 0, \dots, m_k \; and \; k = 1, \dots, r$, the inequality

$$\Phi_1(l_1, \dots, l_r) \leq \Phi_2(l_1, \dots, l_r)$$

holds, then it follows that $T_1 \leq_{ST} T_2$ for all distribution functions $F_1, \dots, F_r$.

Samaniego and Navarro [13] further proved a theorem, which aids in the comparison of two systems having different orders. For such comparisons, some irrelevant components are considered and added to the systems. The following proved result appeared as Theorem 3.1 in Samaniego and Navarro [13] .

Theorem 2. Let Φ be the survival signature of m-order coherent system, having r types of components and suppose it has to be compared with some system of order m+1. An irrelevant component of type-k is added to m-order coherent system, and let Φ^* be the survival signature of resulting new m+1 order system. Considering m_j components of type j, Samaniego and Navarro [13] established following relations for survival signatures Φ and Φ^*

(i) $For\ 0 \leq l_j \leq m_j,\ j = 1,2,,..,k-1,k,k+1,\dots,r,$

$$\Phi^*(l_1, \dots l_{k-1}, 0, l_{k+1}, \dots, l_r) = \Phi(l_1, \dots l_{k-1}, 0, l_{k+1}, \dots, l_r)$$

(ii) $For\ 0 \leq l_j \leq m_j, j = 1,2,,..,k-1,k,\dots,r, and\ for\ 1 \leq l_k \leq m_k,$

$$\Phi^*(l_1, \dots l_{k-1}, l_k, \dots, l_r) = \left(\frac{l_k}{m_k+1}\right) \Phi(l_1, \dots l_{k-1}, l_k - 1, \dots, l_r) + \left(\frac{m_k - l_k + 1}{m_k + 1}\right) \Phi(l_1, \dots l_{k-1}, l_k, \dots, l_r)$$

(iii) $For\ 0 \leq l_j \leq m_j,\ j = 1,2,,..,k-1,k,k+1 \dots,r,$

$$\Phi^*(l_1, \dots l_{k-1}, m_k+1, l_{k+1}, \dots, l_r) = \Phi(l_1, \dots l_{k-1}, m_k, l_{k+1}, \dots, l_r).$$

Samaniego and Navarro [13] also adopted a generalized distorted distribution technique for comparing two systems. They employed a dual distortion function, $\bar{Q}(u_1, u_2, \dots, u_r)$ and distortion function $Q(u_1, u_2, \dots, u_r)$ in this technique. These functions satisfy the following properties:

(i) $\bar{Q}(u_1, u_2, \dots, u_r)$ is a continuous increasing function
(ii) $\bar{Q}(u_1, u_2, \dots, u_r) = 0$ if $u_i = 0 \; \forall \; i \in \{1,2,..,r\}$
(iii) $\bar{Q}(u_1, u_2, \dots, u_r) = 1$ if $u_i = 1 \; \forall \; i \in \{1,2,..,r\}$.
(iv) $Q(u_1, u_2, \dots, u_r) = 1 - \bar{Q}(1-u_1, 1-u_2, \dots, 1-u_r)$

The survival function $\bar{F}_T(t)$ of coherent system having r types of components can be expressed as-

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_r(t)),$$

where $\overline{F_l}$ is the reliability function of components of type l. The lifetimes T_1 and T_2 of the two coherent systems with r types of components can be compared using the distortion function as discussed below. The following proved result appeared as Theorem 4.1. in Samaniego and Navarro [13].

Theorem 3. Let $F_1, F_2, \dots, F_r$ be the distribution functions of the components of type 1, type 2,…, type r respectively. Samaniego and Navarro [13] proved that if $\bar{Q}_1$ and $\bar{Q}_2$ be the dual distortion functions of two considered systems, then

(i) $T_1 \leq_{ST} T_2$ holds for all $F_1, \dots, F_r$ if and only if $\bar{Q}_1 \leq \bar{Q}_2$ in $(0,1)^r$;

(ii) $T_1 \leq_{HR} T_2$ holds for all $F_1, \dots, F_r$ if and only if $\bar{Q}_2/\bar{Q}_1$ is decreasing in $(0,1)^r$;

(iii) $T_1 \leq_{LR} T_2$ holds for all $F_1, \dots, F_r$, if the distributions of T_1 and T_2 are absolutely continuous, and if $\gamma(u_1, u_2, \dots, u_r, v_2, \dots, v_r)$ is decreasing in $u_1, u_2, \dots, u_r$ and increasing (decreasing) in v_r in $(0,1)^r \times (0,\infty)^{r-1}$ and $F_1 \leq_{LR} F_i$ $(\geq_{LR})$ for i = 2, …, r where

$$\gamma(u_1, u_2, \dots, u_r, v_2, \dots, v_r) = \frac{D_1\bar{Q}_2(u_1,u_2,\dots,u_r)+\sum_{i=2}^{r} v_i D_i \bar{Q}_2(u_1,u_2,\dots,u_r)}{D_1\bar{Q}_1(u_1,u_2,\dots,u_r)+\sum_{i=2}^{r} v_i D_i \bar{Q}_1(u_1,u_2,\dots,u_r)},$$

$D_i\bar{Q}_j$ denotes the partial derivatives of $\bar{Q}_j$ about i^{th} component for $i \in \{1, \dots, r\}$ and $j \in \{1,2\}$ and u_r denotes components' reliability function of type r and v_r denotes the ratio of pdfs of components of type r to the type 1.

3. Analysis and Discussion

The purpose of this article is to compare the bridge systems having multiple types of components. The survival signature tool is used to compare the considered systems in three different senses (stochastic, hazard rate, and likelihood ratio ordering). The bridge system as shown in Figure 1 has two units x_{11}, x_{21} of type 1 and three components namely x_{12}, x_{22}, x_{32} of type 2. The second considered system as shown in Figure 2 has again two components of type 1 and three components type 2, but at different positions. The bridge system (Figure 3) having three types of components is also investigated in this study.

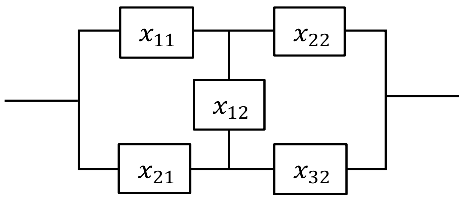

Figure 1: *System A (five-component bridge system)*

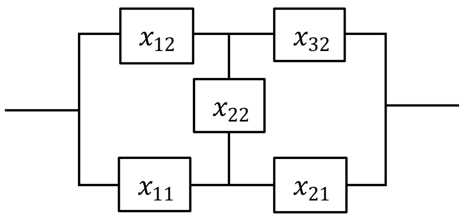

Figure 2: *System B (five-component bridge system with changed positions of components)*

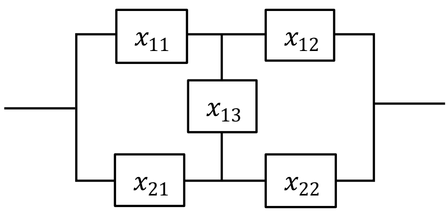

Figure 3: *System C (five-component bridge system containing three types of components)*

3.1. Comparison of two bridge systems with two types of components at different positions

Theorem 4. Consider two bridge systems of order five with two types of components at different positions. Let T_1, T_2 be the lifetimes of bridge systems A and B (Figure 1 and Figure 2) respectively. Then, T_1 is smaller than T_2 in usual stochastic order i.e., $T_1 \leq_{ST} T_2$.

Proof. Let $\Phi_1(l_1, l_2)$ and $\Phi_2(l_1, l_2)$ be the survival signatures of the systems A and B respectively. These systems have two and three components of type 1 and type 2 respectively. The general expression of the survival signature $\Phi(l_1, l_2)$ for considered systems is as follows

$$\Phi(l_1, l_2) = \binom{2}{l_1}^{-1} \binom{3}{l_2}^{-1} \sum_{x \epsilon s_{l_1, l_2}} \phi(x)$$

where s_{l_1, l_2} is set of all state vectors of the system.

Table 1: *Survival signature Φ_1 of the system A*

$\Phi_1(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_3 = 3$
$l_1 = 0$	0	0	0	0
$l_1 = 1$	0	1/3	1	1
$l_1 = 2$	0	2/3	1	1

Table 2: *Survival signature Φ_2 of the system B*

$\Phi_2(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_3 = 3$
$l_1 = 0$	0	0	1/3	1
$l_1 = 1$	0	0	2/3	1
$l_1 = 2$	1	1	1	1

As discussed in Theorem 1, the survival signatures $\Phi_1(l_1, l_2)$ and $\Phi_2(l_1, l_2)$ given in Table 1 and Table 2 are non-comparable because $\Phi_1(0,2) < \Phi_2(0,2)$ and $\Phi_1(1,2) > \Phi_1(1,2)$. Thus, the domination of survival signature is not possible for the considered systems. To compare these systems, we need to do further analysis. Let $\bar{F}_{T_1}(t)$, $\bar{F}_{T_2}(t)$ be the survival functions of the bridge systems A and B with components distribution function $F_1(t)$ and $F_2(t)$. The difference between survival function $\bar{F}_{T_2}(t)$ and $\bar{F}_{T_1}(t)$ is given by

$$\bar{F}_{T_2}(t) - \bar{F}_{T_1}(t) = \sum_{l_1=0}^{2} \sum_{l_2=0}^{3} (\Phi_2(l_1, l_2) - \Phi_1(l_1, l_2)) \binom{2}{l_1}\binom{3}{l_2} F_1(t)^{2-l_1} \bar{F}_1(t)^{l_1} F_2(t)^{3-l_2} \bar{F}_2(t)^{l_2}. \quad (1)$$

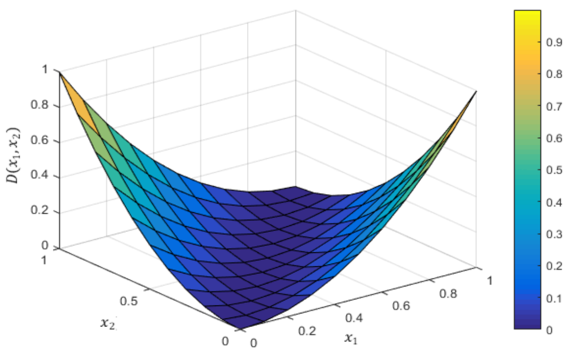

Figure 4: *The difference function $D(x_1, x_2)$.*

To simplify the system's comparison, we consider the variable $\bar{F}_1(t) = 1 - F_1(t)$ as x_1 and $\bar{F}_2(\mathrm{t}) = 1 - F_2(t)$ as x_2. So, the pair (x_1, x_2) belongs to unit square as $t \in [0, \infty)$. The above difference in Equation (1), is taken as $D(x_1, x_2)$ and can be represented as-

$$D(x_1, x_2) = {x_1}^2(1-x_2)^3 - 2x_1x_2(1-x_1)x_2(1-x_2)^2 + {x_1}^2x_2(1-x_2)^2 + {x_2}^2(1-x_1)^2(1-x_2) - 2x_1{x_2}^2(1-x_1)(1-x_2) + {x_2}^3(1-x_1)^2 .$$

The difference function $D(x_1, x_2)$ illustrated in Figure 4 has clearly non-negative values for each value of x_1 and x_2. i.e., $D(x_1, x_2) \geq 0,\ \forall\, x_1, x_2 \in [0,1]$. This implies that $\bar{F}_{T_2}(t) \geq \bar{F}_{T_1}(t)$. Hence the lifetime T_1 is smaller than lifetime T_2 in usual stochastic order i.e., $T_1 \leq_{ST} T_2$ holds for all $\bar{F}_1(t), \bar{F}_2(t)$.

3.2. Comparison of bridge systems using distortion functions

In this part, the systems A and B are compared as per stochastic, hazard rate and likelihood ratio ordering, by using their distortion functions.

Theorem 5. Let T_1, T_2 be the lifetimes of the bridge systems A and B (Figure 1 and Figure 2) respectively. These systems have two types of components with the distribution functions $F_1(t), F_2(t)$ and reliability function $\bar{F}_1(t), \bar{F}_2(t)$. The lifetime of system A is smaller than the lifetime of system B in usual stochastic order but not in hazard rate and likelihood ratio order.

Proof. Let $\bar{Q}_1$ and $\bar{Q}_2$ be dual distortion functions of systems A and B respectively. We have,

$$\begin{aligned}\bar{Q}_2(x_1, x_2) - \bar{Q}_1(x_1, x_2) \\ &= (1-x_2)^3{x_1}^2 - 2x_1x_2(1-x_2)^2(1-x_1) + {x_1}^2x_2(1-x_2)^2 + (1-x_1)^2{x_2}^2(1-x_2) \\ &\quad - 2x_1{x_2}^2(1-x_1)(1-x_2) + {x_2}^3(1-x_1)^2.\end{aligned}$$

Figure 4 indicates that $\bar{Q}_2(x_1, x_2) \geq \bar{Q}_1(x_1, x_2)\ \forall\, x_1, x_2 \in [0,1]$. Using Theorem 3, we can say that the system lifetime T_1 is smaller than system lifetime T_2 in usual stochastic order. i.e., $T_1 \leq_{ST} T_2$ hold for all $\bar{F}_1(t), \bar{F}_2(t)$.

Let R be the ratio of $\bar{Q}_2$ to $\bar{Q}_1$ i.e.,

$$R(x_1, x_2) = \frac{\bar{Q}_2}{\bar{Q}_1}$$

Figure 5 exhibits that the ratio $R(x_1, x_2)$ is neither increasing nor decreasing in x_1, x_2 in $(0,1)^2$. Data presented in Table 3 confirms the same. Using Theorem 3, we can say the system lifetime T_1 is not smaller than system lifetime T_2 in hazard rate order i.e., $T_1 \nleq_{HR} T_2$. Figure 6, further shows that these two bridge systems are not hazard rate ordered when the components of type-1 and type-2 follow exponential and Weibull distribution respectively.

Table 3: *The ratio* $R(x_1, x_2)$

$x_1 \rightarrow$ $x_2 \downarrow$	0.00010	0.09999	0.19988	0.29977	0.39966	0.49955	0.59944	0.69933
0.00010	1.000	499.908	999.330	1498.767	1998.220	2497.687	2997.169	3496.667
0.09999	458.761	1.000	1.235	1.635	2.086	2.566	3.068	3.588
0.19988	861.762	1.220	1.000	1.076	1.236	1.436	1.663	1.912
0.29977	1239.023	1.567	1.073	1.000	1.039	1.129	1.251	1.397

0.39966	1611.871	1.939	1.216	1.037	1.000	1.024	1.085	1.172
0.49955	1998.480	2.331	1.390	1.121	1.023	1.000	1.017	1.063
0.59944	2417.177	2.754	1.588	1.231	1.081	1.017	1.000	1.013
0.69933	2889.461	3.225	1.812	1.365	1.162	1.061	1.013	1.000

To compare the hazard rate ordering of system A and component of type 1, the ratio $R_{x_1}^1(x_1, x_2)$ is computed. We get

$$R_{x_1}^1 = \frac{\bar{Q}_1(x_1, x_2)}{x_1} = 2x_2(1-x_2)^2(1-x_1) + 6{x_2}^2(1-x_1)(1-x_2) + 2{x_2}^3(1-x_1) + 2x_1x_2(1-x_2)^2 + 3x_1{x_2}^2(1-x_2) + x_1{x_2}^3.$$

Figure 7 indicates that the ratio $R_{x_1}^1(x_1, x_2)$ increases with increase in x_2, but it decreases with increase in x_1 in $(0,1)^2$. Therefore, the lifetimes of system A and type 1 components are not comparable in the hazard rate order, i.e., $T_1 \nleq_{HR} X_1$ where X_1 indicates the type 1 component's lifetime. Similarly, for hazard rate order comparison of system A and the type 2 components, the ratio $R_{x_2}^1$ is evaluated. We obtain

$$R_{x_2}^1 = \frac{\bar{Q}_1(x_1, x_2)}{x_2} = 2x_1(1-x_2)^2(1-x_1) + 6x_1x_2(1-x_1)(1-x_2) + 2x_1{x_2}^2(1-x_1) + 2{x_1}^2(1-x_2)^2 + 3{x_1}^2x_2(1-x_2) + {x_1}^2{x_2}^2.$$

Here, the ratio $R_{x_2}^1(x_1, x_2)$ decreases with increase in x_1, but it neither increases nor decreases with increase in x_2 in $(0,1)^2$. Therefore, $T_1 \nleq_{HR} X_2$, where X_2 is type 2 component's lifetime. In the same manner, system B is compared with type 1 and type 2 components in hazard rate order by evaluating the ratios $R_{x_1}^2$ and $R_{x_2}^2$ respectively. We have

$$R_{x_1}^2 = \frac{\bar{Q}_2(x_1, x_2)}{x_1} = \frac{{x_2}^2}{x_1}(1-x_2)(1-x_1)^2 + \frac{{x_2}^3}{x_1}(1-x_1)^2 + 4(1-x_1){x_2}^2(1-x_2) + 2{x_2}^3(1-x_1) + x_1(1-x_2)^3 + 3x_1x_2(1-x_2)^2 + 3x_1(1-x_2){x_2}^2 + x_1{x_2}^3$$

and

$$R_{x_2}^2 = \frac{\bar{Q}_2(x_1, x_2)}{x_2} = x_2(1-x_2)(1-x_1)^2 + (1-x_1)^2{x_2}^2 + 4x_1x_2(1-x_1)(1-x_2) + 2x_1{x_2}^2(1-x_1) + \frac{{x_1}^2(1-x_2)^3}{x_2} + 3{x_1}^2(1-x_2)^2 + 3{x_1}^2x_2(1-x_2) + {x_1}^2{x_2}^2.$$

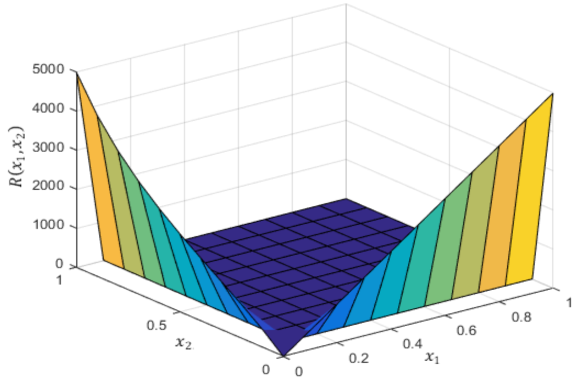

Figure 5: *The Graphical interpretation of the function* $R(x_1, x_2)$

Here, the ratio $R_{x_1}^2(x_1, x_2)$ increases with x_2 but it is not monotonic in x_1 in $(0,1)^2$. Thus, $T_2 \nleq_{HR} X_1$. The ratio $R_{x_2}^2(x_1, x_2)$ increases with increase in x_1 but decreases with increase in x_2 in $(0,1)^2$. Therefore, $T_2 \nleq_{HR} X_2$.

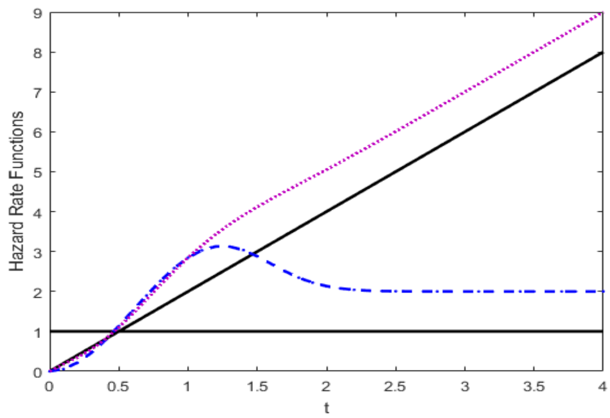

Figure 6: *Hazard rate functions of the bridge systems (A (dash), B (dot)) and their components (dark lines). Type 1 and Type 2 components follow exponential and Weibull distribution* $(a = 2, b = 1)$ *respectively for* $t > 0$

Let X_1, X_2 be the lifetimes of the components of type 1 and type 2 with respective pdfs $f_1(t)$, $f_2(t)$. The components of Type 1 and type-2 are assumed to be exponentially (mean = 1) and Weibull $(a = 2, b = 1)$ distributed respectively. The ratio $\frac{f_2(t)}{f_1(t)}$ is increasing in t as shown in Figure 8. Hence, we get that X_1 is smaller than X_2 in likelihood ratio ordering i.e., $X_1 \leq_{LR} X_2$. For likelihood ratio ordering comparison of systems A and B, as per Theorem 3, we have function $\Upsilon\left(x_1, x_2, \frac{f_2(t)}{f_1(t)}\right)$ as-

$$\Upsilon\left(x_1, x_2, \frac{f_2(t)}{f_1(t)}\right) = \frac{\begin{array}{c} v(2{x_2}^2(x_1-1)^2 - 2x_1{x_2}^2(x_1-1) - 2x_2(x_2-1)(x_1-1)^2 + 8x_1x_2(x_1-1)(x_2-1)) \\ -2x_1(x_2-1)^3 + 6x_1x_2(x_2-1)^2 - 2x_1{x_2}^2(x_2-1) + 2{x_2}^2(x_1-1)(x_2-1) \end{array}}{\begin{array}{c} v(2{x_1}^2(x_2-1)^2 - 2{x_1}^2x_2(x_2-1) - 2x_1(x_1-1)(x_2-1)^2 + 8x_1x_2(x_1-1)(x_2-1)) \\ -2{x_2}^3(x_1-1) + 2x_1x_2(x_2-1)^2 - 2x_2(x_1-1)(x_2-1)^2 + 6{x_2}^2(x_1-1)(x_2-1) \end{array}}$$

where $\frac{f_2(t)}{f_1(t)} = v$.

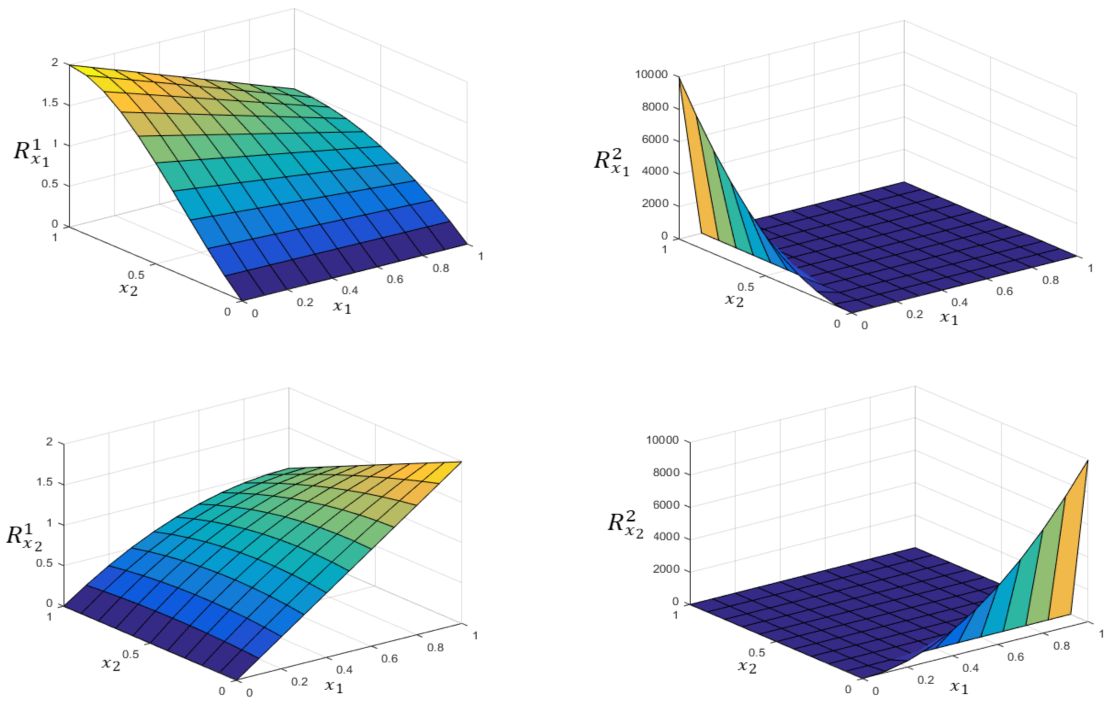

Figure 7: *The Ratio $R^1_{x_1}$, $R^2_{x_1}$, $R^1_{x_2}$, $R^2_{x_2}$*

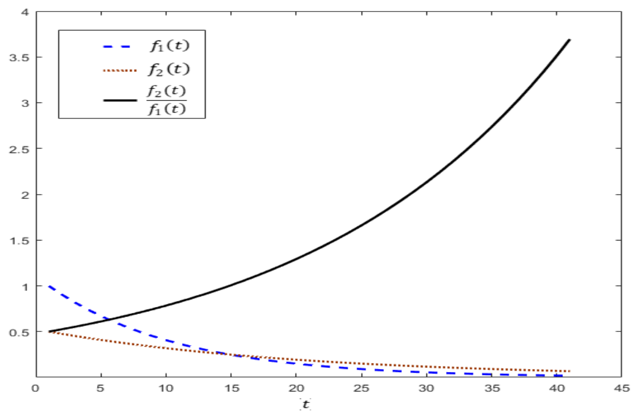

Figure 8: *Likelihood ratio ordering of the components of type 1 and type 2*

Table 4 indicates that the function $\Upsilon(x_1, x_2, 0.0001)$ is increasing in x_1 for the particular value of x_2. But we can see the function is neither increasing nor decreasing in x_2 for any particular values of x_1. In Table 5, the function $\Upsilon(x_1, 0.09999, v)$ is increasing in x_1 for the particular values of v. Table 5 further shows that the function is increasing in v for $x_1 = 0.0001$, but it is decreasing in v for $x_1 =$ $0.09999, 0.19998, 0.29977$. Hence, we get that the function $\Upsilon\left(x_1, x_2, \frac{f_2(t)}{f_1(t)}\right)$ is increasing in x but not monotonic in $\frac{f_2(t)}{f_1(t)}$ in set $(0,1)^2 \times (0, \infty)$. Therefore, these considered bridge systems are not likelihood ratio ordered. i.e., $T_1 \nleq_{LR} T_2$.

Table 4: *The function* $\Upsilon(x_1, x_2, 0.0001)$

$x_1 \rightarrow$ $x_2 \downarrow$	0.00010	0.09999	0.19988	0.29977	0.39966	0.49955	0.59944	0.69933
0.00010	1.000	908.936	1665.738	2306.254	2855.383	3331.387	3747.970	4115.610
0.09999	0.083	1.000	1.965	2.985	4.062	5.203	6.412	7.697
0.19988	0.138	0.549	1.000	1.497	2.049	2.664	3.355	4.135
0.29977	0.173	0.414	0.687	1.000	1.361	1.783	2.283	2.885
0.39966	0.193	0.349	0.530	0.744	1.000	1.312	1.701	2.199
0.49955	0.200	0.304	0.428	0.579	0.764	1.000	1.307	1.725
0.59944	0.193	0.264	0.349	0.454	0.587	0.762	1.000	1.343
0.69933	0.173	0.219	0.276	0.346	0.438	0.561	0.735	1.000

Table 5: *The function* $\Upsilon(x_1, 0.09999, v)$

$x_1 \rightarrow$ $x_2 \downarrow$	0.00010	0.09999	0.19988	0.29977	0.39966	0.49955	0.59944	0.69933
0.00010	0.083	1.000	1.965	2.985	4.062	5.203	6.412	7.697
0.09999	0.175	1.000	1.715	2.349	2.923	3.451	3.945	4.413
0.19988	0.266	1.000	1.540	1.963	2.312	2.611	2.876	3.116
0.29977	0.358	1.000	1.410	1.704	1.932	2.119	2.279	2.422
0.39966	0.449	1.000	1.311	1.519	1.672	1.795	1.898	1.989
0.49955	0.541	1.000	1.232	1.379	1.484	1.565	1.634	1.694
0.59944	0.633	1.000	1.168	1.270	1.340	1.395	1.440	1.479
0.69933	0.724	1.000	1.115	1.182	1.228	1.263	1.291	1.316

3.3. Comparison of two bridge systems with different number of components

Theorem 6. Suppose T_1, T_4 be the lifetimes of the bridge systems A and D shown in Figure 1 and Figure 9 respectively. The system D has six components, where type 1 components are x_{11}, x_{21} and x_{31} and type 2 components are x_{12}, x_{22} and x_{32}. Then the lifetime T_4 is smaller than T_1 in usual stochastic order. i.e., $T_4 \leq_{ST} T_1$.

Proof. Let $\Phi_4(l_1, l_2)$ be the survival signature of the system D. The survival signature $\Phi_1(l_1, l_2)$ of system A is already discussed and given in Table 1. An independent irrelevant component of type $k = 1$ is added to system A, and let us suppose that $\Phi_1^*(l_1, l_2)$ be the survival signature of new resulting system of order 6. Using Theorem 2, we have

(i) $For\ 0 \leq l_2 \leq m_2$

$$\Phi_1^*(0, l_2) = \Phi_1(0, l_2)$$

(ii) For $1 \leq l_1 \leq m_1$ and $0 \leq l_2 \leq m_2$

$$\Phi_1^*(l_1, l_2) = \left(\frac{l_1}{m_1 + 1}\right)\Phi_1(l_1 - 1, l_2) + \left(\frac{m_1 - l_1 + 1}{m_1 + 1}\right)\Phi_1(l_1, l_2)$$

(iii) For $0 \leq l_2 \leq m_2$

$$\Phi_1^*(m_1 + 1, l_2) = \Phi_1(m_1, l_2).$$

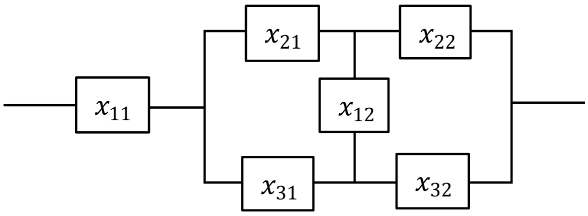

Figure 9: *System D (six-component bridge system)*

Tables 6 and 7 indicate that the survival signature Φ_1^* is greater than Φ_4 for all values of l_1, l_2 i.e., $\Phi_1^*(l_1, l_2) \geq \Phi_4(l_1, l_2)\ \forall\ l_1, l_2 \in \{0,1,2,3\}$. Thus, the lifetime T_4 is smaller than T_1 in usual stochastic order, i.e., $T_4 \leq_{ST} T_1$.

Table 6: *Survival signature Φ_1^* of the bridge system A with irrelevant component of type-1*

$\Phi_1^*(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_2 = 3$
$l_1 = 0$	0	0	0	0
$l_1 = 1$	0	2/9	2/3	2/3
$l_1 = 2$	0	4/9	1	1
$l_1 = 3$	0	2/3	1	1

Table 7: *Survival signature Φ_4 of the bridge system D*

$\Phi_4(l_1, l_2)$	$l_2 = 0$	$l_2 = 1$	$l_2 = 2$	$l_2 = 3$
$l_1 = 0$	0	0	0	0
$l_1 = 1$	0	0	0	0
$l_1 = 2$	0	2/9	2/3	2/3
$l_1 = 3$	0	2/3	1	1

3.4. Comparison of lifetimes of two bridge systems with two and three types of components

Theorem 7. Consider two bridge systems A and C, shown in Figures 1 and 3. Let T_1 and T_3 be the respective lifetimes of systems A and C. Type 1, type 2 and type 3 components are assumed to be *iid* with reliability functions $\bar{F}_1$, $\bar{F}_2$ and $\bar{F}_3$ respectively. Then $T_1 \leq_{ST} T_3$ if $\bar{F}_2(t) \leq \bar{F}_3(t)$.

Proof. Let $\Phi_1(l_1, l_2)$ and $\Phi_3(l_1, l_2, l_3)$ be the survival signature of bridge systems A and C respectively. Here, system C contains two components of type 1, two components of type 2, and one component of type 3. The survival signature $\Phi_3(l_1, l_2, l_3)$ can be written as $\Phi_3(l_1, l_2, l_3) = \binom{2}{l_1}^{-1} \binom{2}{l_2}^{-1} \binom{1}{l_3}^{-1} \sum_{x \epsilon s_{l_1, l_2, l_3}} \phi(x)$, and is given in Table 8. For comparison of bridge systems A and C, we have added an irrelevant component of type 3 ($k = 3$) to system A. Using Theorem 2, we have survival signature $\Phi_1^*(l_1, l_2, l_3)$ of resulting 6-components system as:

(i) For $0 \leq l_j \leq m_j;\ j = 1,2$

$$\Phi_1^*(l_1, l_2, 0) = \Phi(l_1, l_2, 0)$$

(ii) For $0 \leq l_j \leq m_j\ ;\ j = 1,2$

$$\Phi_1^*(l_1, l_2, m_3 + 1) = \Phi(l_1, l_2, m_3).$$

Similarly, we have added one component of type 2 ($k = 2$) which is irrelevant in nature to system C. Using Theorem 2, the survival signature Φ_3^* of the resultant 6-components system is given by

(i) For $0 \le l_j \le m_j$; $j = 1,3$

$$\Phi_3^*(l_1, 0, l_3) = \Phi_3(l_1, 0, l_3)$$

(ii) For $0 \le l_j \le m_j$; $j = 1,3$ and $1 \le l_2 \le m_2$

$$\Phi_3^*(l_1, l_2, l_3) = \frac{l_2}{3}\Phi_3(l_1, l_2 - 1, l_3) + \frac{3 - l_2}{3}\Phi_3(l_1, l_2, l_3)$$

(iii) For $0 \le l_j \le m_j$; $j = 1,3$

$$\Phi_3^*(l_1, m_2 + 1, l_3) = \Phi_3(l_1, m_2, l_3).$$

Table 8: *The survival signature $\Phi_3(l_1, l_2, l_3)$ of the system C*

l_1	l_2	l_3	$\Phi_3(l_1, l_2, l_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
0	2	0	0
0	2	1	0
1	0	0	0
1	0	1	0
1	1	0	1/2
1	1	1	1
1	2	0	1
1	2	1	1
2	0	0	0
2	0	1	0
2	1	0	1
2	1	1	1
2	2	0	1
2	2	1	1

Table 9 shows that the survival signatures Φ_1^* and Φ_3^* are identical for all the combinations of l_1, l_2, l_3 except two cases. The survival signature Φ_1^* and Φ_3^* are not dominated in any sense since $\Phi_1^*(1,1,1) < \Phi_3^*(1,1,1)$ but $\Phi_1^*(1,2,0) > \Phi_3^*(1,2,0)$. So, the comparison of systems A and C needs further analysis. Let $\bar{F}_{T_1}(t), \bar{F}_{T_3}(t)$ be the respective reliability functions of systems A and C. We have

$$\bar{F}_{T_3}(t) - \bar{F}_{T_1}(t) = \sum_{l_1=0}^{2}\sum_{l_2=0}^{3}\sum_{l_3=0}^{1}[(\Phi_3^*(l_1, l_2, l_3) - \Phi_1^*(l_1, l_2, l_3)]\binom{2}{l_1}\binom{3}{l_2}\binom{1}{l_3} F_1(t)^{2-l_1}\bar{F}_1(t)^{l_1}F_2(t)^{3-l_2}\bar{F}_2(t)^{l_2}F_3(t)^{1-l_3}\bar{F}_3(t)^{l_3}$$

Using survival signature given in Table 9, we get

$$\bar{F}_{T_3}(t) - \bar{F}_{T_1}(t) = -2\,F_1(t)\bar{F}_1(t)F_2(t)\bar{F}_2(t)^2F_3(t) + 2\,F_1(t)\bar{F}_1(t)F_2(t)^2\bar{F}_2(t)\bar{F}_3(t) \tag{2}$$

Table 9: *The survival signature Φ_1^* and Φ_3^* of systems after adding an irrelevant component of type-3 and type-2 respectively to system A and system C*

l_1	l_2	l_3	$\Phi_1^*(l_1, l_2, l_3)$	$\Phi_3^*(l_1, l_2, l_3)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
0	2	0	0	0
0	2	1	0	0
0	3	0	0	0
0	3	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	1/3	1/3
1	1	1	1/3	2/3
1	2	0	1	2/3
1	2	1	1	1
1	3	0	1	1
1	3	1	1	1
2	0	0	0	0
2	0	1	0	0
2	1	0	2/3	2/3
2	1	1	2/3	2/3
2	2	0	1	1
2	2	1	1	1
2	3	0	1	1
2	3	1	1	1

To simplify the comparison process, we have taken variable $\bar{F}_1(t)$ =$1 - F_1(t)$ as x_1, $\bar{F}_2(t)$ =$1 - F_2(t)$ as x_2 and $\bar{F}_3(t)$ =$1 - F_3(t)$ as x_3. The 3-tuple (x_1, x_2, x_3) lies in the unit cube as t varies from 0 to ∞. For $t \in [0, \infty)$, the difference $\bar{F}_{T_3}(t) - \bar{F}_{T_1}(t)$ given in Equation (2) can be written as the multivariable function $D(x_1, x_2, x_3)$ as

$$D(x_1, x_2, x_3) = -2x_1(1-x_1)(1-x_2)(1-x_3)y^2 + 2x_1x_2x_3(1-x_1)(1-x_2)^2$$
$$= 2x_1x_2(1-x_1)(1-x_2)(x_3-x_2).$$

If $x_2 \le x_3$ or x_3 = 1 then $D(x_1, x_2, x_3) \ge 0$. In addition, $D(x_1, x_2, x_3) = 0 \; if \; x_1, x_2 = 1 \; or \; x_2 = x_3$. This implies that the system's lifetime T_1 is smaller than T_3 in usual stochastic order if the components lifetime of type 2 is less than the component lifetime of type 3. i.e., $T_1 \le_{ST} T_3$ if $\bar{F}_2(t) \le \bar{F}_3(t)$.

4. Conclusion

The bridge structures are generally used in the design and production industry. The comparative study of such systems is crucial to ensure system productivity and to distinguish the system that performs well. Comparing bridge systems having *iid* and multiple types of components without knowing their component's distribution is very challenging. In this paper, we have seen that the lifetime of bridge system A is smaller than the lifetime of bridge system B in usual stochastic order.

However, the lifetimes of these systems (Figure 1 and Figure 2) are not found to be hazard rate and likelihood ratio ordered. Further, coherent systems A (five order) and D (six order) are compared stochastically by adding irrelevant components. It is found that the lifetime of system D is smaller than A in usual stochastic order. For stochastic comparison of lifetimes of bridge systems A and C, a result has been derived by imposing some conditions on the survival function of its components. This study compares bridge systems by considering different cases with the aid of survival signature. There is further scope to analyse the reliability characteristics and compare the combination of higher-order multi-state bridge systems with different types of components.

Conflict of Interest Declaration: The authors have no conflicts of interest to declare.

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Classical and Bayesian Estimation of Parameter of $SS_E(\epsilon)$-distribution Under Type-II Censored Data

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Abstract

In this present piece of work, we have considered a lifetime distribution based on trigonometric function called $SS_E(\epsilon)$-distribution and discuss its various properties which have not been added previously by host as well as any other authors. This distribution is useful and a good contribution in research under trigonometric function. We are deriving some more useful properties such as moments, conditional moments, mean deviation about mean, mean deviation about median, order statistics etc. Estimation of parameter has been done for both classical and Bayesian paradigms under Type-II censored sample. Simulation study has also been carried out to know the progress of the estimators in the sense of having smallest risk (over the sample space) at the long-run use.

Keywords: $SS_E(\epsilon)$-distribution, Type-II censoring, Bayes estimator, MLE, Gauss-Laguerre method, risk function

1. Introduction

In statistical literature, there are several lifetime distributions available, for example exponential, gamma, Weibull, Lindley distribution etc. In past studies, calculations can only be handled when the expressions corresponding to various properties obtained in the nice closed form and when this was not achieved then rarely preferred. But in this modern era due to the advancement of computational facilities this problem have been resolved almost. Mostly, algebraic and exponential functions have been used to develop the new transformation and sometimes authors see gap in trigonometric, inverse and logarithmic type transformations. Keeping this in mind, the considered transformation is the good contribution in support of filling such gap. As we aware that the use of a single model is not found suitable in every aspect, therefore to adopt a suitable baseline model is also a quite tedious job. Study explores that exponential distribution is preferably used as a lifetime distribution but the extensive use of it is restrictive in the sense of its constant hazard rate. For simplicity and flexibility, we are also using here exponential distribution as a baseline distribution In these days, many authors are introducing transformation techniques to get a new lifetime distribution with the help of available baseline distributions some of which are popular as power transformation proposed by [6], sine square distribution by [1], [20] introduced quadratic rank transmutation map (QRTM), sinofarm distribution by [23], DUS transformation proposed by [10], minimum-guarantee distribution proposed by [11], CS transformation by [3], new Sine-G family based on [13] proposed by [16], new extension of Lindley distribution given by [17], PCM transformation by [12] and many more. In such continuation, [13] have proposed a new transformation known as SS-transformation by using sine function which is given by

$$F(x) = \sin\left(\frac{\pi}{2}G(x)\right) \tag{1}$$

Where $G(x)$ is the baseline's cumulative distribution function (cdf) and the accompanying probability density function (pdf) are

$$f(x) = \frac{\pi}{2}g(x)\cos\left(\frac{\pi}{2}G(x)\right) \tag{2}$$

They have utilized baseline distribution as exponential distribution and named as SS exponential ($SS_E(\epsilon)$)-distribution and having the following form of its pdf is

$$f(x) = \frac{\pi}{2}\epsilon \times e^{-\epsilon x}\sin\left(\frac{\pi}{2}e^{-\epsilon x}\right) \quad ; \ (x,\epsilon) > 0 \tag{3}$$

and its cdf in compact form is

$$F(x) = \cos\left(\frac{\pi}{2}e^{-\epsilon x}\right) \quad ; \ (x,\epsilon) > 0 \tag{4}$$

The reported compact forms of reliability function and hazard rate function respectively are

$$R(x) = 1 - \cos\left(\frac{\pi}{2}e^{-\epsilon x}\right) \tag{5}$$

and

$$h(x) = \frac{\pi}{2}\epsilon \times e^{-\epsilon x}\cot\left(\frac{\pi}{2}e^{-\epsilon x}\right) \tag{6}$$

Figures 1, 2 and 3 presents the shape of pdf, cdf and hazard rate function of $SS_E(\epsilon)$-distribution. And Figure 3, claims that the nature of hazard rate function of the $SS_E(\epsilon)$-distribution is increasing which is different from baseline distribution.
The article is constructed as follows, introductory part have been shown in Section (1), statistical properties discussed in Section (2), estimation of parameter presented in Section (3), comparison of estimators in Section (4) and concluding remarks regarding the work quoted in Section (5).

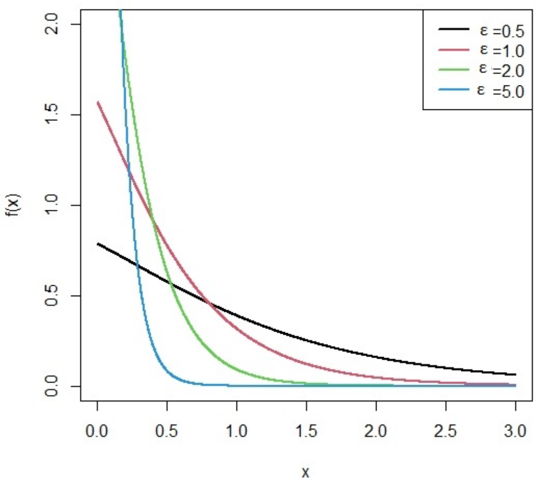

Figure 1: *Plots of pdf of $SS_E(\epsilon)$-distribution for various choices of parameter ϵ.*

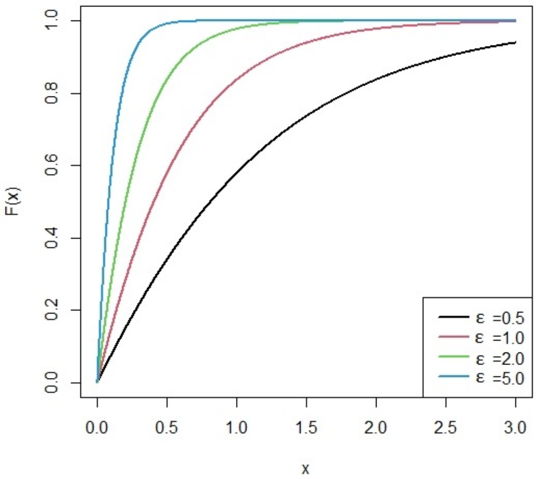

Figure 2: *Plots of cdf of $SS_E(\epsilon)$-distribution for various choices of parameter ϵ.*

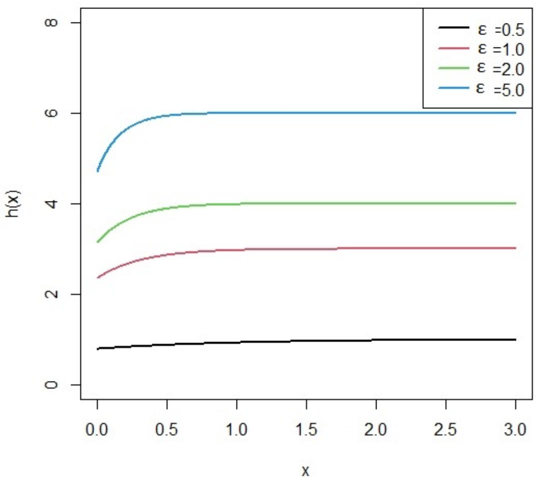

Figure 3: *Plots of hazard rate function of $SS_E(\epsilon)$-distribution for various choices of parameter ϵ.*

2. Statistical Properties

In this section, we are discussing some statistical properties of $SS_E(\epsilon)$-distribution which have not derived yet namely mean deviation about mean, mean deviation about median, order statistics etc. Firstly, we have discussed two lemma which are-

Statement (Lemma-1)

$$\xi_1(\epsilon, r, \zeta) = \int_0^\infty x^r e^{-\zeta x} \times \sin\left(\frac{\pi}{2} e^{-\epsilon x}\right) dx = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \times \left(\frac{\pi}{2}\right)^{2k+1} \left[\frac{r!}{((2k+1)\epsilon + \zeta)^{r+1}}\right]$$

Proof:

$$\begin{aligned}\xi_1(\epsilon, r, \zeta) &= \int_0^\infty x^r e^{-\zeta x} \sin\left(\frac{\pi}{2} e^{-\epsilon x}\right) dx = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \left[\int_0^\infty x^r e^{-((2k+1)\epsilon+\zeta)x} dx\right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \times \left(\frac{\pi}{2}\right)^{2k+1} \left[\frac{r!}{((2k+1)\epsilon + \zeta)^{r+1}}\right]\end{aligned}$$

The r^{th} order moment about origin of $SS_E(\epsilon)$-distribution have already obtained by [13]. Here, we obtain the same by using lemma 1 and is

$$E(X^r) = \frac{\pi}{2}\epsilon \times \xi_1(\epsilon, r, \epsilon) \tag{7}$$

on putting $r = 1, 2, 3, 4$ in (7), we get the first four raw moments of $SS_E(\epsilon)$-distribution and are

$$E(X) = \frac{\pi}{2}\epsilon \times \xi_1(\epsilon, 1, \epsilon)\ ;\ E(X^2) = \frac{\pi}{2}\epsilon \times \xi_1(\epsilon, 2, \epsilon)$$
$$E(X^3) = \frac{\pi}{2}\epsilon \times \xi_1(\epsilon, 3, \epsilon)\ ;\ E(X^4) = \frac{\pi}{2}\epsilon \times \xi_1(\epsilon, 4, \epsilon)$$

And, first four central moments are calculated by the following relations,

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2 \\ \mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \\ \mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4\end{aligned}$$

On using above relations of central moments, we can obtain the measures of skewness and kurtosis, viz., β_1, γ_1 and β_2, γ_2 respectively by the following expressions

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \implies \gamma_1 = \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}}$$
$$\beta_2 = \frac{\mu_4}{\mu_2^2} \implies \gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^2} - 3$$

Statement (Lemma-2)

$$\begin{aligned}\xi_2(\epsilon, r, \zeta, t) &= \int_t^\infty x^r e^{-\zeta x} \times \left[\sin\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right] dx \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{r} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \times \left[\frac{r! \times e^{-((2k+1)\epsilon+\zeta)t} \times (((2k+1)\epsilon + \zeta)t)^l}{l!((2k+1)\epsilon + \zeta)^{r+1}}\right]\end{aligned}$$

Proof:

$$\begin{aligned}\xi_2(\epsilon, r, \zeta, t) &= \int_t^\infty x^r e^{-\zeta x} \left[\sin\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right] dx = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \times \int_t^\infty x^r e^{-((2k+1)\epsilon+\zeta)x} dx \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{r} \frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \times \left[\frac{r! \times e^{-((2k+1)\epsilon+\zeta)t} \times [((2k+1)\epsilon + \zeta)t]^l}{l!\,[(2k+1)\epsilon + \zeta]^{r+1}}\right]\end{aligned}$$

2.1. Conditional Moments

The conditional moment of r^{th} order is represented by $E(X^r|X > r)$ then by using lemma 2, we get

$$E(X^r|X > x) = \frac{\pi}{2} \times \frac{\epsilon \times \xi_2(\epsilon, r, \epsilon, x)}{\left[1 - cos\left(\frac{\pi}{2}e^{-\epsilon x}\right)\right]} \tag{8}$$

2.2. Quantile Function

Using (4), we get the quantile function of order p $(Q(p))$ is

$$\cos\left(\frac{\pi}{2}e^{-\epsilon Q(p)}\right) = p \implies e^{-\epsilon Q(p)} = \frac{2}{\pi}\cos^{-1}(p)$$

Therefore,

$$Q(p) = -\frac{1}{\epsilon}\ln\left(\frac{2}{\pi}\cos^{-1}(p)\right) \tag{9}$$

2.3. Median

On putting $p = \frac{1}{2}$ in equation (9) we will easily get the median of $SS_E(\epsilon)$-distribution and if M_d be the median of $SS_E(\epsilon)$-distribution then the expression is

$$M_d = -\frac{1}{\epsilon}\ln\left(\frac{2}{3}\right) \tag{10}$$

which is same expression as obtained by [13].

Table 1: *Mean, median, variance, skewness and kurtosis of $SS_E(\epsilon)$-distribution for different values of ϵ.*

ϵ	Mean	Median	Variance	Skewness(γ_1)	Kurtosis(γ_2)
0.2	0.25814	2.02733	0.25695	4.13316	19.53054
0.7	0.25379	0.57924	0.16079	2.73800	6.85106
1	0.23603	0.40547	0.11153	2.53429	5.44002
2	0.17377	0.20273	0.04342	2.28171	3.84498
5	0.09108	0.08109	0.00919	2.12107	2.91886
10	0.05023	0.04055	0.00253	2.06265	2.60720
15	0.03446	0.02703	0.00118	2.04642	2.45738

Table 1 shows that the mean, median, variance, skewness (γ_1) and kurtosis (γ_2) of the $SS_E(\epsilon)$-distribution with pdf (3) for different choices of parameter ϵ. The values of mean, median and variance of $SS_E(\epsilon)$-distribution are decreases as values of parameter ϵ increases this shows that mean, median and variance of the $SS_E(\epsilon)$-distribution inversely related to parameter ϵ. Since, $\gamma_1 > 0$ and $\gamma_2 > 0$ so the nature of $SS_E(\epsilon)$-distribution has positively skewed and leptokurtic distribution for considered choices of parameter.

2.4. Mean deviation about mean and median

The mean deviation (MD) about mean is another measure of dispersion and is defined as,

$$\phi_1(x) = \int_0^\infty |x - \mu| f(x)dx$$

where μ is the mean of $SS_E(\epsilon)$-distribution, then

$$\phi_1(x) = \int_0^\mu (\mu - x)f(x)dx + \int_\mu^\infty (x - \mu)f(x)dx$$

$$\implies \phi_1(x) = 2\mu \times F(\mu) - 2\mu + 2\left[\int_\mu^\infty xf(x)dx\right]$$

where notations have their usual meanings then by using lemma 2, we get

$$\int_{\mu}^{\infty} xf(x)dx = \frac{\pi}{2}\epsilon \times \ \xi_2(\epsilon, 1, \epsilon, \mu)$$

Therefore,

$$\phi_1(x) = 2\mu \times F(\mu) - 2\mu + \pi\epsilon \times \xi_2(\epsilon, 1, \epsilon, \mu) \tag{11}$$

In the similar way, MD about median is

$$\begin{aligned}\phi_2(x) &= \int_0^{\infty} |x - M| f(x)dx = \int_0^{M} (M - x) f(x)dx + \left[\int_M^{\infty} (x - M) f(x)dx\right] \\ &= -\mu + 2\int_M^{\infty} xf(x)dx\end{aligned}$$

Now, by lemma 2, we have

$$\int_{M}^{\infty} xf(x)dx = \frac{\pi}{2}\epsilon \times \xi_2(\epsilon, 1, \epsilon, M)$$

Finally,

$$\phi_2(x) = -\mu + \pi\epsilon \times \xi_2(\epsilon, 1, \epsilon, M) \tag{12}$$

2.5. Order Statistics

Let us take random sample of size n from the $SS_E(\epsilon)$-distribution say, $\{X_1, X_2, \dots, X_n\}$ and associated order statistics is $X_{(1)} < X_{(2)} < ... < X_{(r)}$, then pdf of r^{th} order statistics is

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} \times F^{r-1}(x) \times f(x) \times [1 - F(x)]^{n-r}$$

$$\implies f_r(x) = \frac{n!}{(r-1)!(n-r)!} \times \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} F^{r+i+1}(x) \times f(x) \tag{13}$$

Now, using (3) and (4) in 13), we have

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} \times \frac{\pi}{4}\epsilon \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \times \sin\left(\pi e^{-\epsilon x}\right) \left[\cos\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right]^{r+i} \tag{14}$$

and corresponding cdf of r^{th} order statistics is

$$F_r(x) = \sum_{i=r}^{n} \binom{n}{i} F^i(x) \times [1 - F(x)]^{n-i} = \sum_{i=r}^{n}\sum_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} (-1)^j F^{i+j}(x) \tag{15}$$

Using equation (4) in (15), we obtain the expression of cdf of r^{th} order statistic of $SS_E(\epsilon)$-distribution as follows

$$F_r(x) = \sum_{i=r}^{n}\sum_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} (-1)^j \left\{\cos\left(\frac{\pi}{2} e^{-\epsilon x}\right)\right\}^{i+j} \tag{16}$$

3. Estimation of Parameter

In this section, we have discussed the estimation of parameter ϵ of $SS_E(\epsilon)$-distribution for Type-II censored data under both Classical and Bayesian paradigms. It is observed that, it is not possible to obtain the failure times of all the test units placed on a life testing experiment because of the associated costs such as cost of per unit is high or limitations on experimental time etc. Therefore, such situations are handled by removal of test units before the actual failure occurs and are termed as censoring scheme. Since, the removal of these units can be done in various possible ways, these are further known as various type of censoring scheme. The two widely used censoring schemes are Type-I and Type-II censoring schemes. Here, we consider Type-II censoring only. Let $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ be r-ordered Type-II right censored random observations obtained from n units placed on a life testing experiment where each unit has its lifetime and follows $SS_E(\epsilon)$-distribution having pdf (3) with largest $(n-r)$ lifetimes have been censored, then the likelihood function is given by [4] is

$$L_C(\epsilon|\underline{X}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_{(i)};\epsilon)\left(1-F(x_{(r)};\epsilon)\right)^{n-r} \tag{17}$$

Several authors have been done their work in this direction, [18] have been discussed Bayesian estimation of parameter under Type-II censored data, [9] worked on classical and Bayesian estimation of reliability estimation of Maxwell distribution under Type-II censored data, [22] have discussed the Bayesian estimation of exponentiated gamma parameter and reliability function under Type-II censored data for asymmetric loss function, [7] presents the statistical evidences of Type-II censored data, [19] have been derived the Bayesian estimation techniques of system reliability for Weibull distribution under Type-II censored data, [8] discussed the comparison between same Bayesian estimation methods for the parameter of exponential distribution based on Type-II censored data. [5] have been discussed the estimation procedure for new lifetime models under classical and Bayesian set-up in the presence of Type-II censored sample. [2] studied the various properties of Pareto distribution using Type-II hybrid censored sample data.

3.1. Classical Estimation

Using (17), the likelihood function of the $SS_E(\epsilon)$-distribution under Type-II censoring scheme is

$$L_C(\epsilon|\underline{X}) = \frac{n!}{(n-r)!}\epsilon^r e^{-\epsilon\sum_{i=1}^{r} x_{(i)}}\prod_{i=1}^{r}\sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r} \tag{18}$$

and taking logarithm on both sides of (18), we get

$$\ln L_C = \ln\frac{n!}{(n-r)!} + r\ln\epsilon + \epsilon\sum_{i=1}^{r} x_{(i)} + \sum_{i=1}^{r}\ln\left[\sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\right] + (n-r)\ln\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right] \tag{19}$$

On differentiating (19) w.r.to ϵ and equate the resultant to zero, we get

$$\frac{d\ln L_C}{d\epsilon} = \frac{r}{\epsilon} + \sum_{i=1}^{r} x_{(i)} - \frac{\pi}{2}\sum_{i=1}^{r} x_{(i)}e^{-\epsilon x_{(i)}}\left[\cot\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\right] + (n-r)\left[\frac{\pi}{2}\times\frac{x_{(r)}e^{-\epsilon x_{(r)}}\sin\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)}{1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)}\right] = 0 \tag{20}$$

The above equation cannot be solved analytically. So, we use numerical approximation technique through R software to solve them numerically in terms of ϵ i.e. $\hat{\epsilon}_{MC}$ which maximizes the equation (18).

3.2. Bayesian Estimation

In Bayesian paradigm, posterior probability is an effect of two components prior probability and likelihood function, and calculated from the statistical model for the observed data. The prior distribution of the parameters is assumed before the data observed. There are different kinds of prior distribution of parameters defined as proper and improper priors. Another way to define the priors based on available advanced information is known as informative and non-informative priors.

Here, we use informative prior as a Gamma(a, b) prior for ϵ of $SS_E(\epsilon)$-distribution and having the following form

$$\pi(a,b) = \frac{b^a}{\Gamma a}\epsilon^{a-1}e^{-b\epsilon} \quad ;\epsilon > 0,\ a,b > 0 \tag{21}$$

where, hyper-parameters are a and b. If two information's which are independent in nature on ϵ (say prior mean and prior variance are known) are provided, they can be obtained, for more details see [21], [14], [15]. The mean and variance of the prior distribution (21) are $\frac{a}{b}$ and $\frac{a}{b^2}$ respectively. Thus, we take $M = \frac{a}{b}$ and $V = \frac{a}{b^2}$ giving $b = \frac{M}{V}$ and $a = \frac{M^2}{V}$. The informative gamma prior behaves like non-informative prior if hyper-parameters changes i.e. if we fixed prior mean and taking large prior variance.
The posterior density of ϵ given the sample observations $\underline{X}$ is given below

$$\psi_C(\epsilon|\underline{X}) = \frac{L_C(\epsilon|\underline{X}) \times \pi(a,b)}{\int_0^\infty L_C(\epsilon|\underline{X}) \times \pi(a,b)d\epsilon} \tag{22}$$

By using equation (21) and (18) in (22), posterior density of ϵ given $\underline{X}$ under Type-II censoring is

$$\psi_C(\epsilon|\underline{X}) = \frac{\epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)}\left\{\prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}\right\}}{\int_0^\infty \epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)}\left\{\prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r}\right\}d\epsilon} \tag{23}$$

The expressions for considered loss functions namely squared error loss function (SELF) and general entropy loss function (GELF) having the following forms

$$L_S(\hat{\epsilon}_{SC}, \epsilon) = (\hat{\epsilon}_{SC} - \epsilon)^2 \tag{24}$$

$$L_G(\hat{\epsilon}_{GC}, \epsilon) = \left(\frac{\hat{\epsilon}_{GC}}{\epsilon}\right)^c - c\ln\left(\frac{\hat{\epsilon}_{GC}}{\epsilon}\right) - 1 \tag{25}$$

If $\hat{\epsilon}_{GC}$ is a Bayes estimator of ϵ for Type-II censoring under GELF then, we get

$$\hat{\epsilon}_{GC} = \left\{\frac{\int_0^\infty \epsilon^{r+a-c-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r} d\epsilon}{\int_0^\infty \epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r} d\epsilon}\right\}^{-\frac{1}{c}} \tag{26}$$

Putting $c = -1$ in equation (26), we get the Bayes estimator $\hat{\epsilon}_{SC}$ of ϵ for Type-II censoring, we get

$$\hat{\epsilon}_{SC} = \left\{\frac{\int_0^\infty \epsilon^{r+a}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r} d\epsilon}{\int_0^\infty \epsilon^{r+a-1}e^{-\epsilon(\sum_{i=1}^r x_{(i)}+b)} \times \prod_{i=1}^r \sin\left(\frac{\pi}{2}e^{-\epsilon x_{(i)}}\right)\left[1-\cos\left(\frac{\pi}{2}e^{-\epsilon x_{(r)}}\right)\right]^{n-r} d\epsilon}\right\} \tag{27}$$

The above equations (26) and (27) are not solvable analytically. Therefore, we propose some numerical approximation technique to get the solution. Basically, we have used here Gauss-Laguerre quadrature formula to obtain the solution.

4. Comparison of Estimators

In this section, we compare the performance of the considered estimators ($\hat{\epsilon}_{MC}$, $\hat{\epsilon}_{SC}$, $\hat{\epsilon}_{GC}$) of parameter ϵ of $SS_E(\epsilon)$-distribution in the presence of Type-II censoring scheme in terms of lowest risks (expected loss over Ω) under GELF. It is clear that, the expressions of risk function are not obtained in implicit form. So, we use Gauss-Laguerre quadrature formula to obtain the estimators ($\hat{\epsilon}_{SC}$ and $\hat{\epsilon}_{GC}$) of parameter ϵ for computing the risks under Type-II censored data. To know the performance of estimator in long run use, we simulate 20,000 samples for different sample size n and different effective sample size r (for Type-II censoring) from $SS_E(\epsilon)$-distribution with different choices of values of parameter ($\epsilon = 1.5,\ 2.0,\ 3.0$) and loss parameter $c = \pm 2$.

Tables 2, 3 and 4 represents the risks for the variations in hyper-parameters (variation in prior variance ($V = 0.5,\ 1.0,\ 2.0,\ 5.0,\ 80$) for fixed prior mean ($M = 1.0,\ 2.0,\ 3.0$)) when true value of the parameter $\epsilon = 2$ for sample size $n = 30$ with different censoring schemes $r = 12,\ 18,\ 24$ and 30.

Tables 5 and 6 shows the variation in n and r with minimum prior variance (high confidence level $V = 0.5$) and prior mean ($M = 2.0$) for the true value of $\epsilon = 1.5$ and 3, respectively.

Tables 2, 3 and 4 presents the simulated risks under GELF for variation in prior variance (high to low confidence level) with fixed prior mean (M). We see that the risks under GELF for the Bayes estimators of ϵ under SELF and GELF are increases as values of prior variance increases (high to low confidence level) and if prior mean increases then the risks under GELF decreases for the Bayes estimators under SELF and GELF for Type-II censored samples. Bayes estimator under GELF $\hat{\epsilon}_{GC}$ outperforms MLEs ($\hat{\epsilon}_{MC}$) and SELF ($\hat{\epsilon}_{SC}$) under Type-II censored sample when under estimation is more serious as compared to over estimation ($c = -2$) and when over estimation is more serious as compared to under estimation ($c = +2$), then Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperforms MLEs ($\hat{\epsilon}_{MC}$) and GELF ($\hat{\epsilon}_{GC}$) under Type-II censored sample. It is also noted that when prior variance is large (low confidence level i.e. very weak information about the parameter ϵ) then classical estimator MLEs ($\hat{\epsilon}_{MC}$) performs better than the Bayes estimators under SELF ($\hat{\epsilon}_{SC}$) and GELF ($\hat{\epsilon}_{GC}$).

Tables 5 and 6 shows that the variation in sample size n and corresponding different Type-II censoring schemes r for the true values of parameter $\epsilon = 1.5$ and 3. Table 5 provides simulated risks of the Bayes estimators ($\hat{\epsilon}_{SC}$) of ϵ under SELF outperforms MLEs ($\hat{\epsilon}_{MC}$) and Bayes estimators of ϵ under GELF ($\hat{\epsilon}_{GC}$) in both cases under estimation is more serious than over estimation and vice-versa. While Table 6 provides Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) outperforms the Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) and MLE ($\hat{\epsilon}_{MC}$) for the situation when under estimation is more serious than over estimation but in reverse case, Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperforms MLEs ($\hat{\epsilon}_{MC}$) and Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) for the true value of parameter $\epsilon = 3$. It is also observed that the risks of all estimators of ϵ for Type-II censored sample decreases with increase in the value of n and r for all considered values of the parameter ϵ.

Table 2: *Risks of the estimators of ϵ under GELF when prior variance varies for fixed $n = 30$, $r(r = 12, 18, 24)$, $\epsilon = 2.0$, $M = 1$ and $c = \pm 2$.*

V	scheme r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
0.5	12	0.15180	0.13386	0.11519	0.20147	0.09545	0.13667
	18	0.09915	0.09065	0.08189	0.12088	0.07236	0.07057
	24	0.07407	0.06885	0.06373	0.08397	0.05817	0.05724
	30	0.05938	0.05558	0.05214	0.06572	0.04751	0.04789
1	12	0.15180	0.13616	0.11989	0.20147	0.12083	0.13550
	18	0.09915	0.08988	0.08482	0.12088	0.08641	0.08943
	24	0.07407	0.06883	0.06587	0.08397	0.06675	0.06850
	30	0.05938	0.05605	0.05410	0.06572	0.05494	0.05612
2	12	0.15180	0.14083	0.13321	0.20147	0.15280	0.16792
	18	0.09915	0.09324	0.08994	0.12088	0.09933	0.10516
	24	0.07407	0.06992	0.06806	0.08397	0.07367	0.07693
	30	0.05938	0.05718	0.05594	0.06572	0.05952	0.06165
10	12	0.15180	0.14373	0.13906	0.20147	0.17322	0.17442
	18	0.09915	0.09650	0.09440	0.12088	0.10864	0.11646
	24	0.07407	0.07278	0.07155	0.08397	0.07953	0.08381
	30	0.05938	0.05919	0.05836	0.06572	0.06330	0.06606
80	12	0.15180	0.15085	0.14829	0.20147	0.18860	0.19630
	18	0.09915	0.09847	0.09735	0.12088	0.11827	0.12777
	24	0.07407	0.07445	0.07382	0.08397	0.08530	0.09051
	30	0.05938	0.06022	0.05979	0.06572	0.06677	0.07010

Table 3: *Risks of the estimators of ϵ under GELF when prior variance varies for fixed $n = 30$, $r(r = 12, 18, 24)$, $\epsilon = 2.0$, $M = 2$ and $c = \pm 2$.*

V	scheme r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
0.5	12	0.15180	0.05785	0.05776	0.20147	0.06075	0.06133
	18	0.09915	0.05112	0.05095	0.12088	0.05416	0.05772
	24	0.07407	0.04445	0.04421	0.08397	0.04712	0.04948
	30	0.05938	0.03881	0.03858	0.06572	0.04070	0.04237
1	12	0.15180	0.08699	0.08659	0.20147	0.09806	0.09889
	18	0.09915	0.06802	0.06764	0.12088	0.07527	0.08079
	24	0.07407	0.05566	0.05534	0.08397	0.06073	0.06412
	30	0.05938	0.04725	0.04700	0.06572	0.05079	0.05314
2	12	0.15180	0.11490	0.11311	0.20147	0.13630	0.14083
	18	0.09915	0.08291	0.08192	0.12088	0.09455	0.10148
	24	0.07407	0.06399	0.06330	0.08397	0.07033	0.07421
	30	0.05938	0.05327	0.05274	0.06572	0.05719	0.05974
5	12	0.15180	0.13415	0.13238	0.20147	0.17091	0.17531
	18	0.09915	0.09104	0.09008	0.12088	0.10704	0.11543
	24	0.07407	0.06982	0.06916	0.08397	0.07840	0.08300
	30	0.05938	0.05707	0.05665	0.06572	0.06254	0.06558
80	12	0.15180	0.15018	0.14771	0.20147	0.19933	0.19672
	18	0.09915	0.09963	0.09857	0.12088	0.11883	0.12845
	24	0.07407	0.07551	0.07482	0.08397	0.08598	0.09119
	30	0.05938	0.06064	0.06018	0.06572	0.06703	0.07033

Table 4: *Risks of the estimators of ϵ under GELF when prior variance varies for fixed $n = 30$ and $r(r = 12, 18, 24)$, $\epsilon = 2.0$, $M = 3$ and $c = \pm 2$.*

V	schemes r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
	12	0.15180	0.11285	0.12353	0.20147	0.18003	0.18618
0.5	18	0.09915	0.08463	0.09185	0.12088	0.12879	0.14174
	24	0.07407	0.06730	0.07252	0.08397	0.09833	0.10731
	30	0.05938	0.05613	0.06017	0.06572	0.07954	0.08627
	12	0.15180	0.10204	0.11257	0.20147	0.16348	0.16861
1	18	0.09915	0.07569	0.08181	0.12088	0.11215	0.12486
	24	0.07407	0.06083	0.06489	0.08397	0.08519	0.09338
	30	0.05938	0.05096	0.05389	0.06572	0.06833	0.07411
	12	0.15180	0.10845	0.11640	0.20147	0.16802	0.16768
2	18	0.09915	0.07780	0.08184	0.12088	0.10803	0.11971
	24	0.07407	0.06115	0.06362	0.08397	0.07953	0.08651
	30	0.05938	0.05096	0.05263	0.06572	0.06339	0.06810
	12	0.15180	0.12592	0.12934	0.20147	0.17881	0.17903
5	18	0.09915	0.08871	0.09023	0.12088	0.11345	0.12417
	24	0.07407	0.06864	0.06947	0.08397	0.08334	0.08942
	30	0.05938	0.05580	0.05640	0.06572	0.06568	0.06971
	12	0.15180	0.14540	0.14786	0.20147	0.18968	0.19429
80	18	0.09915	0.09905	0.09951	0.12088	0.12069	0.13089
	24	0.07407	0.07306	0.07313	0.08397	0.08471	0.09015
	30	0.05938	0.05921	0.05997	0.06572	0.06597	0.06942

Table 5: *Risks of the estimators of ϵ under GELF when $V = 0.5$, $M = 2$ and $c = \pm 2$ for true value of $\epsilon = 1.5$.*

V	scheme r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
	6	0.30733	0.11180	0.12843	0.58496	0.17749	0.19775
15	9	0.20172	0.09625	0.10710	0.30346	0.14726	0.17215
	12	0.14916	0.08295	0.09059	0.20110	0.12254	0.14004
	15	0.12036	0.07335	0.07906	0.15104	0.10453	0.11761
	8	0.22597	0.09958	0.11208	0.35915	0.15420	0.15351
20	12	0.14976	0.08279	0.09040	0.20231	0.12191	0.13939
	16	0.11135	0.06989	0.07502	0.13774	0.09803	0.10978
	20	0.09013	0.06117	0.06495	0.10558	0.08244	0.09104
	12	0.15112	0.08386	0.09189	0.20662	0.12435	0.12456
30	18	0.09839	0.06540	0.06998	0.11985	0.09049	0.10073
	24	0.07255	0.05290	0.05584	0.08325	0.06937	0.07586
	30	0.05884	0.04529	0.04738	0.06545	0.05727	0.06186
	24	0.07548	0.05425	0.05723	0.08712	0.07144	0.07669
60	36	0.04909	0.03924	0.04075	0.05379	0.04825	0.05158
	48	0.03676	0.03096	0.03186	0.03932	0.03655	0.03855
	60	0.02966	0.02581	0.02643	0.03120	0.02961	0.03098

Table 6: *Risks of the estimators of ϵ under GELF when $V = 0.5$, $M = 2$ and $c = \pm 2$ for true value of $\epsilon = 3.0$.*

V	scheme r	c =-2			c=2		
		MLE	SELF	GELF	MLE	SELF	GELF
15	6	0.30882	0.18838	0.14822	0.57630	0.11627	0.20923
	9	0.20002	0.14235	0.11626	0.29734	0.09092	0.07627
	12	0.14878	0.11418	0.09590	0.19835	0.07558	0.06526
	15	0.12139	0.09769	0.08385	0.14948	0.06649	0.05869
20	8	0.22674	0.15471	0.12462	0.35157	0.09761	0.16869
	12	0.14766	0.11417	0.09571	0.19338	0.07540	0.06492
	16	0.11053	0.09059	0.07813	0.13388	0.06245	0.05537
	20	0.08972	0.07659	0.06737	0.10351	0.05458	0.04936
30	12	0.15287	0.11776	0.09863	0.20241	0.07752	0.12301
	18	0.09955	0.08390	0.07307	0.11933	0.05891	0.05277
	24	0.07382	0.06505	0.05808	0.08374	0.04801	0.04407
	30	0.05927	0.05340	0.04848	0.06563	0.04099	0.03822
60	24	0.07358	0.06458	0.05742	0.08556	0.04785	0.06514
	36	0.04797	0.04366	0.04011	0.05294	0.03478	0.03278
	48	0.03608	0.03355	0.03141	0.03875	0.02795	0.02676
	60	0.02952	0.02785	0.02637	0.03116	0.02385	0.02303

5. Conclusion

In this paper, we have been consider a lifetime distribution by using sine function which has proposed by [13]. We have also discussed some statistical properties of the considered distribution such as conditional moments, mean deviation about mean, mean deviation about median and derived expressions of the pdf and cdf of r^{th} order statistics. Mean, median and variance are inversely related to the parameter ϵ of $SS_E(\epsilon)$-distribution and the distribution has positively skewed and leptokurtic nature. We have developed classical and Bayesian estimation procedure for estimation of parameter ϵ under Type-II censored data. And also check the workout of the estimators at the long-run by performing simulation study. The Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperforms MLE ($\hat{\epsilon}_{MC}$) and Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) for the true value of parameter $\epsilon = 1.5$ whatever the seriousness i.e. over estimation is more serious than under estimation and reversely. In all other considered cases, Bayes estimator under GELF ($\hat{\epsilon}_{GC}$) outperforms MLE ($\hat{\epsilon}_{MC}$) and Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) when under estimation is more serious than over estimation but in reverse case Bayes estimator under SELF ($\hat{\epsilon}_{SC}$) outperform MLE ($\hat{\epsilon}_{MC}$) and Bayes estimator under GELF ($\hat{\epsilon}_{GC}$). Finally, we see that risks under GELF decreases as sample informations (n & r) increases.

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Statistical properties and estimation procedures for a new flexible two parameter lifetime distribution

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Abstract

In this article, a new transformation technique based on the cumulative distribution function is proposed, the proposed transformation technique is very useful to generate a class of lifetime distribution. The various statistical properties of the proposed transformation method are studied. Further, the proposed technique is illustrated by considering exponential distribution as a baseline distribution. Various statistical properties such as survival and hazard rate, moments, mean deviation about mean and median, order statistics, moment generating function (MGF), Bonferroni's, and Lorenz curves, entropy, stress-strength reliability have been discussed. Different classical estimation methods are used to estimate the unknown parameters. Finally, two real data sets are considered to justify the use of the proposed distribution in real scenario.

Keywords: Transformation technique, statistical properties, classical method of estimation, and application.

1. Introduction

In lifetime analysis, various transformation techniques are used to propose the new probability distribution by adding an additional parameter to the baseline probability distribution. The significance of these probability distributions is categorized according to their use and appropriateness of different hazard rates viz. increasing, decreasing, constant, bathtub, and upside-down bathtub (UBT). Modeling of the real-life data set is based on the nature of the hazard rate function. For example, the exponential distribution is the most suitable choice whenever data exhibits the pattern of constant hazard rate. However, underlying data exhibits a non-constant hazard rate then other generalized lifetime distributions such as Weibull, Gamma, Extended Exponential, Generalized Exponential, Lindley distributions, and many others are frequently used to desirable data. To know more about monotone and non-monotone hazard rates, see [17], [32], [4], [5], [14], and [8], etc.

In statistical literature, various method has been suggested by the several authors to generate a new flexible model, viz. [26], [24], [19], and [30]. The beta generated model is used by [15] who uses the beta distribution to develop the beta generated distributions. [11] propose the Kumarswamy-G family of distributions.[12] propose a new class of distribution by adding two more parameters. [2] introduce a new method for generating families of distributions called the T-X family. Recently, the quantile function is used to generate the T-X family of distributions by [1]. For another development in the family of distributions see, [18], [23], [20], and [25], etc. These methods are most popular to propose flexible and appropriate models. Here notable thing is that all the methods of transformation discussed above introduce one additional parameter. Unquestionably, the addition of an extra parameter increases the flexibility but at the same time, it also increases complexity in further statistical inference.

Motivated by the above-mentioned literature, this article aims to propose a new transformation technique to generate the class of distributions. The proposed transformation is illustrated with an exponential baseline model and named a new two-parameter lifetime model. The proposed model through this transformation will be considered as an alternative to the Gamma, Weibull, and Extension of Exponential distributions by using the transformation method.

Let G(x) is the cumulative distribution function (CDF) of any baseline distribution, then the CDF of new distribution is proposed by,

$$F(x) = \frac{G(x)}{G(x) + (1 + G(x))^{\alpha}} \qquad for \;\; x \in \Re \quad and\; \alpha \geq 0 \tag{1}$$

Clearly, $F(x)$ is the distribution function as it satisfy the condition to be a CDF.

$(i) \quad \lim_{x \to -\infty} F(x) = 0 \qquad (ii) \quad \lim_{x \to \infty} F(x) = 1 \qquad (iii) \quad F'(x) = f(x)$

where $f(x)$ and $g(x)$ are the probability distribution function (PDF) of proposed and baseline distribution function respectively.

(iv) It is well known that $0 \leq G(x) \leq 1$, which implies that $0 \leq F(x) \leq 1$.
(v) Clearly, $F(x)$ is a continuous function.

Now, the associated probability distribution function (PDF) $f(x)$ for (1) is,

$$f(x) = \frac{g(x)\{(1 - G(x) + \alpha G(x))\}\left\{(1 - G(x))^{(\alpha-1)}\right\}}{\left\{G(x) + (1 - G(x))^{\alpha}\right\}^{2}} \tag{2}$$

The survival and hazard rates are:

$$S(x) = \frac{(1-G(x))^{\alpha}}{G(x) + (1-G(x))^{\alpha}} \tag{3}$$

$$h(x) = \frac{g(x)\{1-G(x) + \alpha G(x)\}}{(1-G(x))G(x) + (1-G(x))^{\alpha+1}} \tag{4}$$

To illustrate the above transformation, let us assume exponential distribution as the base line distribution. The PDF of the exponential distribution is given by,

$$g(x) = \begin{cases} \lambda e^{-\lambda x} & for \;\; x \geq 0,\, \lambda > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and the associated CDF is:

$$G(x) = 1 - e^{-\lambda x} \qquad\qquad for\; x \geq 0,\, \lambda > 0 \tag{6}$$

here, λ is rate parameter of exponential distribution.

Then by transformation (1), the CDF of the new flexible distribution is,

$$F(x) = \frac{1-e^{-\lambda x}}{1-e^{-\lambda x} + e^{-\lambda \alpha x}} \qquad for \quad x \geq 0, \alpha \geq 0, \lambda > 0 \tag{7}$$

The PDF, survival and hazard rate function of proposed distribution are given as:

$$f(x) = \frac{\lambda e^{-\lambda\alpha x}\left(\alpha + e^{-\lambda x} - \alpha e^{-\lambda x}\right)}{\left(1 - e^{-\lambda x} + e^{-\lambda\alpha x}\right)^2} \qquad for \quad x \geq 0, \alpha \geq 0, \lambda > 0 \tag{8}$$

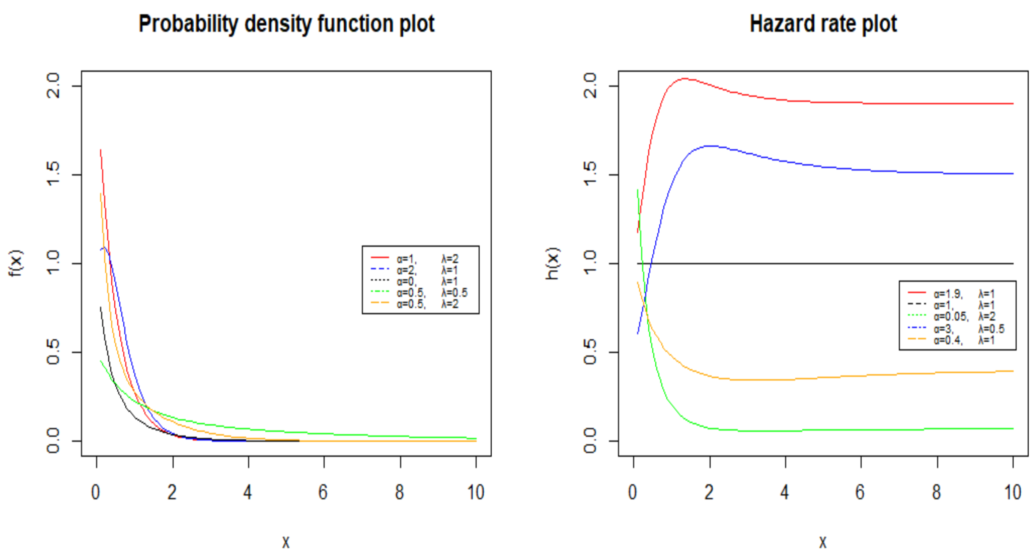
$$S(x) = \frac{e^{-\lambda\alpha x}}{\left(1 - e^{-\lambda x}\right) + e^{-\lambda\alpha x}} \tag{9}$$

and

$$h(x) = \frac{\lambda\left[\alpha + e^{-\lambda x} - \alpha e^{-\lambda x}\right]}{\left[1 - e^{-\lambda x} + e^{-\lambda\alpha x}\right]} \tag{10}$$

respectively.

The principal objective of this paper is to propose a new transformation method and derive its various statistical properties. Specifically, we substitute (6) into (1) to get the CDF of the proposed distribution and the corresponding PDF is obtained by substituting (5) and (6) into (2). Our motivation to construct a new model is: (i) it is applicable for modeling increasing, decreasing, and constant hazard shape which provides a good fit for real data sets; (ii) In our proposed model if we put $\alpha = 1$ then our proposed model reduced to baseline model; (iii) The proposed model can be considered as a good alternative model for fitting the positive data with a longer tail and (iv) The proposed model provides a better fit than some well-known lifetime models to real data sets. As, the proposed model is an alternative to Weibull, Gamma, and Extended Exponential, thus the proposed model might be a good choice for the researcher. Also, we have considered different methods of estimation to estimate the unknown value of the parameter. To check the applicability, AIC and BIC's are also constructed for the parameters of the proposed model. A simulation study has been performed to appraise the performance of the proposed estimation methods. Further, we have considered two real data set to illustrate the superiority of the proposed model and study. The plots of pdf and hazard rate function of new flexible two-parameter lifetime distribution for various values of α and λ are shown in Figure 1(a) and 1(b) respectively. From the Figure 1(b), the proposed distribution has an increasing, decreasing, and constant hazard rate.



(a) (b)

Figure 1: *(a) PDF of proposed distribution. (b) Hazard function of proposed distribution.*

The content of the rest of the paper is organized as follows: In section 2, we have discussed some statistical properties such as raw moments, moment generating function (MGF), mean deviations, Bonferroni and Lorenz curves, Rényi entropy, s- entropy, cumulative residual entropy, order statistics and reliability. In section 3, different method of estimation of the proposed distribution is studied. Simulation studies are carried out, in section 4, to compare the behaviour and performance of the different estimators. In section 5, the proposed model is fitted with some competing models using two real data sets and finally, the conclusions are summarised in section 6.

2. Statistical properties of new flexible two parameter lifetime distribution

In this section, we have discussed various statistical properties of our proposed new two parameter lifetime distribution like moments, moment generating function (MGF), mean deviation about mean and median, order statistics, reliability, Renyi entropy and Shannon entropy.

2.1. Raw Moments:

The r^{th} moment about origin of the distribution with PDF (8) is obtained by,

$$\mu_r'=E\left(X^r\right)=\int_0^\infty x^r f(x)dx=\int_0^\infty x^r\frac{\lambda e^{-\lambda\alpha x}\left(\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right)}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)^2}dx$$

After simplification the above integral, we get,

$$\mu_r'=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha\Gamma\left(r+1\right)}{\lambda^r(\alpha+i-j+\alpha j)^{r+1}}+\frac{(1-\alpha)\,\Gamma\left(r+1\right)}{\lambda^r(1+\alpha+i-j+\alpha j)^{r+1}}\right] \tag{11}$$

The respective four moments about origin can be obtained by putting $r=1,2,3,$ and 4. For $r=1$ we get mean (μ) of the distribution and is given by the following expression,

$$\mu=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha}{\lambda(\alpha+i-j+\alpha j)^2}+\frac{(1-\alpha)}{\lambda(1+\alpha+i-j+\alpha j)^2}\right]$$

For, $r=2,3,$ and 4 we can compute μ_2', μ_3', and μ_4' by putting these values in equation number (11). The variance of the proposed model can be obtained using the expression,

$$V\left(X\right)=E\left(X^2\right)-\left(E\left(X\right)\right)^2$$

Similarly, we can find other moment based, skewness, and the kurtosis of the distribution.

2.2. Moment generating function (MGF):

The moment generating function (MGF) for the proposed distribution with PDF (8) is given by;

$$M_X\left(t\right)=\int_0^\infty\frac{e^{tx}\lambda e^{-\lambda\alpha x}\left[\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right]}{\left[1-e^{-\lambda x}+e^{-\lambda\alpha x}\right]^2}dx$$

After simplification,

$$=\alpha\sum_{r=0}^{\infty}\sum_{J=0}^{\infty}\sum_{i=0}^{J}(-1)^{i}\binom{J}{i}(J+1)\frac{(t-\alpha\lambda)^{r}}{r!}\frac{\Gamma(r+1)}{\lambda^{r}(J-i+i\alpha)^{r+1}}+$$

$$\lambda(1-\alpha)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k}\,(n+1)\binom{n}{k}\frac{(t-\lambda-\alpha\lambda)^{m}}{m!}\frac{\Gamma(m+1)}{((n-k+k\alpha)\lambda)^{m+1}} \quad (12)$$

2.3. Mean Deviation (MD)

The mean deviation is the mean of the deviations. It can be calculated from the mean, median, and mode. It shows how far all the observations from the middle, on average are. The mean deviation about mean and mean deviation about median is defined as,

$$\delta_1(x)=\int_0^{\infty}|\mu-x|\,f(x)\,dx$$

and,

$$\delta_2(x)=\int_0^{\infty}|x-M|f(x)dx$$

Respectively, here $\mu = E(X)$ is the mean and $M = median(X)$ is the median of the distribution. After simplification the mean deviation about mean and the mean deviation about median are given as;

$$\delta_1(x)=2\mu F(\mu)-2\mu+2\int_{\mu}^{\infty}xf(x)\,dx \quad (13)$$

Now, the integral is computed as,

$$\int_{\mu}^{\infty}xf(x)\,dx=\int_{\mu}^{\infty}x\frac{\lambda e^{-\lambda\alpha x}\left[\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right]}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)^{2}}dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}(i+1)\times$$

$$\left[\lambda\alpha\int_{\mu}^{\infty}xe^{-(\alpha+i-j+\alpha j)\lambda x}dx+\lambda(1-\alpha)\int_{\mu}^{\infty}xe^{-(1+\alpha+i-j+\alpha j)\lambda x}dx\right]$$

By the definition of complementary incomplete gamma function and for any integer n,

$$\Gamma(n,x)=\textstyle\int_x^{\infty}t^{n-1}e^{-t}dt$$
$$\Gamma(n,ax)=\textstyle\int_x^{\infty}t^{n-1}e^{-at}dt$$

$$\Gamma(n,x)=\int_x^{\infty}t^{n-1}e^{-t}dt=(n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}=(n-1)!e^{-x}e_{n-1}(x)$$

here, $e_n(x)$ is the exponential sum function. The above expression can be expressed as;

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{j}\binom{i}{j}(i+1)\times$$

$$\left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda\mu}}{(\alpha+i-j+\alpha j)^{2}\lambda}(1+(\alpha+i-j+\alpha j)\lambda\mu)\right.$$
$$\left.+\frac{(1-\alpha))e^{-(1+\alpha+i-j+\alpha j)\lambda\mu}}{(1+\alpha+i-j+\alpha j)^{2}\lambda}(1+(\alpha+i-j+\alpha j)\lambda\mu)\right] \quad (14)$$

By using equation (13) and (14), the mean deviation $\delta_1(x)$ about mean is;

$$\delta_1(x) = 2\mu\, F(\mu) - 2\mu + 2\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda\mu}}{(\alpha+i-j+\alpha j)^2\lambda}\left(1+(\alpha+i-j+\alpha j)\lambda\mu\right)\right.$$
$$\left.+\frac{(1-\alpha)\,e^{-(1+\alpha+i-j+\alpha j)\lambda\mu}}{(1+\alpha+i-j+\alpha j)^2\lambda}\left(1+(1+\alpha+i-j+\alpha j)\lambda\mu\right)\right] \tag{15}$$

Similarly, the mean deviation $\delta_2(x)$ about median is given by,

$$\delta_2(x) = -\mu + 2\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda\mu}}{(\alpha+i-j+\alpha j)^2\lambda}\left(1+(\alpha+i-j+\alpha j)\lambda\mu\right)\right.$$
$$\left.+\frac{(1-\alpha)\,e^{-(1+\alpha+i-j+\alpha j)\lambda\mu}}{(1+\alpha+i-j+\alpha j)^2\lambda}\left(1+(1+\alpha+i-j+\alpha j)\lambda\mu\right)\right] \tag{16}$$

2.4. Bonferroni and Lorenz curves

The Bonferroni [7] and Lorenz curves [21] is used to measure the inequality in the distribution of quantity in the area of economics as in term of income and wealth. The Bonferroni and Lorenz curves have various applications not only in the area of economics to study income and poverty but also in other areas like demography, medicine, insurance and reliability. Lorenz curves cannot be defined if the mean of the distribution is zero or infinite. The Bonferroni and Lorenz curves are given by,

$$B(P) = \frac{1}{P\mu}\int_0^q xf(x)dx \tag{17}$$

and

$$L(x) = \frac{1}{\mu}\int_0^q xf(x)dx \tag{18}$$

where $\mu = E(x)$ and $q = F^{-1}(p)$ respectively. Now, the integral quantity in RHS is simplified as;

$$\int_q^{\infty} xf(x)\,dx = \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\times$$
$$\left[\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda q}}{(\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q)\right.$$
$$\left.+\frac{(1-\alpha))e^{-(1+\alpha+i-j+\alpha j)\lambda q}}{(1+\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q)\right]$$

Hence, the Bonferroni and Lorenz curves for the new distribution are obtained as;

$$B(P) = \frac{1}{P} - \frac{1}{P\mu}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left\{\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda q}}{(\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q)\right.\right.$$
$$\left.\left.+\frac{(1-\alpha)e^{-(1+\alpha+i-j+\alpha j)\lambda q}}{(1+\alpha+i-j+\alpha j)^2\lambda}(1+(1+\alpha+i-j+\alpha j)\lambda q)\right\}\right] \tag{19}$$

and

$$L(p) = 1 - \frac{1}{\mu}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left\{\frac{\alpha e^{-(\alpha+i-j+\alpha j)\lambda q}}{(\alpha+i-j+\alpha j)^2\lambda}(1+(\alpha+i-j+\alpha j)\lambda q)\right.\right.$$

$$+\frac{(1-\alpha)e^{-(1+\alpha+i-j+\alpha j)\lambda q}}{(1+\alpha+i-j+\alpha j)^2\lambda}(1+(1+\alpha+i-j+\alpha j)\lambda q)\Big\}\Bigg] \tag{20}$$

respectively.

2.5. Renyi entropy

Rényi entropy [28] is a most popular measure of average amount of uncertainty of a random variable X. If X is a random variable with probability distribution function $f(x)$ then the Rényi entropy is defined as

$$\mathcal{J}_R(\gamma)=\frac{1}{1-\gamma}\,log\left\{\int f^{\gamma}(x)\,dx\right\} \tag{21}$$

where, $\gamma>0$ and $\gamma\neq 1$. Now, from equation (8) we get,

$$\int_0^{\infty} f^{\gamma}(x)\,dx=\int_0^{\infty}\left\{\frac{\lambda e^{-\lambda\alpha x}\left[\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right]}{(1-e^{-\lambda x}+e^{-\lambda\alpha x})^2}\right\}^{\gamma}dx$$

$$=\lambda^{\gamma}\alpha^{\gamma+l-j}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{j+k+l}\binom{-2\gamma}{i}\binom{\gamma}{J}\binom{\mathrm{i}}{k}\binom{j}{l}\frac{1}{(\gamma\alpha+i-k+\alpha k+j)\lambda}$$

By putting the above value in the equation (21) we get,

$$\mathcal{J}_R(\gamma)=\frac{\gamma}{1-\gamma}log\lambda+\left(\frac{\gamma+l-j}{1-\gamma}\right)log\alpha$$
$$+\frac{1}{1-\gamma}log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{j+k+l}\binom{-2\gamma}{i}\binom{\gamma}{J}\binom{\mathrm{i}}{k}\binom{j}{l}\frac{1}{(\gamma\alpha+i-k+\alpha k+j)\lambda}\right] \tag{22}$$

2.6. s-Entropy

Shannon entropy was proposed by [29], and is a particular case of Rényi entropy as $\gamma\to 1$. It can be defined as $E\left[-logf(X)\right]$.

$$logf(X)=log\lambda-\alpha\lambda x+log\alpha+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\alpha^n n}\left(e^{-\lambda x}-\alpha e^{-\lambda x}\right)^n+2\sum_{m=1}^{\infty}\frac{\left(e^{-\lambda x}-e^{-\lambda\alpha x}\right)^m}{m}$$

After using the result based on series expansion of $log(1+x)$ and $log(1-x)$ in the above expression, the expression for Shannon entropy is given by,

$$E\left[-logf(X)\right]=-log\lambda+\lambda\alpha E\left(X\right)-log\alpha-\sum_{n=1}^{\infty}\sum_{l=0}^{n}\frac{(-1)^{n+l+1}}{\alpha^n n}\binom{n}{l}E\left(e^{-\lambda nx}\right)$$
$$-2\sum_{m=1}^{\infty}\sum_{s=0}^{m}\frac{(-1)^s}{m}\binom{m}{s}E\left(e^{-\lambda mx+\lambda sx-\lambda\alpha sx}\right) \tag{23}$$

The equation number (23) can be computed with the help of following results;

$$E\left(e^{-\lambda nx}\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(i+1)\left[\frac{\lambda\alpha}{(n+\alpha+i-j+\alpha j)\lambda}+\frac{\lambda(1-\alpha)}{(n+\alpha+1+i-j+\alpha j)\lambda}\right] \tag{24}$$

and

$$E\left(e^{-\lambda mx+\lambda sx-\lambda\alpha sx}\right)=\sum_{k=0}^{\infty}\sum_{t=0}^{k}(-1)^t\binom{k}{t}(k+1)\times \left[\frac{\alpha}{(m-s+\alpha s+k+\alpha-t+\alpha t)}+\frac{1-\alpha}{(m-s+\alpha s+\alpha+1+k-t+\alpha t)}\right] \tag{25}$$

respectively.
Corr: 1. The cumulative residual entropy [27] defined as,

$$\mathcal{J}_C=-\int \Pr(X>x)\log\left(\Pr(X>x)\right)dx$$

$$=-\int_0^{\infty}\frac{e^{-\lambda\alpha x}}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)}\left(-\lambda\alpha x-log\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)\right)dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\binom{i}{j}\left[\frac{\lambda\alpha}{\left((\alpha+i-j+\alpha j)\lambda\right)^2}-\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(-1)^l}{k}\binom{k}{l}\frac{1}{(\alpha+i-j+\alpha j+k-l+\alpha l)}\right]$$

2.7. Order Statistics

Suppose that $X_1, X_2, \ldots\ldots X_n$ is a random sample of size n from the proposed continuous probability distribution function (PDF), $f(x)$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. We know that the probability density function of r^{th} order statistics $X_{r:n}$, say $f_r(x)$, where the population PDF and CDF are $f(X)$ and $F(x)$ respectively, is given as follows,

$$f_r(x)=\frac{n!}{(r-1)!\,(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}F^{r+i-1}(x)\,f(x) \tag{26}$$

Consequently, using the equation (7) and (8) in (26) we get,

$$f_r(x)=\frac{n!}{(r-1)!\,(n-r)!}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\left[\frac{1-e^{-\lambda x}}{1-e^{-\lambda x}+e^{-\lambda\alpha x}}\right]^{r+i-1} \times\left[\frac{\lambda e^{-\lambda\alpha x}\left(\alpha+e^{-\lambda x}-\alpha e^{-\lambda x}\right)}{\left(1-e^{-\lambda x}+e^{-\lambda\alpha x}\right)^2}\right] \tag{27}$$

and corresponding r^{th} order statistics of CDF $F_r(x)$ is,

$$F_r(x)=\sum_{j=r}^{n}\sum_{m=0}^{n-j}\binom{n}{j}\binom{n-j}{m}(-1)^m F^{j+m}(x) \tag{28}$$

Hence from the equation (7), equation (28) can be written as,

$$F_r(x)=\sum_{j=r}^{n}\sum_{m=0}^{n-j}(-1)^m\binom{n}{j}\binom{n-j}{m}\left(\frac{1-e^{-\lambda x}}{1-e^{-\lambda x}+e^{-\lambda\alpha x}}\right)^{j-m} \tag{29}$$

2.8. Reliability

In this section, we have discussed about reliability of a component. In the context of reliability, the stress-strength model explains the life of a component which has a random strength X that is subjected to random stress Y. The component will fail if the stress applied to it exceeds the strength and the component will work properly whenever $X > Y$. So, $P[X > Y]$ is a measure of component reliability. It has various number of applications in many areas such as in science, engineering etc. In the field of stress-strength model there has been number of works as regarded

estimation of reliability R when X and Y are independent random variable belonging to the same univariate family of distribution.
Here, we derive the reliability R when X and Y are independent random variables from the proposed distribution with parameter (α_1, λ_1) and (α_2, λ_2) respectively. So, the reliability R is defined as bellow,

$$
\begin{aligned}
R=P\left(X>Y\right)&=\int_0^\infty f_X\left(x,\alpha_1,\lambda_1\right)\ F_Y(x,\alpha_2,\lambda_2)dx\\
&=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{i}\sum_{l=0}^{k}(-1)^{j+1}\binom{i}{j}\binom{k}{l}(i+1)\left[\lambda_1\alpha_1\left\{\frac{1}{(\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l)\,\lambda_2}\right.\right.\\
&\left.-\frac{1}{(\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l+1)\,\lambda_2}\right\}+\lambda_1\left(1-\alpha_1\right)\\
&\times\left\{\frac{1}{(1+\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l)\,\lambda_2}\right.\\
&\left.\left.-\frac{1}{(1+\alpha_1+i-j+\alpha_1 j)\,\lambda_1+(k-l+\alpha_2 l+1)\,\lambda_2}\right\}\right]
\end{aligned}
\tag{30}
$$

3. Methods of estimation

In this section, we will discuss different method of estimation namely, maximum likelihood estimation (MLE), maximum product spacing estimation (MPS), least square estimation (LSE) and weighted least square estimation (WLSE), Cramer-von-Mises estimation (CVME), Anderson Darling estimation (ADE) to estimate the unknown parameter of the considered model.

3.1. Maximum likelihood estimation (MLE)

Let $x_1, x_2, \ldots\ldots\ldots x_n$ be the random samples of size n from the proposed distribution. Then the log-likelihood function of the proposed distribution is given as,

$$
logL\ = \mathrm{n}\log\lambda - \ \alpha\lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}log\left(\alpha+e^{-\lambda x_i}-\alpha e^{-\lambda x_i}\right)-2\sum_{i=1}^{n}log\left(1-e^{-\lambda x_i}+e^{-\lambda\alpha x_i}\right)
\tag{31}
$$

Now, differentiating equation (31) with respect to parameters α and λ we get,

$$
\frac{\partial logL}{\partial\alpha}=-\ \lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{1-e^{-\lambda x_i}}{\left(\alpha+e^{-\lambda x_i}-\alpha e^{-\lambda x_i}\right)}+2\sum_{i=1}^{n}\frac{\lambda x_i e^{-\lambda\alpha x_i}}{\left(1-e^{-\lambda x_i}+e^{-\lambda\alpha x_i}\right)}
\tag{32}
$$

and

$$
\frac{\partial logL}{\partial\lambda}=\frac{n}{\lambda}-\alpha\sum_{i=1}^{n}x_i+\sum_{i}^{n}\frac{(\alpha-1)\,x_i e^{-\lambda x_i}}{\left(\alpha+e^{-\lambda x_i}-\alpha e^{-\lambda x_i}\right)}-2\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}-\alpha x_i e^{-\lambda\alpha x_i}}{\left(1-e^{-\lambda x_i}+e^{-\lambda\alpha x_i}\right)}
\tag{33}
$$

Now, putting the equation (32) and (33) equal to zero, we have two non-liner likelihood equations. After solving these equations, we get MLEs $\widehat{\alpha}$ and $\widehat{\lambda}$ of parameters α and λ. These equations are not in closed form consequently it cannot be solved analytically. Therefore, Newton-Raphson method is used to get the MLE's of the parameters α and λ.

3.2. Maximum Product Spacing Estimation (MPSE)

This method is one of the most popular method of estimation and it was developed by [9]. Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be the ordered sample of size n and we define spacing as,

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x,\alpha,\lambda)dx \qquad ; i = 1,2,3\ldots\ldots\ldots(n+1)$$

$$= F\left(x_{(i)},\alpha,\lambda\right) - F\left(x_{(i-1)},\alpha,\lambda\right)$$

Where, initial conditions are $F\left(x_{(0)}\alpha,\lambda\right) = 0$, $F\left(x_{(n+1)}\alpha,\lambda\right)$ and sum of all the spacing will be zero.
We are taking the observation from the proposed distribution, now from the equation (7) the $D_i's$ are defined as,

$$D_i = \frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}} - \frac{1-e^{-\lambda x_{(i-1)}}}{1-e^{-\lambda x_{(i-1)}}+e^{-\lambda\alpha x_{(i-1)}}} \qquad ; for\ all\ i = 1,2,\ldots.n \tag{34}$$

For, i = $2,3,\ldots.$n. The MPS estimator $\widehat{\alpha}_{mps}$ and $\widehat{\lambda}_{mps}$ of α and λ are obtained by maximising the geometric mean of the differences,

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{1/(n+1)}$$

after taking logarithm of G we get,

$$logG = \left(\frac{1}{n+1}\right)\sum_{i=1}^{n+1} logD_i \tag{35}$$

substituting the value of D_i from the equation (34) in equation (35) we get,

$$logG = \left(\frac{1}{n+1}\right)\sum_{i=1}^{n+1} log\left[\frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}} - \frac{1-e^{-\lambda x_{(i-1)}}}{1-e^{-\lambda x_{(i-1)}}+e^{-\lambda\alpha x_{(i-1)}}}\right] \tag{36}$$

It may be noted that from the equation (36) we can get the derivatives $\frac{\partial logL}{\partial\alpha}$, $\frac{\partial logL}{\partial\lambda}$ and set it equal to zero, the equation, thus obtained, cannot solved analytically, therefore, the same numerical technique may be used to obtain the solution.

3.3. Least Squares Estimation (LSE)

This method is most popular method [31]. Let $x_{(1)} < x_{(2)} \ldots \cdots < x_{(n)}$ be ordered sample of size n from proposed distribution. LSEs $\widehat{\alpha}_{ls}$ and $\widehat{\lambda}_{ls}$ of parameters α and λ are obtained by minimizing

$$Z(\alpha,\lambda) = \sum_{i=1}^{n}\left(F\left(x_{(i)},\alpha,\lambda\right) - E\left[F\left(x_i\right)\right]\right)^2$$

where, $E\left[F\left(x_i\right)\right] = \frac{i}{n+1} \quad ; i = 1,2,\ldots\ldots(n+1)$
then,

$$Z(\alpha,\lambda) = \sum_{i=1}^{n}\left(\frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}} - \frac{i}{n+1}\right)^2 \tag{37}$$

In order to minimize $Z(\alpha,\lambda)$given in (37), we differentiate equation (37) with respect to α and λ

and equating to zero, which results of the following equations,

$$\frac{\partial Z(\alpha,\lambda)}{\partial(\alpha)} = \sum_{i=1}^{n} F'_{\alpha}\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right) - \frac{i}{n+1}\right) = 0 \tag{38}$$

$$\frac{\partial Z(\alpha,\lambda)}{\partial(\lambda)} = \sum_{i=1}^{n} F'_{\lambda}\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right) - \frac{i}{n+1}\right) = 0 \tag{39}$$

The above two non-linear equations cannot be solved analytically, therefore, numerical technique is used for solution.

3.4. Weighted Least Squares Estimation (WLSE)

The estimation procedure to obtain the estimates of the parameters through WLSE is quite similar to the LSE with a slight change that it minimizes the weighted sum of squared deviation between true and expected CDF at observed ordered sample points, where weights are inversely proportional to the $var\left[F\left(x_{(i)}\right)\right]$. Thus, WLSE is obtained by minimizing

$$W(\alpha,\lambda) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i)},\alpha,\lambda\right) - \frac{i}{n+1}\right]^2$$

$$W(\alpha,\lambda) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{1-e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}+e^{-\lambda\alpha x_{(i)}}} - \frac{i}{n+1}\right]^2 \tag{40}$$

To get the WLSE estimates $\widehat{\alpha}_{WLS}$ and $\widehat{\lambda}_{WLS}$ of parameters α and λ, differentiate equation (40) with respect to α and λ and equating to zero, which results of the following equations

$$\frac{\partial W(\alpha,\lambda)}{\partial\alpha} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} F'_{\alpha}\left(x_{(i)},\alpha,\lambda\right)\left[F\left(x_{(i)},\alpha,\lambda\right) - \frac{i}{n+1}\right]^2 = 0 \tag{41}$$

$$\frac{\partial W(\alpha,\lambda)}{\partial\lambda} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} F'_{\lambda}\left(x_{(i)},\alpha,\lambda\right)\left[F\left(x_{(i)},\alpha,\lambda\right) - \frac{i}{n+1}\right]^2 = 0 \tag{42}$$

Again, equation (41) and (42) cannot be solved analytically, therefore, numerical technique is used to secure the solution.

3.5. Cramer-von-Mises Estimation (CVME)

This method of estimation is proposed by [22], the method is the minimum distance method based on the difference between empirical and cumulative distribution functions. See, [10, 13] for more detail about this method. The CVM estimator of the parameters are obtained by minimizing

$$C(\alpha,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{(i)},\alpha,\lambda\right) - \frac{2i-1}{2n}\right)^2 \tag{43}$$

To get the CVM estimates $\widehat{\alpha}_{CVM}$ and $\widehat{\lambda}_{CVM}$ of parameters α and λ, differentiate equation (43) with respect to α and λ and equating to zero, which results of the following equations

$$\frac{\partial C(\alpha,\lambda)}{\partial\alpha} = \sum_{i=1}^{n} F'_{\alpha}\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right) - \frac{2i-1}{2n}\right) \tag{44}$$

$$\frac{\partial C(\alpha,\lambda)}{\partial\lambda} = \sum_{i=1}^{n} F'_{\lambda}\left(x_{(i)},\alpha,\lambda\right)\left(F\left(x_{(i)},\alpha,\lambda\right) - \frac{2i-1}{2n}\right) \tag{45}$$

Again, equation (44) and (45) cannot be solved analytically, therefore, same numerical technique is used to obtain the solution.

3.6. Anderson-Darling Method of Estimation (ADE)

This method is based on the minimization criteria of Anderson-Darling statistic [3] . The ADE estimate can be obtained by minimizing the following equation,

$$A(\alpha,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\left(F\left(x_{(i)},\alpha,\lambda\right)\right) + log\left(\overline{F}\left(x_{(n+1-i)},\alpha,\lambda\right)\right)\right] \tag{46}$$

where, $\overline{F}\left(x_{(n+1-i)},\alpha,\lambda\right) = 1 - F\left(x_{(i)},\alpha,\lambda\right)$. Therefore, the AD estimates $\widehat{\alpha}_{ADE}$ and $\widehat{\lambda}_{ADE}$ of the parameters α and λ can be obtained as the solutions of the partial differentiation based on equation (46) using same iterative procedure.

4. Simulation Study

In this section, the Monte Carlo simulation study has been performed to assess the performance of the different estimators obtained via different method of estimation viz., MLE, MPS, LSE, WLSE, CVME, and ADE.

In order to perform simulation, the random sample for the for the different variation of the sample size, and parameters. In particular, $n = 10, 20, \ldots, 100$ and $\alpha = (0.75, 1, 1.5, 2.5)$, $\lambda = 0.5$ are chosen. The estimators obtained via considered methods are not assumed any explicit mathematical form and not yield closed form solution, therefore N-R method is used to secure estimates of the parameters. The average estimate, MSE of the parameters using the above methods reported in Table 1-4 based on $N = 5000$ replication using the following formula.
Average estimates:

$$\widehat{\alpha}_{AE} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\alpha}_i \quad , \quad \widehat{\lambda}_{AE} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\lambda}_i$$

Mean square error:

$$\widehat{\alpha}_{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\alpha_i - \widehat{\alpha})^2 \quad , \quad \widehat{\lambda}_{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(\lambda_i - \widehat{\lambda}\right)^2$$

From Table 1,2,3 and 4 it has been overserved that the MSE of the parameter decreases as the sample size increases, which ensures the consistency of the proposed estimators. It is important to mention that ML and MPS methods are based on likelihood and others are based on a distance measure. Further, if we fixed $\lambda = 0.5$ and varied $\alpha = (0.5, 1.0,\ 1.5,\ 2.0, 4.0)$ the Table 1 and 2 show that, in likelihood-based methods for shape parameter α and scale parameter λ, MPS and MLE perform well respectively. Furthermore, in the considered distance measure there is no trend for the shape parameter however there is a specific trend for the scale parameter viz.

$$CVME < ADE < WLSE < LSE$$

Next, if we fixed $\alpha = 0.5$ and varied $\lambda = (0.5, 1.0,\ 1.5,\ 2.0, 4.0)$ the Table 3 and 4 show that in likelihood-based methods MPS is better than MLE for shape parameter α and MLE is better than MPS for scale parameter λ. Further, in the considered distance measure there is the same conclusion as in the previous case. Also, we have calculated coverage probability and average length of 95% confidence interval. The MSEs of point estimates and average lengths of interval estimates decreases with increasing sample sizes.

Table 1: *Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter α.*

			MLE	MPS	LSE	WLSE	CVME	ADE		
n	α	λ	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	CP	AL
10	0.5	0.5	0.7480(0.4486)	0.4206(0.1025)	0.5093(0.1580)	0.5280(0.1919)	0.8315(0.6497)	0.6123(0.2148)	0.9021	2.7740
	1		1.2587(0.9402)	0.6711(0.3066)	0.7418(0.3047)	0.7731(0.3167)	1.2324(0.8992)	0.9776(0.4341)	0.8964	4.9517
	1.5		1.6001(1.1336)	0.8334(0.6915)	0.9000(0.6562)	0.9409(0.6356)	1.4972(1.0294)	1.2206(0.6386)	0.8746	6.4410
	2		1.8439(1.3678)	0.9495(1.2598)	1.0172(1.2853)	1.0650(1.2287)	1.7136(1.2342)	1.3941(0.9913)	0.8578	7.4948
	4		3.7018(7.0880)	1.7501(6.3543)	1.7955(6.0933)	1.9273(5.8014)	3.1482(5.4229)	2.6206(4.9758)	0.8352	16.6406
20	0.5	0.5	0.6214(0.1700)	0.4564(0.0725)	0.5348(0.1216)	0.5457(0.1189)	0.6761(0.2575)	0.5729(0.1198)	0.9226	1.5587
	1		1.1372(0.5215)	0.7987(0.2494)	0.9064(0.3177)	0.9336(0.3121)	1.1634(0.6126)	1.0094(0.3567)	0.9032	3.0567
	1.5		1.5862(0.8163)	1.0880(0.4970)	1.1987(0.5313)	1.2552(0.5276)	1.5510(0.8429)	1.3760(0.5689)	0.8882	4.4341
	2		1.8794(0.9814)	1.2730(0.9129)	1.3964(0.8732)	1.4651(0.8341)	1.8158(1.0075)	1.6155(0.8123)	0.8668	5.3555
	4		3.7750(5.0184)	2.4191(4.3008)	2.6459(3.9490)	2.8126(3.8230)	3.5429(4.4899)	3.1448(3.8077)	0.8588	11.8705
30	0.5	0.5	0.5635(0.0856)	0.4568(0.0491)	0.5153(0.0726)	0.5248(0.0710)	0.5975(0.1170)	0.5386(0.0694)	0.9206	1.1230
	1		1.0698(0.3216)	0.8346(0.1949)	0.9365(0.2681)	0.9641(0.2614)	1.1028(0.4134)	0.9939(0.2584)	0.9148	2.2968
	1.5		1.5375(0.5993)	1.1729(0.4137)	1.2994(0.4845)	1.3492(0.4785)	1.5419(0.6836)	1.4047(0.4811)	0.8964	3.4581
	2		1.9344(0.8011)	1.4569(0.7000)	1.5892(0.7153)	1.6585(0.6753)	1.8931(0.8565)	1.7430(0.6775)	0.8848	4.4816
	4		3.7371(3.9179)	2.7059(3.5130)	2.9295(3.3280)	3.1099(3.2099)	3.5529(3.6898)	3.2870(3.2222)	3.2870	9.5240
50	0.5	0.5	0.5431(0.0457)	0.4764(0.0316)	0.5157(0.0420)	0.5238(0.0399)	0.5615(0.0558)	0.5288(0.0395)	0.9352	0.8225
	1		1.0231(0.1800)	0.8725(0.1339)	0.9481(0.1800)	0.9704(0.1757)	1.0426(0.2285)	0.9806(0.1635)	0.9112	1.6688
	1.5		1.5300(0.4326)	1.2806(0.3240)	1.3832(0.3893)	1.4262(0.3804)	1.5297(0.4835)	1.4483(0.3800)	0.9000	2.6398
	2		1.9594(0.5990)	1.6229(0.5192)	1.7557(0.6009)	1.8135(0.5593)	1.9498(0.7049)	1.8444(0.5491)	0.8974	3.4954
	4		3.7675(2.7696)	3.0364(2.5694)	3.2692(2.6738)	3.4104(2.5143)	3.6703(2.9441)	3.4859(2.9441)	0.8862	7.3997
100	0.5	0.5	0.5165(0.0202)	0.4820(0.0168)	0.5029(0.0207)	0.5095(0.0197)	0.5241(0.0236)	0.5102(0.0193)	0.9428	0.5450
	1		1.0021(0.0787)	0.9188(0.0689)	0.9607(0.0833)	0.9770(0.0788)	1.0061(0.0919)	0.9800(0.0760)	0.932	1.1384
	1.5		1.5070(0.2048)	1.3649(0.1768)	1.4444(0.2228)	1.4698(0.2080)	1.5181(0.2491)	1.4720(0.2020)	0.9288	1.8126
	2		1.9760(0.3487)	1.7767(0.3169)	1.8749(0.3911)	1.9127(0.3543)	1.9750(0.4278)	1.9167(0.3448)	0.9174	2.4772
	4		3.8139(1.6466)	3.3752(1.6005)	3.5492(1.8477)	3.6490(1.6550)	3.7598(1.9505)	3.6633(1.6357)	0.8970	5.2742

Table 2: ***Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter λ.***

			MLE	MPS	LSE	WLSE	CVME	ADE		
n	α	λ	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	CP	AL
	0.5		0.6673(0.3265)	0.8401(0.5701)	0.7561(0.3530)	0.7459(0.3447)	0.6104(0.2124)	0.6804(0.2809)	0.9219	2.0061
	1		0.6841(0.2847)	0.9104(0.5775)	0.8337(0.3811)	0.8189(0.3685)	0.6635(0.2175)	0.7269(0.2686)	0.9680	2.1378
10	1.5	0.5	0.7356(0.3258)	1.0049(0.7196)	0.9328(0.4947)	0.9116(0.4639)	0.7372(0.2748)	0.7988(0.3227)	0.8746	6.4410
	2		0.7680(0.3239)	1.0590(0.7175)	1.0069(0.5788)	0.9860(0.5528)	0.7879(0.3119)	0.8585(0.3748)	0.9986	2.3944
	4		0.8259(0.6516)	1.3265(8.4690)	1.1403(0.9113)	1.1075(0.8641)	0.8682(0.4780)	0.9547(0.6153)	0.9970	2.7113
	0.5		0.5911(0.1410)	0.6685(0.2076)	0.6183(0.1171)	0.6086(0.1128)	0.5557(0.0896)	0.5900(0.1039)	0.9344	1.3010
	1		0.6122(0.1256)	0.7151(0.1965)	0.6716(0.1326)	0.6586(0.1255)	0.5976(0.0963)	0.6321(0.1106)	0.9628	1.3512
20	1.5	0.5	0.6344(0.1332)	0.7579(0.2170)	0.7201(0.1628)	0.7022(0.1527)	0.6357(0.1144)	0.6691(0.1322)	0.9812	1.3977
	2		0.6649(0.1426)	0.8052(0.2433)	0.7676(0.1929)	0.7475(0.1792)	0.6739(0.1319)	0.7096(0.1506)	0.9944	1.4626
	4		0.6755(0.1574)	0.8551(0.2959)	0.8163(0.2518)	0.7882(0.2262)	0.7021(0.1670)	0.7426(0.1912)	0.9944	1.5231
	0.5		0.5717(0.0816)	0.6169(0.1016)	0.5886(0.0696)	0.5813(0.0685)	0.5487(0.0573)	0.5721(0.0656)	0.9416	1.0205
	1		0.5866(0.0731)	0.6539(0.1015)	0.6241(0.0777)	0.6119(0.0731)	0.5766(0.0613)	0.6009(0.0687)	0.9608	1.0440
30	1.5	0.5	0.6022(0.0815)	0.6845(0.1172)	0.6557(0.0961)	0.6403(0.0882)	0.6023(0.0746)	0.6254(0.0813)	0.9756	1.0649
	2		0.6132(0.0725)	0.7068(0.1125)	0.6810(0.1005)	0.6630(0.0905)	0.6227(0.0755)	0.6457(0.0823)	0.9922	1.0844
	4		0.6352(0.0870)	0.7579(0.1514)	0.7297(0.1385)	0.7029(0.1204)	0.6580(0.1005)	0.6818(0.1090)	0.9916	1.1660
	0.5		0.5393(0.0373)	0.5636(0.0414)	0.5522(0.0379)	0.5457(0.0368)	0.5297(0.0338)	0.5428(0.0362)	0.9502	0.7331
	1		0.5622(0.0387)	0.6023(0.0484)	0.5852(0.0420)	0.5759(0.0390)	0.5581(0.0359)	0.5717(0.0375)	0.9634	0.7580
50	1.5	0.5	0.5658(0.0381)	0.6163(0.0498)	0.5989(0.0469)	0.5866(0.0423)	0.5687(0.0395)	0.5816(0.0410)	0.9702	0.7622
	2		0.5742(0.0390)	0.6324(0.0543)	0.6132(0.0504)	0.5995(0.0454)	0.5802(0.0415)	0.5934(0.0433)	0.9840	0.7795
	4		0.5918(0.0438)	0.6686(0.0695)	0.6471(0.0646)	0.6280(0.0558)	0.6066(0.0510)	0.6199(0.0528)	0.9880	0.8447
	0.5		0.5203(0.0167)	0.5316(0.0174)	0.5262(0.0167)	0.5221(0.0164)	0.5154(0.0157)	0.5217(0.0164)	0.9472	0.4975
	1		0.5350(0.0165)	0.5558(0.0186)	0.5471(0.0177)	0.5413(0.0168)	0.5342(0.0161)	0.5404(0.0165)	0.9632	0.5027
100	1.5	0.5	0.5340(0.0158)	0.5609(0.0189)	0.5487(0.0180)	0.5418(0.0168)	0.5343(0.0163)	0.5413(0.0166)	0.9696	0.5028
	2		0.5404(0.0170)	0.5717(0.0213)	0.5594(0.0207)	0.5512(0.0188)	0.5438(0.0185)	0.5504(0.0185)	0.9716	0.5169
	4		0.5577(0.0217)	0.5990(0.0299)	0.5866(0.0294)	0.5739(0.0258)	0.5674(0.0250)	0.5725(0.0255)	0.9776	0.5664

Table 3: *Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter α.*

			MLE	MPS	LSE	WLSE	CVME	ADE		
n	α	λ	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\alpha}$	CP	AL
10	0.5	0.5	0.7480(0.4486)	0.4206(0.1025)	0.5093(0.1580)	0.5280(0.1919)	0.8315(0.6497)	0.6123(0.2148)	0.9021	2.7740
		1	0.8849(0.7309)	0.4884(0.1380)	0.5704(0.1908)	0.5877(0.2157)	0.9441(0.8618)	0.7115(0.3390)	0.9412	3.3487
		1.5	0.9648(0.8586)	0.5286(0.1485)	0.6263(0.2364)	0.6443(0.2600)	1.0448(1.0950)	0.7730(0.4007)	0.9656	3.6730
		2	1.0314(0.9580)	0.5621(0.1575)	0.6512(0.2421)	0.6718(0.2728)	1.0759(1.1039)	0.8143(0.4252)	0.9838	3.9668
		4	1.2500(2.3613)	0.6568(0.3859)	0.8041(0.6648)	0.8251(0.7489)	1.3927(3.0527)	0.9708(1.0589)	0.9870	5.0005
20	0.5	0.5	0.6214(0.1700)	0.4564(0.0725)	0.5348(0.1216)	0.5457(0.1189)	0.6761(0.2575)	0.5729(0.1198)	0.9226	1.5587
		1	0.6754(0.2220)	0.4931(0.0856)	0.5840(0.1686)	0.5932(0.1650)	0.7406(0.3659)	0.6208(0.1553)	0.9380	1.7072
		1.5	0.7226(0.2984)	0.5248(0.1091)	0.6145(0.2036)	0.6300(0.2045)	0.7796(0.4392)	0.6630(0.2110)	0.9516	1.8383
		2	0.7451(0.3109)	0.5400(0.1096)	0.6327(0.2048)	0.6442(0.1985)	0.8047(0.4506)	0.6790(0.2101)	0.9734	1.9020
		4	0.7940(0.4595)	0.5710(0.1594)	0.6835(0.3378)	0.6970(0.3382)	0.8773(0.7492)	0.7230(0.3191)	0.9782	2.0525
30	0.5	0.5	0.5635(0.0856)	0.4568(0.0491)	0.5153(0.0726)	0.5248(0.0710)	0.5975(0.1170)	0.5386(0.0694)	0.9206	1.1230
		1	0.6090(0.1140)	0.4915(0.0579)	0.5558(0.0986)	0.5649(0.0946)	0.6459(0.1643)	0.5806(0.0926)	0.9366	1.2204
		1.5	0.6233(0.1264)	0.5021(0.0626)	0.5699(0.1114)	0.5811(0.1096)	0.6631(0.1879)	0.5934(0.1025)	0.9462	1.2543
		2	0.6472(0.1460)	0.5200(0.0696)	0.5951(0.1299)	0.6052(0.1288)	0.6929(0.2193)	0.6162(0.1179)	0.9616	1.3059
		4	0.6558(0.1619)	0.5262(0.0768)	0.6018(0.1511)	0.6120(0.1465)	0.7013(0.2553)	0.6237(0.1299)	0.9624	1.3278
50	0.5	0.5	0.5431(0.0457)	0.4764(0.0316)	0.5157(0.0420)	0.5238(0.0399)	0.5615(0.0558)	0.5288(0.0395)	0.9352	0.8225
		1	0.5649(0.0533)	0.4946(0.0344)	0.5370(0.0496)	0.5460(0.0485)	0.5851(0.0674)	0.5512(0.0475)	0.9448	0.8580
		1.5	0.5708(0.0528)	0.4996(0.0335)	0.5442(0.0527)	0.5530(0.0504)	0.5932(0.0724)	0.5570(0.0476)	0.9534	0.8675
		2	0.5871(0.0639)	0.5130(0.0391)	0.5587(0.0616)	0.5666(0.0582)	0.6091(0.0848)	0.5716(0.0569)	0.9600	0.8971
		4	0.5911(0.0676)	0.5163(0.0413)	0.5650(0.0671)	0.5721(0.0632)	0.6164(0.0926)	0.5770(0.0607)	0.9542	0.9009
100	0.5	0.5	0.5165(0.0202)	0.4820(0.0168)	0.5029(0.0207)	0.5095(0.0197)	0.5241(0.0236)	0.5102(0.0193)	0.9428	0.5450
		1	0.5237(0.0203)	0.4886(0.0165)	0.5133(0.0211)	0.5179(0.0199)	0.5349(0.0245)	0.5184(0.0194)	0.9504	0.5511
		1.5	0.5341(0.0223)	0.4979(0.0174)	0.5226(0.0227)	0.5279(0.0217)	0.5447(0.0267)	0.5283(0.0212)	0.9526	0.5642
		2	0.5384(0.0231)	0.5020(0.0179)	0.5250(0.0230)	0.5312(0.0222)	0.5472(0.0270)	0.5319(0.0215)	0.9584	0.5689
		4	0.5401(0.0245)	0.5035(0.0189)	0.5267(0.0243)	0.5333(0.0235)	0.5490(0.0286)	0.5339(0.0231)	0.9542	0.5704

Table 4: *Average estimate, corresponding MSE (in brackets), coverage probability (CP), and average length (AL) of the parameter λ.*

			MLE	MPS	LSE	WLSE	CVME	ADE		
n	α	λ	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\lambda}$	CP	AL
		0.5	0.6673(0.3265)	0.8401(0.5701)	0.7561(0.3530)	0.7459(0.3447)	0.6104(0.2124)	0.6804(0.2809)	0.9219	2.0061
		1	1.0886(0.5368)	1.3732(0.9025)	1.2920(0.7307)	1.2710(0.7011)	1.0419(0.4598)	1.1508(0.5634)	0.8852	3.2606
10	0.5	1.5	1.4210(0.7104)	1.7915(1.0438)	1.7105(0.9571)	1.6764(0.9080)	1.3752(0.6680)	1.5178(0.7507)	0.8530	4.2231
		2	1.6901(0.8701)	2.1557(1.0643)	2.0771(1.0523)	2.0381(1.0064)	1.6672(0.8659)	1.8324(0.8789)	0.8380	5.0404
		4	3.2462(3.9636)	4.1322(4.5452)	3.9455(4.4978)	3.8684(4.2918)	3.1487(3.8866)	3.5183(3.9629)	0.7940	9.5658
		0.5	0.5911(0.1410)	0.6685(0.2076)	0.6183(0.1171)	0.6086(0.1128)	0.5557(0.0896)	0.5900(0.1039)	0.9344	1.3010
		1	1.0851(0.3558)	1.2222(0.4673)	1.1608(0.3942)	1.1436(0.3778)	1.0436(0.3130)	1.1061(0.3461)	0.9110	2.3593
20	0.5	1.5	1.5361(0.6136)	1.7292(0.7719)	1.6564(0.6916)	1.6260(0.6685)	1.4886(0.5637)	1.5723(0.6165)	0.8800	3.3108
		2	1.9234(0.7698)	2.1675(0.9012)	2.0903(0.8724)	2.0557(0.8501)	1.8759(0.7469)	1.9870(0.7958)	0.8742	4.1137
		4	3.7479(3.0296)	4.2350(3.4680)	4.0701(3.4265)	4.0019(3.3306)	3.6457(3.0171)	3.8748(3.1226)	0.8592	8.0157
		0.5	0.5717(0.0816)	0.6169(0.1016)	0.5886(0.0696)	0.5813(0.0685)	0.5487(0.0573)	0.5721(0.0656)	0.9416	1.0205
		1	1.0743(0.2605)	1.1607(0.3147)	1.1155(0.2511)	1.1002(0.2447)	1.0393(0.2135)	1.0819(0.2347)	0.9234	1.9081
30	0.5	1.5	1.5641(0.4701)	1.6897(0.5472)	1.6331(0.4934)	1.6088(0.4780)	1.5219(0.4264)	1.5833(0.4543)	0.9028	2.7595
		2	1.9950(0.6302)	2.1572(0.7043)	2.0966(0.7009)	2.0667(0.6790)	1.9525(0.6221)	2.0341(0.6486)	0.8992	3.4823
		4	3.9617(2.5757)	4.2871(2.8736)	4.1582(2.7998)	4.0980(2.7071)	3.8725(2.5033)	4.0339(2.6208)	0.8880	6.9524
		0.5	0.5393(0.0373)	0.5636(0.0414)	0.5522(0.0379)	0.5457(0.0368)	0.5297(0.0338)	0.5428(0.0362)	0.9502	0.7331
		1	1.0420(0.1413)	1.0903(0.1542)	1.0657(0.1376)	1.0532(0.1358)	1.0219(0.1250)	1.0470(0.1329)	0.9324	1.4151
50	0.5	1.5	1.5299(0.2733)	1.6014(0.2941)	1.5685(0.2765)	1.5505(0.2730)	1.5038(0.2540)	1.5416(0.2676)	0.9228	2.0756
		2	2.0063(0.4690)	2.1016(0.4961)	2.0582(0.4837)	2.0352(0.4692)	1.9732(0.4503)	2.0240(0.4639)	0.9120	2.7205
		4	4.0068(1.8972)	4.1955(2.0007)	4.1190(1.9678)	4.0726(1.9180)	3.9486(1.8315)	4.0461(1.8783)	0.9094	5.3994
		0.5	0.5203(0.0167)	0.5316(0.0174)	0.5262(0.0167)	0.5221(0.0164)	0.5154(0.0157)	0.5217(0.0164)	0.9472	0.4975
		1	1.0314(0.0626)	1.0541(0.0651)	1.0440(0.0639)	1.0368(0.0626)	1.0226(0.0605)	1.0359(0.0620)	0.9350	0.9803
100	0.5	1.5	1.5243(0.1433)	1.5584(0.1472)	1.5423(0.1442)	1.5319(0.1424)	1.5105(0.1379)	1.5308(0.1413)	0.9526	0.5642
		2	2.0210(0.2438)	2.0665(0.2494)	2.0475(0.2470)	2.0310(0.2421)	2.0053(0.2368)	2.0293(0.2416)	0.9316	1.9212
		4	4.0084(0.9497)	4.0983(0.9621)	4.0687(0.9915)	4.0342(0.9895)	3.9851(0.9550)	4.0317(0.9820)	0.9282	3.7997

5. Real data application

In this section, the applicability of the proposed flexible extension has been discussed based on two survival data sets. The data set-I is taken from [6] which represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The second data set has been obtained from [16] which consists of survival times, in week, of 33 patients suffering from acute Myelogenous Leukaemia. The summary of both data sets is given in Table 5.
Further, to show the superiority of the proposed model, the following well-known lifetime models are taken.

1. Extension of exponential distribution with pdf

$$f(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}e^{\left(1-(1+\lambda x)^{\alpha}\right)} \qquad ; x > 0,\ \alpha > 0,\ \lambda > 0$$

2. Weibull distribution with pdf

$$f(x) = \frac{\alpha}{\lambda^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \qquad ; x \geq 0,\ \alpha > 0,\ \lambda > 0$$

3. Gamma distribution with pdf

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x} \qquad ; x > 0,\ \alpha > 0, \lambda > 0$$

where, α is shape and λ is the scale parameter.
The superiority of the proposed extension with the above considered families of the distribution are shown with the help of model criterion tools. Hence, the criterion like p-value, Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), and Kolmogorov-Smirnov (KS) test are taken.
To compare the models, Table 6-7, contains the values of the parameters estimated by the maximum likelihood, AIC, BIC, and KS statistics with the p-value for fitted data sets. From the K-S test statistics or associated p-value, it may be seen that the proposed model provides better fit than Weibull, Gamma, and Extension of Exponential models. Also, a similar result is concluded on the basis of negative of Log-likelihood which is higher than other three. Also based on the relative model selection criteria it is observed that the proposed model has smaller AIC and BIC in comparison to other thee considered models. Hence, the proposed mode is more suitable for considered real phenomena.

Further, the plots of the empirical cumulative distribution function (ECDF) and the fitted CDF for the considered two data set are shown in Figure 2. From Figure 2, it is concluded that the proposed model fits better to considered real data in comparison to other competitive models. Hence it may be taken as the alternative to the several lifetime models.

Table 5: *Summary of the considered data sets.*

Data	Min.	Q1	Median	Mean	Q3	SD	Skewness	Kurtosis	Max.
I	0.080	1.080	1.560	1.837	2.303	1.215	1.754	7.151	7.000
II	1.00	4.00	22.00	40.88	65.00	46.703	1.164	3.122	156.00

6. Conclusion

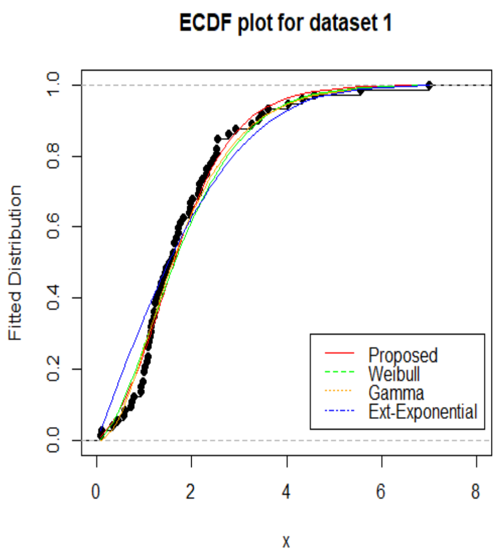
In this article, we have introduced a new transformation technique to generate the class of lifetime distributions. Further, the purposed transformation technique is illustrated via exponential distribution as baseline distribution and named a new flexible two-parameter lifetime distribution. The

Table 6: *MLE, AIC, BIC and KS statistics with the p-value for the data set-I.*

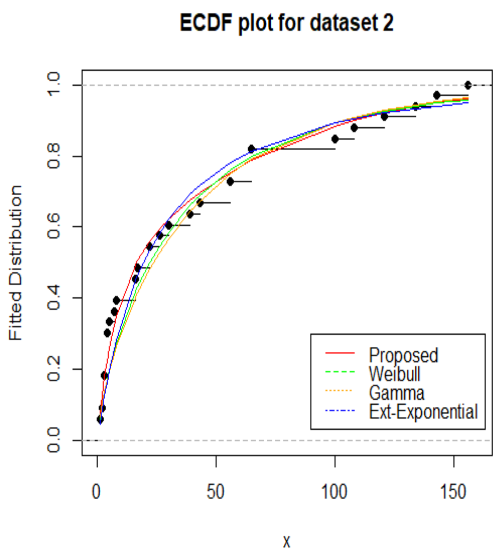
	ML Estimates			KS-Test			
Distributions	$\hat{\alpha}$	$\hat{\lambda}$	-LogL	Statistics	p-value	AIC	BIC
Proposed	8.2801	0.1257	102.4141	0.09658	0.5127	208.8283	213.3816
Weibull	1.6172	2.0558	104.0168	0.11346	0.3121	212.0336	216.5869
Gamma	2.4379	1.3273	102.9648	0.10372	0.4207	209.9296	214.4829
EE	5.0262	0.0731	109.6485	0.19617	0.0078	223.2970	227.8504

Table 7: *MLE, AIC, BIC and KS statistics with the p-value for the data set-II.*

	ML Estimates			KS-Test			
Distributions	$\hat{\alpha}$	$\hat{\lambda}$	-LogL	Statistics	p-value	AIC	BIC
Proposed	0.2659	0.0763	151.7045	0.10004	0.8959	307.4091	310.4021
Weibull	0.7764	35.3613	153.5868	0.13668	0.5684	311.1737	314.1667
Gamma	0.6877	0.0168	153.6737	0.13900	0.5466	311.3473	314.3403
EE	0.4897	0.0998	153.7430	0.13920	0.5440	311.4860	314.4790



(a)



(b)

Figure 2: *(a) ECDF and fitted CDF plot of proposed distribution for data 1. (b) ECDF and fitted CDF plot of proposed distribution for data 2.*

proposed distribution has an increasing, decreasing, and constant hazard nature see Figure 1(b). Next, the different distributional properties are derived viz. mean, moments, moment generating function (MGF), mean deviation about mean and median, Bonferroni and Lorenz curves, Renyi entropy, s-entropy, cumulative residual entropy, *rth* order statistics, and reliability. The unknown parameter of the proposed model is estimated by different methods of estimation namely MLE, MPS, LSE, WLSE, CVME, and ADE. To compare the performances of different estimators obtained via different estimation methods Monte Carlo simulation study has been performed in Table 1,2,3 and 4. The superiority of the present study and model has been illustrated by constructing two real data sets. From the Table 6,7 and Figure 2, It is observed that the proposed new flexible two-parameter lifetime distribution provides better fits to the considered data sets among the most popular distributions viz., Weibull, Gamma, and extension of exponential distribution. Therefore, we may conclude that the proposed model might be considered an alternative to other considered models.

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A DIFFERENT INITIATIVE TO FIND AN OPTIMAL SOLUTION TO THE TRIANGULAR FUZZY TRANSPORTATION PROBLEM BY IMPLEMENTING THE ROW-COLUMN MAXIMA METHOD

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Abstract

In this paper, we discussed an issue in fuzzy transportation problem, which involves fuzzy costs, fuzzy supply, and fuzzy product needs. The goal of this article is to convey the item from point of origin to point of destination at the least possible cost. For fuzzy transportation problems with balance and unbalance types, the proposed technique provides a superior optimal. Transportation costs, supply, and demand are represented by generalized triangular fuzzy numbers using this proposed named Row - Column Maxima Method (RCMM). A numerical example of a fuzzy transportation problem is illustrated and the solution is compared with the outcomes of other approaches. This method reduces iterations and which help to understand and implement easily in real life applications.

Keywords: Fuzzy set, Fuzzy Number, Triangular fuzzy number, Fuzzy Transportation problem, RCM- Method, Fuzzy optimal solution.

1. Introduction

In 1941, Hitchcock had his initial idea regarding the transportation problem. In 1965, L.A. Zadeh [17] created fuzzy set theory and successfully applied it to a variety of fields. There is a need to send products from various origins (Factories) to various destinations in a variety of real-world situations (warehouses). The decision maker's goal is to figure out how much product to order. Many distribution challenges in today's actual world, such as in business or industrial settings are imprecise in nature due to parameter variances. However, due to some unavoidable circumstances, all of these elements of the transportation problem may not be precisely understood in real time. In 1978, the fuzzy decision-making method was introduced. Zimmermann developed a variety of fuzzy optimization algorithms for TP and FTP [18]. Hitchcock [5] was the first to come up with the basic transportation problem. To handle the totally fuzzy transportation problem, Dhanaseker et al. [4] presented the Hungarian-Modi technique. Muthuperumal et al. [9] offered an algorithmic solution to the problem of unbalanced triangular fuzzy transportation. Senthil Kumar et al. [13] suggested the Harmonic Mean Way as a new method for solving the Generalized Fuzzy Transportation problem. A new strategy for finding an optimal solution to Generalized Fuzzy Transportation Problems was proposed by Srinivasaro Thota and Raja [16]. Fuzzy Transportation Problem By Using Triangular Fuzzy Numbers With Ranking Using Area Of Trapezium, Rectangle, And Centroid At Different Levels Of -Cut was discussed by Ambadas Deshmukh et al[1]. Balasubramanian et al. [2], [3] explored utilizing a ranking function to solve the Fuzzy Transportation Problem. Srinivasan et al. [14] established a method for handling fully fuzzy transportation problems in which the materials are transformed, and this method is straightforward to evaluate and can rank many forms of triangular fuzzy numbers. This study by Ladji Kane et al. [8] addressed a Simplified approach for Solving Transportation Problems with Triangular Fuzzy Numbers in Fuzzy Environments. Purushoth kumar et al. [10] proposed employing the diagonal optimum method to address fully fuzzy transportation problems. Indira Singuluri et al. [6] proposed their strategies to address a novel transportation approach to solving type-2 triangular intuitionistic fuzzy transportation problems.

In this study, we offer a new method for solving the fuzzy transportation problem called the RCM method, which assumes supply, demand, and unit transportation cost as triangular fuzzy integers. It provides a minimal value when compared to other approaches such as the NWCM [North-West Corner Method], LCM [Least Cost Method], VAM [Vogel's Approximation Method], and RMM [Row Minima Method]. Finally, an example is provided to aid in the comprehension of the method.

The remainder of this work is arranged in the following manner. Present the fundamental definitions and mathematical constructions of transportation problems in section 2. Present a new algorithm to handle the fully fuzzy transportation problem in section 3. The proposed approach is illustrated numerically in Section 4. The conclusion and future study is presented in section 5.

2. Preliminaries

Definition 2.1[17]

Let U is a collection of elements indicated by u then a fuzzy set $\mathcal{P}$ is a set of ordered pairs in U: $\mathcal{P} = \{(u,\ \mu_{\mathcal{P}}(u)) |\ u \in \mathrm{U}\}$, where the membership function or grade of membership of u in $\mathcal{P}$ is $\mu_{\mathcal{P}}(u)$:U $\longrightarrow$ [0,1].

Definition 2.2[4]

P is a fuzzy set of real numbers that is defined on the universal set of real numbers. If R's membership function satisfies the following properties, R is said to be a fuzzy number.

1. $\mu_{\mathcal{P}}(u)$ is a piecewise continuous
2. $\mathcal{P}$ is convex. $\mu_{\mathcal{P}}(\delta u_1 + (1-\delta)u_2) \geq \min(\mu_{\mathcal{P}}(u_1),\ \mu_{\mathcal{P}}(u_2))\,, \forall u_1,\ u_2 \in \mathcal{R}\ \&\ \forall\delta \in [0,\ 1]$.
3. $\mathcal{P}$ is Normal.

Definition 2.3[4]

If the membership function $\mathcal{P}: \mathcal{R} \rightarrow [0,1]$ of a fuzzy number P on R satisfies the following characteristics, it is said to be a triangular fuzzy number (TFN) or linear fuzzy number.

$$\mathcal{P}(u) = \begin{cases} \dfrac{u-p_1}{p_2-p_1}, \text{for } p_1 \le u \le p_2 \\ 1 \quad , \quad u = p_2 \\ \dfrac{p_3-u}{p_3-p_2}, \text{for } p_2 \le u \le p_3 \\ 0, \text{ elsewhere} \end{cases}$$

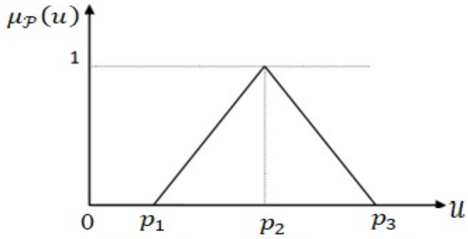

Fig 1.Triangular fuzzy number

2.4 Arithmetic Operation on Fuzzy Numbers [4]:

The operations that can be performed on triangular fuzzy numbers are as follows: Then, if $\mathcal{P} = (p_1, p_2, p_3)$ and $\mathcal{Q} = (q_1, q_2, q_3)$.

(i) Addition: $\mathcal{P} + \mathcal{Q} = (p_1 + q_1, p_2 + q_2, p_3 + q_3)$.

(ii) Subtraction: $\mathcal{P} - \mathcal{Q} = (p_1 - q_3, p_2 - q_2, p_3 - q_1)$.

(iii) Multiplication: $\mathcal{P} \times \mathcal{Q} = (p_1q_1, p_2, p_3q_3)$.

2.5 MATHEMATICAL CONSTRUCTION [9]:

A fuzzy transportation problem can be expressed mathematically as follows:

$$\text{Minimize (Total cost) } \mathcal{Z} = \sum_{i=1}^{m} c_{ij} \sum_{j=1}^{n} x_{ij}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1,2, \dots\dots\dots, m \text{(Fuzzy Supply constraints)}$$
$$\sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1,2, \dots\dots\dots, n \text{(Fuzzy Demand constraints)}$$
$$x_{ij} \ge 0, \quad i = 1,2, \dots\dots\dots, m \text{ and } j = 1,2, \dots\dots\dots, n$$

Where m: Total number of sources, n: Total number of destinations

Notations:

s_i: The product's fuzzy availability at i^{th} the source.
d_j: The product's fuzzy demand at j^{th} destination.
c_{ij}: The fuzzy transportation cost of transporting one unit of commodity from i^{th}source to j^{th} destinations.
x_{ij}: To minimize total fuzzy transportation, a fuzzy quantity is delivered from i^{th}source to j^{th}destination (or fuzzy decision variables).
$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$: The fuzzy cost of transporting one unit of the product from i^{th}source to the j^{th} destination.
$\sum_{i=1}^{m} s_i$: The product's total fuzzy availability
$\sum_{j=1}^{n} d_j$: The product's total fuzzy demand

	Destination 1	**Destination 2**	**…**	**Destination n**	**Supply**
Source 1	$c_{11}x_{11}$	$c_{12}x_{12}$	…	$c_{1n}x_{1n}$	s_1
Source 2	$c_{21}x_{21}$	$c_{22}x_{22}$	…	$c_{2n}x_{2n}$	s_2
⋮	⋮	⋮	⋮	⋮	⋮
Source m	$c_{m1}x_{m1}$	$c_{m2}x_{m2}$	…	$c_{mn}x_{mn}$	s_m
Demand	d_1	d_2	…	d_n	$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$

2.6 Balanced and unbalanced FTP [11]:

Balanced fuzzy transportation problem: The total fuzzy supply is equal to total fuzzy demand

$$\text{i. e. } \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

Unbalanced fuzzy transportation problem:

The total fuzzy supply is not equal to total fuzzy demand

$$\text{i. e. } \sum_{i=1}^{m} s_i \neq \sum_{j=1}^{n} d_j$$

2.7 To Modify Unbalanced FTP to Balanced FTP: An Unbalanced FTP may occur in two different forms: (i) Excess of availability, (ii) Shortage in availability.

We now discuss these two cases by considering the usual m- sources, n- destinations FTP with the condition that $\sum_{i=1}^{m} s_i \neq \sum_{j=1}^{n} d_j$

Case1: (Excess of Availability, i.e. $\sum s_i \geq \sum d_j$)

The general FTP may be stated as follows:

Minimize (Total cost) $Z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}c_{ij}$

Subject to the constraints

$\sum_{j=1}^{n} x_{ij} \leq s_i, \quad i = 1,2,\dots\dots\dots,m$(Fuzzy Supply constraints)

$\sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1,2,\dots\dots\dots,n$(Fuzzy Demand constraints)

and $x_{ij} \geq 0, \quad i = 1,2,\dots\dots\dots,m$ and $j = 1,2,\dots\dots\dots,n$

The problem will possess a fuzzy feasible solution if $\sum s_i \geq \sum d_j$. In the first constraints, the introduction of slack variable $x_{i,n+1}(i = 1,2,\dots\dots\dots,m)$ gives

$$\Rightarrow \sum_{j=1}^{n} x_{ij} + x_{i,n+1} = s_i, \quad i = 1,2,\dots\dots\dots,m$$

$$\Rightarrow \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} + x_{i,n+1}\right) = \sum_{i=1}^{m} s_i$$

$$\Rightarrow \sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}\right) + \sum_{i=1}^{m} x_{i,n+1} = \sum_{i=1}^{m} s_i$$

$$\Rightarrow \sum_{j=1}^{n} \boldsymbol{d_j} + \sum_{i=1}^{m} x_{i,n+1} = \sum_{i=1}^{m} s_i \quad (\because \sum_{i=1}^{m} x_{ij} = d_j$$

$$\Rightarrow \sum_{i=1}^{m} x_{i,n+1} = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} \boldsymbol{d_j} = \text{Excess of Availability}$$

If this excess availability is denoted by d_{n+1}, the modified FTP, can be reformulated as:

Minimize $Z = \sum_{i=1}^{m}\sum_{j=1}^{n+1} x_{ij}c_{ij}$,

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} + x_{i,n+1} = s_i, \quad i = 1,2,\dots\dots\dots,m$$

$\sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1,2,\dots\dots\dots,n+1$ and $x_{ij} \geq 0$, for all i and j

and $c_{i,n+1} = 0$, for $i = 1,2,\dots\dots\dots,m$ and $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n+1} d_j$

This is clearly the balanced FTP and thus can be easily solved by fuzzy transportation algorithm.

Working Rule: If $\sum s_i \geq \sum d_j$, avoid using a fake row or column when converting to balance. Let see $\omega = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$. The difference ω added to the demand (d_1, d_2, d_3) minimum. Reconstruct the provided Fuzzy transportation table using $(d_1 + \omega_1, d_2 + \omega_2, d_3 + \omega_3)$.

Case2: (Shortage in Availability, i.e. $\sum s_i \leq \sum d_j$)

In this case, the general FTP becomes:

Minimize $\mathcal{Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1,2, \dots \dots \dots, m$$
$$\sum_{i=1}^{m} x_{ij} \leq d_j, \quad j = 1,2, \dots \dots \dots, n$$

and $x_{ij} \geq 0, \quad i = 1,2, \dots \dots \dots, m; j = 1,2, \dots \dots \dots, n$

Now, introducing the slack variable $x_{m+1,j} (j = 1,2, \dots \dots \dots, n)$ in the second constraint, we get

$$\Rightarrow \sum_{i=1}^{m} x_{ij} + x_{m+1,j} = d_j, \quad j = 1,2, \dots \dots \dots, n$$
$$\Rightarrow \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} + x_{m+1,j} \right) = \sum_{j=1}^{n} d_j$$
$$\Rightarrow \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) + \sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} d_j$$
$$\Rightarrow \sum_{i=1}^{m} s_i + \sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} d_j \quad (\because \sum_{j=1}^{n} x_{ij} = s_i$$
$$\Rightarrow \sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} d_j - \sum_{i=1}^{m} s_i = \text{Shortage in availability } s_{m+1}, \text{ say}$$

Thus the modified FTP, in this case becomes:

$$\text{Minimize } \mathcal{Z} = \sum_{i=1}^{m+1} \sum_{j=1}^{n} x_{ij} c_{ij},$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1,2, \dots \dots \dots, m+1$$

$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} = d_j, \quad j = 1,2, \dots \dots \dots, n$ and $x_{ij} \geq 0$, for all i and j

where $c_{m+1,j} = 0$, for $j = 1,2, \dots \dots \dots, n$ and $\sum_{i=1}^{m+1} s_i = \sum_{j=1}^{n} d_j$

This is clearly the balanced FTP and thus can be easily solved by fuzzy transportation algorithm.

Working Rule: If $\sum_{i=1}^{m} s_i \leq \sum_{j=1}^{n} d_j$, avoid using a fake row or column when converting to balance. Let see $\omega = \sum_{i=1}^{m} d_j - \sum_{j=1}^{n} s_i$. The difference ω added to the supply (s_1, s_2, s_3) minimum. Reconstruct the provided Fuzzy transportation table using $(s_1 + \omega_1, s_2 + \omega_2, s_3 + \omega_3)$.

2.8 Fuzzy Feasible Solution [9]:

A fuzzy feasible solution is any set of fuzzy non negative allocations $x_{ij} (x_{ij} \geq 0)$ that fulfills (in the sense equivalent) the row and column requirements.

2.9 Fuzzy Basic Feasible Solution [9]:

If the number of positive allocations is exactly equal to $(m + n - 1)$, a fuzzy feasible solution to a fuzzy transportation problem with m origins and n destinations is said to be fuzzy basic feasible solution.

2.10 Fuzzy Optimal Solution [9]:

If the entire fuzzy transportation cost is minimized, a fuzzy feasible solution is said to be fuzzy optimum.

Theorem 2.11 [11]: (Existence of Fuzzy feasible solution)

A necessary and sufficient condition for the existence of feasible solution of a fuzzy transportation problem is $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ $(i = 1,2, \dots \dots \dots, m; j = 1,2, \dots \dots \dots, n)$.

Proof: The condition is necessary: Let there exist a feasible solution to the fuzzy transportation problem. Then,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} s_i, \quad (1)$$

$$\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} d_j, \quad (2)$$

From equation (1) and (2), we get

$$\Leftrightarrow \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

The condition is sufficient: Let $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j = \mathcal{K}$ (say).

If $\mu_i \neq 0$ be any real number such that $x_{ij} = \mu_i d_j \forall\, i, j$, then μ_i is given by

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \mu_i d_j = \mu_i \sum_{j=1}^{n} d_j = \mathcal{K}\mu_i$$

$$\Rightarrow \mu_i = \frac{1}{\mathcal{K}} \sum_{j=1}^{n} x_{ij} = \frac{s_i}{\mathcal{K}} \left(\because \sum_{j=1}^{n} x_{ij} = s_i\right)$$

Thus, $x_{ij} = \mu_i d_j = \frac{s_i d_j}{\mathcal{K}} \geq 0$, since $s_i > 0, d_j > 0 \;\forall\, i, j$. Hence a Fuzzy feasible solution exists.

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3. Proposed algorithm

In this paper, we proposed Row-Column maxima method [RCMM] to find optimum solution and this result compared with NWCM, LCM, RMM, VAM methods.

Step 1: Check to see if the given FTP is balanced or not.

Case1: If $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$. then go to step 3.

Case2: If $\sum_{i=1}^{m} s_i \neq \sum_{j=1}^{n} d_j$, possible, avoid using a fake row or column when converting to balanced. Let see $(i)\omega = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j \; if \sum_{j=1}^{n} d_j < \sum_{i=1}^{m} s_i$ or $(ii)\omega = \sum_{i=1}^{m} d_j - \sum_{j=1}^{n} s_i \; if \sum_{i=1}^{m} s_i < \sum_{j=1}^{n} d_j$.

Step 2: The difference ω will be divided into three parts $(\omega_1, \omega_2, \omega_3)$ such that $\omega = \sum_{i=1}^{3} \omega_i$ and added to the supply (s_1, s_2, s_3) or demand (d_1, d_2, d_3) minimum. Reconstruct the provided Fuzzy transportation table using $(s_1 + \omega_1, s_2 + \omega_2, s_3 + \omega_3)/(d_1 + \omega_1, d_2 + \omega_2, d_3 + \omega_3)$.

Step 3: For each row, find the difference between the first and second maximum values and use that value instead of the first maximum value.

Step 4: After completing step 3, calculate the difference between the 1st and 2nd maximum values and use that value to replace the 1st maximum value in each column.

Step 5: Choose the fuzzy cost's minimum value in either a row or a column. Then determine the minimum supply and demand value and assign it.

Step 6: After step 5, delete the row/column in which supply/demand has reached its limit.

Step 7: Steps 5 – 6 should be repeated until (m + n-1) cells have been allotted.

Step 8: Calculate the minimum Fuzzy Transportation Cost. That is,

Total Cost= $\sum_{i=1}^{m} c_{ij} \sum_{j=1}^{n} x_{ij}$.

4. Numerical Example

A manufacturing company produces diesel engines in 10 cities $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}$ and they are purchased by ten trucking companies $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}$. The table below indicates how many engines are required by $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}$. It also displays the cost of transportation per engine from origin to destination. The corporation wants to maintain the total transportation cost to a minimum.

Table 1: *Triangular Fuzzy Transportation Problem*

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	Supply
C_1	(3,5,7)	(4,6,8)	(3,6,9)	(11,12,13)	(2,3,4)	(7,8,9)	(5,6,7)	(12,14,16)	(1,4,7)	(7,8,9)	(15,20,25)
C_2	(2,5,8)	(5,7,9)	(4,5,6)	(7,8,9)	(14,16,18)	(12,13,14)	(5,7,9)	(0,1,2)	(2,4,6)	(6,7,8)	(5,10,15)
C_3	(5,6,7)	(4,6,8)	(11,13,15)	(3,6,9)	(14,15,16)	(2,3,4)	(8,9,10)	(4,8,16)	(9,10,11)	(14,16,18)	(25,30,35)
C_4	(9,10,11)	(2,5,7)	(2,3,4)	(3,5,7)	(10,15,20)	(5,6,7)	(7,8,9)	(1,3,5)	(3,6,9)	(10,11,12)	(40,45,50)
C_5	(8,9,10)	(2,4,6)	(8,10,12)	(6,8,10)	(3,6,7)	(10,12,14)	(1,4,7)	(14,16,18)	(1,2,3)	(8,9,10)	(90,95,100)
C_6	(6,7,8)	(12,13,14)	(14,16,18)	(1,2,3)	(1,3,5)	(3,5,7)	(6,8,10)	(2,4,6)	(7,8,9)	(13,15,17)	(70,75,80)
C_7	(5,6,7)	(12,14,16)	(13,15,17)	(5,6,7)	(0,1,2)	(11,13,15)	(14,16,18)	(2,4,6)	(7,9,11)	(3,5,7)	(50,55,60)
C_8	(16,18,20)	(1,3,5)	(7,8,9)	(8,10,12)	(3,6,9)	(4,5,6)	(10,11,12)	(3,6,9)	(14,15,16)	(4,6,8)	(65,70,75)
C_9	(4,6,8)	(1,2,3)	(2,4,6)	(11,12,13)	(1,2,3)	(2,4,6)	(3,5,7)	(5,6,7)	(8,9,10)	(4,5,6)	(85,90,95)
C_{10}	(7,8,9)	(5,7,9)	(6,8,10)	(9,11,13)	(4,6,8)	(14,15,16)	(11,12,13)	(14,16,18)	(0,2,4)	(2,3,4)	(55,60,65)
Demand	(15,20,25)	(40,45,50)	(25,30,35)	(5,10,15)	(50,55,60)	(70,75,80)	(90,95,100)	(85,90,95)	(65,70,75)	(55,60,65)	

Applying the proposed algorithm [RCMM]:

Step 1:

$\sum_{i=1}^{m} s_i = (500,550,600)$ and $\sum_{j=1}^{n} d_j = (500,550,600)$.

$\Rightarrow \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ (Total supply = Total demand).

Since the given Fuzzy Transportation Problem is balanced. So go to step 3,

Step 3:

In first row, First maximum value = (12,14,16)

Second maximum value = (11,12,13)

The difference between 1st and 2nd maximum value

i.e., $(12,14,16) - (11,12,13) = (-1,2,5)$

Then replace the subtracted value instead of the first maximum value

i.e., $(12,14,16) = (-1,2,5)$

Similarly, apply step 3 other 2nd, 3rd up to 10th row, then we get table 2.

Table 2: *Row-wise Difference Table*

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	Supply
C_1	(3,5,7)	(4,6,8)	(3,6,9)	(11,12,13)	(2,3,4)	(7,8,9)	(5,6,7)	**(-1,2,5)**	(1,4,7)	(7,8,9)	(15,20,25)
C_2	(2,5,8)	(5,7,9)	(4,5,6)	(7,8,9)	**(0,3,6)**	(12,13,14)	(5,7,9)	(0,1,2)	(2,4,6)	(6,7,8)	(5,10,15)
C_3	(5,6,7)	(4,6,8)	(11,13,15)	(3,6,9)	(14,15,16)	(2,3,4)	(8,9,10)	(4,8,16)	(9,10,11)	**(-2,1,4)**	(25,30,35)
C_4	(9,10,11)	(2,5,7)	(2,3,4)	(3,5,7)	**(-2,4,10)**	(5,6,7)	(7,8,9)	(1,3,5)	(3,6,9)	(10,11,12)	(40,45,50)
C_5	(8,9,10)	(2,4,6)	(8,10,12)	(6,8,10)	(3,6,7)	(10,12,14)	(1,4,7)	**(0,4,8)**	(1,2,3)	(8,9,10)	(90,95,100)
C_6	(6,7,8)	(12,13,14)	**(-3,1,5)**	(1,2,3)	(1,3,5)	(3,5,7)	(6,8,10)	(2,4,6)	(7,8,9)	(13,15,17)	(70,75,80)
C_7	(5,6,7)	(12,14,16)	(13,15,17)	(5,6,7)	(0,1,2)	(11,13,15)	**(-3,1,5)**	(2,4,6)	(7,9,11)	(3,5,7)	(50,55,60)
C_8	**(0,3,6)**	(1,3,5)	(7,8,9)	(8,10,12)	(3,6,9)	(4,5,6)	(10,11,12)	(3,6,9)	(14,15,16)	(4,6,8)	(65,70,75)
C_9	(4,6,8)	(1,2,3)	(2,4,6)	**(1,3,5)**	(1,2,3)	(2,4,6)	(3,5,7)	(5,6,7)	(8,9,10)	(4,5,6)	(85,90,95)
C_{10}	(7,8,9)	(5,7,9)	(6,8,10)	(9,11,13)	(4,6,8)	(14,15,16)	(11,12,13)	**(-2,1,4)**	(0,2,4)	(2,3,4)	(55,60,65)
Demand	(15,20,25)	(40,45,50)	(25,30,35)	(5,10,15)	(50,55,60)	(70,75,80)	(90,95,100)	(85,90,95)	(65,70,75)	(55,60,65)	

Step 4: In table 2, apply the step 4 of proposed algorithm

In first column, First maximum value = (9,10,11), Second maximum value = (8,9,10)

The difference between 1st and 2nd maximum value = $(9,10,11) - (8,9,10) = (-1,1,3)$

Then replace the subtracted value instead of the first maximum value. $[(9,10,11) = (-1,1,3)]$ Similarly, apply step 4 other 2nd, 3rd up to 10th column, then we get table 3.

Table 3: *Column-wise difference table*

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	**Supply**
C_1	(3,5,7)	(4,6,8)	(3,6,9)	**(-2,1,4)**	(2,3,4)	(7,8,9)	(5,6,7)	(-1,2,5)	(1,4,7)	(7,8,9)	(15,20,25) - (5,10,15) = (0,10,20)
C_2	(2,5,8)	(5,7,9)	(4,5,6)	(7,8,9)	(0,3,6)	(12,13,14)	(5,7,9)	(0,1,2)	(2,4,6)	(6,7,8)	(5,10,15)
C_3	(5,6,7)	(4,6,8)	(11,13,15)	(3,6,9)	**(6,9,12)**	(2,3,4)	(8,9,10)	**(-5,2,13)**	(9,10,11)	(-2,1,4)	(25,30,35)
C_4	**(-1,1,3)**	(2,5,7)	(2,3,4)	(3,5,7)	(-2,4,10)	(5,6,7)	(7,8,9)	(1,3,5)	(3,6,9)	(10,11,12)	(40,45,50)
C_5	(8,9,10)	(2,4,6)	(8,10,12)	(6,8,10)	(3,6,7)	(10,12,14)	(1,4,7)	(0,4,8)	(1,2,3)	(8,9,10)	(90,95,100)
C_6	(6,7,8)	(12,13,14)	(-3,1,5)	(1,2,3)	(1,3,5)	(3,5,7)	(6,8,10)	(2,4,6)	(7,8,9)	**(1,4,7)**	(70,75,80)
C_7	(5,6,7)	**(-2,1,4)**	**(-2,2,6)**	(5,6,7)	(0,1,2)	(11,13,15)	(-3,1,5)	(2,4,6)	(7,9,11)	(3,5,7)	(50,55,60)
C_8	(0,3,6)	(1,3,5)	(7,8,9)	(8,10,12)	(3,6,9)	(4,5,6)	(10,11,12)	(3,6,9)	**(3,5,7)**	(4,6,8)	(65,70,75)
C_9	(4,6,8)	(1,2,3)	(2,4,6)	(1,3,5)	(1,2,3)	(2,4,6)	(3,5,7)	(5,6,7)	(8,9,10)	(4,5,6)	(85,90,95)
C_{10}	(7,8,9)	(5,7,9)	(6,8,10)	(9,11,13)	(4,6,8)	**(-1,2,5)**	**(-1,1,3)**	(-2,1,4)	(0,2,4)	(2,3,4)	(55,60,65)
Demand	(15,20,25)	(40,45,50)	(25,30,35)	(5,10,15)	(50,55,60)	(70,75,80)	(90,95,100)	(85,90,95)	(65,70,75)	(55,60,65)	

Step 5: Follow step 5 of the outlined procedure in table 4 to assign the initial allocation.

Table 4: *First allocation table*

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	**Supply**
C_1	(3,5,7)	(4,6,8)	(3,6,9)	**(5,10,15)** (-2,1,4)	(2,3,4)	(7,8,9)	(5,6,7)	(-1,2,5)	(1,4,7)	(7,8,9)	(15,20,25) - (5,10,15) = (0,10,20)
C_2	(2,5,8)	(5,7,9)	(4,5,6)	(7,8,9)	(0,3,6)	(12,13,14)	(5,7,9)	(0,1,2)	(2,4,6)	(6,7,8)	(5,10,15)
C_3	(5,6,7)	(4,6,8)	(11,13,15)	(3,6,9)	(6,9,12)	(2,3,4)	(8,9,10)	(-5,2,13)	(9,10,11)	(-2,1,4)	(25,30,35)
C_4	(-1,1,3)	(2,5,7)	(2,3,4)	(3,5,7)	(-2,4,10)	(5,6,7)	(7,8,9)	(1,3,5)	(3,6,9)	(10,11,12)	(40,45,50)
C_5	(8,9,10)	(2,4,6)	(8,10,12)	(6,8,10)	(3,6,7)	(10,12,14)	(1,4,7)	(0,4,8)	(1,2,3)	(8,9,10)	(90,95,100)
C_6	(6,7,8)	(12,13,14)	(-3,1,5)	(1,2,3)	(1,3,5)	(3,5,7)	(6,8,10)	(2,4,6)	(7,8,9)	(1,4,7)	(70,75,80)
C_7	(5,6,7)	(-2,1,4)	(-2,2,6)	(5,6,7)	(0,1,2)	(11,13,15)	(-3,1,5)	(2,4,6)	(7,9,11)	(3,5,7)	(50,55,60)
C_8	(0,3,6)	(1,3,5)	(7,8,9)	(8,10,12)	(3,6,9)	(4,5,6)	(10,11,12)	(3,6,9)	(3,5,7)	(4,6,8)	(65,70,75)
C_9	(4,6,8)	(1,2,3)	(2,4,6)	(1,3,5)	(1,2,3)	(2,4,6)	(3,5,7)	(5,6,7)	(8,9,10)	(4,5,6)	(85,90,95)
C_{10}	(7,8,9)	(5,7,9)	(6,8,10)	(9,11,13)	(4,6,8)	(-1,2,5)	(-1,1,3)	(-2,1,4)	(0,2,4)	(2,3,4)	(55,60,65)
Demand	(15,20,25)	(40,45,50)	(25,30,35)	(5,10,15)	(50,55,60)	(70,75,80)	(90,95,100)	(85,90,95)	(65,70,75)	(55,60,65)	

Step 6: Using step 6 of the proposed method, remove T_4 from table 4, and then the new reduction indicated in table 5, and again execute steps 5 to 6 for the second allocation shown in table 6.

Table 5: *New Reduced Table*

	T_1	T_2	T_3	T_5	T_6	T_7	T_8	T_9	T_{10}	**Supply**
C_1	(3,5,7)	(4,6,8)	(3,6,9)	(2,3,4)	(7,8,9)	(5,6,7)	(-1,2,5)	(1,4,7)	(7,8,9)	(0,10,20)
C_2	(2,5,8)	(5,7,9)	(4,5,6)	(0,3,6)	(12,13,14)	(5,7,9)	(0,1,2)	(2,4,6)	(6,7,8)	(5,10,15)
C_3	(5,6,7)	(4,6,8)	(11,13,15)	(6,9,12)	(2,3,4)	(8,9,10)	(-5,2,13)	(9,10,11)	(-2,1,4)	(25,30,35)
C_4	(-1,1,3)	(2,5,7)	(2,3,4)	(-2,4,10)	(5,6,7)	(7,8,9)	(1,3,5)	(3,6,9)	(10,11,12)	(40,45,50)
C_5	(8,9,10)	(2,4,6)	(8,10,12)	(3,6,7)	(10,12,14)	(1,4,7)	(0,4,8)	(1,2,3)	(8,9,10)	(90,95,100)
C_6	(6,7,8)	(12,13,14)	(-3,1,5)	(1,3,5)	(3,5,7)	(6,8,10)	(2,4,6)	(7,8,9)	(1,4,7)	(70,75,80)
C_7	(5,6,7)	(-2,1,4)	(-2,2,6)	(0,1,2)	(11,13,15)	(-3,1,5)	(2,4,6)	(7,9,11)	(3,5,7)	(50,55,60)
C_8	(0,3,6)	(1,3,5)	(7,8,9)	(3,6,9)	(4,5,6)	(10,11,12)	(3,6,9)	(3,5,7)	(4,6,8)	(65,70,75)
C_9	(4,6,8)	(1,2,3)	(2,4,6)	(1,2,3)	(2,4,6)	(3,5,7)	(5,6,7)	(8,9,10)	(4,5,6)	(85,90,95)
C_{10}	(7,8,9)	(5,7,9)	(6,8,10)	(4,6,8)	(-1,2,5)	(-1,1,3)	(-2,1,4)	(0,2,4)	(2,3,4)	(55,60,65)
Demand	(15,20,25)	(40,45,50)	(25,30,35)	(50,55,60)	(70,75,80)	(90,95,100)	(85,90,95)	(65,70,75)	(55,60,65)	

Table 6: *Second allocation table*

	T_1	T_2	T_3	T_5	T_6	T_7	T_8	T_9	T_{10}	**Supply**
C_1	(3,5,7)	(4,6,8)	(3,6,9)	(2,3,4)	(7,8,9)	(5,6,7)	(-1,2,5)	(1,4,7)	(7,8,9)	(0,10,20)
C_2	(2,5,8)	(5,7,9)	(4,5,6)	(0,3,6)	(12,13,14)	(5,7,9)	**(5,10,15)** (0,1,2)	(2,4,6)	(6,7,8)	(5,10,15)
C_3	(5,6,7)	(4,6,8)	(11,13,15)	(6,9,12)	(2,3,4)	(8,9,10)	(-5,2,13)	(9,10,11)	(-2,1,4)	(25,30,35)
C_4	(-1,1,3)	(2,5,7)	(2,3,4)	(-2,4,10)	(5,6,7)	(7,8,9)	(1,3,5)	(3,6,9)	(10,11,12)	(40,45,50)
C_5	(8,9,10)	(2,4,6)	(8,10,12)	(3,6,7)	(10,12,14)	(1,4,7)	(0,4,8)	(1,2,3)	(8,9,10)	(90,95,100)
C_6	(6,7,8)	(12,13,14)	(-3,1,5)	(1,3,5)	(3,5,7)	(6,8,10)	(2,4,6)	(7,8,9)	(1,4,7)	(70,75,80)
C_7	(5,6,7)	(-2,1,4)	(-2,2,6)	(0,1,2)	(11,13,15)	(-3,1,5)	(2,4,6)	(7,9,11)	(3,5,7)	(50,55,60)
C_8	(0,3,6)	(1,3,5)	(7,8,9)	(3,6,9)	(4,5,6)	(10,11,12)	(3,6,9)	(3,5,7)	(4,6,8)	(65,70,75)
C_9	(4,6,8)	(1,2,3)	(2,4,6)	(1,2,3)	(2,4,6)	(3,5,7)	(5,6,7)	(8,9,10)	(4,5,6)	(85,90,95)
C_{10}	(7,8,9)	(5,7,9)	(6,8,10)	(4,6,8)	(-1,2,5)	(-1,1,3)	(-2,1,4)	(0,2,4)	(2,3,4)	(55,60,65)
Demand	(15,20,25)	(40,45,50)	(25,30,35)	(50,55,60)	(70,75,80)	(90,95,100)	(85,90,95) -(5,10,15) =(70,80,90)	(65,70,75)	(55,60,65)	

Step 7: Using Steps 5 to 6 of the proposed technique once again, all allocations are made as indicated in Table 7.

Table 7: *Final allocations of fuzzy transportation table*

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
C_1	(3,5,7)	(4,6,8)	(3,6,9)	**(5,10,15)** (-2,1,4)	(2,3,4)	(7,8,9)	(5,6,7)	**(0,10,20)** (-1,2,5)	(1,4,7)	(7,8,9)
C_2	(2,5,8)	(5,7,9)	(4,5,6)	(7,8,9)	(0,3,6)	(12,13,14)	(5,7,9)	**(5,10,15)** (0,1,2)	(2,4,6)	(6,7,8)
C_3	(5,6,7)	(4,6,8)	(11,13,15)	(3,6,9)	(6,9,12)	(2,3,4)	(8,9,10)	(-5,2,13)	(9,10,11)	**(25,30,35)** (-2,1,4)
C_4	**(15,20,25)** (-1,1,3)	(2,5,7)	(2,3,4)	(3,5,7)	(-2,4,10)	(5,6,7)	(7,8,9)	**(15,25,35)** (1,3,5)	(3,6,9)	(10,11,12)
C_5	(8,9,10)	(2,4,6)	(8,10,12)	(6,8,10)	(3,6,7)	(10,12,14)	**(15,25,35)** (1,4,7)	(0,4,8)	**(65,70,75)** (1,2,3)	(8,9,10)
C_6	(6,7,8)	(12,13,14)	**(25,30,35)** (-3,1,5)	(1,2,3)	(1,3,5)	(3,5,7)	(6,8,10)	**(35,45,55)** (2,4,6)	(7,8,9)	(1,4,7)
C_7	(5,6,7)	**(40,45,50)** (-2,1,4)	(-2,2,6)	(5,6,7)	**(0,10,20)** (0,1,2)	(11,13,15)	(-3,1,5)	(2,4,6)	(7,9,11)	(3,5,7)
C_8	(0,3,6)	(1,3,5)	(7,8,9)	(8,10,12)	(3,6,9)	**(5,30,55)** (4,5,6)	**(-10,10,30)** (10,11,12)	**(-40,0,40)** (3,6,9)	(3,5,7)	**(20,30,40)** (4,6,8)
C_9	(4,6,8)	(1,2,3)	(2,4,6)	(1,3,5)	**(30,45,60)** (1,2,3)	**(25,45,65)** (2,4,6)	(3,5,7)	(5,6,7)	(8,9,10)	(4,5,6)
C_{10}	(7,8,9)	(5,7,9)	(6,8,10)	(9,11,13)	(4,6,8)	(-1,2,5)	**(55,60,65)** (-1,1,3)	(-2,1,4)	(0,2,4)	(2,3,4)

As a result, $(m + n - 1) = (10 + 10 - 1 = 19,$ cells are assigned and we have a feasible solution. Then find the minimum fuzzy transportation cost.

Step 8: Calculate the minimum Fuzzy Transportation Cost. Total cost $Z = \sum_{i=1}^{m} \mathcal{C}_{ij} \sum_{j=1}^{n} \mathcal{X}_{ij}$.
⇒Z = (5,10,15) (-2,1,4) + (0,10,20) (-1,2,5) + (5,10,15) (0,1,2) + (25,30,35) (-2,1,4) + (15,20,25) (-1,1,3) + (15,25,35) (1,3,5) + (15,25,35) (1,4,7) + (65,70,75) (1,2,3) + (25,30,35) (-3,1,5) + (35,45,55) (2,4,6) + (40,45,50) (-2,1,4) + (0,10,20) (0,1,2) + (5,30,55) (4,5,6) + (-10,10,30) (10,11,12) + (-40,0,40) (3,6,9) + (20,30,40) (4,6,8) + (30,45,60) (1,2,3) + (25,45,65) (2,4,6) + (55,60,65) (-1,1,3)
Z = (-160, 1440, 3930)

4.1 Result and discussion:

The fuzzy transportation cost Z of the given FTP is a TFN as given below:

Z = (-160, 1440, 3930).

The result can be explained (Refer to Fig. 2) as follows:

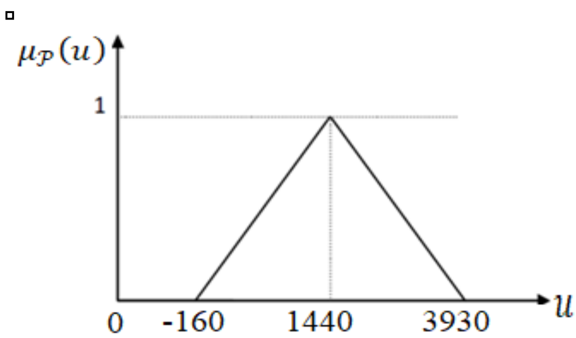

Fig. 2 fuzzy transportation cost

The least amount of the minimum total transportation cost is -160.
The most possible amount of the minimum total transportation cost is 1440.

The greatest amount of the minimum total transportation cost is 3930. i.e., the minimum total transportation cost will always be greater than -160 and less than 1440, and highest chances are that the minimum total transportation cost will be 3930.
The above result was verified by MATLAB.

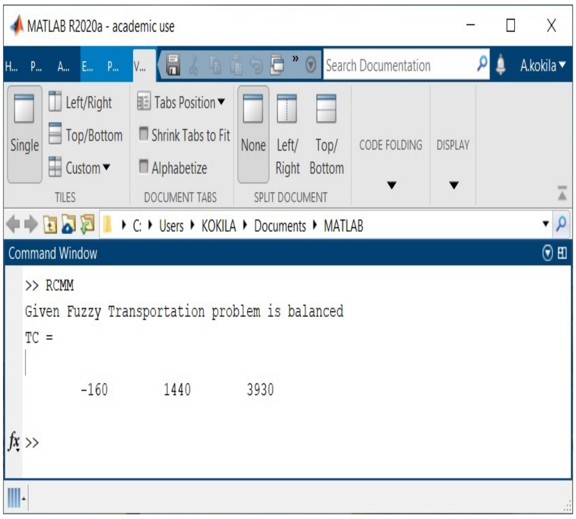

Fig 3. Result of the Fuzzy Transportation problem

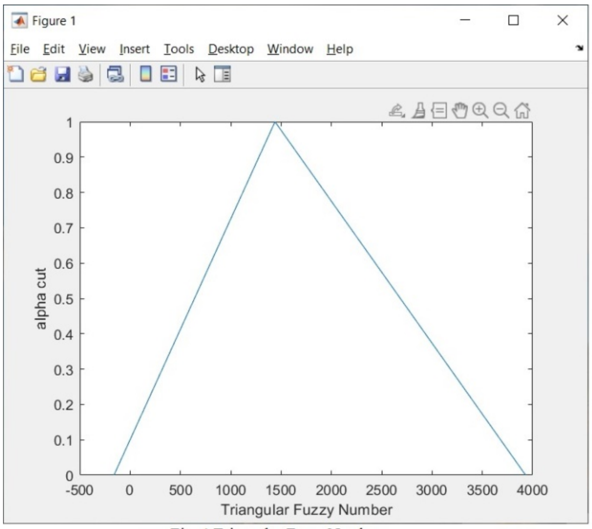

Fig. 4 Triangular Fuzzy Number

Table 8: *Comparative results of NWCM, LCM, RMM, VAM and proposed method (RCMM) for example 1*

Numerical example	NWCM	LCM	RMM	VAM	Proposed method (RCMM)
1	(-570,4050,11080)	(445,2160,4755)	(455,2110,4775)	(-70,1760,4260)	(-160,1440,3930)

The comparative results in table 8 are also depicted using bar graphs and the results are given in the Figure 5.

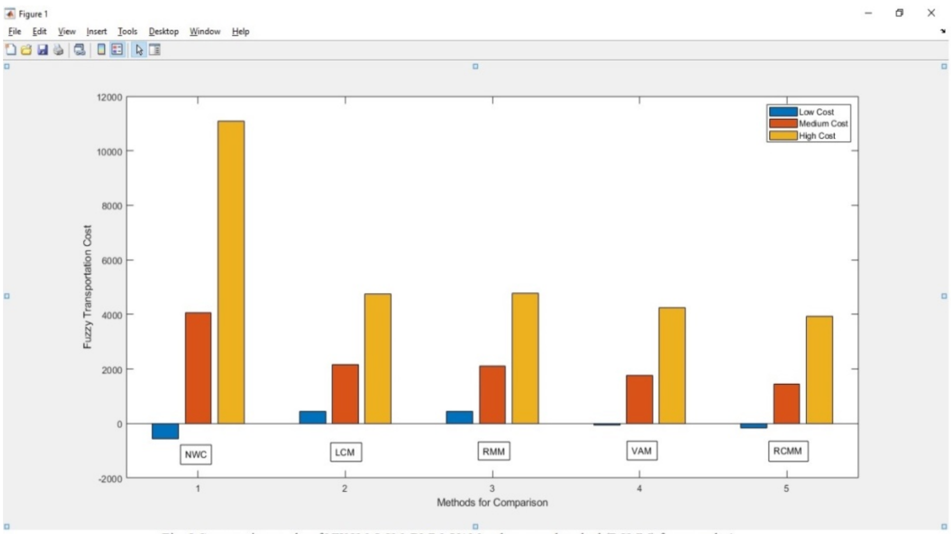

Fig. 5 Comparative results of NWCM, LCM, RMM, VAM and proposed method (RCMM) for example 1

4.2 Comparison of results: The numerical examples 2, 3, 4, 5, 6 are taken from the referred journals 1, 8, 9, 10, 14 respectively, and it is verified with our proposed method and the existing methods NWCM, LCM, RMM, VAM.

Table 9: *Comparative results of NWCM, LCM, RMM, VAM and proposed method (RCMM) for example 2 to 6*

Numerical examples	NWCM	LCM	RMM	VAM	Proposed method	RCMM
2[Ref:1]	(68,176,316)	(62,150,258)	(62,150,258)	(52,149,274)	(62, 150, 258)	(-2,83,208)
3[Ref:8]	(1850,6609,12882)	(1790,6609,12998)	(1900,8292,19080)	(1850,6609,12882)	(3532, 6609, 9852)	(-462,3249,8798)
4[Ref:9]	(40,1230.3560)	(-120,1210,3860)	(140,1250,3220)	(120,1210,3140)	(270,1210,2750)	(-1140,500,3760)
5[Ref:10]	(125,1000,2950)	(-175,850,2925)	(-275,950,3450)	(-25,850,2625)	(-75,850,2750)	(-175,350,2425)
6[Ref:14]	(-270,4285,10470)	(160,2455,5470)	(-330,2290,6500)	(-25,2220,5455)	(1825,2455,3085)	(-340,1025,4180)

5. Conclusion and future study

Our proposed method uses the comparison table to find the best initial feasible solution to the balanced and unbalanced fuzzy transportation problems. We compared our strategy to others and discovered that ours is the most effective. This technique considers the entire fuzzy cost of each origin and destination for allotment, allowing for a reduction in iterations to provide the best basic feasible solution to FTP. In addition, the proposed method is used to achieve the best solution for an unbalanced TP by converting it to a balanced TP without the need of a dummy source/destination, saving time and space. The proposed method is simple to implement and can be used to solve a variety of fuzzy transportation problems, including minimizing the total transportation costs. In the future, this technique might be expanded to fuzzy multiple objective transportation problems and used to solve real-world transportation problems using fuzzy numbers.

Conflict of interest: There are no conflicts of interest declared by the authors.

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A METHOD FOR GENERATING LIFETIME MODELS AND ITS APPLICATION TO REAL DATA

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Abstract

In the present work, we are going to propose a new transformation called Beta transformation. The new model includes the exponential distribution as a special case and it is known as Beta transformed exponential(BTE) distribution. We have been obtained its various statistical properties such as moments, moment generating function, median, hazard rate function, entropies, and order statistics. Parameters of BTE distribution are estimated by the method of maximum likelihood, Cramer-von-Mises and method of least square. Monte Carlo simulation is performed in order to investigate the performance of these estimates. Finally, two data sets have been analyzed to show how the proposed model works in practice.

Keywords: Cramer-von-Mises method, Exponential distribution, Hazard rate function, Method of maximum likelihood, , Method of least squares.

1. Introduction

The development of new methods of expanding the existing distributions is quite rich in the literature of distribution theory. There are several methods to propose new distributions by the use of some baseline distribution in statistical literature. This has been done through different approaches.
In Statistical literature no. of transformations are available to produce new cumulative distribution function (cdf) corresponding to a given cdf. Suppose, we have a cdf F(x), then the associated proposed cdf will be $G_i(x)$.

- The most popular among them is the power transformation initiated by Gupta et al. (1998) having the form
$$G_1(x) = [F(x)]^{\alpha}; \alpha > 0$$
- Quadratic rank transformation map (QRTM) proposed by Shaw and Buckley (2007) having the form
$$G_2(x) = (1+\lambda)F(x) - \lambda F^2(x); \mid \lambda \mid \leq 1$$
- DUS transformation proposed by Kumar et al. (2015) having the form
$$G_3(x) = \frac{e^{F(x)} - 1}{e - 1}; e = exp(1)$$
- SS-transformation proposed by Kumar et al. (2015) having the form
$$G_4(x) = sin(\frac{\pi}{2}F(x))$$

- Minimum Guarantee (MG)-distribution proposed by Kumar et al. (2017) having the form

$$G_5(x) = e^{1-\frac{1}{F(x)}}$$

- Log-transformation proposed by Maurya et al. (2016) and having the form

$$G_6(x) = 1 - \frac{\ln(2 - F(x))}{\ln 2}$$

- Transformation based on the generalization of Kumar et al. (2015) called GDUS transformation proposed by Maurya et al. (2017) having the form

$$G_7(x) = \frac{e^{F^{\alpha}(x)} - 1}{e - 1}; \alpha > 0$$

- New transformation initiated by Kyurkchiev (2017) to develop a sigmoid family of functions for Verhulst Logistic function is

$$G_9(x) = \frac{2F(x)}{1 + F(x)}$$

- New trigonometry based transformation called PCM proposed by Kumar et al. (2021) and having the form

$$G_{10}(x) = tan(\frac{\pi}{4}F(x))$$

The lifetime of a system can be modeled with statistical distributions that can be used in modeling lifetime data; among them, the most popular are gamma and weibull distributions. The proposed model contains several lifetime distributions as its special cases that are very flexible and able to accommodate different types of data sets since the probability density function and hazard rate can take on different forms such as increasing, decreasing, and constant shapes, and the potentiality of this model has been tested statistically by using it to model some real life data set.
In this article, We have decided to propose a new transformation known as beta transformation for $x \in \Re$ is given below

$$G(x) = \begin{cases} \frac{\beta}{\beta-1}[1 - \beta^{-F(x)}] & \text{if } \beta > 0, \beta \neq 1 \\ F(x) & \text{if } \beta = 1 \end{cases} \tag{1}$$

Where, $G(x)$ and $F(x)$ are the cdfs of the proposed transformation and baseline distribution. On differentiating (1) w.r.t. x, we get the probabilty density function (pdf) $g(x)$ and is given by

$$g(x) = \begin{cases} \frac{\beta \log \beta}{\beta-1} f(x)\beta^{-F(x)} & \text{if } \beta > 0, \beta \neq 1 \\ f(x) & \text{if } \beta = 1 \end{cases} \tag{2}$$

For $\beta \neq 1$, $g(x)$ is a weighted version of $f(x)$, where the weight function

$$w(x) = \beta^{-F(x)},$$

and g(x) can be written as

$$g(x) = \frac{f(x)w(x)}{c}.$$

where constant $c = E(w(X))$,
Here $c = \frac{\beta-1}{\beta \log \beta}$.

The survival reliability function(sf) $S(x)$ and the hazard rate function(hrf) $h(x)$ are obtained as

$$S(x) = \begin{cases} \frac{\beta^{1-F(x)}-1}{\beta-1} & \text{if } \beta \neq 1 \\ 1-F(x) & \text{if } \beta = 1 \end{cases} \tag{3}$$

and

$$h(x) = \begin{cases} f(x)\frac{\log\beta\beta^{1-F(x)}}{\beta^{1-F(x)}-1} & \text{if } \beta \neq 1 \\ \frac{f(x)}{S(x)} & \text{if } \beta = 1 \end{cases} \tag{4}$$

Lifetime models are used to explain the life of a system or device. These models are used in reliability, engineering, biological field, insurance, etc. The motivations for introducing our beta transformation model is that it is efficient to analyze lifetime data and very easy method of inducting an additional parameter to a family of distributions functions. It improve the characteristics, bring more flexibility to the given family and provide better fits than the other models having the same or higher number of parameters. The proposed method is very interesting with a closed form for the cdf and capable of modeling heavy tailed data sets.
The aim of this article is to introduce a transformation that yields new distributions by using a given baseline distribution. It contains only one new parameter other than the parameters involved in the baseline distribution. To illustrate the usefulness of this new transformation, We choose exponential as the baseline distributions in the present work.
The rest of this work is as follows. In Section 2, We introduce a special sub-case of (1), called a beta transformed exponential(BTE) distribution by considering exponential model as a parent distribution. Some mathematical properties are derived in Section 3. Certain characterizations of the proposed distribution are provided in Section 4. Estimation of parameter has been carried out in Section 5, Simulation study have been discussed in Section 5. Illustrate the flexibility of models using two real-life data sets discussed in Section 7. Finally, the article is concluded in Section 8.

2. Beta transformed exponential distribution

In this section, a sub model of the beta transformed family, called the beta transformed exponential (BTE) distribution is introduced. Let $G(x;\theta)$ be cdf of the exponential random variable given by $G(x;\theta) = 1 - e^{-\theta x}; x, \theta > 0$. Using this in equation(1), then the cdf of the BTE for $x > 0$ with the shape and scale parameters as $\beta > 0$ and $\theta > 0$ has the following form

$$G(x) = \begin{cases} \frac{\beta}{\beta-1}[1-\beta^{e^{-\theta x-1}}] & \text{if } \beta \neq 1 \\ 1-e^{-\theta x} & \text{if } \beta = 1 \end{cases} \tag{5}$$

The pdf g(x) is given by

$$g(x) = \begin{cases} \frac{\theta\log\beta}{\beta-1}e^{-\theta x}\beta^{e^{-\theta x}} & \text{if } \beta \neq 1 \\ \theta e^{-\theta x} & \text{if } \beta = 1 \end{cases} \tag{6}$$

The survival reliability function $S(x)$ and the hazard rate function(hrf) $h(x)$ are obtained as

$$S(x) = \begin{cases} \frac{\beta^{e^{-\theta x}}-1}{\beta-1} & \text{if } \beta \neq 1 \\ e^{-\theta x} & \text{if } \beta = 1 \end{cases} \tag{7}$$

and

$$h(x) = \begin{cases} \frac{\theta\log\beta}{\beta^{e^{-\theta x}}-1}e^{-\theta x}\beta^{e^{-\theta x}} & \text{if } \beta \neq 1 \\ \theta & \text{if } \beta = 1 \end{cases} \tag{8}$$

We have the following results for a general distribution function $F(x)$.

Table 1: *Behavior of the hazard functions of the three distributions.*

Parameter	Gamma	Weibull	BTE
$\beta = 1$	θ	θ	θ
$\beta > 1$	Increasing from 0 toθ	Increasing from 0 to ∞	Decreasing from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ
$\beta < 1$	Decreasing from ∞ toθ	Decreasing from ∞ to 0	Increasing from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ

- If $f(x)$ is a decreasing function, and $\beta \geq 1$, then $g(x)$ is a decreasing function.
- If $f(x)$ is a decreasing function, and $f(x)$ is log-convex, then for $\beta \geq 1$, the hazard rate function $h(x)$ is a decreasing function.

It can be easily seen that $f(x;\beta,\theta)$ is a unimodal function with mode at $\frac{(log(log\beta))}{\theta}$.
Here note that,
$\lim_{x\to 0} h(x) = \frac{\beta\theta\log\beta}{\beta-1}$,and
$\lim_{x\to\infty} h(x) = \theta$.
We have the following cases:

- When $\beta < 1$, $h(x)$ is an increasing function increases from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ
- When $\beta > 1$, $h(x)$ is an decreasing function decreases from $\frac{\beta\theta\log\beta}{\beta-1}$ to θ
- When $\beta = 1$, $h(x)$ is a constant function.

By taking the second derivative of $f(x;\beta,\theta)$, it easily follows that the pdf of $BTE(\beta,\theta)$ is log-convex if $\beta > 1$ and log-concave if $\beta < 1$;
Table 1 provides the comparison of the hazard function of the BTE distribution with the corresponding hazard functions of Weibull and Gamma distributions. In all these cases the shape and scale parameters are assumed to be β and θ, respectively. It is clear from Table 1 that the hazard function of the BTE distribution is a decreasing or an increasing function depending on the shape parameter similarly as the Gamma and Weibull distributions, the ranges are quite different.

Figure 1 and 2 provides the plots of the pdf and hrf of the model for different values of β when $\theta = 1$

BTE distribution for $\beta > 1$, $\frac{\beta\log\beta}{\beta-1}$ is a decreasing function from 1 to 0, as β varies from 1 to ∞. If $X \sim BTE(\beta,\theta)$, then BTE distribution has the following mixture representation:

$$X = \begin{cases} X_1 \; with \; probability & \text{if } \frac{\log\beta}{\beta-1} \\ X_2 \; with \; probability & \text{if } 1 - \frac{\log\beta}{\beta-1}, \end{cases} \tag{9}$$

where X_1 and X_2 have the following pdfs:

$$f_{X_1(x)} = \theta e^{-\theta x}; x > 0 \tag{10}$$

$$f_{X_2(x)} = \frac{\log\beta}{\beta - 1 - \log\beta}\theta e^{-\theta x}(\beta^{e^{-\theta x}} - 1); x > 0, \tag{11}$$

respectively. From (9), as β approaches 1, X behaves like an exponential distribution, and as β increases, it behaves like X_2.

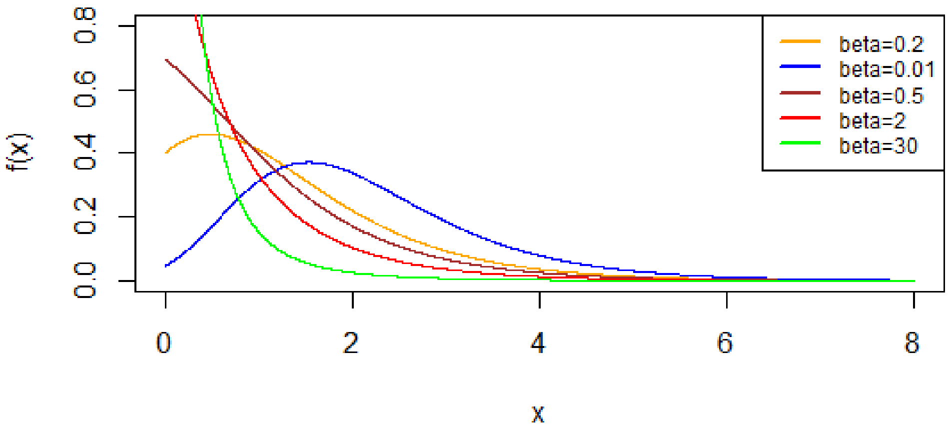


Figure 1: *plot of pdf of distribution*

3. The basic mathematical properties

This section provides some mathematical properties of proposed distribution.

3.1. Quantile function

The q^{th} quantile x_q of the BTE random variable is given by

$$x_q = -\frac{1}{\theta} \log \left[1 + \frac{\log(1 - \frac{q(\beta-1)}{\beta})}{\log \beta} \right]. \tag{12}$$

3.2. Moments

In this subsection, we intend to derive the moments and the moment generating function of the BTE distribution. Let X follow (6), then, the r^{th} moment of X is derived as

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x; \beta, \theta) dx, \tag{13}$$

using(6)in(13), we get

$$\mu_r' = \frac{r!}{\theta^r(\beta - 1)} \sum_{k=1}^{\infty} \frac{(\ln \beta)^k}{k! k^n} \tag{14}$$

Furthermore, a general expression for the moment generating function (mgf) of the BTE random variable X is given by

$$M_X(t) = \frac{\theta}{(\beta - 1)} \sum_{k=0}^{\infty} \frac{(\ln \beta)^{k+1}}{k!} \left[\frac{1}{\theta + \theta k - t} \right]; t < \theta \tag{15}$$

3.3. Sample Generation

The method to generate a sample is the inverse CDF transformation method. If X is $U(0, 1)$ with CDF $F(x)$, then by the transformation, we generate the sample from the equation $G(x) = U$

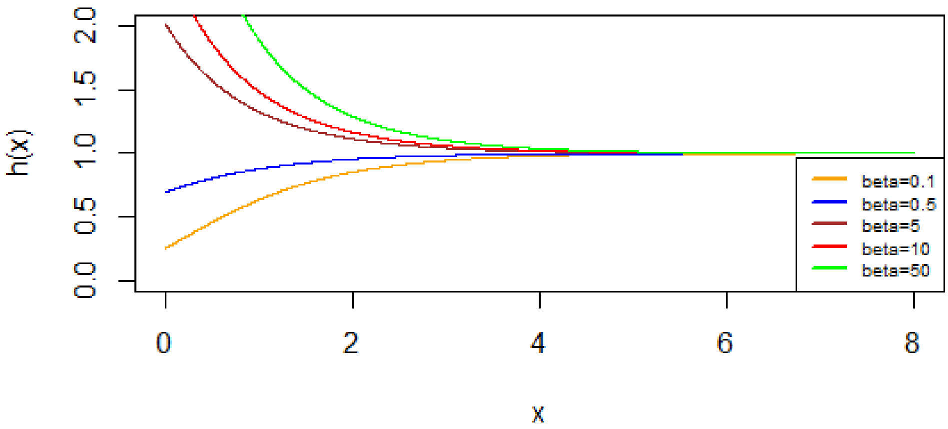


Figure 2: *plot of hrf of distribution*

implies $x = G^{-1}(U)$ of BTE distribution

$$x = -\frac{1}{\theta}\log\left[1 + \frac{\log(1 - \frac{U(\beta-1)}{\beta})}{\log\beta}\right]. \tag{16}$$

3.4. Order Statistics

Order statistics are used in applied fields of statistics such as reliability and lifetime testing. Let $X_1, X_2, ..., X_n$ be a random sample from $BTE(\beta,\theta)$. Also, let $X_{(1)}, X_{(2)}, ..., X_{(n)}$, denote the corresponding order statistics. Then the pdf and cdf of k^{th} order statistics, are given by

$$\begin{aligned} f_X(x) &= \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}\,[1-F(x)]^{n-k}\,f(x) \\ &= \frac{n!}{(k-1)!(n-k)!}\frac{\theta\log\beta}{\beta-1}e^{-\theta x}\beta^{e^{-\theta x}}[\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]]^{k-1} \\ &\left[1-[\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]]\right]^{n-k} \end{aligned} \tag{17}$$

and

$$\begin{aligned} F_X(x) &= \sum_{j=k}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)[F(x)]^j[1-F(x)]^{n-j} \\ &= \sum_{j=k}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\left[\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]\right]^j \\ &\left[1-[\frac{\beta}{\beta-1}[1-\beta^{-F(x)}]]\right]^{n-j} \end{aligned} \tag{18}$$

respectively.
The pdf of the minimum and maximum of order statistics are obtained by putting $X = X_1$ and $X = X_n$ respectively in equation (6).

3.5. Entropy

The entropy of a random variable measures the variation of the uncertainity. A large value of entropy indicates the greater uncertainty in the data. The concept of entropy is important in different areas such as physics, probability and statistics, communication theory, and economics, etc. Several measures of entropy have been studied and compared in the literature.
If X is an absolute continuous random variable with $RE_X(\rho)$ for $\rho > 0$ and $\rho \neq 1$, is defined as

$$RE_X(\rho) = \frac{1}{1-\rho} \log \left[\int_{-\infty}^{\infty} f(x)^{\rho} dx \right] \tag{19}$$

From equation(19), we get

$$RE_X(\rho) = \frac{\rho}{1-\rho} \log \left(\frac{\theta \log \beta}{\beta - 1} \right) + \frac{1}{1-\rho} \log \left(\sum_{k=0}^{\infty} \frac{\ln(\beta)^k}{k! \rho (k+1)} \right). \tag{20}$$

4. Characterization of Beta transformed exponential distribution

In this section, we present certain characterizations of the BTE distribution based on a simple relationship between two truncated moments. This characterization result employs a theorem due to Glanzel (1987), which stated as follows:

Theorem 1. Let (Ω,F, $\mathbf{P}$) be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ ($a = -\infty$, $b = \infty$ might as well be allowed). Let $X : \Omega \to \mathbf{H}$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on $\mathbf{H}$ such that

$$E[q_2(X)|X \geq x] = E[q_1(X)|X \geq x]\eta(x), x\epsilon\mathbf{H},$$

is defined with some real function η. Assume that q_1, q_2 are continuous functions, η has continuous derivative and F is twice continuously differentiable and strictly monotone function on the set $\mathbf{H}$. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of $\mathbf{H}$. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x \mathcal{C} \mid \frac{\eta^{'}(\mu)}{\eta(\mu)q_1(u) - q_2(u)} \mid \exp(-s(u))du,$$

where the function s is a solution of the differential equation $s^{'} = \frac{\eta^{'} q_1}{\eta q_1 - q_2}$ and $\mathcal{C}$ is a constant, chosen to make $\int_{\mathbf{H}} dF = 1$.

Proposition 1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x) = \beta^{-e^{-\theta x}}$ and $q_2(x) = q_1(x)e^{-\theta x}$ for $x > 0$. The random variable X has pdf (6) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{\theta + 1}{\theta} e^{-x}, x > 0$$

Proof. Let X be a random variable with pdf (6), then
$(1 - F(x))E[q_1(X) \mid X \geq x] = \frac{\log \beta}{\beta - 1} e^{-\theta x}, x > 0$
and
$(1 - F(x))E[q_2(X) \mid X \geq x] = \frac{\theta \log \beta}{\beta - 1} \frac{e^{-x(\theta+1)}}{\theta+1}, x > 0$
and finally
$\eta(x)q_1(x) - q_2(x) = \frac{q_1 e^{-x}}{\theta} > 0, for x > 0.$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta q_1(x) - q_2(x)} = -(1+\theta), x > 0$$

and hence
$s(x) = -(1+\theta)x, x > 0$, or $e^{-s(x)} = e^{(1+\theta)x}, x > 0$. Now, in view of Theorem 1, X has density(6). ■

Corollary 1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 1. The pdf of X is (6) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta q_1(x) - q_2(x)} = -(1+\theta), x > 0$$

Remark 1. The general solution of the differential equation in Corollary 1 is

$$\eta(x) = e^{-(1+\theta)x} \left[\int (1+\theta)[q_1(x)]^{-1} q_2(x) e^{(1+\theta)x} dx + D \right]$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 1 with $D = 0$. However, it should be also noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

5. Estimation and simulation

In this section, we use the method of maximum likelihood, method of Cramer-von-Mises and ordinary least square method for estimation of parameters of BTE distributions.

5.1. Method of Maximum Likelihood Estimation

This is an extensively used method initiated by C.F. Gauss and elaborative study initiated by Prof. R. A. Fisher to obtain the estimator of the unknown parameter of the distribution. If $X_1, X_2 ..., X_n$ be a set of random observations from the population $BTE(\beta, \theta)$ distribution having pdf $g(x; \beta, \theta)$, then its log likelihood function will be as follows

$$\log L = n \log \theta + n \log \left(\frac{\log \beta}{\beta - 1} \right) - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} e^{-\theta x_i} \log(\beta). \tag{21}$$

The likelihood equations are,

$$\frac{\partial \log L}{\partial \beta} = \frac{n(\beta - 1 - \beta \log \beta)}{\beta(\beta - 1) \log \beta} + \frac{1}{\beta} e^{-\theta x_i} = 0, \tag{22}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-\theta x_i} = 0. \tag{23}$$

The MLE of β and θ can be obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as the Newton-Raphson algorithm.

5.2. Method of Cramer-von Mises

Cramer-von-Mises type minimum distance estimators are based on minimizing the distance between the theoretical and empirical cumulative distribution functions. Macdonald(1971) provided empirical evidence that the bias of these estimators is smaller than the bias of other minimum distance estimators. The Cramer-von-Mises estimators, $\hat{\beta}_{CME}$ and $\hat{\theta}_{CME}$ are the values of β and θ minimizing

$$C(\beta,\theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(t_i \mid \beta,\theta) - \frac{2i-1}{2n} \right]^2.$$

Differentiating the above equation partially, with respect to the parameters β and θ respectively and equating them to zero, we get the normal equations. Since the normal equations are non-linear, we can use iterative method to obtain the solution.

5.3. Method of Least-Square Estimation

The least square estimators were proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. Here, we apply the same technique for the BTE distribution. The least square estimators of the unknown parameters β and θ of BTE distribution can be obtained by minimizing

$$\sum_{i=1}^{n} \left[F(t_i \mid \beta,\theta) - \frac{i}{n+1} \right]^2.$$

with respect to unknown parameters β and θ.

5.4. Simulation study

We conduct Monte Carlo simulation studies to compare the performance of the estimators discussed in the previous sections and the process is repeated 1000 times. We evaluate the performance of the estimators based on bias and mean squared error. Methods are compared for sample sizes $n = 500, 700$ and 1000.

For each estimate we calculate the mean-squared error. The statistics are obtained using the following formulae.

$MSE(\hat{\beta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\beta} - \beta)^2$ $\quad MSE(\hat{\theta}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\theta} - \theta)^2$

The estimates, and the mean square errors (MSE) of the parameter estimates for the Maximum likelihood estimation procedure, method of Cramer-von-Mises and method of least squares are presented in Tables 1-3.

From Tables , we note that the maximum likelihood method performs well for estimating the model parameters. Also, as the sample size increases, the MSEs of the average estimates of maximum likelihood estimates decrease as expected.

The following observations can be drawn from the Tables 1-3.

1. All the estimators show the property of consistency, i.e. the MSE decreases as the sample size increases.
2. The MSE of $\hat{\beta}$ decreases with an increasing n for all the method of estimations.
3. The MSE of $\hat{\theta}$ decreases with an increasing n for all the method of estimations.
4. The MSE of $\hat{\beta}$ and $\hat{\theta}$ generally increases with an increasing beta and theta for any given n in all methods of estimation.
5. In terms of MSE, all the methods of estimation produce smaller MSE for $\hat{\beta}$ compared to that of $\hat{\theta}$.

Table 2: *Simulation result for $\beta = 0.5$ and $\theta = 0.1$.*

n	$Est.$	MLE	CVM	LSE
500	$\hat{\beta}$	0.5173	0.5087	0.5245
	$\hat{\theta}$	0.1004	0.1004	0.1021
	$MSE(\hat{\beta})$	5.247×10^{-5}	7.3129×10^{-5}	7.4119×10^{-5}
	$MSE(\hat{\theta})$	0.0209	0.0244	0.0311
700	$\hat{\beta}$	0.5171	0.5222	0.5306
	$\hat{\theta}$	0.1000	0.1003	0.0994
	$MSE(\hat{\beta})$	3.702×10^{-5}	7.0427×10^{-5}	5.4085×10^{-5}
	$MSE(\hat{\theta})$	0.0150	0.0220	0.0260
1000	$\hat{\beta}$	0.5013	0.5062	0.5845
	$\hat{\theta}$	0.1004	0.1005	0.0972
	$MSE(\hat{\beta})$	2.495×10^{-5}	4.0516×10^{-5}	3.9591×10^{-5}
	$MSE(\hat{\theta})$	0.0093	0.0129	0.0187

Table 3: *Simulation result for $\beta = 0.9$ and $\theta = 0.5$.*

n	$Est.$	MLE	CVM	LSE
500	$\hat{\beta}$	0.9315	0.9527	0.9941
	$\hat{\theta}$	0.5044	0.5049	0.4966
	$MSE(\hat{\beta})$	0.0019	0.0029	0.0025
	$MSE(\hat{\theta})$	0.0943	0.1899	0.1144
700	$\hat{\beta}$	0.9196	0.9489	0.9046
	$\hat{\theta}$	0.5018	0.5044	0.5011
	$MSE(\hat{\beta})$	0.0013	0.0025	0.0016
	$MSE(\hat{\theta})$	0.0664	0.1047	0.0556
1000	$\hat{\beta}$	0.9260	0.9310	0.9120
	$\hat{\theta}$	0.5004	0.4948	0.5034
	$MSE(\hat{\beta})$	0.0009	0.0014	0.0014
	$MSE(\hat{\theta})$	0.0436	0.0503	0.0484

6. Applications

In this section, we consider two real life data sets to illustrate the importance of the proposed distribution. The model parameters are estimated by the method of maximum likelihood and compare the fit of the BTE distribution with the following distributions: KuE,EW,W and E models.

(a) Kumaraswamy Exponential (KuE) distribution having pdf

$$f(x;\theta,\beta,c) = \theta\beta c e^{-cx}(1-e^{cx})^{\theta-1}[1-(1-e^{-cx})^{\theta}]^{\beta-1}; x > 0, \theta, \beta, c > 0. \tag{24}$$

(b) Exponentiated Weibull (EW) distribution having pdf

$$f(x;\theta,\beta,c) = \theta\beta^{\theta} c x^{\theta-1} e^{-}(\beta x)^{\theta}(1-e^{-(\beta x)^{\theta}})^{c-1}; x > 0, \theta, \beta, c > 0. \tag{25}$$

(c)Weibull (W) distribution having pdf

$$f(x;\theta,\beta) = \beta\theta^{\beta} x^{\beta-1} e^{(-\theta x)^{\beta}}; x > 0, \theta, \beta > 0. \tag{26}$$

Table 4: *Simulation result for $\beta = 1.5$ and $\theta = 1$.*

n	$Est.$	MLE	CVM	LSE
500	$\hat{\beta}$	1.5861	1.7673	1.6562
	$\hat{\theta}$	1.004	0.9999	1.0142
	$MSE(\hat{\beta})$	0.0101	0.0235	0.0218
	$MSE(\hat{\theta})$	0.3336	0.9141	0.8282
700	$\hat{\beta}$	1.5470	1.5931	1.5058
	$\hat{\theta}$	1.005	1.0114	1.0263
	$MSE(\hat{\beta})$	0.0075	0.0128	0.0174
	$MSE(\hat{\theta})$	0.2346	0.4939	0.2132
1000	$\hat{\beta}$	1.5278	1.5810	1.5941
	$\hat{\theta}$	1.003	0.9929	1.0107
	$MSE(\hat{\beta})$	0.0048	0.0052	0.0152
	$MSE(\hat{\theta})$	0.1388	0.1505	0.3750

Table 5: *The descriptive statistics of Data set.*

Min	1st Q	Median	Mean	3rd Q	Max
0.30	17.50	40.00	46.33	60.00	154.00

(d) Exponential (E) distribution having pdf

$$f(x;\theta) = \theta e^{-\theta x}; x > 0, \theta > 0 \tag{27}$$

The values of the log-likelihood functions$-\ln(L)$, AIC(Akaike Information Criterion), AICC(Akaike Information Criterion with correction) and BIC(Bayesian Information Criterion) are calculated for the five distributions in order to verify which distribution fits better to data. The better distribution corresponds to smaller $-\ln(L)$, AIC, AICC and BIC values. Here, $AIC = -2\ln(L) + 2k$, $AICC = -2\ln(L) + (\frac{2kn}{n-k-1})$ and $BIC = -2\ln(L) + k\ln(n)$; where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. The K-S distance $D_n = sup_x|F(x) - F_n(x)|$,where,$F_n(x)$ is the empirical distribution. Kolmogorov-Smirnov (K-S) statistic is computed to compare the fitted models.
The required computations are carried out in the R-language introduced by R Development Core Team (2019).

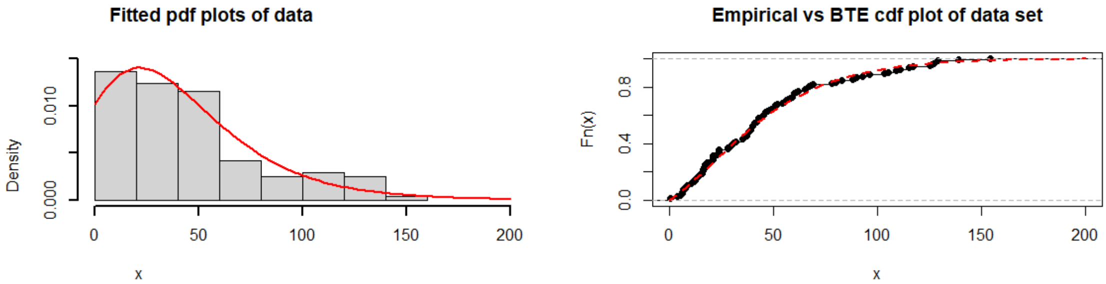
6.1. Data set 1

The first real data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 taken from Lee (1992). The data are:
(0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0). The data is skewed-to-the right with skewness =1.0432 and kurtosis =0.4021
The descriptive statistics of the above data set are given in Table 4. The values in Table 5 shows that the BTE distribution leads to a better fit to the other four models.

Figure 3, shows the fitted density curves, Empirical and the fitted cumulative distribution functions for the data set 1.

Table 6: *Maximum likelihood parameter estimates and goodness of fit for various models fitted for the Data set.*

Model	parameter estimates	log L	AIC	AICC	BIC	K-S	p-value
BTE	$\hat{\beta} = 0.131$ $\hat{\theta} = 0.033$	-579.155	1162.309	1162.411	1167.901	0.0534	0.8802
KuE	$\hat{\theta} = 1.651$ $\hat{\beta} = 0.098$ $\hat{c} = 0.231$	-583.314	1172.63	1172.83	1181.02	0.1152	0.0803
EW	$\hat{\theta} = 1.393$ $\hat{\beta} = 0.017$ $\hat{c} = 0.798$	-579.879	1165.76	1165.96	1174.15	0.0664	0.6606
W	$\hat{\beta} = 1.306$ $\hat{\theta} = 0.019$	-580.024	1164.05	1164.15	1169.64	0.0588	0.7967
E	$\hat{\theta} = 0.022$	-585.128	1172.26	1172.29	1175.05	0.1206	0.0594



(a) Fitted pdf plots of Data set 1

(b) Empirical and the fitted cumulative distribution functions for the data set 1

Figure 3. *Histogram with fitted pdf's (left) and Empirical cdf with fitted cdf's (right) for the data set 1.*

6.2. Data set 2

Here we consider the data set of the life of fatigue of Kelvar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. The data sets are taken from Andrews and Herzberg (1985). The data are:
(0.0251, 0.6751, 1.0483, 1.4880, 1.8808, 2.2460, 3.4846, 0.0886, 0.6753, 1.0596, 1.5728, 1.8878, 2.2878, 3.7433, 0.0891, 0.7696, 1.0773, 1.5733, 1.8881, 2.3203, 3.7455, 0.2501, 0.8375, 1.1733 1.7083, 1.9316, 2.3470, 3.9143, 0.3113, 0.8391, 1.2570, 1.7263, 1.9558, 2.3513, 4.8073, 0.3451, 0.8425, 1.2766, 1.7460, 2.0048, 2.4951, 5.4005, 0.4763, 0.8645, 1.2985, 1.7630, 2.0408, 2.5260, 5.4435, 0.5650, 0.8851, 1.3211, 1.7746, 2.0903, 2.9941, 5.5295, 0.5671, 0.9113, 1.3503, 1.8275, 2.1093, 3.0256, 6.5541, 0.6566, 0.9120, 1.3551, 1.8375, 2.1330, 3.2678, 9.0960, 0.6748, 0.9836, 1.4595, 1.8503, 2.2100, 3.4045). The data is skewed-to-the right with skewness =1.9794 and kurtosis =5.160
The descriptive statistics of the above data set are given in Table 6. The values in Table 7 shows that the BTE distribution leads to a better fit to the other four models.

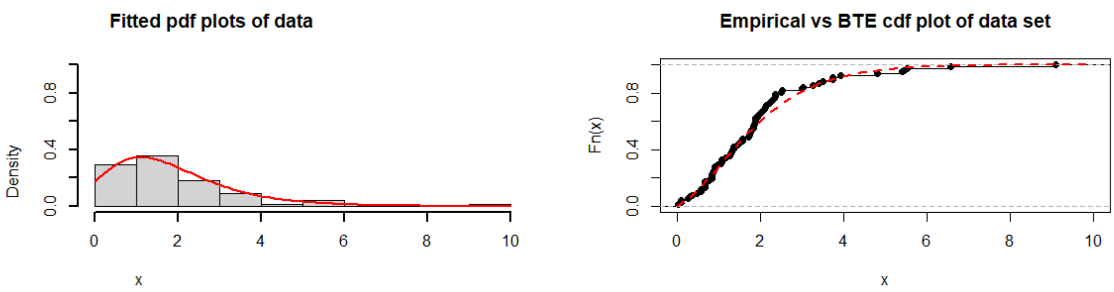
Figure 4, shows the fitted density curves, Empirical and the fitted cumulative distribution functions for the data set 2.

Table 7: *The descriptive statistics of Data set.*

Min	1st Q	Median	Mean	3rd Q	Max
0.025	0.905	1.736	1.959	2.296	9.096

Table 8: *Maximum likelihood parameter estimates and goodness of fit for various models fitted for the Data set.*

Model	parameter estimates	log L	AIC	AICC	BIC	K-S	p-value
BTE	$\hat{\beta} = 0.070$ $\hat{\theta} = 0.873$	-121.410	246.820	246.984	251.481	0.099	0.4167
EW	$\hat{\theta} = 1.101$ $\hat{\beta} = 0.609$ $\hat{c} = 1.443$	-122.166	250.332	250.665	257.324	0.0992	0.4160
W	$\hat{\beta} = 1.326$ $\hat{\theta} = 0.469$	-122.526	249.052	249.219	253.714	0.1098	0.2968
E	$\hat{\theta} = 0.510$	-127.114	256.228	256.282	258.559	0.5120	0.0266



(c) Fitted pdf plots of Data set 2

(d) Empirical and the fitted cumulative distribution functions for the data set 2

Figure 4. *Histogram with fitted pdf's (left) and Empirical cdf with fitted cdf's (right) for the data set 2.*

7. Concluding remarks

In this paper,we have proposed beta transformation in order to get a transformed distribution of some available baseline distribution. Beta transformation of $exp(\theta)$ distribution has been considered to check its application to the real problem called the Beta transformed (BTE) distribution. In the present work, we have provide expressions for the quantiles,moments,moment generating function,hazard rates,entropies and order statistics. The model parameters are estimated by maximum likelihood, Cramer-von Mises and least squares method. We have performed an extensive simulation study to compare these methods. We have compared estimators with respect to mean-squared error. The simulation results show that maximum likelihood estimators is the best performing estimator in terms of MSE. The next best performing estimator is the least square estimator followed by the Cramer-von Mises estimator.
Two real data sets are analyzed to show the importance and flexibility of this distribution.

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Second Order Sliding Mode Control for Robust Performance of the Systems

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Abstract

An integral PID control sliding surface with first order filter is proposed in this paper to the systems with single-input single-output (SISO). In this The developed sliding mode controller results well, even though there are differences in the model of the system via parametric uncertainty. To verify its applicability to disturbances, the presented work validates the controller performance with the application of an external load. An integral and filtered type sliding surface has advantages in terms of the stability of the systems. The proposed controller properties of stability and robustness are proven by the Lyapunov's stability theorem. By the adoption of switching gain with predetermined parameters of system, the chattering problem phenomenon is greatly minimized. Therefore, the proposed controller in this work is appropriate for extended use in real world systems. In this method proposed control is verified using simulation examples and results for its performance. It will be compared to a similar controller shown in the previous literature work.

Keywords: Integral sliding mode control, Robustness, Stability, Uncertain systems

1. Introduction

Most real-world applications involve non-linear systems, but for analytical and control purposes these are approximated by linear systems. The control for systems composed of the specifications of parameter inaccuracy, that is, the structured uncertainty of the system, the neglected dynamics of unstructured uncertainty, and the generally approximated time delay impose serious challenges to controller design. [1]. The nonlinear controller design techniques, like feedback linearization and sliding mode control are proved to be promising and applicable in control issues includes only an approximate linear description of the system [2, 3]. The sliding mode control (SMC), recommended in initial phase of the early 1950s, validated with ability to handle framework uncertainties and outside disturbances with greater strength [4]-[6]. The dynamic behaviour of system can be modified with the system specifications by the suitable selection of switching of oscillatory function with the SMC method.

In literature, one of the major application of sliding mode control is to limit the effects of external disturbance present in th uncertain systems. control, as presented in the international literature developed earlier in Russia [5]. There are so many sliding mode theories are available in the literature. In the initial study, the focus is on conventional or traditional sliding modes. Traditional SMCs use approximate system models to provide a systematic design procedure [7].Therefore, they are widely used in industries with applications including power electronic converters, position or speed control and robotics, space technology applications, and power converters [8]. Conventional SMCs are popular because of their robustness to modelling errors and their insensitivity to external disturbances and parameter changes [9]. However, in many practical applications, the problem in the control action of vibrations known as chattering occurs because of the SMC design. Chattering is a high-frequency (theoretically infinite frequency) switching in control input because of the unmodeled system dynamics. The high frequency oscillations in control signal called chattering occurs due to discontinuous control term is dangerous, specially, in the systems with mechanical parts. The chattering causes undesired overuse of the actuators and final control elements and also results in system instability [10]. The advances in digital control technology has impacted attention of highly robust controller such as SMCs because it can be easily implemented in digital systems such personal computer or can be implemented in discrete domain [11]- [12]. However, if SMC is designed in discrete mode, the discrete control law of discontinuous or switching term, not only induces chattering phenomenon but it drives the system to be unstable due to infinite sampling rate, and the sampling rate due to infinity may be distant. This can be answered by making the discontinuous term value very small [11]. In literature, for state regulation [13, 14, 15, 25] or for set-point tracking [16, 17, 7, 11, 27] either continuous or discrete SMCs are designed . In literature, it is common that, the concerned researchers have developed a continuous-time sliding mode controller (CSMC) or discrete time sliding mode controller (DSMC)that tracks the setting value considering specific application [33]. Among them, Tannuri et al. [19] and Lee et al. [20] reviewed the positioning control system application, and Orr et al. [21] and Lu et al. [22] has prepared a CSMC for spacecraft applications. tn the Mihoub et al. [23] work furnished, a DSMC with the phase variable state model of second order, for tracking of semi-batch reactors. Eker's research mainly focuses on use of traditional SMC or second order SMC for the speed control application of electromechanical system [16, 17, 26]. Recent contribution by Furat and Eker in development of second order integral SMC for the speed control of electromechanical system through experimental application [24] for the reduction of chattering including robustness to disturbances and uncertainties. In this work, a simple SMC algorithm based on the PID with a first-order filter sliding surface was developed. This developed algorithm is used to tune a general system with second order behaviour. Considering the basic second order model (or an identified second order model), an equivalent or continuous controller is designed with the help of sliding surface parameters and model parameters. It is easy to synthesize and implement a new simple sliding-mode controller with the help of filter parameter λ and PID parameters like K_p, K_i, K_d which can consider for plant uncertainties. In meeting the sliding condition of controller of the closed loop system, the system behaviour and the robust stability are investigated. The scheme presented in this paper is further extended to systems capable of handling the inverse response process. The control application for the FOPDT framework is additionally included as a unique case in a similar manner. The usefulness and applicability of the method proposed is being carefully studied and assessed through several general processes.It also includes performance comparison with few current sliding mode control methods as reliable evaluation criteria.

In the real system instead, the controllers are used in a continuous time domain, as we use microprocessors or computer systems in general. Recently, among the researchers involved in introducing continuous SMC to a discrete time SMC. In the literature, it was discovered that much of the work had been completed in a different way for the design of a continuous SMC. The limitations of the Continuous SMC is some extent removed using the DTSMC approach[33] The paper is organized as follows, the section II includes the description of electromechanical system with mathematical model while section III focuses on the integral sliding surface. Further part of the paper is organized in following manner. The nest section describes the system for transformation of the system with lower order and higher order into the general second order system models. section III introduces the design of sliding surface and derives overall control law, whereas section IV provides a typical examples for continuous and discrete SMC. The typical controllers are compared to the proposed controller to test its control capabilities and usefulness in a closed loop. Section V presents conclusions and future directions for work.

2. Description of Systems

In general case, second order system is represented as

$$\frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{C_n}{s^2 + A_n s + B_n} \tag{1}$$

where, ζ is damping factor, ω_n is natural frequency of oscillation of system, and K is gain of system. Then the given system is required to translate in second order system given by the above Eq. 1. Let us discuss the case of system with low order and in subsequent subsection the case of higher order systems be also considered.

2.1. First order plus delay time systems

The first order plus delay time (FOPDT) model of a system is considered as

$$\frac{Y(s)}{U(s)} = \frac{ke^{-t_d s}}{\tau s + 1} \tag{2}$$

where, the term τ represents time constant, t_d represents time delay, and k represents steady-state gain. As the time delays become too small in comparison with time constant τ, then a system model may become modified by approximation as [7]:

$$\frac{Y(s)}{U(s)} = \frac{k}{(\tau s + 1)(t_d s + 1)} = \frac{C_n}{s^2 + A_n s + B_n} \tag{3}$$

Here in above case, the Taylor series approximation in case of time delay $e^{-t_d s} = 1/(t_d s + 1)$ is used. As this is common in the control theory to use Taylor series approximation for the delay time during the design of control system [5].

2.2. Higher order plus delay time systems

Now transfer function model of higher order plus delay time system is considered as,

$$G_P^1(s) = \frac{b_0}{s^q + a_1^1 s^{q-1} + a_2^1 s^{q-2} + ... + a_q^1} e^{-t_d s}, \tag{4}$$

where, a_j^1 $(j = 1, 2, \cdots, q)$ are constant coefficients of the polynomial. The delay time term $e^{-t_d s}$ is replaced by the first order Taylor approximation with $1/(1 + t_d s)$. After approximation, the transfer function in equation (4) can be written as

$$G_p(s) = \frac{b}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_n}, \tag{5}$$

where a_j $(j = 0, 1, 2, \cdots, n)$ represents the constant coefficients. The conversion of any high order system model by first order plus dead time model by approximation is a regular practice. As a matter of fact, all the qualities of higher order process are included in the FOPDT model, but it is sufficient to provide an explanation to the effective dead time, overall time constant, and process gain of system of this type [28]. There are three unknown parameters are needed to create a reasonable FOPDT model to be approximate, namely τ, t_d and k should be determined steady-state gain. Let the transfer function of lower model is denoted by

$$l(s) = \frac{Y(s)}{U(s)} = \frac{ke^{-t_d s}}{\tau s + 1}, \tag{6}$$

and for higher order it is denoted by $h(s) = G_p(s)$. In the literature, the higher order and lower order models at certain places tried to fit the Nyquist plots, but it was unsuccessful [29].

$$\begin{aligned} l(0) &= h(0) \\ |l(j\omega_c)| &= |h(j\omega_c)| \\ \angle l(j\omega_c) &= \angle h(j\omega_c) \end{aligned} \tag{7}$$

where, ω_c represents phase crossover frequency. As a result, the FOPDT model parameters may be determined with the help of [28],

$$\begin{aligned} k &= h(0) \\ \tau &= \frac{\sqrt{(\frac{h(0)}{|h(j\omega_c|})^2 - 1}}{\omega_c} \\ t_d &= \frac{\pi - tan^{-1}(\tau\omega_c)}{\omega_c} \end{aligned} \tag{8}$$

Now from Eq. 3 by getting the above three values of constant parameters, it is easy to obtain the specified structure.

3. Sliding Mode Control Approaches

3.1. Continuous SMC

For continuous SMC, the sliding surface for PID controller with first order filter is defined by:

$$\sigma(t) = \left[K_p + \frac{K_i}{s} + sK_d\right]^{n-1} \Psi(E(s)) \tag{9}$$

where $\Psi(E(s))$ is the Laplace domain tracking error filter, $\Psi(E(s)) = 1/(\lambda s + 1)E(s)$ and 'n' is the order of system. In this, terms K_p, K_i K_d and λ are the parameters used for tuning the controller, these supports in defining the sliding surface $\sigma(t)$ and determined by designer. The sliding surface can be used to determine the how well the system perform. Designing a control law has the purpose of guaranteeing the output of plant response $y(t)$ equal to the set value of reference $r(t)$ for the remaining time, which means the value of error and derivatives of all errors must be equal to zero. In SMC law, the main purpose is to reduce the error signal $e(t)$ to move towards the defined sliding surface also it must stay along with it towards origin. By putting the value of $\Psi(E(s)) = 1/(\lambda s + 1)E(s)$ in Eq. (9), results in

$$\sigma(t) = \frac{1}{\lambda s + 1} K_p E(s) + \frac{K_i}{s(\lambda s + 1)} E(s) + \frac{1}{\lambda s + 1} K_d s E(s) \tag{10}$$

. The transfer function of model is second order, means the term $n = 2$.

The tracking error, in mathematical way may be represented by the equation

$$e(t) = r(t) - y(t) \tag{11}$$

. where, reference input is represented by $r(t)$, $e(t)$ represents error signal, and plant output is represented by $y(t)$. The Second Derivative of above Eq. 11 is

$$\ddot{e}(t) = \ddot{r}(t) - \ddot{y}(t) \tag{12}$$

Generally, from Eq. 3, $\ddot{y}(t) = -A_n\dot{y}(t) - B_n y(t) + C_n u(t) + D(t, u(t))$.
Substituting value of $\ddot{y}(t) = -A_n\dot{y}(t) - B_n y(t) + C_n u(t) + D(t, u(t))$ into the Eq. 12, therefore

$$\ddot{e}(t) = \ddot{r}(t) - [-A_n\dot{y}(t) - B_n y(t) + C_n u(t) + D(t, u(t))] \tag{13}$$

$$\ddot{e}(t) = \ddot{r}(t) + A_n\dot{y}(t) + B_n y(t) - C_n u(t) - D(t, u(t)) \tag{14}$$

The sliding surface second-order derivative which is taken from Eq. 10 is determined with multiplication on both side of equation by $'s(\lambda s + 1)'$. Hence, Eq. 10 may get modified as

$$s(\lambda s + 1)\sigma(t) = sK_p E(s) + K_i E(s) + s^2 K_d E(s) \tag{15}$$

. By modifying above Eq. 15 in the time domain and represented as

$$\ddot{\sigma}(t) = \frac{K_p}{\lambda}\dot{e}(t) + \frac{K_i}{\lambda}e(t) + \frac{K_d}{\lambda}\ddot{e}(t) - \frac{\sigma\dot{(t)}}{\lambda} \tag{16}$$

. In view of the Eq. 13, we know that, $\ddot{e}(t) = \ddot{r}(t) + A_n\dot{y}(t) + B_n y(t) - C_n u(t) - D(t, u(t))$. Put this in 16, now it is written as,

$$\ddot{\sigma}(t) = \frac{K_p}{\lambda}\dot{e}(t) + \frac{K_i}{\lambda}e(t) + \frac{K_d}{\lambda}[\ddot{r}(t) + A_n\dot{y}(t) + B_n y(t) - C_n u(t) - D(t, u(t))] - \frac{\sigma\dot{(t)}}{\lambda} \tag{17}$$

. When condition $\sigma(t)$=$\dot{\sigma}(t)$ and $\ddot{\sigma}(t) = 0$ with $u(t) = u_{eq}(t)$ is determined, then controller algorithm designed in the form of second-order SMC is primarily established by the equivalent control concept. The control of a system at its nominal parameters is achieved by equivalent control, if $D(t, u(t)) = 0$, given by the steps :
Step 1
As $\ddot{\sigma}(t) = 0$, put in Eq. 17

$$\frac{K_p}{\lambda}\dot{e}(t) + \frac{K_i}{\lambda}e(t) + \frac{K_d}{\lambda}[\ddot{r}(t) + A_n\dot{y}(t) + B_n y(t) - C_n u(t)] - \frac{\sigma\dot{(t)}}{\lambda} = 0, \tag{18}$$

. Step 2
Replace u by u_{eq} in Eq. 18

$$\frac{K_p}{\lambda}\dot{e}(t) + \frac{K_i}{\lambda}e(t) + \frac{K_d}{\lambda}[\ddot{r}(t) + A_n\dot{y}(t) + B_n y(t) - C_n u_{eq}(t)] - \frac{\sigma\dot{(t)}}{\lambda} = 0. \tag{19}$$

Step 3
Obtain u_{eq} from above Eq. 19

$$u_{eq}(t) = \frac{1}{K_d C_n}(K_p\dot{e}(t) + K_i e(t) + K_d\ddot{r}(t) + K_d A_n\dot{y}(t) + K_d B_n y(t)) + \frac{1}{K_d C_n}(-\frac{K_p}{\lambda}e^{-\frac{t}{\lambda}}e(t) - K_i e(t) + K_i e^{-\frac{t}{\lambda}}e(t) - \frac{K_d}{\lambda}e^{-\frac{t}{\lambda}}\dot{e}(t)). \tag{20}$$

The above value is named equivalent controller. The form of input control to the conventional SMC is:

$$u(t) = u_{eq}(t) + u_{sw}(t) \tag{21}$$

Now we take the switching control, here three switching controls are taken as represented by

$$u_{sw}(t) = k_{sw}r^2(t)\tilde{e}(t)sgn\left(\frac{k_{sf}}{\tilde{e}(t)}\dot{\sigma}(t)\right) + \frac{1}{K_d C_n}sgn(\sigma(t)) \tag{22}$$

where the term k_{sw} is a positive gain employed for reduction of high frequency oscillations known as chattering, by preserving the tracking efficiency, and considering that $r(t) \neq 0$ the setpoint and $\tilde{e}(t)$ is the corrected error given by:

$$\tilde{e}(t) = \epsilon_1 sgn(e(t)) \qquad if|e(t)| \leq \epsilon_1 \tag{23}$$

and

$$\tilde{e}(t) = e(t)) \qquad if|e(t)| \geq \epsilon_1 \tag{24}$$

where, ϵ_1 is a number with small positive value used for avoiding the situation of zero division. At the time os starting, when $t = 0$, the amount of error present in the switching control gain is maximum, so the switching control law provides the maximum control signal. As time approaches infinity, the error value tends to zero. This means that $lim_{t\to\infty} u_{sw}(t) \cong 0$. Depending on the uncertainty of a given time or the error due to external disturbance of the load, the amount of switching control increases and converges to the setpoint more quickly. As the sliding surface represents a functional variable of error signal, condition $\sigma(t) = \dot{\sigma}(t) = 0$ is determined by slight variations near zero, if the error value tends to zero.

3.2. Discrete SMC

The design of the DSMC required to satisfy the stability condition for the reaching phase and sliding phase as same like the continuous SMC given in the section (III). The concept of the reaching condition [25] ,

$$s(t)s(\dot{t}) \leq 0, i.e. \tag{25}$$

$$\mid s(k+1) \mid < \mid s(k) \mid$$

apply the Lyapunov stability criteria for ideal condition of sliding mode [32]

$$\nu(\dot{t}) < 0, \tag{26}$$

where

$$\nu(t)(t) = \frac{1}{2} s^2(t) \tag{27}$$

which may be written in discrete time as

$$\nu(t)(k+1) - \nu(t)(k) < 0, \tag{28}$$

where

$$\nu(t)(k) = \frac{1}{2} s^2(k) \tag{29}$$

Let us consider the continuous time model of the system represented in the discrete-time model as represented by:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \delta(k) \\ y(k) &= Cx(k) \end{aligned} \tag{30}$$

By defining a state error vector with the equation

$$e(k) = x(k) - y(k) \tag{31}$$

where $e(k)$ is the error signal, and $x(k) = \Re^n$ is the vector of state variables, $u(k) \in \Re$ is the vector input control signal and $y(k) \in \Re$ is the scalar output signal of the system. A, B and C are representing constant value matrices with proper dimensions. The DSMC approach involved in designing the controller have the following steps:

- Determination of a switching function $s(x)$ in such a way that the sliding mode on switching surface $s(x) = 0$ becomes stable.
- Determination of a control law

$$u(k) = \begin{cases} 1, & \text{when } s(k) > 0 \\ -1, & \text{when } s(k) \leq 0 \end{cases} \tag{32}$$

4. Simulation Examples

Example 1: Simulation of sliding mode control is conducted for a brushless DC motor. Results show the successfulness of the controller. The controller is differentiated with existing sliding mode controllers present in literature. For simulation the *Mathworks*^{TM} MATLAB 2019a is used. This paper uses a flat BLDC motor of Maxon's EC 45 with diameter of Φ 45 mm, 30 Watt from Maxon motors [30]. Mathematical models use the parameters that are obtained from the Motor's datasheet as well as othr relevant information. For LDC motors, the mathematical model uses the parameters available in the datasheet [30].

$$G(s) = \frac{1/K_g}{\tau_m \tau_e s^2 + \tau_m s + 1}$$

where K_g, τ_m and τ_e are the constants and required to be determined.

The term τ_e is determined using the relation

$$\tau_e = \frac{L}{3R} = \frac{0.560 \times 10^{-3}}{3 \times 1.10}.$$

Thus,

$$\tau_e = 151.56 \times 10^{-6}$$

The term τ_m is determined using the relation

$$\tau_m = \frac{3R_\phi J}{K_g K_t} = 0.0171$$

where K_e is

$$K_e = \frac{3R_\phi J}{\tau_m K_t} = 0.0763$$

Hence, the DC motor model is represented by transfer function form is

$$G(s) = \frac{13.11}{155.56 \times 0.0171 \times 10^{-6} s^2 + 0.0171 s + 1}$$

or

$$G(s) = \frac{82620}{s^2 + 269.7s + 6302} = \frac{C_n}{1 + A_n + B_n}.$$

The various parameters defined for the controller of proposed here and Furat & Eker [24] are taken as: ksw=200; ksf=0.025; Kp=12; Ki=0.001; Kd=0.0024; with $\lambda = 0.9$; as filter parameter for suggested method. Fig. 1,Fig. 2 and Fig. 3 respectively reveals the output, input and sliding surface responses of the suggested SMC and other considered controllers. Looking at the output response, Furat & Eker provided controller and the controllers implemented here showed speedy and reasonably acceptable response, instead the slow response given by Camacho-2000 and

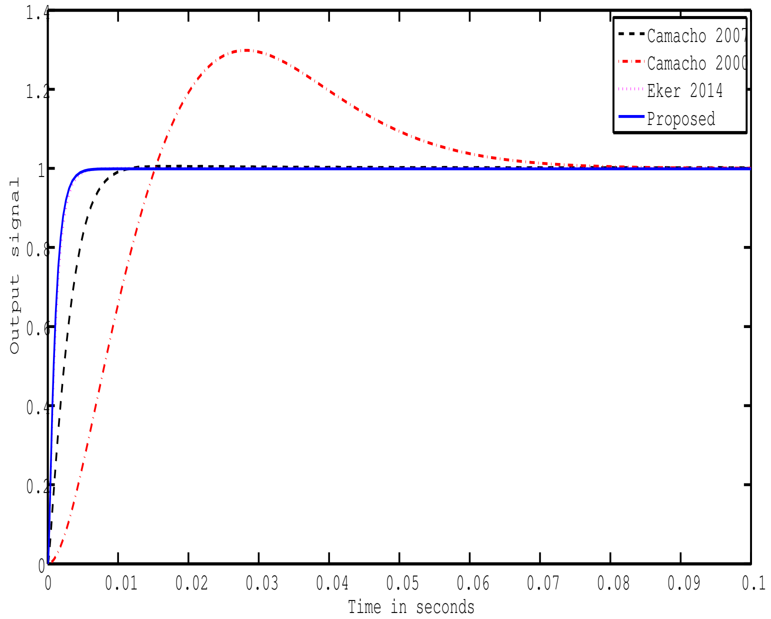

Figure 1: *Output Responses*

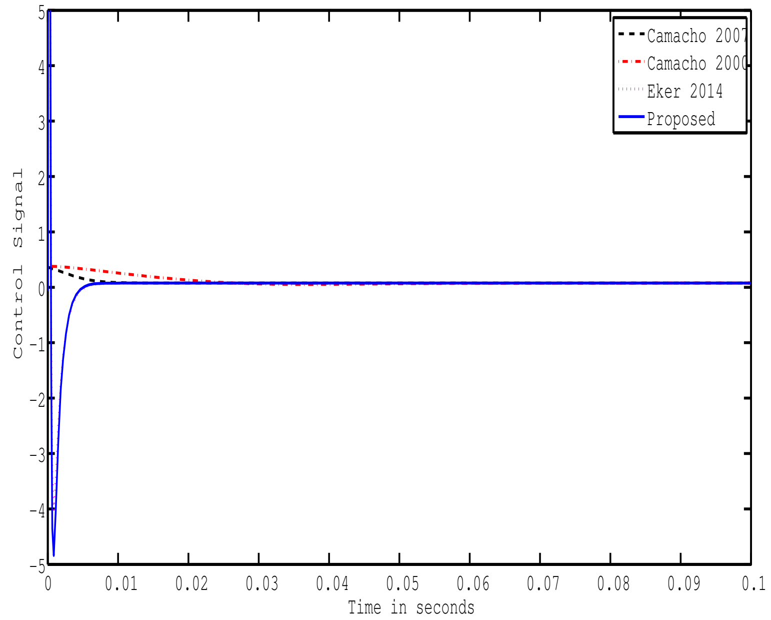

Figure 2: *Input Responses*

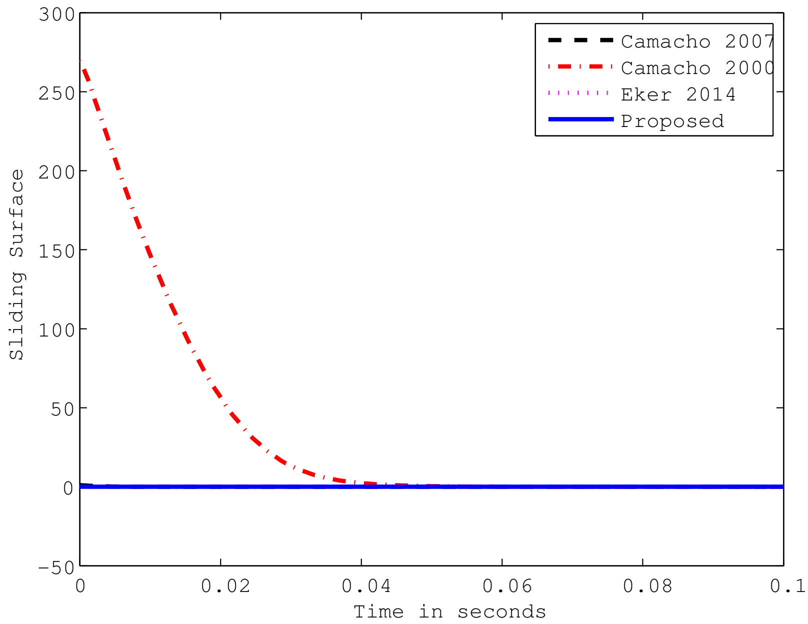

Figure 3: *Error Responses*

Camacho-2007. The suggested response by Camacho-2000 shows high overshoot and unsuitable for applications of electromechanical systems in speed control of the DC motor. The proposed controller provides the smooth response and the stable sliding surface.

At the time $t = 0.05s$, to check the stability and behaviour of all controllers, the output disturbance $d = 0.2r$ is inserted in the system. The controller responses of the controllers are shown in Fig. 4. From the Fig. 4 shows that, controllers provided by Camacho-2000 and Camacho-2007 are not suitable due to poor performance. The controller proposed in this paper provides comparable and preferable performance characteristics.

Example 2: The repeated pole systems are well studied in the literature and are used for design of controller in higher order systems [29].

$$G_p(s) = \frac{1}{(s+1)^5}$$

Using the technique given in section II, the FOPDT parameters of the system are $k = 1$, $\tau = 3.7540$ and $t_d = 2.6566$. The second order model with Taylor approxiamtion for delay time is,

$$\frac{Y(s)}{U(s)} = \frac{0.1003}{s^2 + 0.6428s + 0.1003}$$

4.1. Simulation example of DSMC

Consider the higher order transfer function given in Example 2 reduced in to the third order approximation and represented in state space form [29]

$$G_p(s) = \frac{1}{(s+1)^5}$$

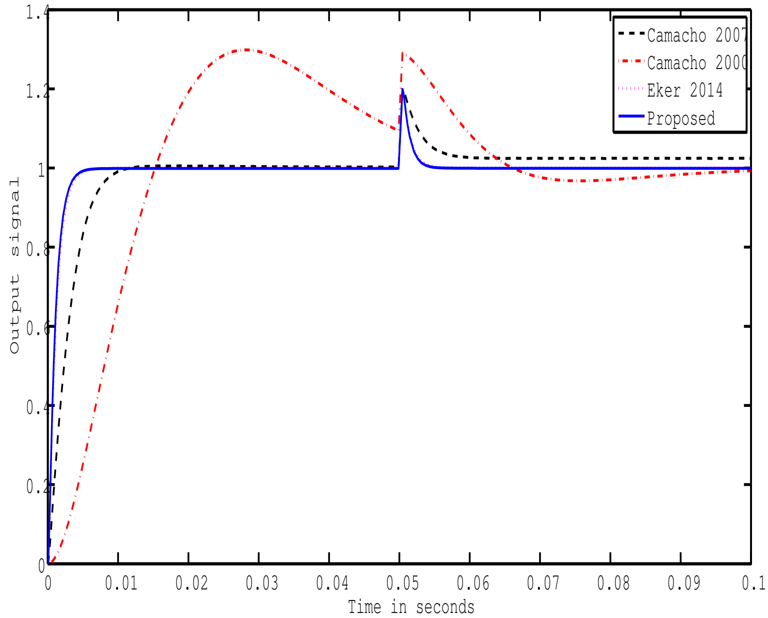

Figure 4: *Output Responses under 20% external disturbance*

An equation in state space can be derived by matched pole-zero method with selected sampling interval of $T_{[}s] = 0.1s$ and may be given as

$$A = \begin{bmatrix} -1.5630 & -1.0140 & -0.2375 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0896 \\ -0.1608 \\ 0.2406 \end{bmatrix}$$

and D=[0].

The parameters used for the prevalent controller Khandekar et.al. , Weibing Gao et.al. & our previous work [31, 32, 33] are: In simulation of [31] switching gain $alpha = 0.4$, $K_t = 1$ and the controller gain matrix ct = [-5.3630 -1.1215 -0.3097]. In simulation of [32] switching gain $alpha = 0.8$, $K_t = 0.8$ and the controller gain matrix ct = [-3.3630 -0.1215 -0.3097]. In simulation of [33] switching gain $alpha = 0.6$, $K_t = 0.8$ and the controller gain matrix ct = [-1.3630 -0.1215 -0.3097]. The performances of DSMC [33] and other controllers are shown in Fig. 5 and Fig. 6 respectively in relation to the output responses and input responses. From these figures, it is observed that the output responses of the controller given by DSMC in [33] controllers gives fast and satisfactory response. It is also observed that the responses are more oscillatory for DSMC of prevalent controllers.

5. Conclusion

According to the results, the integral SMC performs better than the conventional SMC and PID controller in terms of output response. The output response of the integral SMC had no overshoot,faster rise time, and a faster settling time in magnitude. Traditional SMC and PID controllers are unable achieve needs of precise control requirements, resulting in large percentage overshoots and settling times are required for system. The second order integral SMC gives superior performance compared to the conventional SMC or traditional PID controller like

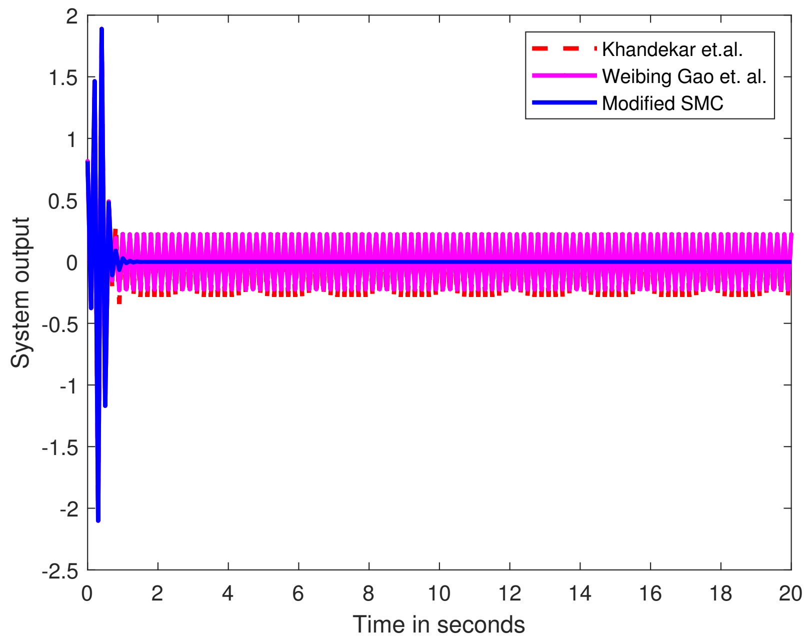

Figure 5: *System output.*

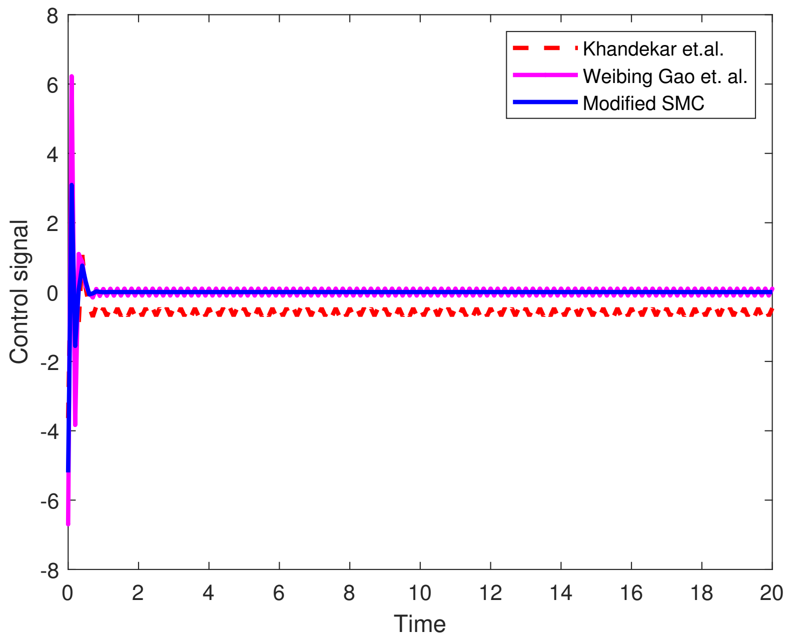

Figure 6: *Control signals*

reducing the overshoot exist in speed, also minimising the rise time and settling time of the system response. Based on the results of simulation, second order integral SMC compared to both conventional SMC techniques gives improved results under nominal parameter or system with uncertainties in the parameters. However, the results obtained for the nominal parameters are better than the results obtained for the system under parametric uncertainties. The conventional SMC simulation results are preferable when the system is at its nominal parameters,but are not acceptable for systems with parametric uncertainty. The second order integral SMC is suitable for systems with uncertain parameters that cannot ne estimated or measured. In case of external disturbance the proposed controller will be useful. Selecting the right sliding surface is critical in the approach to SMC design, also selecting a sliding sirface can significantly reduce the chattering phenomenon, but with an extra work it can be eliminated. The results can be compared to other second order Integral sliding surfaces or by using different control laws. The control approach used in this work is restricted to second order integral SMC, conventional SMC and PID controller, but other control approaches such as higher order SMC, predictive SMC can also be implemented. This work may be further moved forward for the systems with higher than 10% parametric variation with uncertainty in the modification of the control law. This discussed study may be further worked with the applications in real time by designing an experimental setup and DC drive interfacing accessories.

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An Effective Sentiment Analysis in Hindi-English Code-Mixed Twitter Data using Swea Clustering and Hybrid BLSTM-CNN Classification

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Abstract

Sentiment Analysis is the process of examining the individual's emotions. In tweet sentiment analysis, opinions in messages are categorized into positive, negative and neutral categories. A clustering-based classification approach is used to increase the accuracy level and enhance the performance in sentiment classification. The input dataset comprises of Hindi-English code-mixed text data. Initially, the input text data is pre-processed with different pre-processing techniques such as stop word removal, tokenization, Stemming, lemmatization. This effectively pre-processes the data and makes it appropriate for further processing. Afterwards, effective features such as Count Vectors, Modified term frequency-inverse document frequency (MTF-IDF), Feature hashing, Glove feature and Word2vector features are extracted for enhancing the classification performance. Afterwards, Sentiment word embedding-based agglomerative (SWEA) clustering is presented for effective sentiment feature clustering. Finally, a hybrid Bidirectional long short-term memory-convolutional neural network (Hybrid BLSTM-CNN) is used to accurately classify tweet sentiments into positive, negative, and neutral. Here, modified horse herd optimization (MHHO) approach is used for weight optimization in Hybrid BLSTM-CNN. This optimization approach further enhances the performance of classification. The dataset used for the implementation is a Hindi-English mixed dataset. The experimental result significantly improves the different existing approaches in terms of accuracy, precision, recall, and F-measure.

Keywords: Sentiment Analysis, Hindi-English, Twitter, code-mixed text data, modified horse herd optimization

1. Introduction

Over the last two decades and the increase in the population, social media users have also increased tremendously [1]. Thelanguages are mixed in several forms of communication due to different cultures. People started to communicate online to share and express their thoughtson social network websites like Twitter, YouTube, and Facebook. These media are a better platform to interact with each other [2]. Most of the words are used generally in one language, and the translation in another language is not very popular. Hence, when the person utilizes those words in a text, they are like the most fashionable language [3]. This makes a sentence in two different languages and arranges a grammar part of one language.The text in social media isfamiliar with many linguistic variations. In multilingual countries like India, commonly people combine the English and Hindi languages with their native language [4]. The method of switching sentences between the many languages is known as code-mixing, also called code switching [5]. It is amodification phenomenon among many languages, generally two, within a single sentence [6].

Sentiment Analysis (SA) in mixed languagehas gained popularity due to the increasing number of non-English speaking users [7]. SA can offer precise insight from product reviews to capture trending themes to design business models. Today, most institutions depend on SA of social media text to monitor the performance of their products and fetch feedback [8]. SA is used in product reviews and applications like reputation management, social media monitoring, and brand monitoring [9].In addition, people post their suggestions, opinion and results in a huge amount of text information accessible for analysis. Humans can understand the paragraph written in a language they know and identify the paragraph as positive, negative, or neutral feelings [10]. However, the computer doesn't know the languages around the world, and it cannot interpret them. SA overcomes this limitation of the computer by Natural Programming Language (NLP) for recognizing the word and classifying them [11]. A mixed language tweet has many unseen complexities to NLP tasks like language identification, semantic processing, machine translation and Parts-of-Speech (POS) tagging.Hence it is important to develop a technology for mixed language text [12, 13].

Based on the report of KPMG Group in India, users of the Indian language can expand up to 537 million in 2022 [14]. People states their opinion on changing subjects varies from sports to politics and movies [15]. In addition, people express their suggestions in mixed languages like English-Urdu, English-Hindi, English-Tamil, and English-Bengali [16]. Hindi is a national language of India, which is majorly spoken in different states. Due to many Hindi speaking people contributes majorly in several social media about various social activities [17]. The formation of the Hindi language utilized in social networks is mixed with English and accessible in roman scripts [18].Some of the existing sentiment analysis approaches are Naive Bayes, convolutional neural network (CNN) [19], decision tree (DT), support vector machines (SVM), recurrent neural network (RNN), and RF (random forests) [20].

Motivation: The phenomenon of mixed language can learn by analyzing different applications. SA is circumstantial text analysis, finding the social sentiment for better understanding the source. SA on mixed language helps to understand the sentiment of sentences and phrases. Multilingualism is the potential of people to communicate efficiently in many languages. There are three categories of SA approaches: Machine learning (ML), Lexicon based and Deep learning (DL). The Lexicon-based models depend on the predefined rules for determining the sentence from the text. An ML technique utilizes semantic mining for identifying sentiments. ML techniques are semi-supervised, supervised and unsupervised. ML techniques require hand-crafted features extraction, which is time consuming, needs expertise and is expensive. DL models are more efficient in learning features automatically from text and achieve better results. The most commonly used methods in NLP tasks are Recurrent Neural Network (RNN), Convolutional Neural Network (Convnet), and Long Short-Term Memory Network (LSTM). The text representation and the good classification approach are necessary to improveclassification performance. The major contributions of the presented methodology are described as,

- To effectively pre-process the Hindi mixed English tweets, different pre-processing approaches like stop word removal, tokenization, stemming, and lemmatization are utilized.
- To extract the most important features from the pre-processed data, count vectors, MTF-IDF, Feature hashing, Glove feature and Word2vector features are extracted. The effective feature extraction process is a necessary one for accurate sentiment prediction.
- A Sentiment word embedding-based agglomerative clustering approach is presented to cluster the sentiment features effectively. This clustering process further improves the performance of sentiment prediction.

- To accurately classify the sentiments as positive, negative and neutral, a hybrid BLSTM-CNN framework is presented. Here, the modified horse herd optimization approach improves this classification network performance. This MHHO approach is used for updating the optimized weights in the hybrid BLSTM-CNN framework. Finally, the performance of the sentiment analysis is well improved using these combinations of approaches.

2. Related Work

Jhanwar and Das [21] proposed an ensemble method for SA of Hi-En mixed data using DL models. In this work, LSTM and Multinomial Naive Bayes (MNB) were utilized for identifying the Hi-En sentiments. The ensemble model integrated the LSTM and keywords polarity from a probabilistic method for identifying sentiments in inconsistent and sparse mixed data. Both models were averaged and weighted based on the accuracy. The experimental outcomes proved the system's performance compared with other DL models.

Sasidhar et al. [22] used the DL model CNN-BLSTMto identify the emotions expressed via Hi-En mixed languages in many social media sites. To evaluate the detection method, 12,000 CM sentences from various sources have many emotions. A bilingual model was used for generating the vector, and the DL model was used for classification. This model provided higher performance with an accuracy of 83.1%.

Singh et al. [23] presented a unique language and POS tagged database of CM Hi-En tweets. It was based on five happenings in India that led to many twitter activities. The database used in this model has two factors: it was longer than the prior annotated database and was like real-world tweets. Then the POS was trained on this database to show how this database can be utilized. This model has attained a better performance with an F-measure value of 88.6%, but the model has overfitting issue despite the usage of regularization.

Singh and Lefever, [24] proposed two stages for the SA task for Hi-En sentiments. In the first stage, baseline methods and monolingual embedding were initialized. Then, in a second stage, cross lingual embedding's for Hi-En were constructed. The transfer learning-based classifieris trained on En sentiment and implemented on code-mixed information.The task comprises three sentiments, and the experimental outcomes proved that this model improved the results in fully supervised and can utilize as a baseline for a distant base.

Garg and Kamlesh Sharma, [25] presented the model of creating a corpus for Hi-En sentiments. The method utilized for annotating the corpus into 5 categories. The inner agreement measure was computed for positive and negative tweets. This model provided a standard corpus for code switching in Hi-En. The words utilized for sarcasm and slang were also discussed in this work. This model overcame spelling which was inconsistent and misspelt words.

Nagamanjula and Pethalakshmi, [26] developed an innovative model using a logistic adaptive network that depends on a neuro fuzzy inference system (LAN2FIS) to classify sentiments on Twitter data. Here, features were chosen by using the bi-objective optimization scheme. The sentiment analysis using this approach provided enhanced performance. But the analysis was performed only in public reviews. A bigger data set is a necessity to improve the accuracy performance. The performance of the approach can be improved by utilizing a better combination of approaches.

Naresh and Venkata Krishna, [27] presented a proficient sentiment analysis methodology utilizing the machine learning approach. Here, optimization was utilized to enhance the performance ofthe machine learning technique. At first, input data was taken and pre-processed. Next, the optimized data was attained through the feature extraction process. The trained feature data was utilized to categorize the sentiment classes through the machine learning classifier in the final stage. The attained accuracy of this approach was 89.47%. The examination performance of the methodology can be improved by using the deep learning approaches in future.

Kanika Garg and Lobiyal, [28] developed a methodology for evaluating the feature values through KL (Kullback-Leibler) divergence methodology. The features were used for finding the membership values with the neuro fuzzy and fuzzy logic methodology. The obtained accuracy of the methodology was 89.93%. In future work, an effective feature selection and classification approach was suggested to improve the performance.

3. Proposed Methodology

This work presents an effective sentiment analysis in Hindi-English mixed twitter texts. Initially, the input twitters in the combination of Hindi and English texts are collected using the Twitter data set. Afterwards, input texts are effectively pre-processed using different pre-processing techniques like tokenization, stop word removal, lemmatization and stemming. Then, effective combinations of features like count vectors, feature hashing, Glove feature vectors and modified TF-IDF are extracted from the pre-processed data. Subsequently, the extracted sentiment features are clustered using the presented SWEA clustering approach. Finally, the hybrid BLSTM-CNN approach accurately predicts sentiments into negative, positive, and neutral classes. Here, the performance of sentiment prediction is improved by updating the optimized weights through the modified horse herd optimization approach. The schematic diagram of the presented methodology is depicted in figure 1.

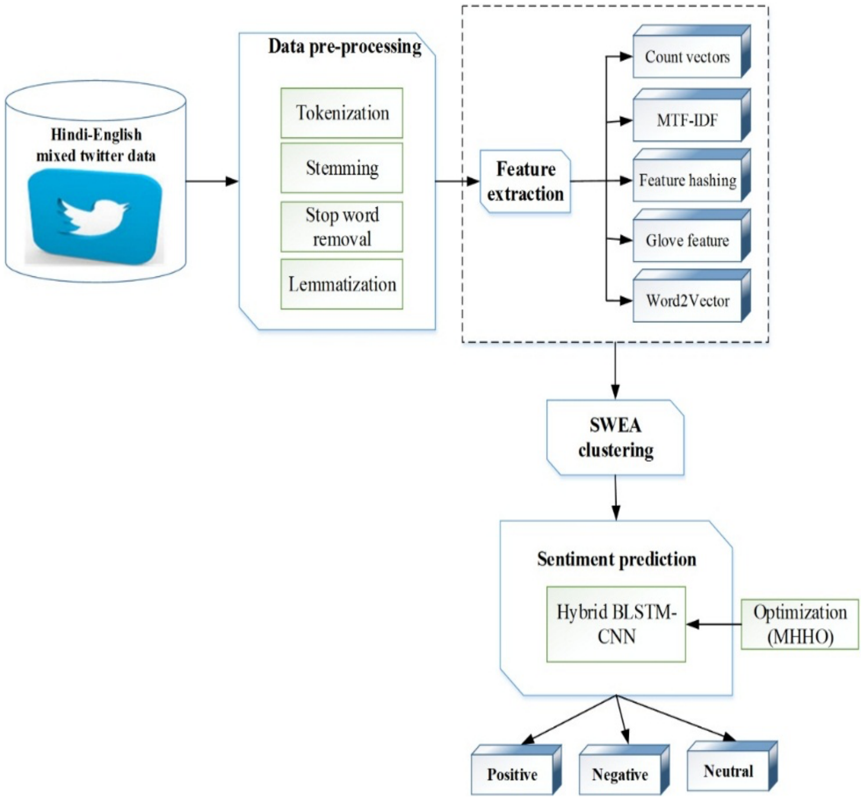

Figure 1: *Schematic diagram of the presented methodology*

The SA process has four stages: (a) Pre-processing is the first stage to clean and prepare data for sentiment analysis. It reduces computational process and feature space, enhancing the system's accuracy. (b) Feature extraction is a second stage to extract the important features for the analysis. (c) Clustering is the third stage to group the data points and (d) Sentiment prediction is the fourth stage to classify the sentiments of data into positive, negative or neutral.

3.1 Pre-processing

Pre-processing is an important process for SA since textual information has unstructured and noisy data. The Pre-processing step is applied to the dataset to improve the quality and classify the text. This step is used to clean the dataset from noise like spelling error correction, disambiguation of ambiguous abbreviations and reduction of repetitive characters. Hence in these cases, pre-processing techniques like stop word removal, tokenization, Stemming and lemmatization can enhance the dataset's quality.

3.1.1 Tokenization

It is complex for machines to understand the context and semantics of a paragraph and sentences. Initially, the tokenization process breaks sentences into punctuations and words called tokenization. For tokenizing the words, Natural Language Toolkit (NLTK) is used.
Example: Input data- "I am playing"
Output data- (I) (am) (playing)

3.1.2 Stop word removal

In the text data processing, the words which have high existence in the documents are called stop words like "is", "am", "and", "the". These words are helpful in sentence formation but do not provide any importance in language processing. Applying these words on lexical resources has less emotional meaning and doesn't affect a sentimental score. Hence these words are filtered from the tweets. The example representation of stop word removal is provided below.
Example: Input data- I am playing
Output data- Playing

3.1.3 Stemming

It is the process of converting the tenses of words to their basic form. This stemming process avoids the processing time of words. The example of the stemming process is described as,
Example: "Playing" to "play","Arguing" to "Argue", "Fishing" to "Fish", "noises" to "noise".

3.1.4 Lemmatization

This is the process of integrating two or three words into one word. This finds the Word morphology and removes the end of the word.

Example: "Matched" to "Match", "taught" to "teach"

3.2 Text Feature extraction

The most important features like count vectors, MTF-IDF, Feature hashing, Glove feature and Word2vector features are extracted from the pre-processed data. The effective feature extraction process is a necessary one for accurate sentiment prediction. The presented feature extraction techniques are described in the subsequent sub-sections.

3.2.1 Count Vectors

Count vectors feature just count the number of word occurrences in a document. Afterwards, the count value is utilized as a weight in the feature vector. This feature extraction process counts the words and outputs in integers. The count feature vector is similar to the TF-IDF feature. In TF-IDF feature extraction, a score of the words are computed, and in the case of count vectors, floats are computed from the words. The count vectors are illustrated in the subsequent example shown in table 1.
Example data: {“Good”, “Boy”, “Play”, “Well”, “Top”, “Good”, “Student”, “School”}

Table 1: *Example illustration of output Count Vectors*

Data	Good	Boy	Play	Well	Top	Student	School
Count vectors	2	1	1	1	1	1	1

3.2.2 Feature hashing

This is the process of changing separate tokens into their corresponding integers. This process generates the dictionary of words. After the process completion of dictionary generation, the words in the dictionary are transformed into respective hash values. This hash value representation is utilized to find the feature is already utilized in the process or not. In this feature hashing, the model results in the group of columns for every text data in rows and one column for every hashed feature.

The feature hashing is equivalent to one hot encoding process and results in the hashes to mention each text word. Example illustration is provided in table 2.

Table 2: *Example of feature hashing*

Word	**Hash1**	**Hash2**
Blue	0	0
Black	0	1
Red	1	0
Pink	1	1

The main objective of using feature hashing is to decrease the size of features. This can be used to mention the texts in variable length to the feature vectors in equivalent numeric size and attains the reduced size of features.

3.2.3 Word2Vec feature

In the Word2Vec feature extraction model, the input is word data, and the output data is vector data. This makes the words into their corresponding numeric form. A similar form of words has the same embedding. This word2vec feature extraction process combines a continuous bag of words (CBoW) and the skip-gram process. Here, CBoW provides the occurrence probability of words. This makes the output target vector for the words in embedding. The word2vec feature extraction model is represented in figure 2.

The continuous bag of words is utilized for finding the target text, and the skip-gram process finds the target context data from the predicted words.

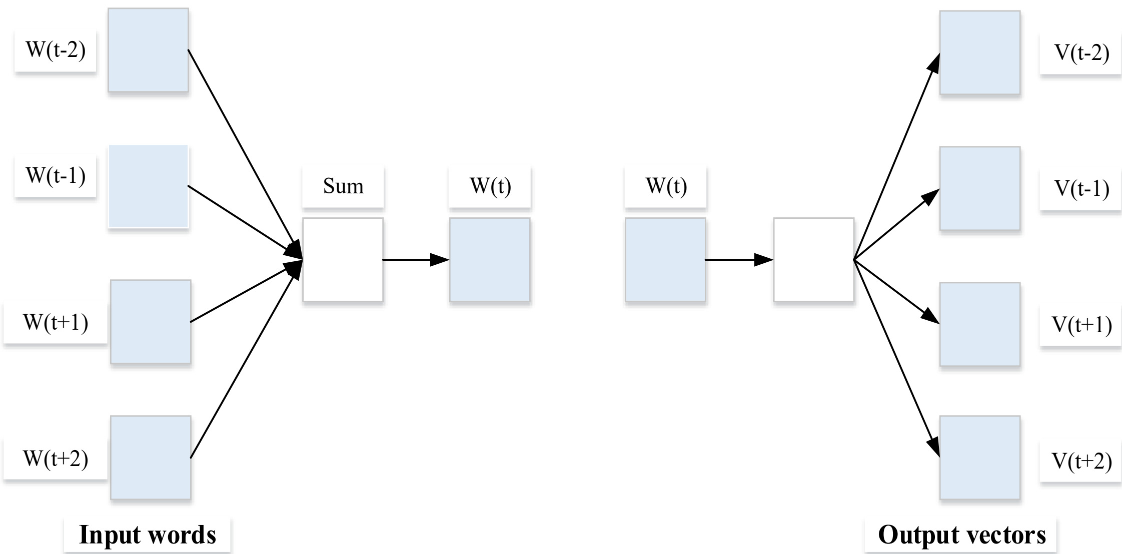

Figure 2: *Word2Vec feature extraction model*

3.2.4 Modified TF-IDF (MTF-IDF)

This feature extraction approach is utilized to find the importance of particular words in a document. Here, the term frequency of a particular word is computed by the number of times that the particular word occurred in the document to the whole number of words in that same document. This is utilized to remove unnecessary words like “a”, “an”, “the” in the document. The modified form of the TF-IDF feature is computed here to advance the feature extraction process. At first, the term frequency is computed through the subsequent condition (1).

$$\overline{T}_F'' = \frac{m_{l,m}}{\sum_p m_{p,m}} \tag{1}$$

Here, $\overline{T}_F''$ signifies the extracted term frequency feature vector, $m_{l,m}$ signifies the number of the word t_l occurs in the document d_m, and the denominator portion in condition (1) describes the addition of word occurrences of the document. The inverse document frequency is computed through the subsequent condition (2),

$$\overline{Idf}'' = \log\left[\left(\frac{\overline{M}}{Df_l}\right)+1\right] \tag{2}$$

Here, $\overline{Idf}''$ signifies the IDF feature vector, $\overline{M}$ signifies the total amount of documents in the dataset and Df_l signifies the feature vector of documents. The TF-IDF feature extraction process is enhanced by updating the weight calculated using the term frequency. The improved form of feature vector based on the weight of term frequency is computed by the subsequent condition (3).

$$\overline{W}_T = \frac{\overline{T}_F'' + \overline{Idf}''}{2} \tag{3}$$

Here, $\overline{T}_F''$ signifies the term frequency of words in each tweet, $\overline{W}_T$ signifies the weight of the word frequency, and $\overline{Idf}''$ signifies the feature vector computed by the condition (3) and. The higher weight words are considered as a important word. This weight factor is updated in the TF-IDF feature extraction process to attain the modified TF-IDF feature. This computed weight factor is updated in the subsequent condition (4),

$$\overline{M}_{Tf-Idf} = \overline{T}_F'' \times \log\left[\left(\frac{M}{FD_l}\right) * \overline{W}_T + 1\right] \tag{4}$$

Here, $\overline{W}_T$ signifies the updated weight factor computed by the condition (3), the first term in condition (4) describes the term frequency and the next term describes the IDF. Based on this MTF-IDF feature extraction using the condition (4), improved form of feature vectors is attained.

Significance of MTF-IDF: The word frequency of each tweet data are considered for the calculation of weight and it is updated in the condition (4). The average word frequency of words is computed through the condition (3). The condition (3) computed the word frequency weights and it is also updated in condition (4) to improve the process of feature extraction. This weight parameter is used to predict the relevancy of word related to the category. This updated feature extraction process improves the decision on classification and increases the output accuracy. This way of feature extraction makes easier the process of sentiment classification. In existing TF-IDF, only the term frequency and IDF of tweets only considered. This general TF-IDF provides the feature vector counts of all the words in the considered data. But, the improved form of TF-IDF provides the optimal feature vector to the existing to further improve the process of classification. The updated weights improve the TF-IDF feature extraction and it enhances the further classification performance. This feature extraction process effectively supports the accurate sentiment classification as positive, neutral and negative.

3.2.5 Glove feature vectors for words representation

This model is the representation of the text of the global word vector. The idea used in this feature extraction process is a word by word estimation of the counting matrix. The correlation among the random words is determined by finding the ratio between their occurrence probabilities. The relation of co-occurrence is described in the subsequent conditions (5),

$$G_F = \left((K_l - K_m)^{\bar{t}} K_P\right) \tag{5}$$

Here, G_F represents the glove feature and K represents the counting matrix. This creates the counting matrix for words between tweets (l, m) to the random words (p). This condition creates the transposable feature matrix based on its transpose ($\bar{t}$). Moreover, the additive shift is utilized in this process as described by condition (6),

$$\log(G_F) \Rightarrow \log(1 + G_F)$$
(6)

Here, this logarithmic condition (6) maintains the sparsity of G_F and avoids the divergences when calculating the co-occurrence matrix.

3.3 Sentiment word embedding based agglomerative (SWEA) clustering

Sentiment word embedding based agglomerative clustering is a hierarchical based clustering approach. It is otherwise called bottom-up clustering. In this clustering, every data point is considered a separate cluster is, known as a leaf. At first, the input data points are clustered arbitrarily. Afterwards, the distance among two pair clusters are computed and, based on the shortest distance of the data points in the clusters, are grouped further. This process is repeated until getting the one optimal form of cluster. The process of clustering formation is stopped when reaching one optimal cluster known as root. The Euclidean distance between the considered data points are calculated and described in the subsequent condition (7),

$$\overline{E}_D = \sqrt{\sum_{k=1}^{m} \left(y_k - z_k\right)^2} \tag{7}$$

Here, y_k and z_k represents the two data points, $\overline{E}_D$ represents the Euclidean distance. A distance matrix is generated according to the estimated distance between data points. Afterwards,a connection between the clusters is computed through the linkage criteria. This linkage criterion computes the shortest distance among the data points in the clusters. It is computed by the subsequent condition (8),

$$\overline{S}_D(Y,Z) = \underset{y\in Y, z\in Z}{Min} \overline{E}_{D(y,z)} \tag{8}$$

Here, $\overline{S}_D(Y,Z)$ represents the linkage criterion among the data points (Y,Z), $\overline{E}_{D(y,z)}$ represents the Euclidean distance amongst data points. This criterion is utilized to merge the clusters. Furthermore, the distance matrix is updated after this merging. This process is continued until reaching one optimal form of cluster. This process of clustering provides the tree based model for clustering. Here, the word to vector model is utilized for word embeddings. Every neighbouring word is semantically correlated in the word embedding-based clustering process. This clustering process considers that the centroid of the clusters is associated with the neighbouring data point with the minimum local density. The local density of the data points are computed by the subsequent condition (9),

$$\delta_k = \sum_m \chi\left(D_{lm} - D_{cut-off}\right) \tag{9}$$

Here, D_{lm} represents the distance among data points, $D_{cut-off}$ represents the cut-off distance and $\chi(\cdot)$ is represented in condition (10),

$$\chi(\cdot) = \begin{cases} 1 & if \;\; D_{lm} < D_{cut-off} \\ 0 & else \end{cases} \tag{10}$$

The local density measure δ_k is equivalent to the total quantity of data points that are closest to the data point k than $D_{cut-off}$. Moreover, ρ_k is computed by the subsequent condition (11),

$$\rho_k = \begin{cases} \underset{l\cdot\delta_m-\delta_l}{Min}\left(D_{lm}\right) & if \;\; D_{lm} < D_{cut-off} \\ \underset{l}{Max}\left(D_{lm}\right) & else \end{cases} \tag{11}$$

The Word embedding based clusters are formed as per the illustrations of δ_k and ρ_k. The maximum value of δ_k and ρ_k is considered as a cluster centre. The flow diagram of SWEA clustering is depicted in figure 3.

According to the steps provided in figure 3, sentiment features are clustered. This process clusters the features effectively and is given as an input to the hybrid BLSTM-CNN framework.

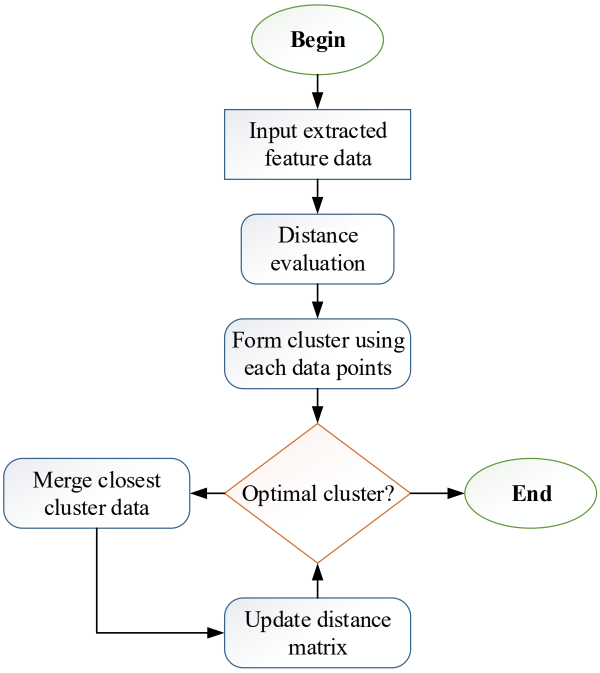

Figure 3: *Flow diagram of SWEA clustering*

3.4 Hybrid BLSTM-CNN

In the presented approach, hybrid BLSTM-CNN is utilized to get the entire context information to predict the accurate output. Combining BLSTM and the CNN layers can provide important information to the output layer through deep learning. Here, the CNN is utilized in the hybrid form is to attain enhanced performance. Here, the weights are updated optimally to increase the performance of classification. The presented hybrid framework comprises CNN layers and the BLSTM layers. The schematic diagram of hybrid BLSTM-CNN is depicted in figure 4.

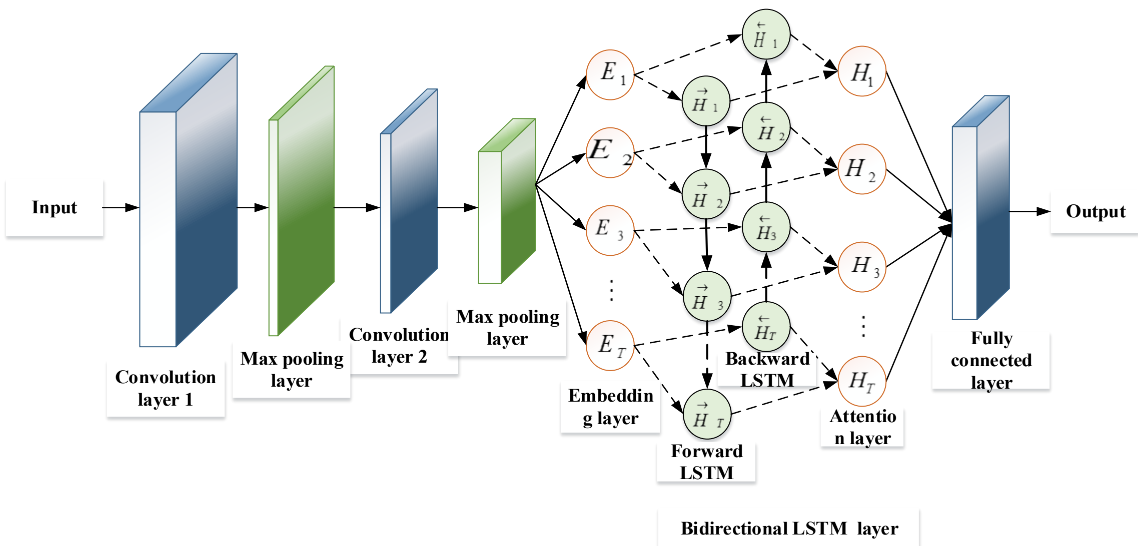

Figure 4: *Structure of hybrid BLSTM-CNN*

3.4.1 Weight optimization

The main motivation of using a modified horse herd optimization algorithm for optimizing the weight is the hierarchical organization of horse herds. The optimization approach is modified here by updating the weight factor in condition (10).In the hierarchical organization, horse A is more dominant than horse B. Similarly, horse B is more dominant than horse C. The optimized weights are attained through the presented optimization approach. The weight of the presented hybrid framework is described by the subsequent condition (12).

$$\Delta \overline{W_t} = -\frac{z\lambda}{m'}\overline{W_t} - \frac{z}{m'}\frac{\partial \overline{b}}{\partial \overline{W_t}} + N\Delta \overline{W_t}(n') \quad (12)$$

The first priority weight of the optimization technique is used to update the hybrid BLSTM-CNN. Here, the hierarchical order is attained through the fitness estimation of horse herds. The number of horses (M) and function $(P)$are considered as per the corresponding conditions.

$$\overline{H} = \{\overline{h}_1, \overline{h}_2, \overline{h}_3 \ldots. \overline{h}_M\} \quad (13)$$

$$\overline{P} = \overline{H} \rightarrow \{1,2,3 \ldots. M\}$$

(14)

Here, $\overline{H}$ represents the horse herd. The fitness of the horse herd is computed by the subsequent condition (13). If $\overline{F}(\overline{h}_u) < \overline{F}(\overline{h}_v)$, where $u \neq v$ and $u, v \in \{1,2,3 \ldots m\}$ then

$$\overline{P}(\overline{h}_u) > \overline{P}(\overline{h}_v) \quad (15)$$

If $\overline{F}(\overline{h}_u) = \overline{F}(\overline{h}_v)$, where $u \neq v$ and $u, v \in \{1,2 \ldots M\}$ then

$$\left[\overline{P}(\overline{h}_u) - \overline{P}(\overline{h}_v)\right](u - v) > 0$$

(16)

Similarly, the prioritized orders of every horse in the herd are estimated through the subsequent condition (17),

$$\widetilde{O}_P(\overline{h}_u) = \frac{\overline{P}(\overline{h}_u)}{M}$$

(17)

Here, $\widetilde{O}_P(\overline{h}_u)$ signifies the prioritized order of the horses. Afterwards, the weighted average value is estimated for each horse position in the herd, considered a centre of the horse herd. The centre evaluation of each horse herd through this process is computed in the condition (18),

$$\overline{h}_C = \frac{\sum_{u=1}^{M} y_u \overline{h}_{u,rank}}{\sum_{u=1}^{M} \overline{h}_{u,rank}} \quad (18)$$

Here, $\overline{h}_C$ represents the centre of the horse herd. The distance between the location of the horse and the herd centre is computed by the condition (19),

$$\hat{d}_{(horse,herd)} = \sqrt{\sum_{v=1}^{M} \left(horse_v - \overline{h}_C\right)^2} \quad (19)$$

Here, $\hat{d}$ signifies the evaluated distance between the horse location and herd centre.Then, the velocity function is updated if one particular horse is fitted to the group of horse herd through the condition (20),

$$\widetilde{V}_{u,v}^{T+1} = \widetilde{V}_{u,v}^{T} + \overline{h}_{u,rank} * \overline{W_F} * \left(\overline{h}_{center,v}^{T} - y_{u,v}^{T}\right) \quad (20)$$

Here, $\overline{W_F}$ signifies the weight factor. This factor is updated in the optimization approach instead of arbitrary numbers in0 and 1. Here, T signifies the current iteration and $T+1$ signifies the newest iteration. The weight factor updated in condition (20) is computed by the condition (21),

$$\overline{W}_F = \bar{e}\left(\frac{\overline{CI}}{\bar{I}_{\max} - 1}\right) \tag{21}$$

Here, $\overline{CI}$ represents the current iteration, $\bar{e}$ represents the exponential term, and $\bar{I}_{\max}$ signifies the maximum of iteration. The horse memory matrix (S) comprisesa number of rows equal to the horse memory pool (HMP) and has the P columns.

$$S_u^{T+1} = \begin{bmatrix} S_{1,u,1}^{T+1} & \cdots & S_{1,u,P}^{T+1} \\ \vdots & \vdots & \vdots \\ S_{HMP,u,1}^{T+1} & \cdots & S_{HMP,u,1}^{T+1} \end{bmatrix} T+1 \tag{22}$$

The memory matrix is updated by the subsequent condition described as,

$$G_{best} = S_{N,u,v}^{T+1} = y_{u,v}^{T+1} * \eta_D(0, \delta_d) \tag{23}$$

Here, η_D signifies the normal distribution and δ_d signifies the standard deviation. Subsequently, the global best position is attained in maximum iteration $K+1$. This optimization process is significant for updating the optimal weight in thehybrid BLSTM-CNN framework. The attained global best positions (G_{best}) are considered optimal weight to update the hybrid BLSTM-CNN framework.

3.4.2 Layers in hybrid CNN-LSTM

The presented framework comprises CNN and BLSTM layers described in the subsequent sub-sections.

- **Convolutional layer**

This layer performsthe convolution operation on the input data of the deep learning classifier. The operation of the convolution layer is represented in the subsequent condition (22),

$$S_F(l,m) = (L * N)(l,m)\sum\sum T(l+u\ ,\ m+v)N(u,v) \tag{24}$$

Here, T represents the input matrix, N represents the 2-dimensional filter of size (u,v) and S_F represents the output of a 2D feature map.

- **Pooling layer**

This layer is otherwise known as a down sampling layer. In this layer, the dimensionality of the features is reduced for the output of the convolutional layer. Here, the biggest feature value is taken for the average pooling operation.

- **BLSTM layer**

The bidirectional LSTM layer is utilized to obtain a high level of features. The bidirectional LSTM is the advanced feature learning of LSTM. This BLSTM layer performs deep learning features in both forward and backward processes. The concatenation of BLSTM layer feature learning is described in the subsequent condition (25)

$$\overline{H}_{Total} = \overrightarrow{H}_t + \overleftarrow{H}_t \tag{25}$$

Here, $\overrightarrow{H}_t$ signifies the forward pass outputs and $\overleftarrow{H}_t$ signifies the backward pass outputs. The forward and backward process outputs are combined in this condition. The output of the final bidirectional LSTM layer is connected with the fully connected layer.

- **Fully connected layer**

The feature matrix attained from the previous layer is flattened and given as an input to this layer. It acts as an interface between the layers with features deep learning to the output decision. This layer is updated with the softmax activation function.

Softmax: This softmax activation function is utilized to predict the output class probability values accurately. This is computed by the subsequent condition (26),

$$\overline{Y}_t = \frac{e^{x_t}}{\sum_{t=1}^{n} e^{x_t}} \tag{26}$$

Here, $\overline{Y}_t$ represents the system output. According to this process, sentiment on Hindi-English mixed twitter text is predicted and classified into positive, negative and neutral.

4. Results and Discussion

This section provides the experimental results of the presented methodology in an effective sentiment classification on Hindi-English code mixed twitter data. The presented methodology is implemented in the PYTHON working platform. In order to achieve better results, the SWEA clustering approach and Hybrid BLSTM-CNN framework are presented. The performance of the developed approach is examined with the different existing methodologies to prove the effectiveness of the developed approach.

4.1 Dataset description: Hindi-English code-mixed twitter dataset

The dataset consists of English-Hindi mixed social media contents. Here, Twitter data is gathered in this database. This dataset contains the twitter data in both Hindi and English language mixed form. The total number of tweets in the dataset is 10500 with 5249 user location. The available total 10500 numbers of tweets is in the Hindi-English code mixed language.

4.2 Performance metrics

In this section, different performance evaluations are provided to analyze the effectiveness of the presented methodology. Various performance metrics like accuracy, precision, recall, F-measure, and Error rate are described in the subsequent subsections,

4.2.1 Accuracy

This performance measure evaluates the overall accurateness of the sentiment classification.This performance measure evaluates the proportion of accurately predicted classes among the total number of classes. It is computed through the subsequent condition (27),

$$A''_Y = \frac{\overline{T}_{(+ive)} + \overline{T}_{(-ive)}}{\overline{T}_{(+ive)} + \overline{F}_{(+ive)} + \overline{F}_{(-ive)} + \overline{T}_{(-ive)}} \tag{27}$$

Here, A''_Y signifies the accuracy, $\overline{T}_{(+ive)}$ signifies the true positive, $\overline{T}_{(-ive)}$ signifies the true negative, $\overline{F}_{(+ive)}$ signifies the false positive, and $\overline{F}_{(-ive)}$ signifies the false negative.

4.2.2 Precision

This performance evaluation characterizes the ratio of accurately predicted tweet classfor the particular sentiment to the total amount of classified tweets in that sentiment. It is computed through the subsequent condition (28),

$$\hat{P}_N'' = \frac{\overline{T}_{(+ive)}}{\overline{T}_{(+ive)} + \overline{F}_{(+ive)}} \tag{28}$$

Here, $\hat{P}_N''$ signifies the precision performance.

4.2.3 Recall

This performance metric computes the proportion of accurately categorized tweets of given sentiment to the total quantity of tweets that are actually under that sentiment category. It is evaluated through the subsequent condition (29),

$$\overline{\mathrm{Re}_L''} = \frac{\overline{T}_{(+ive)}}{\overline{T}_{(+ive)} + \overline{F}_{(-ive)}} \tag{29}$$

Here, $\overline{\mathrm{Re}_L''}$ signifies the recall performance.

4.2.4 F-measure

The F-measure performance is evaluated by incorporating both precision and recall performances. The F-measure performance is computed through the expressed condition (30),

$$\overline{Fm''} = 2 \times \frac{[\overline{\mathrm{Re}_L''} \times \hat{P}_N'']}{[\overline{\mathrm{Re}_L''} + \hat{P}_N'']} \tag{30}$$

4.2.5 Error rate

The error rate performance is computed based on the ratio of difference among the actual value and the predicted value. To illustrate the accuracy of the classifier, the error rate is computed. This performance measure is computed through the subsequent condition (31),

$$E_{rate}'' = \frac{\left|\overline{P}_d' - \overline{A}_d'\right|}{\overline{A}_d'} \tag{31}$$

Here, E_{rate}'' signifies the error rate, $\overline{P}_d'$ signifies the predicted data and $\overline{A}_d'$ signifies the actual data.

4.3 Performance Analysis

In this section, the performance of the presented approach is examined with different existing methodologies. The accuracy performance comparison with different existing approaches is mentioned in table 3. Table 3 provides the accuracy performance comparison. This proved that the presented approach attains enhanced accuracy performance (97.2%) than the existing KL-NF (Kull back-LeiblerNeuro fuzzy) (89.93%) [28], SVM (support vector machine) (88.13%), Multi-class SVM (70.1%), LAN²FIS (logistic adaptive network depends on Neuro-fuzzy inference system) (89%), KNN (K-nearest neighbour) (67%), KNN+SVM (76%), DT (decision tree) (80%), SMO (sequential minimal optimization) +DT (89.47%) methodologies [27]. Moreover, the graphical representation of the accuracy performance comparison is provided in figure 5.

In figure 5, accuracy performance is compared with the different existing methodologies. From this illustration, the presented approach attains higher accuracy performance (97.2%) than the existing KL-NF (89.93) [28], SVM (88.13%), Multi-class SVM (70.1%), LAN^{2}FIS (89%) [26], KNN (67%), KNN+SVM (76%), DT (80%), SMO+DT (89.47%) [27]. This proved that the presented approach provides a significant improvement in regards to accuracy than the existing approaches. The performance comparison on precision is mentioned in table 4.

Table 3: *Performance on accuracy*

Methodology	**Accuracy (%)**
KL-NF	89.93
SVM	88.13
Multi-class SVM	70.1
LAN^{2}FIS	89
KNN	67
KNN+SVM	76
DT	80
SMO+DT	89.47
Proposed	97.2

In table 4, the presented methodology performance on precision is portrayed. This representation depicts that the performance of the presented approach is attaining improved performance in precision (96.8%). The existing methodologies are attained only lesser precision values than the presented approach. Furthermore, the performance examination on precision is depicted in figure 6.

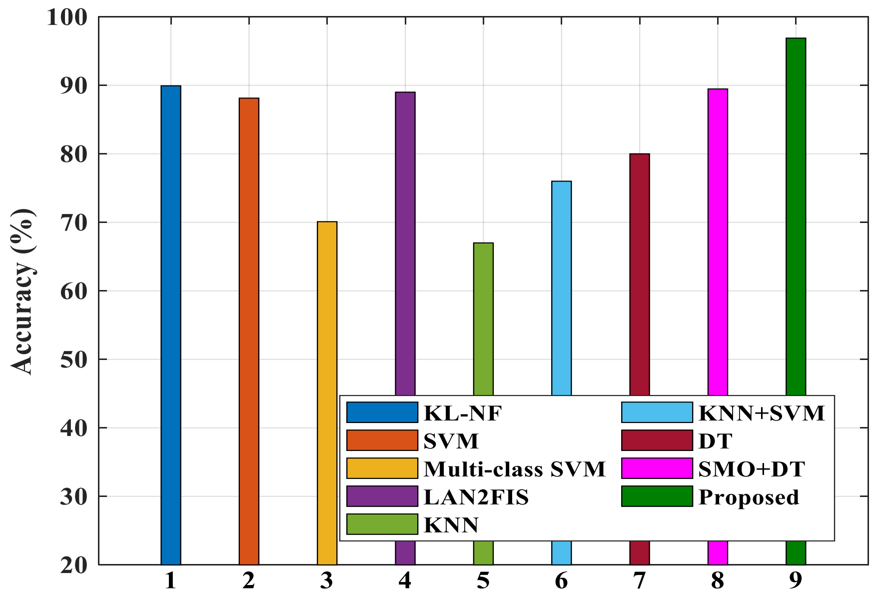

Figure 5: *Comparison examination on accuracy*

Table 4: *Performance comparison on precision*

Methodology	Precision (%)
KL-NF	93.67
SVM	84.15
Multi-class SVM	69.7
LAN²FIS	88.12
KNN	70.5
KNN+SVM	68.45
DT	81.4
SMO+DT	91.6
Proposed	96.8

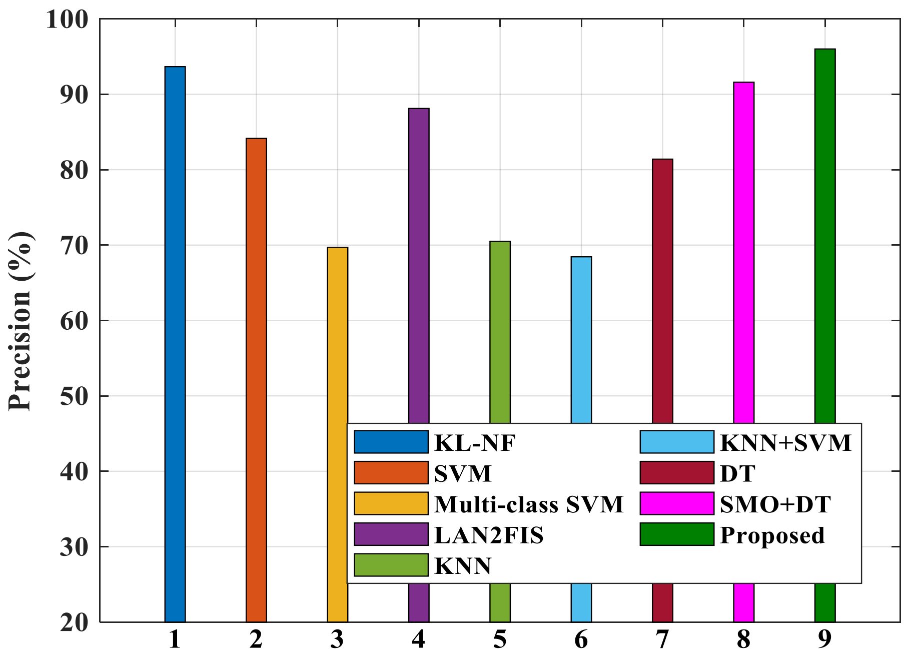

Figure 6: *Comparison examination on precision*

In figure 6, the precision performance is compared with the already existing approaches. The presented approach attains improved precision performance than the existing methodologies. The precision performance of the presented approach is 96.8%, which is higher than the existing approaches like KL-NF (93.67%) [28], SVM (84.15%), Multi-class SVM (69.7%), LAN²FIS (88.12%), KNN (70.5%), KNN+SVM (68.45%), DT (81.4%), SMO+DT (91.6%) [27]. Then, the comparative analysis on recall performance is provided in table 5.

Table 5: *Comparison analysis on recall*

Methodology	Recall (%)
KL-NF	93.01
SVM	89.76
Multi-class SVM	70.1
LAN²FIS	89.96
KNN	69.3
KNN+SVM	68.14
DT	81.4
SMO+DT	89.5
Proposed	**96.5**

In table 5, recall performance analysis is compared. This demonstrates that the presented approach providesenhanced recall (96.5%) than the various existing techniques. This further proved the effectiveness of the presented approach in accurate sentiment classification on Hindi English mixed tweets. Moreover, the performance comparison on recall is portrayed in figure 7.

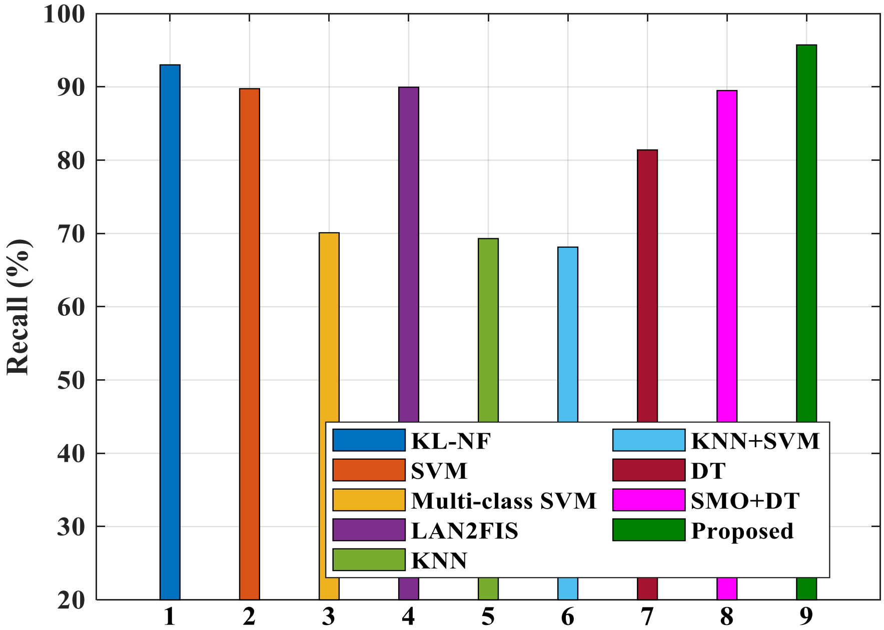

Figure 7: *Comparison examination on recall*

In figure 7, the performance comparison on recall is depicted. This proved that the developed approach attaining improved performance in regards to recall than the existing methodologies. The recall performance of the presented approach is 96.5%, which is significantly higher than the existing approaches like LAN²FIS (89.96%) [26], KL-NF (93.01%) [28], SVM (89.76%), KNN (69.3%), Multi-class SVM (70.1%), KNN+SVM (68.14%), SMO+DT (89.5%), DT (81.4%) [27]. Moreover, the comparison analysis of precision, F-measure and recall performances with varying feature extractions is given in table 6.

Table 6: *Performance comparison based on feature extraction*

Techniques	Recall (%)	Precision (%)	F-measure (%)
Key word extraction with TF-IDF	72.85	59.28	65.37
Key word extraction with WF-TF-IDF	85.57	69.47	76.68
Proposed with MTF-IDF	96.5	96.8	97

Table 7: *Performance examination on Error rate*

Methodology	Error rate (%)
SVM	17.81
Multi-class SVM	29.12
LAN²FIS	11
Proposed	**4.2**

In table 6, the performance comparison is provided by varying feature extractions. This proved that the presented approach provides improved performance with MTF-IDF feature extractionthan the existing TF-IDF and weight frequency based TF-IDF (WF-TF-IDF) [29]. The Comparison analysis on error rate is mentioned in table 7.

Table 7 compares the error rate results to the existing schemes. Here, the presented approach attains a lesser error value (4.2%) than the existing approaches like SVM (17.81%), Multi-class SVM (29.12%), and LAN²FIS (11%) [26]. Lesser error value increases the accuracy level of the developed approach in accurate sentiment classification. Furthermore, the performance comparison on error rate is depicted in figure 8.

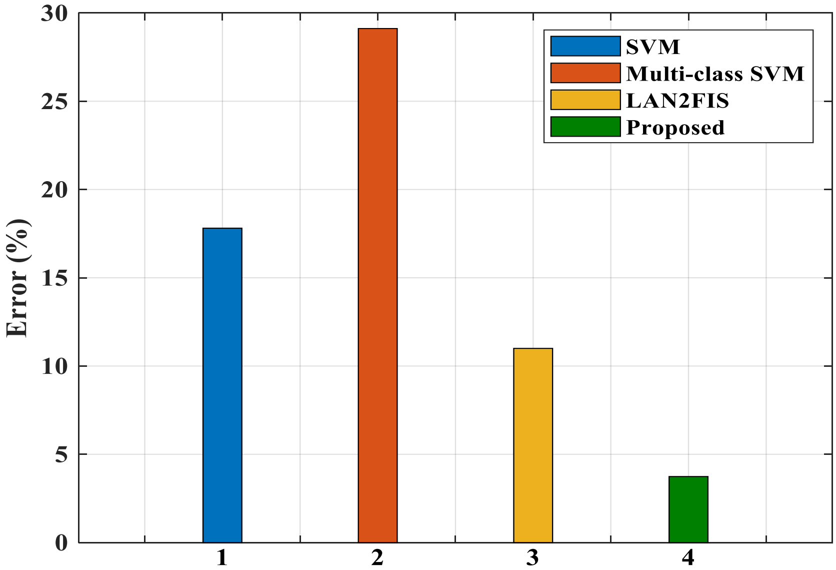

Figure 8: *Performance examination on the Error rate*

Figure 8 compares the error rate shown in sentiment classification using different approaches. The presented approach attains a lesser error rate, and it proves the efficiency of the presented approach in sentiment classification. The error rate of the presented approach is (4.2%), which is significantly lesser than the different existing classifier approaches like SVM (17.81%), Multi-class SVM (29.12%), and LAN²FIS (11%) [26]. This demonstrates that the presented approach attains improved performance than the existing methodologies.Then the F-measure performance comparison is provided in table 8.

Table 8: *Comparison analysis on F-measure*

Methodology	F-measure (%)
KL-NF	93.33
SVM	88.63
Multi-class SVM	69.9
LAN²FIS	89.03
KNN	67.9
KNN+SVM	77.56
DT	80.95
SMO+DT	96.3
Proposed	**97.0**

Table 8 compares the F-measure performance of the presented methodology with the different existing approaches. The F-measure of the presented approach is 97.0%, which is very much higher than the existing approaches. These performance evaluations are increasing the efficiency of the presented approach in sentiment classification. Furthermore, the performance comparisonon F-measure is depicted in figure 9.

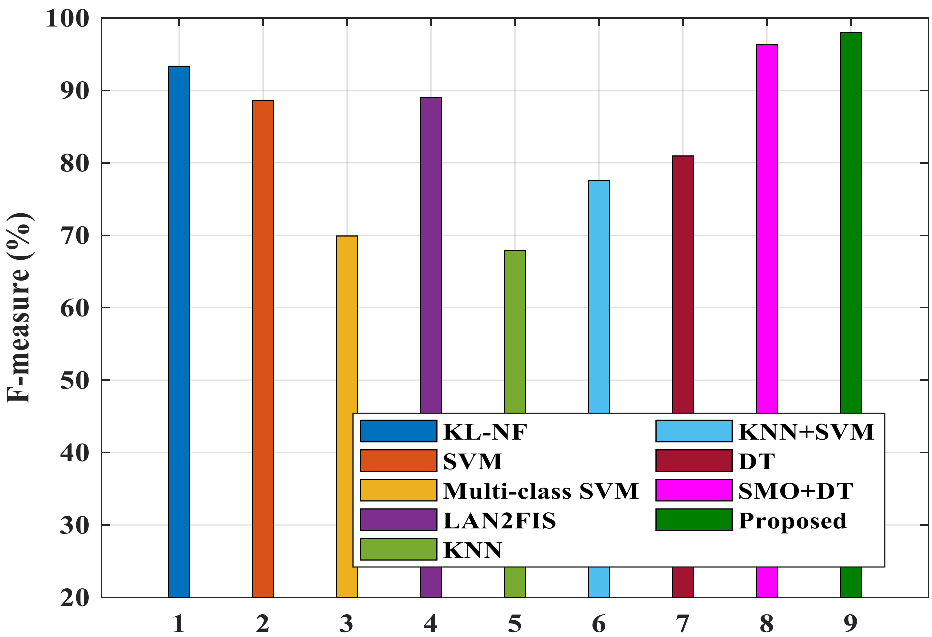

Figure 9: *Performance examination on F-measure*

Figure 9 compares the F-Measure performance with the existing approaches. The attained F-measure performance of the presented approach is 97.0%, which is significantly enhanced than the different existing methodologies like KL-NF (93.33) [28], SVM (88.63%), Multi-class SVM (69.9%),

LAN^{2}FIS (89.03%) [26], KNN (67.9%), KNN+SVM (77.56%), DT (80.95%), SMO+DT (96.3%) [27]. Performancecomparisons are proved the effectiveness of the presented framework in sentiment classification.

5. Conclusion

This paper presented an effective classification of sentiments in Hindi-English mixed twitter data. At first, the input twitter data is pre-processed with Stemming, stop word removal, tokenization, and lemmatization processes. Subsequently, effective features like Glove feature, count vectors, Feature hashing, Modified term frequency-inverse document frequency, and Word2vector features are extracted for improved sentiment analysis. Then the extracted features are clustered by utilizing sentiment word embedding based agglomerative clustering. Lastly, a hybrid Bidirectional long short term memory-convolutional neural network (Hybrid BLSTM-CNN) is utilized for accurately categorizing the tweet sentiments into positive, negative and neutral. Here, the modified horse herd optimization (MHHO) approach is utilized to update the optimized weights in Hybrid BLSTM-CNN. The presented methodology effectively predicts the sentiments in Twitter data. The experimental results of the presented Hybrid BLSTM-CNN framework in sentiment analysis provided improved performance than the different existing approaches in regards to accuracy (97.2%), precision (96.8), Error rate (4.2%), recall (96.5%), and F-measure (97.0%).

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Confidence intervals for the reliability characteristics via different estimation methods for the power Lindley model

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Abstract

In this article, classical and Bayes interval estimation procedures have been discussed for the reliability characteristics, namely mean time to system failure, reliability function, and hazard function for the power Lindley model and its special case. In the classical part, maximum likelihood estimation, maximum product spacing estimation are discussed to estimate the reliability characteristics. Since the computation of the exact confidence intervals for the reliability characteristics is not directly possible, then, using the large sample theory, the asymptotic confidence interval is constructed using the above-mentioned classical estimation methods. Further, the bootstrap (standard-boot, percentile-boot, students t-boot) confidence intervals are also obtained. Next, Bayes estimators are derived with a gamma prior using squared error loss function and linex loss function. The Bayes credible intervals for the same characteristics are constructed using simulated posterior samples. The obtained estimators are evaluated by the Monte Carlo simulation study in terms of mean square error, average width, and coverage probabilities. A real-life example has also been illustrated for the application purpose.

Keywords: Point estimation, Interval estimation of RC, MCMC method.
2000 AMS Classification: 60E05, 62M09, 62F15.

Abbreviations

AIC	:	Akaike information criterion
ACIs	:	Asymptotic confidence intervals
BCIs	:	Bootstrap confidence intervals
BIC	:	Bayesian information criterion
CDF	:	Cumulative distribution function
CIs	:	Confidence intervals
$\mathcal{C}$	:	Coverage probability
DFR	:	Decreasing failure rate
HF	:	Hazard function
HPD	:	Highest posterior density
IFR	:	Increasing failure rate
KS	:	Kolmogrov Smirnov
LD	:	Lindley distribution
LLF	:	Linex loss function
MCMC	:	Markov Chain Monte Carlo method
MTSF	:	Mean time to system failure
MLE	:	Maximum likelihood estimation
MPSE	:	Maximum product spacing estimation
p-boot	:	Percentile bootstrap
PLD	:	Power Lindley distribution
PDF	:	Probability density distribution
RC	:	Reliability characteristics
RF	:	Reliability function
SELF	:	Squared error loss function
s-boot	:	Standard bootstrap
t-boot	:	Student's t-bootstrap
$\mathcal{W}$	:	Width of the intervals

1. Introduction

The study of the reliability characteristics, MTSF and RF, HF having great importance to study the aging pattern of any lifetime phenomenon. The aging pattern of lifetime products are varying in nature and hence modeled by suitable probability distribution. In this context, exponential distribution is the most exploited model to describe the inherent characteristics of the data. Although, its uses are restricted to the constant failure rate data. Alternatively, one parameter LD is also a good choice to analyze several survival/reliability data. The latter model received more consideration of several researchers because LD having IFR. LD was proposed by [14] as a counter example of fiducial statistics. The LD has been extensively used by several researchers to draw the inferences for the parameters using complete and censored information. For reference, the readers may be see in [1], [10], [13], [17] & [18] and the cited references therein. Let a random variable Y follow LD with parameter β, then the variable $X = Y^{1/\alpha}$ has the PLD. PLD was proposed by [9]. The PDF, CDF of PLD are, respectively, given by;

$$f(x,\alpha,\beta) = \frac{\alpha\beta^2}{(1+\beta)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \quad ; x \geq 0, \alpha, \beta > 0 \tag{1}$$

$$F(x,\alpha,\beta) = 1 - \left(1 + \frac{\beta}{1+\beta}x^\alpha\right)e^{-\beta x^\alpha} \tag{2}$$

where, $x \in \mathcal{R}^+$, $\alpha(> 0)$ is the shape parameter and $\beta(> 0)$ is the scale parameter. The parameter α involves the additional flexibility in terms of hazard rate as it has IFR for $\alpha > 1$ and DFR for $\alpha < 1$. PLD has been extensively used for estimation and prediction purpose and possesses all similar property as LD for $\alpha = 1$.

The theory of classical point estimation is based on the MLE because it assumes all optimum property such as consistency, sufficiency, efficiency, etc but sometimes it leads absurd result, especially for J-shaped distribution or unbounded range of distribution. Therefore, in such cases the MPSE might be better alternative. Moreover, the MLE required joint density function and MPSE required product spacing function. Whereas, the Bayes point estimation theory combines prior information and sample information supplied by likelihood function. Hence, Bayes paradigm involves the updating form of likelihood function. An important element, in Bayes estimation theory, is the loss function. The most popular one is SELF, which can be easily justified on grounds of minimum variance-unbiased estimation theory. However, the weakness of this loss function is that it is symmetric and provides an equal weight to the overestimation and underestimation of the same magnitude. But, in some real-life situation, specially in reliability analysis overestimation can lead to more severe or less severe consequences than underestimation, or vice versa. Thus, the use of asymmetric loss function is recommended. Also, use of symmetric loss function may be inappropriate as has been recognized by [4] and [22]. Thus, a number of asymmetric loss functions are available in literature, and one of the most widely used asymmetric loss function is the Linex loss function, originally proposed by [22] and popularized by [23] which has been found to be appropriate in the situation where overestimation is more serious than underestimation or vice-versa. Let, $\hat{\theta}$ be the estimate of the parameter θ and $\Delta = (\hat{\theta} - \theta)$ defines the deviation between estimated and true value of θ. The linex loss function (LLF) may be expressed as;

$$L(\Delta) \propto \left(e^{\psi\Delta} - \psi\Delta - 1\right); \qquad \psi \neq 0 \tag{3}$$

where ψ is the loss parameter which reflects the direction and degree of asymmetry. The loss parameter ψ allows different shapes of this loss function. If $\psi > 0$, then the linex loss function is quite asymmetric about zero with overestimation being more costly than underestimation and vice-versa. For ψ closes to zero, then this loss function is approximately squared error loss and therefore almost symmetric. Several authors have used this loss function in various estimation and prediction problems.

The focus of this paper is to consider the classical and Bayesian interval estimation of the MTSF, RF and HF for the PLD and its special case LD, and to develop a guideline for choosing the best estimation method that gives better estimates and CIs for RC, which would be of deep interest to applied statisticians/engineers. In classical estimation MLE and MPSE have been discussed for the estimation of the reliability characteristics. The Bayes estimators are derived under gamma informative prior using SELF and LLF. It is observed that the posterior expectation are turned in implicit form. Therefore, MCMC technique has been used to obtain the Bayes estimates based on posterior samples. Besides, ACIs using MLE and MPSE and Bayes credible/HPD are discussed. Further, different BCIs namely, standard bootstrap (s-boot), percentile bootstrap (p-boot) and student t-bootstrap (t-boot) of the reliability characteristics are proposed. To the best of our knowledge, no attempt has been made to study the aforementioned estimators, as well as CIs based on reliability characteristics for the PLD. The present work aims to fill this gap.

Evaluation of the different confidence intervals for the parameters as well RC associated with any lifetime distribution have great advantages in different fields e.g. engineering, industry, clinical trial study to predict the possible values of the lower and upper bound to achieve some standard benchmark. For example, in reliability theory several applications may be found in measuring stress level applied on a particular system. Minimum/maximum value of the stress level beyond the certain range of stress-level affects the working mode of the system/equipment. Further, the same may be seen in case of power supply in any electronic device, the minimum/maximum power supply beyond the specified limits leads the fail to functioning the electric circuit. Similarly, in industry, the experimenter may be interested to predict the quality of goods between certain limits. If the quality of lots lies in that interval then the practitioner may be interested to send/accept the lot in market otherwise reject the lot. Acceptance/rejection of the lot may lead certain level of confidence coefficient. Further, in context of the survival analysis, cancer patients are treated with drug with specified limit of doses, if a particular patient does not receive certain amount of drug/doses would causes the death of the patient. Motivated with this fact and variety of application of the confidence interval, several confidence intervals estimation for the RC have been proposed and studied for LD and PLD. Since, exact classical CIs for the considered characteristics can not be obtained because of unavailability of exact pivotal quantity, therefore, the asymptotic and bootstrap approaches have been employed to overcome the same difficulties. The similar difficulty has been encountered with the construction of Bayes interval. Therefore, the approximate Bayes interval has been constructed for RC based on generated posterior samples. The underlying RC for PLD are listed as follows;

- **MTSF:** The mean time to system failure is the simply mean of the PLD is given by;

$$\mu = \frac{(\alpha + \alpha\beta + 1)\ \Gamma\left(\frac{1}{\alpha}\right)}{\alpha^2 \beta^{\frac{1}{\alpha}} (1+\beta)} \tag{4}$$

- **RF:** It is the probability that the system performs beyond the certain time t, for PLD it is given as

$$R(t) = \left(1 + \frac{\beta}{1+\beta} t^{\alpha}\right) e^{-\beta t^{\alpha}} \tag{5}$$

- **HF:** The instantaneous failure of any system is defined by its HF

$$h(t) = \frac{\alpha\, \beta^2\, (1+t^{\alpha})\, t^{\alpha-1}}{(1+\beta+\beta t^{\alpha})} \tag{6}$$

The RC for the LD can be obtained by putting $\alpha = 1$ in the above expressions, respectively.

The reminder of the paper is organized as follows: Section 2, describes different methods of classical estimation. The problem of ACIs based on MLEs and MPSEs are discussed in Section of 3. The BCIs for RC are described in Section 4. Bayes estimation procedure along with Bayes computation technique have been discussed in Section 5. Section 6 presents the comparison among the classical and Bayes estimators using Monte Carlo simulations. A real-life data set has been used for illustrative purpose in Section 7. Lastly, Section 8 concludes the findings of the considered study.

2. Classical methods of estimation

2.1. Maximum likelihood estimation

In this section, we consider the classical estimation of the RC discussed in previous section. For this purpose, first we obtain the MLE of the parameters and then the MLE of the RC can be constructed by using invaraince property. Let $X_1, X_2, \cdots, X_n$ are the n iid units from the Equation (1) put on a life test. The log-likelihood function ($\ln(\alpha, \beta|x) = \mathbf{L}$) based on all n observations is

$$\mathbf{L} = n\ln(\alpha) + 2n\ln(\beta) - n\ln(1+\beta) - \beta\sum_{i=1}^{n} x_i^{\alpha} + \sum_{i=1}^{n}\ln(1+x_i^{\alpha}) + (\alpha-1)\sum_{i=1}^{n}\ln x_i \tag{7}$$

The MLEs of the parameters α, β are obtained by solving the derivatives of $\mathbf{L}$ w. r. t. α and β respectively. Let $\hat{\alpha}_m$, $\hat{\beta}_m$ be the MLEs of the parameters then the MLEs of the RC are obtained as

$$\hat{\mu}_m = \frac{\Gamma\left(\frac{1}{\hat{\alpha}_m}\right)\ (\hat{\alpha}_m + \hat{\alpha}_m\hat{\beta}_m + 1)}{\hat{\alpha}_m^2\,\hat{\beta}_m^{\frac{1}{\hat{\alpha}_m}}\ (1+\hat{\beta}_m)},\ \hat{R}(t)_m = \left(1 + \frac{\hat{\beta}_m}{1+\hat{\beta}_m}\, t^{\hat{\alpha}_m}\right) e^{-\hat{\beta}_m t},\ \hat{h}(t)_m = \frac{\hat{\alpha}_m\,\hat{\beta}_m^2\,(1+t^{\hat{\alpha}_m})t^{\hat{\alpha}_m - 1}}{1+\hat{\beta}_m + \beta t^{\hat{\alpha}_m}} \tag{8}$$

2.2. Maximum product spacing estimation

The maximum product spacing method is introduced by [5], [6] as an alternative to MLE for the estimation of the unknown parameters of continuous univariate distributions. The maximum product spacing method was also derived independently by [15] as an approximation to the Kullback-Leibler measure of information. To motivate our choice, [6] proved that this method is as efficient as the MLEs and consistent under more general conditions.

Let us define the spacing function as the difference of the two consecutive CDFs

$$\mathcal{D}_i = F(x_i) - F(x_{(i-1)}) = \left(1 + \frac{\beta}{1+\beta}x_{i-1}^{\alpha}\right) e^{-\beta x_{i-1}^{\alpha}} - \left(1 + \frac{\beta}{1+\beta}x_i^{\alpha}\right) e^{-\beta x_i^{\alpha}} \tag{9}$$

such that $\sum \mathcal{D}_i = 1$,

MPSE method chooses α, β which maximizes the geometric mean of the spacing defined in equation (9)

$$\mathcal{G} = \left(\prod_{i=1}^{n+1}\mathcal{D}_i\right)^{\frac{1}{n+1}} \tag{10}$$

The equation (10) defines the alternative likelihood function using spacing. The MPS estimates can be obtained with the help of above equation by maximizing w.r.t. the parameters using iterative procedure. Once MPSE of the parameters say $\hat{\alpha}_p, \hat{\beta}_p$ are obtained, the MPSE of the reliability characteristics are obtained by simply using the invariance property.

$$\hat{\mu}_p = \frac{\Gamma\left(\frac{1}{\hat{\alpha}_p}\right)\ (\hat{\alpha}_p + \hat{\alpha}_p\hat{\beta}_p + 1)}{\hat{\alpha}_p^2\,\hat{\beta}_p^{\frac{1}{\hat{\alpha}_p}}\ (1+\hat{\beta}_p)},\ \hat{R}(t)_p = \left(1 + \frac{\hat{\beta}_p}{1+\hat{\beta}_p}\, t^{\hat{\alpha}_p}\right) e^{-\hat{\beta}_p t},\ \hat{h}(t)_p = \frac{\hat{\alpha}_p\,\hat{\beta}_p^2\,(1+t^{\hat{\alpha}_p})t^{\hat{\alpha}_p - 1}}{1+\hat{\beta}_p + \beta t^{\hat{\alpha}_p}} \tag{11}$$

respectively.

3. Asymptotic confidence intervals for RC

3.1. ACIs using the usual likelihood function

In most of the two parameter lifetime distributions the construction of exact confidence intervals (CIs) usually is not an easy task due to implicit form of the MLEs. Therefore, $100(1-\tau)\%$ ACIs may be considered based on asymptotic distribution of MLEs. It is noted that $\sqrt{n}\left(\Theta-\hat{\Theta}\right) \sim AN(0, I(\hat{\Theta}))$, where $\Theta = (\alpha, \beta)$ and $\hat{\Theta}$ is the estimate of Θ. Hence, for this purpose the Fisher Information matrix is computed as;

$$I(\hat{\alpha}, \hat{\beta}) = \mathbf{E}\begin{pmatrix} -\dfrac{\partial^2 \mathbf{L}}{\partial \alpha^2} & -\dfrac{\partial^2 \mathbf{L}}{\partial \alpha \partial \beta} \\ \\ -\dfrac{\partial^2 \mathbf{L}}{\partial \beta \partial \alpha} & -\dfrac{\partial^2 \mathbf{L}}{\partial \beta^2} \end{pmatrix}_{(\hat{\alpha}, \hat{\beta})} \tag{12}$$

where;

$$\frac{\partial^2 \mathbf{L}}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \beta \sum_{i=1}^{n} (\ln x_i)^2 x_i^{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha} (\ln x_i)^2}{1+x_i^{\alpha}} \left(1 - \frac{x_i^{\alpha}}{1+x_i^{\alpha}}\right) + 2\sum_{i=1}^{n} x_i^{\alpha} \ln x_i + \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2 \left\{(\alpha-1)\right\}$$

$$\frac{\partial^2 \mathbf{L}}{\partial \alpha \partial \beta} = \frac{\partial^2 \mathbf{L}}{\partial \beta \partial \alpha} = \sum_{i=1}^{n} x_i^{\alpha} \ln x_i, \qquad \frac{\partial^2 \mathbf{L}}{\partial \beta^2} = \frac{-2n}{\beta^2} + \frac{n}{(1+\beta)^2}.$$

All the above derivatives are evaluated at $(\hat{\alpha}, \hat{\beta})$. The above matrix given in equation (17) can be inverted to obtain the estimate of the asymptotic variance- covariance matrix of the MLEs and diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\beta})$ provides asymptotic variance of α and β respectively. Then using large sample theory, two sided $100(1-\tau)\%$ approximate confidence interval for α, β is constructed as

$$\hat{\alpha} \pm Z_{1-\frac{\tau}{2}} \sqrt{var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{1-\frac{\tau}{2}} \sqrt{var(\hat{\beta})}.$$

Since, the MLEs of the RC's are constructed easily by applying invariance property of MLE but at the same time similar difficulties arise in construction of CIs for RCs, because no explicit distributions are available for the RCs. As we have seen from previous equations, RCs are the function of parameters α, β. Hence, the intervals for μ, $R(t)$ and $H(t)$ are constructed by applying the concept of Δ-method. The Δ-method is a general approach for computing confidence intervals for functions of maximum likelihood estimates. Let $g(\Theta)$ is any function of Θ such that it is differentiable w.r.t. the parameter(s), then

$$\sqrt{n}\left(g(\Theta) - g(\hat{\Theta})\right) \sim AN(0, \sigma^2_{\Theta}\, g'(\hat{\Theta})^2)$$

CI for μ:

For large sample, it may verified that,

$$\frac{\sqrt{n}\,(\mu - \hat{\mu})}{\sqrt{\sigma^2_{\hat{\mu}}}} \sim AN(0, 1)$$

where, variance of μ ($\sigma^2_{\hat{\mu}}$) is given as;

$$\sigma^2_{\hat{\mu}} = \sigma^2_{\hat{\alpha}} \left(\frac{\partial \mu}{\partial \alpha}\right)^2 + \sigma^2_{\hat{\beta}} \left(\frac{\partial \mu}{\partial \beta}\right)^2,$$

$$\frac{\partial \mu}{\partial \alpha} = \frac{\Gamma\frac{1}{\alpha}(\alpha+\alpha\beta+1)}{\alpha^2\beta^{\frac{1}{\alpha}}(1+\beta)} \left[-\frac{1}{\alpha^2}\Psi(1/\alpha)+\frac{1+\beta}{\alpha+\alpha\beta+1}-\frac{2}{\alpha}+\frac{\log\beta}{\alpha^2}\right]$$
$$\frac{\partial \mu}{\partial \beta} = \frac{\Gamma\frac{1}{\alpha}(\alpha+\alpha\beta+1)}{\alpha^2\beta^{\frac{1}{\alpha}}(1+\beta)} \left[\frac{\alpha}{\alpha+\alpha\beta+1}-\frac{1}{\alpha\beta}-\frac{1}{1+\beta}\right]$$

$\hat{\sigma}^2_{\alpha}$ and $\hat{\sigma}^2_{\beta}$ are the variances of the parameter α, β respectively. The $100(1-\tau)\%$ CIs for the μ is given by,

$$\hat{\mu}_m \pm Z_{\frac{\tau}{2}}\sqrt{\sigma^2_{\hat{\mu}}}$$

CI for R(t):
Similarly, for reliability function $R(t)$,

$$\frac{\sqrt{n}\left(R(t)-\hat{R}(t)\right)}{\sqrt{\sigma^2_{\hat{R}(t)}}} \sim AN(0,1)$$

where, variance of $\hat{R}(t)(\sigma^2_{\hat{R}(t)})$ is given as;

$$\sigma^2_{\hat{R}(t)} = \sigma^2_{\hat{\alpha}}\left(\frac{\partial R(t)}{\partial \alpha}\right)^2 + \sigma^2_{\hat{\beta}}\left(\frac{\partial R(t)}{\partial \beta}\right)^2$$

$$\frac{\partial R(t)}{\partial \alpha} = \left(1+\frac{\beta}{1+\beta}t^{\alpha}\right)e^{-\beta t^{\alpha}}\left[-\beta\, t^{\alpha}\,\log t+\frac{\beta\, t^{\alpha}\,\log t}{1+\beta+\beta\, t^{\alpha}}\right]$$
$$\frac{\partial R(t)}{\partial \beta} = \left(1+\frac{\beta}{1+\beta}t^{\alpha}\right)e^{-\beta t^{\alpha}}\left[-t^{\alpha}+\frac{t^{\alpha}}{(1+\beta)(1+\beta+\beta\, t^{\alpha})}\right]$$

The $100(1-\tau)\%$ CIs for the $R(t)$ is given by,

$$\hat{R}(t)_m \pm Z_{\frac{\tau}{2}}\sqrt{\sigma^2_{\hat{R}(t)}}$$

CI for h(t): Similarly for hazard rate;

$$\frac{\sqrt{n}\left(h(t)-\hat{h}(t)\right)}{\sqrt{\sigma^2_{\hat{h}(t)}}} \sim AN(0,1)$$

where, variance of $\hat{h}(t)$, $\sigma^2_{\hat{h}(t)}$ is given as;

$$\sigma^2_{\hat{h}(t)} = \sigma^2_{\hat{\alpha}}\left(\frac{\partial h(t)}{\partial \alpha}\right)^2 + \sigma^2_{\hat{\beta}}\left(\frac{\partial h(t)}{\partial \beta}\right)^2$$

$$\frac{\partial h(t)}{\partial \alpha} = \frac{\alpha\,\beta^2\,(1+t^{\alpha})\,t^{\alpha-1}}{(1+\beta+\beta t^{\alpha})}\left[\frac{1}{\alpha}+\frac{t^{\alpha}\,\log t}{1+t^{\alpha}}+\log t-\frac{\beta\, t^{\alpha}\,\log t}{1+\beta+\beta\, t^{\alpha}}\right], \frac{\partial h(t)}{\partial \beta} = \frac{\alpha\,\beta^2\,(1+t^{\alpha})\,t^{\alpha-1}}{(1+\beta+\beta t^{\alpha})}\left[\frac{2}{\beta}+\frac{1+t^{\alpha}}{1+\beta+\beta\, t^{\alpha}}\right]$$

The $100(1-\tau)\%$ CIs for the $h(t)$ is given by,

$$\hat{h}(t)_m \pm Z_{\frac{\tau}{2}}\sqrt{\sigma^2_{\hat{h}(t)}}.$$

3.2. ACIs using spacing function

In this section, we have obtained the asymptotic confidence intervals using MPSE. As it was mentioned by [6] that the MPS method has the similar properties as MLE and is asymptotically equivalent. Estimation using MPSE has been also discussed by [19] and they showed mathematically that $\hat{\theta}_{MPS} = \hat{\theta}_{ML} + o(n^{-\frac{1}{2}})$ i.e. (the asymptotic or bootstrap inference around parameters based on MPSE may be carried out by utilizing the ML asymptotic). Utilizing the same concept, as MPSEs do not yield closed form of the estimators, hence the ACIs using MPSE for the parameters have been constructed. Let $I^{'}(\tilde{\alpha}, \tilde{\beta})$ be the observed Fishers information matrix and is defined as

$$I^{'}(\tilde{\alpha}_p, \tilde{\beta}_p) = \begin{bmatrix} -\mathcal{G}'^{''}_{\alpha\alpha} & -\mathcal{G}'^{''}_{\alpha\beta} \\ -\mathcal{G}'^{''}_{\beta\alpha} & -\mathcal{G}'^{''}_{\beta\beta} \end{bmatrix}_{(\tilde{\alpha},\ \tilde{\beta})} \tag{13}$$

The elements of the above matrix are given below

$$\begin{aligned}
\mathcal{G}'^{''}_{\alpha\alpha} &= \frac{1}{n+1}\left[\frac{F(x_1,\alpha,\beta)F''_{\alpha\alpha}(x_1,\alpha,\beta) - \left(F'_{\alpha}(x_1,\alpha,\beta)\right)^2}{F(x_1,\alpha,\beta)^2}\right] \\
&+ \frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}\left\{F''_{\alpha\alpha}(x_i,\alpha,\beta) - F''_{\alpha\alpha}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
&- \frac{1}{n+1}\left[\frac{\left\{F'_{\alpha}(x_i,\alpha,\beta) - F'_{\alpha}(x_{i-1},\alpha,\beta)\right\}^2}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] - \frac{1}{n+1}\left[\frac{\{1-F(x_n,\alpha,\beta)\}F''_{\alpha\alpha}(x_n,\alpha,\beta) + \left\{F'_{\alpha}(x_n,\alpha,\beta)\right\}^2}{\{1-F(x_n,\alpha,\beta)\}^2}\right]
\end{aligned}$$

Similarly, the second derivative of the function $\mathcal{G}$ with respect to β is given by,

$$\begin{aligned}
\mathcal{G}'^{''}_{\beta\beta} &= \frac{1}{n+1}\left[\frac{F(x_1,\alpha,\beta)F''_{\beta\beta}(x_1,\alpha,\beta) - \left(F'_{\beta}(x_1,\alpha,\beta)\right)^2}{F(x_1,\alpha,\beta)^2}\right] \\
&+ \frac{1}{n+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}\left\{F''_{\beta\beta}(x_i,\alpha,\beta) - F''_{\beta\beta}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
&- \frac{1}{n+1}\left[\frac{\left\{F'_{\beta}(x_i,\alpha,\beta) - F'_{\beta}(x_{i-1},\alpha,\beta)\right\}^2}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] - \frac{1}{n+1}\left[\frac{\{1-F(x_n,\alpha,\beta)\}F''_{\beta\beta}(x_n,\alpha,\beta) + \left\{F'_{\beta}(x_n,\alpha,\beta)\right\}^2}{\{1-F(x_n,\alpha,\beta)\}^2}\right]
\end{aligned}$$

and the second derivative of the function $\mathcal{G}$ with respect to α, β is given as:

$$\begin{aligned}
\mathcal{G}'^{''}_{\alpha\beta} = \mathcal{G}'^{''}_{\beta\alpha} &= \frac{1}{m+1}\left[\frac{F(x_1,\alpha,\beta)F''_{\alpha\beta}(x_1,\alpha,\beta) - F'_{\alpha}(x_1,\alpha,\beta)F'_{\beta}(x_1,\alpha,\beta)}{F(x_1,\alpha,\beta)^2}\right] \\
&+ \frac{1}{m+1}\left[\sum_{i=2}^{n}\frac{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}\left\{F''_{\alpha\beta}(x_i,\alpha,\beta) - F''_{\alpha\beta}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
&- \frac{1}{m+1}\left[\frac{\left\{F'_{\alpha}(x_i,\alpha,\beta) - F'_{\alpha}(x_{i-1},\alpha,\beta)\right\}\left\{F'_{\beta}(x_i,\alpha,\beta) - F'_{\beta}(x_{i-1},\alpha,\beta)\right\}}{\{F(x_i,\alpha,\beta) - F(x_{i-1},\alpha,\beta)\}^2}\right] \\
&- \frac{1}{m+1}\left[\frac{\{1-F(x_m,\alpha,\beta)\}F''_{\alpha\beta}(x_m,\alpha,\beta) + \left\{F'_{\alpha}(x_m,\alpha,\beta)\right\}\left\{F'_{\beta}(x_m,\alpha,\beta)\right\}}{\{1-F(x_m,\alpha,\beta)\}^2}\right]
\end{aligned}$$

where,

$$F_{\alpha}'(x_i,\alpha,\beta) = -\left(1+\frac{\beta}{1+\beta}x_i^{\alpha}\right)e^{-\beta x_i^{\alpha}}\left[-\beta\, x_i^{\alpha}\,\log x_i + \frac{\beta\, x_i^{\alpha}\,\log x_i}{1+\beta+\beta\, x_i^{\alpha}}\right],$$

$$F_{\alpha\alpha}''(x_i,\alpha,\beta) = F_{\alpha}'(x_i,\alpha,\beta)\left[-\frac{F_{\alpha}'(x_i,\alpha,\beta)}{1-F(x_i,\alpha,\beta)} + \log x_i + \frac{\beta(1+\beta)\, x_i^{\alpha}\,\log x_i}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha})+1} - \frac{\beta\, x_i^{\alpha}\,\log x_i}{1+\beta+\beta\, x_i^{\alpha}}\right]$$

$$F_{\beta}'(x_i,\alpha,\beta) = -\left(1+\frac{\beta}{1+\beta}x_i^{\alpha}\right)e^{-\beta x^{\alpha}}\left[-x_i^{\alpha} + \frac{x_i^{\alpha}}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha})}\right]$$

$$F_{\beta\beta}''(x_i,\alpha,\beta) = F_{\beta}'(x_i,\alpha,\beta)\left[\frac{F_{\beta}'(x_i,\alpha,\beta)}{1-F(x_i,\alpha,\beta)} - \frac{2+x_i^{\alpha}+2\beta+2\beta\, x_i^{\alpha}}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha})+1} - \frac{1}{1+\beta} - \frac{1+x_i^{\alpha}}{1+\beta+\beta\, x_i^{\alpha}}\right]$$

$$F_{\beta\alpha}''(x_i,\alpha,\beta) = -\frac{F_{\beta}'}{1-F(x_i,\alpha,\beta)} + \frac{2+x_i^{\alpha}+2\beta+2\beta\, x_i^{\alpha}}{(1+\beta)(1+\beta+\beta\, x_i^{\alpha})+1} - \frac{1}{1+\beta} - \frac{1+x_i^{\alpha}}{1+\beta+\beta\, x_i^{\alpha}}.$$

Thus, we can obtain an estimator of the information matrix as I($\hat{\alpha},\hat{\beta}$), where $\hat{\alpha}$=$\hat{\alpha}_p$ and $\hat{\beta}$=$\hat{\beta}_p$ are the MPS estimator of the parameters and $V(\hat{\alpha})$ and $V(\hat{\beta})$ are the diagonal elements of $I^{-1}(\hat{\alpha},\hat{\beta})$ which denotes the variance and covariance matrix. The approximate $(1-\tau)100\%$ confidence intervals for the parameters α and β is, therefore, given as, $\hat{\alpha}\pm Z_{\frac{\tau}{2}}\sqrt{V(\hat{\alpha})}$ and $\hat{\beta}\pm Z_{\frac{\tau}{2}}\sqrt{V(\hat{\beta})}$ respectively, where $Z_{\frac{\tau}{2}}$ is the upper $(\frac{\tau}{2})$ percentile of standard normal distribution. The interval estimate of RC using MPSE can be constructed in same way as discussed in previous subsection.

4. Bootstrap confidence interval

The confidence regions of parameters of a distribution have been determined using aspects of the distribution of the data. In particular, these regions have often been specified by appealing to the central limit theorem and normal approximations. The notion behind bootstrap techniques begins with the concession that the information about the source of the data is insufficient to perform the analysis to produce the necessary description of the distribution of the estimator. Thus, in this section, we considered an alternative procedure to usual method of CIs called as bootstrap method. The bootstrap method of finding confidence interval of parameters of a distribution is a most efficient sampling and re-sampling procedures without need of pivotal quantity, for more detail see, [7-8], [11]. Here, we discuss the different types of bootstrap confidence interval (BCIs), namely standard bootstrap (s-boot), percentile boot (p-boot) and students t-bootstrap (t-boot). The following steps may be used to construct the 95% BCI's.

1. Specify the value of sample size n and model parameters α, β.
2. Generate a sample $x_1, x_2, \cdots, x_n$ from equation (1)
3. Compute MLE $\hat{\alpha}$, $\hat{\beta}$ of α, β using $x_1, x_2, \cdots, x_n$.
4. Again generate bootstrap samples $x_1^*, x_2^*, \cdots, x_n^*$ from equation (1) using $\hat{\alpha}$, $\hat{\beta}$ as a population value and then compute MLE of RC $\hat{\rho} = [\hat{\mu}, \hat{R}(t), \hat{h}(t)]$.
5. Repeat step 2-3, B times and simulate $\hat{\rho}_i^*$; $i = 1, 2, \cdots, B$.

4.1. s-boot

Let $\bar{\rho}^*$ and σ_ρ^* be the sample mean and sample standard deviation of $\hat{\rho}^*, i = 1, 2, \cdots, B$.

$$\bar{\rho}^* = \frac{1}{B}\sum_{i=1}^{B}\hat{\rho}_i^* \quad \text{and} \quad \sigma_\rho^* = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\rho}_i^* - \bar{\rho}^*)^2}$$

respectively. Thus, $100(1-\tau)\%$ s-boot confidence interval for ρ is given by

$$[\hat{\rho}_L^s, \hat{\rho}_U^s] \in [\hat{\rho}^* - Z_{\tau/2}\,\sigma_\rho^* \qquad \hat{\rho}^* + Z_{\tau/2}.\sigma_\rho^*]$$

4.2. p-boot

Let $\hat{\rho}^{*(\delta)}$ be the δ-percentile of $(\hat{\rho}_{(i)}^*;\ i = 1, 2, \cdots, B)$ and $\hat{\rho}^{*(\delta)}$ is such that

$$\frac{1}{B}\sum_{i=1}^{B} I(\hat{\rho}_{(i)}^* \le \hat{\rho}^{*(\delta)}) = \delta \quad : 0 \le \delta \le 1$$

where, $I(.)$ is the indicator function. Then $100(1-\tau)\%$ p-boot confidence interval is given by

$$\left(\hat{\rho}_L^p, \hat{\rho}_U^p\right) \in \left(\hat{\rho}^{*[B\frac{\tau}{2}]}, \hat{\rho}^{*[B\frac{1-\tau}{2}]}\right)$$

4.3. t-boot

The students t-bootstrap confidence interval is obtained by the following additional steps;

- Generate again bootstrap sample $x_1^{**}, x_2^{**}, x_n^{**}$ of size n from equation (1) using $\hat{\rho}^*$.
- Compute MLE of ρ say $\hat{\rho}^{**}$.
- Calculate $\sigma_\rho^{**} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\rho}_i^{**} - \bar{\rho}^{**})^2}$ where $\bar{\rho}^{**} = \frac{1}{B}\sum_{i=1}^{B}\hat{\rho}_i^{**}$
- Compute the statistic $T = \dfrac{\hat{\rho}^{**} - \bar{\rho}^{**}}{\sigma_\rho^{**}}$. The $100(1-\rho)\%$ t-boot confidence interval for ρ is given by

$$\left(\hat{\rho}_L^p,\ \hat{\rho}_U^p\right) \in \left(\bar{\rho}^{**} - T^{\tau/2}\sigma_\rho^{**}, \quad \bar{\rho}^{**} + T^{\tau/2}\,\sigma_\rho^{**}\right)$$

To study the different CIs, we consider their estimated $\mathcal{W}$ and $\mathcal{C}$. For each of the considered methods, the average width of the BCIs is computed based on the B different trials. The average width and coverage probability are given by

$$\mathcal{W} = \frac{\sum_{i=1}^{B}(U_i - L_i)}{B}, \quad \mathcal{C} = \frac{\#\,(L \le \rho \le U)}{B}$$

where L and U are the $100(1-\tau)\%$ CI based on B replicates.

5. Bayesian estimation and credible interval

In this section, the Bayes estimators of the RC have been derived under gamma priors and different loss function as mentioned in Section 1. Let $\underline{X} = (X_1, X_2, X_3, ..., X_n)$ be the random observations of size n from (1). Since, Bayes paradigm combines sample information with prior distribution and provide the updated distribution, termed as posterior distribution, hence, the posterior distribution is derived and the respective Bayes estimates are computed under SELF and LLF. For review of the parametric inference on Bayesian paradigm one may see, [16], [18], [20] etc. Let us consider the priors for α, β are;

$$\pi_1(\alpha) \propto \alpha^{a-1}e^{-b\alpha}, \quad \pi_2(\beta) \propto \beta^{c-1}e^{-d\beta}$$

Since, the considered priors are independent and flexible in nature, hence the joint prior $\pi(\alpha,\beta)$ of (α,β) is given by;

$$\pi(\alpha,\beta) \propto \alpha^{a-1}e^{-b\alpha}\beta^{c-1}e^{-d\beta} \tag{14}$$

where a, b, c and d are the hyper-parameters, assuming to be known and positive. The prior defined in the equation above accommodates the different shapes of other distributions which depends over the values of hyper-parameters. Jeffrey's non-informative prior is also a particular case of the above prior and obtained by assuming $a, b, c, d \to 0$, given by

$$\pi(\alpha,\beta) \propto \frac{1}{\alpha\,\beta}; \quad \alpha, \beta > 0$$

Although, the prior defined above is improper in nature but the resulting posterior always remains proper. The joint posterior distribution is obtained as

$$p(\alpha,\beta|\underline{x}) \propto \alpha^{n+a-1}\beta^{2n+c-1}(1+\beta)^{-n}\exp\left\{-b\alpha - d\beta - \beta\sum_{i=1}^{n}x_i^{\alpha}\right\}\prod_{i=1}^{n}\left\{(1+x_i^{\alpha})x_i^{\alpha-1}\right\} \tag{15}$$

The Bayes estimators of the RC under SELF is the posterior mean and is given by

$$\hat{\Theta}_{sf} = E_p(\Theta|x) \tag{16}$$

where

$$E_p(\Theta|\underline{x}) = K^{-1}\int_{\alpha=0}^{\infty}\int_{\beta=0}^{\infty}\Theta\,\alpha^{n+a-1}\beta^{2n+c-1}(1+\beta)^{-n}\exp\left\{-b\alpha - d\beta - \beta\sum_{i=1}^{n}x_i^{\alpha}\right\}\prod_{i=1}^{n}\left\{(1+x_i^{\alpha})x_i^{\alpha-1}\right\}d\alpha\,d\beta \tag{17}$$

SELF is the most popular and most widely used symmetric loss function, although sometimes in reliability inference SELF does not provide more accurate result due to over and under estimation. The details of LLF is given in Section 1. The Bayes estimates of the considered characteristics under the LLF is obtained by using the following expression.

$$\hat{\Theta}_{lf} = -\frac{1}{\psi}\log\left(E_p\left[e^{-\psi\Theta}|\underset{\sim}{x}\right]\right) \tag{18}$$

provided that the expectation $E_p\left[e^{-\psi\Theta}|\underset{\sim}{x}\right]$ exists and is finite, where $\Theta = [\alpha, \beta, \mu, R(t), h(t)]$ and

$$E_p\left[e^{-\psi\Theta}|\underset{\sim}{x}\right] = K^{-1}\int_{\alpha=0}^{\infty}\int_{\beta=0}^{\infty}\alpha^{n+a-1}\beta^{2n+c}(1+\beta)^{-n}\exp\left\{-b\alpha - d\beta - \beta\sum_{i=1}^{n}x_i^{\alpha} - \psi\Theta\right\}\times\prod_{i=1}^{n}\left\{(1+x_i^{\alpha})x_i^{\alpha-1}\right\}d\alpha\,d\beta. \tag{19}$$

5.1. Bayes computation via Markov Chain Monte Carlo method

From the previous section, It has been observed that the form of Bayes estimators can not be solved analytically. The evaluation of the posterior expectation will be complicated and it will be the ratio of two intractable integrals. In such situations, Markov Chain Monte Carlo (MCMC) technique can be effectively used to generate sample from full conditional posterior distributions. For more detail about MCMC method see, [12], [20], [21]. Thus concept of Gibbs under Metropolis-Hastings (M-H) sampling procedure has been utilized to generate sample from the posterior density function (22) under the assumption that parameters α and β has independent gamma priors with hyper-parameters $(a,\, b)$ and $(c,\, d)$ respectively. To implement Gibbs under M-H algorithm the full conditional posterior densities of α and β are given by;

$$p_1(\alpha|\beta,\underline{x}) \propto \alpha^{n+a-1}e^{-b\alpha}\exp\left\{-b\alpha-\beta\sum_{i=1}^{n}x_i^{\alpha}\right\}\prod_{i=1}^{n}\left\{(1+x_i^{\alpha})x_i^{\alpha-1}\right\} \tag{20}$$

$$p_2(\beta|\alpha,\underline{x}) \propto \beta^{2n+c-1}(1+\beta)^{-n}\exp\left\{-d\beta-\beta\sum_{i=1}^{n}x_i^{\alpha}\right\} \tag{21}$$

The following steps are taken to generate posterior samples from full conditional distribution.

- Start with $j=1$ and initial values (α_0,β_0)
- Using M-H algorithm generate posterior sample for α and β from (25) and (26) respectively, where asymptotic normal distribution of full conditional densities are considered as a proposal.
- Repeat step 2, for all $j=1,2,3,\cdots,N$ and obtained $(\alpha_1,\beta_1),(\alpha_2,\beta_2),...(\alpha_N,\beta_N)$
- Generate the sequence of μ, $R(t)$ and $h(t)$ for specified t by plugin the sequences of $(\alpha_j,\beta_j); j=1,2,\cdots,N$, as

$$\mu_1,\mu_2,\cdots,\mu_N,\quad R_1,R_2,\cdots,R_N,\quad h_1,h_2,\cdots,h_N$$

- The Bayes estimates of the RC under SELF are given by

$$\hat{\mu}_{sf}\approx\frac{1}{N-N_0}\sum_{j=1}^{N-N_0}\mu_j,\quad \hat{R}(t)_{sf}\approx\frac{1}{N-N_0}\sum_{j=1}^{N}R_j,\qquad \hat{h}(t)_{sf}\approx\frac{1}{N-N_0}\sum_{j=1}^{N}h_j$$

- The Bayes estimates under LLF are obtained as;

$$\hat{\mu}_{lf}=-\frac{1}{\psi}\log\left(\frac{1}{N-N_0}\sum_{j=1}^{N}\exp(-\psi\mu_j)\right),\ \hat{R}(t)_{lf}=-\frac{1}{\psi}\log\left(\frac{1}{N-N_0}\sum_{j=1}^{N}\exp(-\psi R_j)\right),\ \hat{h}(t)_{lf}=-\frac{1}{\psi}\log\left(\frac{1}{N-N_0}\sum_{j=1}^{N}\exp(-\psi h_j)\right),$$

  respectively. N_0 is the burn in period of Markov Chain.

5.2. HPD credible interval

After extracting the posterior samples we can easily construct the HPD credible intervals for α and β, see [3]. Therefore for this purpose order $\alpha_1,\alpha_2,...,\alpha_N$ as $\alpha_1<\alpha_2<...<\alpha_N$ and $\beta_1,\beta_2,...,\beta_N$ as $\beta_1<\beta_2<...<\beta_N$. Then $100(1-\tau)\%$ credible intervals of α and β are

$$(\alpha_1,\alpha_{[N(1-\tau)]}),...,(\alpha_{[N\tau]},\alpha_N)\text{and}(\beta_1,\beta_{[N(1-\tau)]}),...,(\beta_{[N\tau]},\beta_N)$$

Using the sequence of μ, $R(t)$ and $h(t)$ the $100(1-\tau)\%$ credible intervals for RC can be constructed by proceeding in similar way. Here $[x]$ denotes the greatest integer less than or equal to x. Then, the HPD credible interval is that interval which has the shortest length.

6. Comparison of estimators by a simulation study

In this section, we carry out a simulation study to assess the performance of the proposed point (classical & Bayesian) and interval estimates (AICs, BCIs & HPD) for PLD and in particular for Lindley distribution. To perform simulation study, a set of sample sizes $n=10,20,30,50,100$ with different parametric combinations $(\alpha,\beta)=(0.75,0.85),(1,0.75),(2.5,1.5),(2,2.5)\,\&(3.5,2)$ are taken. Since, the PLD reduces to LD when $\alpha=1$, therefore the choice $(\alpha,\beta)=(1,0.75)$ among the considered choices corresponds the result for LD. In classical setup, the MLE, MPSE of μ,

$R(t)$ and $h(t)$ have been computed for specified $t = 0.75$. The ACIs based on MLEs and MPSEs are constructed for the considered characteristics. Also, for each design, $B = 1,000$ bootstrap samples with each of size n are drawn from the original sample and BCIs are constructed based on replicated $K = 3000$ times. Next, we discussed the Bayesian estimation procedure for the estimation of the same characteristics using MCMC technique and construct HPD credible interval based on generated posterior samples. The Bayes estimates are reported under informative gamma prior and non-informative prior using SELF and LLF. For LLF, the choices of loss parameter ψ are taken as $(-2, 1.5)$. The negative/positive choices of ψ indicated the departure from symmetry. Average mean square error (MSEs) of the RC for each set up are reported in Tables 4-6. In all the simulation Tables, $(\bullet)_m$, $(\bullet)_p$ denote the estimates obtained via MLE, MPSE and $(\bullet)_{sf}$, $(\bullet)_{lf1}$, $(\bullet)_{lf1}$ denote the Bayes estimates obtained under SELF and LLF ($\psi = -2, \psi = -1.5$) respectively. Tables 4-6 describe the average estimates and MSEs for MTSF, RF and HF obtained via different classical methods of estimation and Bayes estimation method, respectively. Form this simulation study it is noted that the MSEs of the classical estimates obtained through MLE and MPSE methods are very close to each other and more or less similar to the MSEs of the Bayes estimates obtained under non-informative prior. However, the Bayes estimates under informative prior information provide better results in terms of MSE than classical estimates and Bayes estimates with non-informative prior. The MSEs of all the proposed estimates ensure the property of consistency through increasing sample size, also the MSEs of the Bayes estimators under SELF and Bayes estimates under LLF are almost same and the significance differences are very small for all the considered choices of parameters and sample sizes.

The estimated average widths $(\mathcal{W})$ and coverage probabilities $(\mathcal{C})$ of 95% ACIs based on MLE and MPSE, different BCIs and HPDIs of the RC are reported in Tables 7-9. We observe that as the sample size increases, the average widths decreases in all the cases as expected. The width of the Bayes interval is less as compared to the width of ACIs and BCIs. In comparison of ACIs and BCIs, the width of BCIs are lesser and boot-p perform better. Consequently, the smaller width affects the coverage probability. All simulations were performed using programs written in the open source statistical package R. Moreover, among the three methods of BCIs, the average width of p-boot is minimum in most of the cases and the average widths follows the order p-boot $<$ s-boot $<$ t-boot for all the considered variations of the sample size and model parameters. Therefore, we conclude that p-boot shows overall better performance of the BCIs for PLD.

7. Real data analysis

In this section, a real-life data set has been considered to show the applicability of proposed study. The data set is reported by [2] which represents the strength measurements in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. The data set is given below; The average strength of single carbon fiber is 1.451 with standard deviation 0.495. The summary of the data set is presented via box plot in Figure 1, and noted that the median and mean of the data are very close to each other. However, the quartiles are equidistance from median value which indicates the symmetricity of the data. The fitting of the PLD for the above data set is checked by different model selection tools and compared with most popular two parametric probability distributions namely, Weibull distribution (WD), gamma distribution (GD), normal distribution (ND), logistic distribution (LGD) and generalized exponential distribution (GED). The considered selcetion tools are: negative of log-likelihood $-\mathbf{L}$, Akaike information criterion (AIC) [AIC=-2 $\mathbf{L}$+2k], Bayesian information criterion (BIC) [BIC=-2 $\mathbf{L} + k \log n$], Kolmogrov-Smirnov (KS) test [KS=$Sup_x|F_n(x) - \hat{F}(x)|$]. The model would be taken as best with least value of these measures. The values of the considered measures along with the $p-$value are given in Table 1, and observed that PLD has least values of $-\hat{\mathbf{L}}$, AIC, BIC, KS with higher p-value. Further, the estimated density with histogram and empirical cumulative distribution function plots for the different models are displayed in Figure 2 and Figure 3, respectively. From these Figures, it may be easily verified that the PLD might be a better choice as compared to other considered probability distributions.

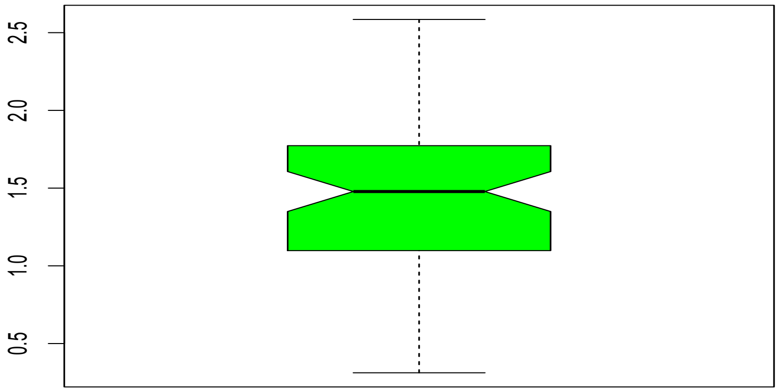

Figure 1: *Summary of the data via box plot.*

Table 1: *Values of different tools for model selection.*

Model	MLE	-LogL	AIC	BIC	KS	P-VALUE
PLD	$\hat{\alpha}$=2.6959 $\hat{\beta}$=0.4864	-48.6797	101.3594	105.8276	0.04054	0.9999
WD	$\hat{a}$=3.2487 $\hat{b}$=1.0171	-49.00054	102.0011	106.4693	0.043752	0.9994
GD	$\hat{\alpha}$=6.9968 $\hat{\beta}$=4.8209	-53.08266	110.1653	114.6335	0.0884	0.6536
ND	$\hat{\mu}$=1.4513 $\hat{\sigma}$=0.4915	-48.90256	101.8051	106.2733	0.037603	0.9992
LGD	$\hat{\mu}$=1.4533 $\hat{s}$=0.2796	-49.40943	102.8189	107.2871	0.047889	0.9974
GED	$\hat{\alpha}$=8.8283 $\hat{\lambda}$=1.8965	-56.66857	117.3371	121.8054	0.11192	0.3531

The classical estimates (MLEs, MPSE) of the RC, μ, $R(t)$ and $h(t)$ are computed for arbitrarily chosen $t = 1.5$. Since in case of real-life data set no any prior information is available, thus one may use most suited non-informative prior which may be proper or improper but it leads proper posterior. Here, we have taken the same non-informative prior where losses are SELF and LLF. The Bayes estimates are calculated under non-informative prior using MCMC method, reported in Table 2. In order to perform Bayes computation using MCMC method, well mixing of the chain has been checked via tuning of the variance of the MLE. To achieve stationarity of the Markov Chain, ($N_0 = 500$) samples (burn in period) are discarded out of 12000 generated posterior deviates. It has been verified that the generated posterior samples are well mixed and assume the stationary property. Further, different interval estimates namely ACIs, BCIs and HPD credible are constructed for the same characteristics, given in Table 3. From Table 3, it is clearly visible that the width of the interval obeys the pattern $\mathcal{W}_m \approx \mathcal{W}_p > \mathcal{W}_{s-boot} > \mathcal{W}_{p-boot} \approx \mathcal{W}_{t-boot} > \mathcal{W}_{Bayes}$.

Table 2: *Estimates of the RC for $t = 1.5$ of the considered data set.*

RC	$\hat{\Theta}_m$	$\hat{\Theta}_p$	$\hat{\Theta}_{sf}$	$\hat{\Theta}_{lf1}$	$\hat{\Theta}_{lf2}$
μ	1.4519	1.4488	1.4563	1.4586	1.4546
$R(t)$	0.4632	0.4752	0.4673	0.4647	0.4647
$h(t)$	1.7200	1.7236	1.7121	1.7520	1.6840

Table 3: *Width of the interval of RC for the considered data set when $t = 1.5$.*

	MLE	MPSE	s-boot	p-boot	t-boot	HPD
RC	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$	$\mathcal{W}$
μ	0.4149	0.4147	0.2318	0.2242	0.2269	0.1860
$R(t)$	0.3346	0.3426	0.0906	0.0846	0.0798	0.0532
$h(t)$	1.8786	1.8626	0.3030	0.3000	0.3221	0.1843

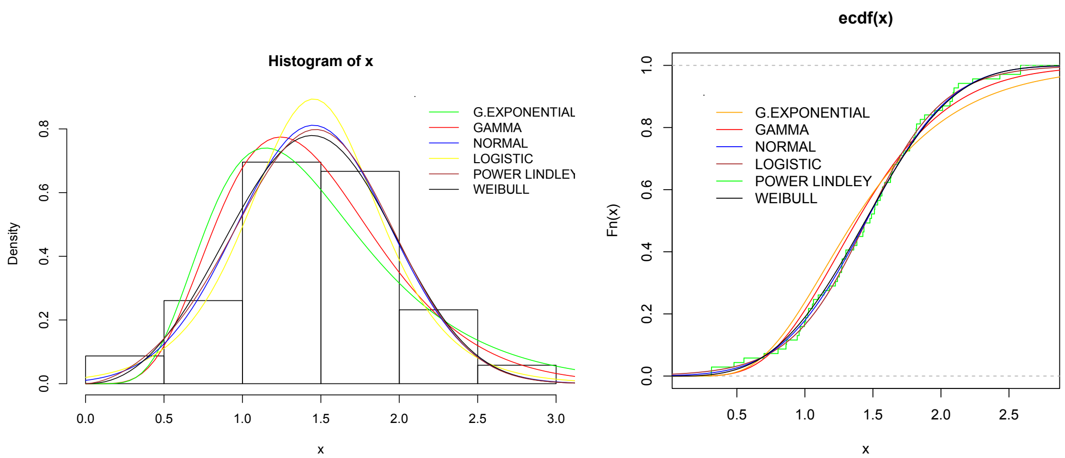

Figure 2: *Estimated density and ECDF plots based on considered data.*

Table 4: *Average MSEs of the classical and Bayes estimators of MTSF and RF.*

n	μ	Classical estimators		Bayes(informative)			Bayes (non-informative)		
		$\hat{\mu}_m$	$\hat{\mu}_p$	$\hat{\mu}_{sf}$	$\hat{\mu}_{lf1}$	$\hat{\mu}_{lf2}$	$\hat{\mu}_{sf}$	$\hat{\mu}_{lf1}$	$\hat{\mu}_{lf2}$
10		0.8280	0.8472	0.6872	0.6972	0.6197	0.6964	0.6988	0.6269
20		0.5430	0.6847	0.5356	0.4203	0.3421	0.5739	0.6125	0.3905
30	2.5445	0.3997	0.4273	0.3416	0.3611	0.2436	0.3434	0.3499	0.2500
50		0.2170	0.2123	0.1856	0.1914	0.1561	0.2016	0.3007	0.1644
100		0.0917	0.0940	0.0851	0.0903	0.0773	0.0915	0.1152	0.0840
10		0.5299	0.5285	0.4791	0.4136	0.3923	0.5175	0.4658	0.2759
20		0.1858	0.2002	0.1813	0.1282	0.1502	0.1946	0.2113	0.1551
30	2.0952	0.1053	0.1230	0.1262	0.1249	0.1094	0.1228	0.2076	0.1389
50		0.0700	0.0780	0.0723	0.0886	0.0661	0.0742	0.0921	0.0674
100		0.0304	0.0313	0.0310	0.0340	0.0298	0.0350	0.0390	0.0331
10		0.0213	0.0233	0.0175	0.0209	0.0167	0.0234	0.0214	0.0221
20		0.0101	0.0109	0.0096	0.0100	0.0094	0.0103	0.0107	0.0100
30	0.6951	0.0068	0.0066	0.0062	0.0064	0.0062	0.0070	0.0072	0.0069
50		0.0040	0.0042	0.0037	0.0037	0.0036	0.0041	0.0042	0.0041
100		0.0020	0.0020	0.0020	0.0020	0.0020	0.0022	0.0022	0.0021
10		0.0089	0.0086	0.0081	0.0083	0.0080	0.0104	0.0323	0.0103
20		0.0045	0.0044	0.0042	0.0042	0.0042	0.0049	0.0049	0.0049
30	0.7621	0.0033	0.0033	0.0030	0.0030	0.0030	0.0031	0.0031	0.0031
50		0.0022	0.0021	0.0020	0.0020	0.0020	0.0022	0.0022	0.0021
100		0.0010	0.0010	0.0009	0.0009	0.0009	0.0010	0.0011	0.0010
10		0.0091	0.0087	0.0083	0.0085	0.0082	0.0100	0.0102	0.0100
20		0.0045	0.0043	0.0043	0.0043	0.0043	0.0045	0.0046	0.0045
30	0.8841	0.0034	0.0033	0.0030	0.0030	0.0030	0.0032	0.0033	0.0032
50		0.0017	0.0018	0.0016	0.0016	0.0016	0.0020	0.0020	0.0020
100		0.0010	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009	0.0009
n	$R(t)$	Classical estimators		Bayes(informative)			Bayes (non-informative)		
		$\hat{R}(t)_m$	$\hat{R}(t)_p$	$\hat{R}(t)_{sf}$	$\hat{R}(t)_{lf1}$	$\hat{R}(t)_{lf2}$	$\hat{R}(t)_{sf}$	$\hat{R}(t)_{lf1}$	$\hat{R}(t)_{lf2}$
10		0.0143	0.0143	0.0136	0.0134	0.0138	0.0144	0.0142	0.0147
20		0.0080	0.0074	0.0069	0.0069	0.0069	0.0079	0.0079	0.0080
30	0.6907	0.0051	0.0049	0.0045	0.0045	0.0046	0.0052	0.0052	0.0053
50		0.0031	0.0033	0.0029	0.0029	0.0032	0.0031	0.0031	0.0031
100		0.0015	0.0015	0.0014	0.0014	0.0014	0.0015	0.0015	0.0015
10		0.0119	0.0116	0.0114	0.0111	0.0116	0.0120	0.0117	0.0122
20		0.0066	0.0068	0.0063	0.0062	0.0064	0.0064	0.0063	0.0064
30	0.7529	0.0045	0.0047	0.0043	0.0042	0.0043	0.0044	0.0045	0.0047
50		0.0027	0.0028	0.0025	0.0025	0.0025	0.0027	0.0027	0.0027
100		0.0013	0.0013	0.0012	0.0012	0.0012	0.0013	0.0013	0.0013
10		0.0190	0.0128	0.0135	0.0137	0.0134	0.0196	0.0199	0.0194
20		0.0089	0.0088	0.0076	0.0077	0.0076	0.0082	0.0082	0.0081
30	0.3909	0.0052	0.0054	0.0051	0.0051	0.0051	0.0053	0.0056	0.0055
50		0.0033	0.0029	0.0028	0.0027	0.0028	0.0033	0.0033	0.0033
100		0.0016	0.0015	0.0016	0.0016	0.0016	0.0017	0.0017	0.0017
10		0.0182	0.0123	0.0133	0.0134	0.0133	0.0187	0.0187	0.0188
20		0.0084	0.0087	0.0075	0.0075	0.0075	0.0087	0.0087	0.0087
30	0.5000	0.0062	0.0061	0.0051	0.0051	0.0051	0.0056	0.0056	0.0056
50		0.0036	0.0038	0.0034	0.0034	0.0034	0.0031	0.0032	0.0031
100		0.0017	0.0018	0.0016	0.0016	0.0016	0.0018	0.0018	0.0018
10		0.0161	0.0131	0.0119	0.0117	0.0121	0.0165	0.0162	0.0168
20		0.0081	0.0072	0.0067	0.0066	0.0068	0.0074	0.0074	0.0075
30	0.6655	0.0058	0.0054	0.0046	0.0045	0.0046	0.0054	0.0054	0.0055
50		0.0031	0.0031	0.0027	0.0027	0.0027	0.0033	0.0033	0.0033
100		0.0016	0.0016	0.0015	0.0015	0.0015	0.0016	0.0016	0.0016

Table 5: *Average MSEs of the classical and Bayes estimators of HF.*

n	$h(t)$	Classical estimators		Bayes(informative)			Bayes (non-informative)		
		$\hat{h}(t)_m$	$\hat{h}(t)_p$	$\hat{h}(t)_{sf}$	$\hat{h}(t)_{lf1}$	$\hat{h}(t)_{lf2}$	$\hat{h}(t)_{sf}$	$\hat{h}(t)_{lf1}$	$\hat{h}(t)_{lf2}$
10		0.0387	0.0392	0.0365	0.0345	0.0321	0.0351	0.0360	0.0378
20		0.0141	0.0195	0.0115	0.0121	0.0111	0.0146	0.0156	0.0139
30	0.4148	0.0079	0.0076	0.0070	0.0073	0.0069	0.0079	0.0082	0.0077
50		0.0048	0.0047	0.0045	0.0046	0.0044	0.0046	0.0047	0.0046
100		0.0020	0.0022	0.0019	0.0019	0.0019	0.0021	0.0022	0.0021
10		0.0394	0.0324	0.0320	0.0365	0.0297	0.0327	0.0375	0.0300
20		0.0155	0.0150	0.0145	0.0153	0.0141	0.0145	0.0153	0.0140
30	0.4257	0.0097	0.0074	0.0094	0.0073	0.0094	0.0096	0.0099	0.0098
50		0.0057	0.0058	0.0052	0.0054	0.0055	0.0056	0.0057	0.0056
100		0.0035	0.0033	0.0025	0.0025	0.0024	0.0026	0.0027	0.0026
10		0.7851	0.7478	0.7298	0.7252	0.7206	0.7124	0.7503	0.7864
20		0.4959	0.4832	0.3240	0.4209	0.2428	0.3953	0.5106	0.3887
30	1.9937	0.2342	0.1563	0.2093	0.2179	0.1739	0.2124	0.2891	0.2176
50		0.1168	0.1320	0.1085	0.1293	0.0976	0.1179	0.1412	0.1055
100		0.0487	0.0432	0.0420	0.0412	0.0477	0.0531	0.0580	0.0503
10		0.5859	0.6898	0.4941	0.4781	0.4453	0.5271	0.6132	0.5805
20		0.5130	0.6181	0.3966	0.4567	0.2981	0.4765	0.5106	0.5461
30	2.4304	0.2681	0.2955	0.2371	0.2319	0.2000	0.2595	0.3539	0.2162
50		0.1771	0.1747	0.1454	0.1727	0.1320	0.1605	0.1532	0.1603
100		0.0978	0.0819	0.0649	0.0713	0.0616	0.0713	0.0773	0.0683
10		0.7301	0.4766	0.4056	0.4707	0.3060	0.6604	0.4941	0.4933
20		0.3194	0.3469	0.2161	0.2860	0.1830	0.2875	0.2577	0.2668
30	1.7233	0.1819	0.1732	0.1228	0.1428	0.1132	0.1325	0.1528	0.1230
50		0.0681	0.0705	0.0640	0.0620	0.0613	0.0773	0.0848	0.0736
100		0.0375	0.0383	0.0335	0.0352	0.0327	0.0373	0.0392	0.0363

Table 6: *Average width and coverage probability of different CIs for MTSF.*

n	μ	ACIs				BCIs						HPD interval			
		MLE		MPSE		s-boot		p-boot		t-boot		Informative Prior		non-informative Prior	
		$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$
10		5.8205	0.913	6.7708	0.971	3.5789	0.861	3.5142	0.875	4.21954	0.885	3.4797	0.873	3.5598	0.866
20		4.0643	0.959	5.3867	0.984	2.5230	0.892	2.4962	0.892	2.84413	0.913	2.3161	0.898	2.3989	0.864
30	2.5445	3.3274	0.963	4.0324	0.989	2.0414	0.915	2.0259	0.926	2.25289	0.935	1.8731	0.896	1.8774	0.889
50		2.5589	0.976	2.9189	0.992	1.6106	0.919	1.5997	0.929	1.74043	0.936	1.3967	0.879	1.4197	0.894
100		1.8102	0.988	1.9574	0.995	1.1540	0.946	1.1468	0.947	1.21691	0.954	0.9679	0.891	0.9877	0.900
10		3.6159	0.944	5.4237	0.978	2.1069	0.885	2.0834	0.886	2.33669	0.893	2.0846	0.870	2.0677	0.887
20		2.5600	0.969	3.1146	0.986	1.5304	0.917	1.5174	0.919	1.64496	0.93	1.3822	0.893	1.3843	0.874
30	2.0952	2.1025	0.980	2.4159	0.990	1.2545	0.917	1.2459	0.921	1.33214	0.933	1.1029	0.876	1.1205	0.899
50		1.6140	0.985	1.7783	0.991	0.9838	0.952	0.9780	0.949	1.03119	0.958	0.8393	0.896	0.8558	0.878
100		1.1449	0.993	1.2156	0.997	0.7020	0.948	0.6976	0.945	0.72394	0.956	0.5945	0.891	0.5993	0.892
10		0.5463	0.947	0.8013	0.994	0.5138	0.902	0.5096	0.905	0.54385	0.913	0.3918	0.863	0.4134	0.795
20		0.3889	0.951	0.4841	0.983	0.3729	0.897	0.3707	0.899	0.38834	0.907	0.2772	0.835	0.2898	0.836
30	0.6951	0.3188	0.945	0.3733	0.981	0.3100	0.917	0.3082	0.922	0.32090	0.931	0.2264	0.842	0.2315	0.822
50		0.2479	0.952	0.2754	0.972	0.2427	0.941	0.2416	0.943	0.24979	0.946	0.1759	0.845	0.1767	0.828
100		0.1752	0.948	0.1865	0.968	0.1736	0.943	0.1726	0.945	0.17659	0.945	0.1240	0.852	0.1244	0.815
10		0.3811	0.943	0.5222	0.986	0.3525	0.897	0.3504	0.895	0.35813	0.894	0.2687	0.861	0.2832	0.823
20		0.2722	0.949	0.3281	0.981	0.2632	0.917	0.2618	0.915	0.26621	0.918	0.1909	0.852	0.1994	0.831
30	0.7621	0.2227	0.952	0.2553	0.975	0.2165	0.934	0.2151	0.931	0.21781	0.934	0.1572	0.843	0.1589	0.833
50		0.1724	0.958	0.1894	0.965	0.1698	0.949	0.1691	0.947	0.17076	0.946	0.1218	0.828	0.1243	0.859
100		0.1224	0.954	0.1291	0.969	0.1212	0.941	0.1207	0.941	0.12160	0.939	0.0865	0.839	0.0872	0.814
10		0.3523	0.951	0.4558	0.989	0.3415	0.894	0.3389	0.891	0.34075	0.886	0.2531	0.810	0.2651	0.801
20		0.2517	0.941	0.2942	0.977	0.2538	0.921	0.2523	0.919	0.25420	0.916	0.1809	0.826	0.1861	0.820
30	0.8841	0.2055	0.944	0.2313	0.973	0.2096	0.926	0.2087	0.929	0.20947	0.929	0.1483	0.812	0.1499	0.813
50		0.1606	0.947	0.1735	0.957	0.1648	0.93	0.1641	0.928	0.16484	0.927	0.1147	0.838	0.1156	0.800
100		0.1141	0.936	0.1188	0.946	0.1171	0.954	0.1166	0.955	0.11693	0.956	0.0808	0.832	0.0812	0.815

Table 7: *Average length and width of the CIs for RF.*

n	$R(t)$	ACIs				BCIs						HPD interval			
		MLE		MPSE		s-boot		p-boot		t-boot		Informative Prior		non-informative Prior	
		$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$
10		0.4153	0.866	0.4235	0.925	0.4541	0.877	0.4444	0.908	0.4199	0.827	0.2728	0.743	0.2789	0.737
20		0.3043	0.898	0.3064	0.927	0.3299	0.919	0.3256	0.936	0.3202	0.896	0.2045	0.776	0.2077	0.749
30	0.6907	0.2519	0.912	0.2532	0.926	0.2705	0.922	0.2679	0.934	0.2655	0.905	0.1713	0.788	0.1730	0.762
50		0.1958	0.919	0.1975	0.928	0.2113	0.938	0.2096	0.943	0.2089	0.928	0.1347	0.755	0.1348	0.758
100		0.1398	0.924	0.1401	0.940	0.1503	0.947	0.1494	0.951	0.1492	0.943	0.0966	0.807	0.0968	0.787
10		0.3662	0.830	0.3880	0.911	0.4028	0.857	0.3914	0.895	0.3545	0.793	0.2437	0.716	0.2480	0.716
20		0.2720	0.879	0.2810	0.920	0.2995	0.906	0.2949	0.932	0.2813	0.880	0.1837	0.743	0.1871	0.756
30	0.7529	0.2252	0.885	0.2303	0.926	0.2483	0.917	0.2454	0.929	0.2378	0.903	0.1548	0.748	0.1552	0.762
50		0.1768	0.902	0.1789	0.922	0.1938	0.930	0.1922	0.942	0.1883	0.917	0.1213	0.751	0.1215	0.756
100		0.1263	0.920	0.1268	0.927	0.1385	0.943	0.1375	0.942	0.1358	0.940	0.0870	0.783	0.0876	0.783
10		0.4871	0.895	0.5628	0.969	0.5093	0.893	0.5060	0.914	0.5158	0.898	0.3202	0.843	0.3310	0.755
20		0.3480	0.931	0.3733	0.964	0.3566	0.914	0.3550	0.915	0.3604	0.908	0.2356	0.812	0.2416	0.809
30	0.3909	0.2843	0.928	0.2991	0.967	0.2897	0.934	0.2885	0.933	0.2921	0.929	0.1953	0.829	0.1989	0.813
50		0.2205	0.945	0.2279	0.967	0.2225	0.944	0.2215	0.946	0.2237	0.944	0.1531	0.829	0.1550	0.822
100		0.1561	0.943	0.1589	0.960	0.1563	0.948	0.1555	0.947	0.1570	0.953	0.1094	0.839	0.1099	0.815
10		0.5178	0.900	0.5944	0.975	0.5191	0.897	0.5148	0.913	0.5177	0.894	0.3279	0.837	0.3460	0.795
20		0.3653	0.929	0.3914	0.974	0.3657	0.910	0.3636	0.913	0.3697	0.908	0.2443	0.836	0.2520	0.818
30	0.5000	0.2976	0.941	0.3120	0.970	0.2980	0.938	0.2964	0.941	0.3019	0.937	0.2028	0.833	0.2077	0.827
50		0.2299	0.952	0.2371	0.961	0.2295	0.931	0.2283	0.931	0.2322	0.930	0.1592	0.825	0.1614	0.856
100		0.1623	0.952	0.1654	0.965	0.1614	0.930	0.1608	0.932	0.1631	0.931	0.1139	0.837	0.1146	0.809
10		0.4261	0.863	0.4923	0.952	0.4643	0.886	0.4548	0.915	0.4351	0.842	0.2857	0.781	0.3002	0.742
20		0.3045	0.884	0.3300	0.943	0.3381	0.926	0.3345	0.938	0.3316	0.909	0.2167	0.812	0.2216	0.795
30	0.6655	0.2504	0.904	0.2661	0.950	0.2762	0.933	0.2739	0.942	0.2736	0.929	0.1794	0.793	0.1823	0.777
50		0.1947	0.915	0.2025	0.931	0.2161	0.945	0.2145	0.949	0.2152	0.940	0.1411	0.809	0.1430	0.786
100		0.1379	0.908	0.1412	0.930	0.1530	0.948	0.1520	0.944	0.1528	0.943	0.1011	0.805	0.1019	0.792

Table 8: *Average width and corresponding coverage probability for different CIs of HF.*

n	$h(t)$	ACIs				BCIs						HPD interval			
		MLE		MPSE		s-boot		p-boot		t-boot		Informative Prior		non-informative Prior	
		$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{C}$
10		0.7809	0.960	0.7237	0.972	0.9071	0.936	0.8213	0.920	1.1156	0.954	0.4329	0.811	0.4433	0.803
20		0.5370	0.977	0.5118	0.980	0.4753	0.945	0.4721	0.920	0.5716	0.959	0.2997	0.840	0.3094	0.822
30	0.6907	0.4361	0.983	0.4213	0.985	0.3573	0.948	0.3567	0.943	0.4119	0.961	0.2449	0.856	0.2487	0.852
50		0.3343	0.984	0.3276	0.991	0.2658	0.951	0.2656	0.937	0.2958	0.966	0.1903	0.846	0.1895	0.839
100		0.2358	0.987	0.2323	0.992	0.1819	0.946	0.1814	0.943	0.1958	0.964	0.1341	0.878	0.1346	0.861
10		0.8098	0.933	0.7739	0.967	0.7706	0.939	0.7213	0.929	1.0156	0.955	0.4422	0.777	0.4517	0.810
20		0.5705	0.963	0.5534	0.981	0.5011	0.938	0.4984	0.928	0.5762	0.953	0.3146	0.813	0.3206	0.834
30	0.7529	0.4635	0.972	0.4522	0.984	0.3834	0.948	0.3823	0.946	0.4241	0.965	0.2605	0.817	0.2590	0.827
50		0.3602	0.977	0.3526	0.989	0.2882	0.951	0.2871	0.948	0.3089	0.967	0.2013	0.822	0.2010	0.820
100		0.2544	0.986	0.2502	0.991	0.2003	0.950	0.1997	0.946	0.2100	0.961	0.1420	0.836	0.1433	0.851
10		3.4746	0.973	3.2738	0.919	16.0037	0.990	7.8693	0.839	13.4845	0.995	2.1102	0.862	2.4974	0.798
20		2.0997	0.948	1.9753	0.928	2.9501	0.989	2.8136	0.905	4.1606	0.997	1.4083	0.836	1.4482	0.830
30	0.3909	1.6443	0.941	1.5721	0.933	2.0297	0.971	1.9845	0.907	2.7617	0.992	1.1319	0.824	1.1537	0.833
50		1.2301	0.949	1.2003	0.934	1.3968	0.974	1.3810	0.924	1.7973	0.994	0.8618	0.836	0.8765	0.832
100		0.8548	0.949	0.8384	0.943	0.9079	0.970	0.9014	0.948	1.0936	0.991	0.6007	0.837	0.6079	0.830
10		3.8633	0.950	3.7965	0.941	9.6634	0.981	6.9379	0.879	11.1590	0.984	2.2629	0.855	2.6232	0.775
20		2.3826	0.955	2.3127	0.946	3.1208	0.986	3.0400	0.914	4.2482	0.993	1.5854	0.846	1.6504	0.821
30	0.5000	1.8798	0.952	1.8309	0.935	2.2095	0.974	2.1795	0.926	2.8706	0.992	1.2797	0.840	1.3125	0.841
50		1.4160	0.953	1.3905	0.938	1.5640	0.961	1.5500	0.928	1.9286	0.986	0.9800	0.820	0.9854	0.831
100		0.9842	0.950	0.9782	0.945	1.0323	0.965	1.0267	0.953	1.2018	0.985	0.6911	0.833	0.6921	0.826
10		2.4993	0.923	2.4870	0.951	4.0126	0.946	3.6154	0.914	5.0554	0.955	1.5872	0.794	1.6935	0.779
20		1.6623	0.933	1.6466	0.954	1.9417	0.960	1.9263	0.936	2.3671	0.977	1.1402	0.818	1.1386	0.821
30	0.6655	1.3441	0.943	1.3374	0.967	1.4932	0.951	1.4867	0.925	1.7524	0.971	0.9148	0.817	0.9246	0.823
50		1.0293	0.951	1.0244	0.956	1.0857	0.954	1.0832	0.940	1.2279	0.969	0.7074	0.842	0.7161	0.819
100		0.7218	0.944	0.7210	0.950	0.7354	0.945	0.7335	0.945	0.8005	0.968	0.5021	0.832	0.5048	0.826

8. Conclusion

In this paper, we have considered the classical and Bayesian point and interval estimation of the reliability characteristics (RC) for the PLD based on complete observations. In classical estimation, MLE, MPSE are discussed for RC. The Bayes estimators are derived with informative and non-informative priors under SELF and LLF for the same characteristics. Further, different CIs, as ACIs based on MLE and MPSE, three BCIs, namely s-boot, p-boot, t-boot and Bayes credible HPD intervals based on generated posterior samples are obtained. The theoretical comparison of the point and interval estimates obtained via different methods of estimation are not feasible. Hence, the Monte Carlo simulation study has been performed to make the extensive comparison in terms of average MSEs and average width of the respective CIs. Monte Carlo simulation results showed that p-boot CIs achieve better performance than the other BCIs and ACIs in terms of width for all the considered choices. Among the methods of estimation, Bayes estimates under informative prior are the best performing estimator in terms of the average MSEs as well as average width of CIs. Coverage probabilities do not follow any specific trend but for shorter length of the CIs, C decreases and reaching to the nominal values. Lastly, a practical data set has been used to illustrate the proposed methodology, and observed that it echoed the same pattern as simulation.

Conflict of interest

The authors declare that they have no conflict of interest.

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Inventory Model with Truncated Weibull Decay Under Permissible Delay in Payments and Inflation Having Selling Price Dependent Demand

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Abstract

For optimal utilization of resources, the inventory models are required in several places such as market yards, production processes, warehouses, oil exploration industries and food vegetable markets. Huge work has been produced by several researchers in inventory models for obtaining optimal ordering quantity and pricing policies. This paper addresses an EOQ model for deteriorating items having Weibull decay under inflation and permissible delay in payments. It is considered that the demand of items is a function of selling price. It is further assumed that the decay of items starts after certain period of time which can be well characterized by truncated Weibull probability model for the life time of the commodity. The optimal ordering and pricing policies of this system are derived and analyzed in the light of the input parameters and costs. Through sensitivity analysis it is demonstrated that the delay in the payments and rate of inflation have significant effect on the optimal policies. This model is very useful in the analyzing market yards where sea foods, vegetables, fruits, edible oils are stored and distributed.

Keywords: EOQ model, selling price depended demand, truncated decay.

1. Introduction

Decay is the major consideration for planning inventory and scheduling orders. The decay is in general random due to various factors such as environmental conditions, type of commodity, storage facility and natural life time. Considering the life time of commodity as random several authors developed various inventory model for deteriorating items with various plausible assumptions. The review on inventory models with deteriorating items is given by [1], [2], [3], [4]. Recently [5], [6], [7], [8], [9], and [10] have developed several inventory models with the assumption that the life time of a commodity is random and follows a specified distribution depending on the nature of commodity. In all these papers they assumed that the decay starts immediately after the procurement. But in many practical situations the deterioration of items in the stock starts only after certain period of time. This type of delay in decay can be characterized by truncated Weibull life time distribution which is often known as three parameter Weibull distribution.

Another basic assumption made by all these authors is that the payments must be made to the supplier immediately after receiving the items. However, it is a common phenomenon that the supplier allows a certain fixed period for finalizing the accounts and does not charge any interest during that period from the retailer. In [11] studied an EOQ model with assumption of permissible delay in payments. His work was extended to deteriorating items by [12]. Later [13], [14], [15] and others have developed EOQ models with permissible delay in payments.

In today's business transaction, the supplier will offer a cash discount to encourage the retailer in addition to allowing a fixed period for settlement of account. In addition to this there is a change in money value over time. Ignoring inflation may leads falsification in the model. Recently [16] has studied Inventory Model with Generalized Pareto life time under permissible delay in payments while deriving the optimal pricing and ordering policies. Considering the inflation several authors have studied various inventory model with permissible delay in payments. However, they assumed the decay is constant or independent of time, but in many practical situations the deteriorating rate is time dependent. An EOQ model with time quadratic demand by [17]. They considered the inflation while determining the optimal policies.

Little work has been reported regarding EOQ models under permissible delay in payments having inflation and selling price dependent demand, which are very useful for analyzing many practical situations arising at market yards, warehouse etc. Hence in this paper we develop and analyze the Economic Order Quantity model with truncated Weibull decay under permissible delay in payments and inflation having selling price dependent demand.

Section (2) of this paper deals with the assumptions of the model and notation. Section (3) is to develop the instantaneous inventory level at any given time t. The optimal ordering and pricing policies of the model are derived in Section (4). Section (5) considers Numerical illustration of the model. The sensitivity analysis is presented in Section (6). Section (7) deals with conclusions.

2. Assumptions

For developing the Economic Order Quantity model, the following assumptions are made

- Deterioration start time is γ.
- Weibull distribution is the life time distribution of the commodity. Its p.d.f is

$$f(t) = \alpha\beta(t-\gamma)^{\beta-1}e^{-\alpha(t-\gamma)^{\beta}}$$

Where α is the scale parameter, β is the shape parameter and γ is the location parameter

The instantaneous deterioration rate is

$$h(t) = \alpha(t-\gamma)^{\beta}, \qquad t \geq \gamma$$

- Demand function is

$$R(p(t)) = a - bp(t) = a - bpe^{rt}$$

Which is selling price dependent demand. Where, a is the fixed demand, $a > 0$, b is the demand parameter, $b > 0$, and $a > b$, $p(t)$ is the selling price of an item at time t and p is the selling price of the item at time $t = 0$.

- Rate of inflation is r, $0 < r < 1$
- Shortages are not allowed.
- Zero lead time.
- During the permissible delay period (M), the account is not settled, the generated sales revenue is deposited in an interest-bearing account. At the end of the trade credit period, the customer pays off for all the units ordered.
- There is no repair or replacement of the deteriorated units during the cycle time.

Notation

H	: Finite horizon length.
$R(p(t))$	: Demand per unit time as a function of selling price.
h	: Holding cost of inventory per unit time after excluding interest.
r	: Rate of inflation.
$p(t) = pe^{rt}$	: Per unit selling price.
$g(t) = ge^{rt}$	: Purchase cost of a unit at time t.

$A(t) = Ae^{rt}$: Per order cost at time t.

I_C : Interest charged per Rs. INR in stock per a year by the supplier.

I_e : Interest earned in Rs. INR per a year.

M : Permissible delay period which is allowed in settling the account.

Q : Order quantity per a cycle.

T : Cycle length

$I(t)$: On-hand inventory at time $t, 0 \leq t \leq T$.

$TC(p, T)$: Total cost over $(0, H)$.

$NP(p, T)$: Net profit rate function over planning period.

3. Inventory Model

Let Q be the inventory level of the system at time $t = 0$. During $(0, \gamma)$ inventory will decrease due to demand and during (γ, T) inventory will decrease due to demand and deterioration. Since no shortages are allowed, at time T the inventory level reaches zero, the stock is replenished instantaneously. The schematic diagram representing the inventory level is shown in Figure-3.1.

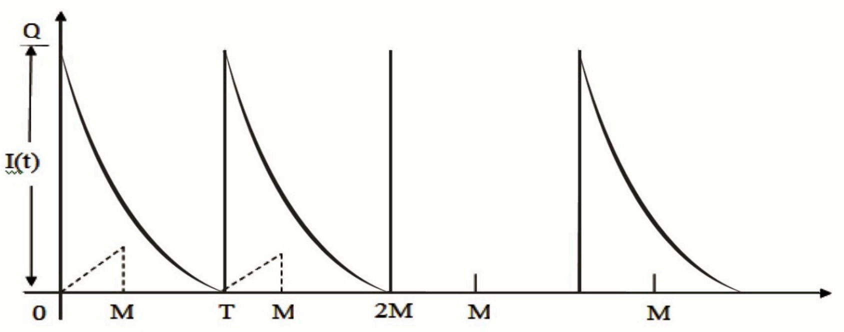


Figure -1: *Schematic diagram representing the inventory level of selling price dependent demand model*

Let $I(t)$ be the on-hand inventory at time t. The differential equations governing the on-hand inventory at time t are

$$\frac{d}{dt}I(t) = -R(p(t)) \qquad 0 \leq t \leq \gamma \qquad (1)$$

$$\frac{d}{dt}I(t) + h(t)I(t) = -R(p(t)) \qquad \gamma \leq t \leq T \qquad (2)$$

where $h(t) = \alpha\beta(t-\gamma)^{\beta-1}$ $\quad \gamma \leq t \leq T$

and $R(p(t)) = a - bp(t) = a - bpe^{rt}$

with initial conditions $I(0) = Q \; and \; I(T) = 0$.

Solving equation (1) and using the initial condition $I(0) = Q$, we get

$$I(t) = Q - at + \frac{bp}{r}(e^{rt} - 1) \qquad 0 \leq t \leq \gamma \qquad (3)$$

Solving equation (2) and using the initial condition $I(T) = 0$, we get

$$I(t) = e^{-\alpha(t-\gamma)^{\beta}}\left[a\int_{t}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{t}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}\,du\right] \qquad \gamma \le t \le T \tag{4}$$

Equating equations (3) and (4) when $t = \gamma$, we get

$$Q = a\gamma - \frac{bp}{r}(e^{r\gamma} - 1) + a\int_{\gamma}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{\gamma}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}du \tag{5}$$

Substituting Q in equation (3), we get

$$I(t) = a(\gamma - t) + \frac{bp}{r}(e^{rt} - e^{r\gamma}) + a\int_{\gamma}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{\gamma}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}du \qquad 0 \le t \le \gamma \tag{6}$$

Since the length of time intervals are all the same, we have

$$I(jT+t) = \begin{cases} a(\gamma - t) + \dfrac{bp}{r}(e^{rt} - e^{r\gamma}) + a\displaystyle\int_{\gamma}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{\gamma}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}\,du & 0 \le t \le \gamma \\ e^{-\alpha(t-\gamma)^{\beta}}\left[a\displaystyle\int_{t}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{t}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}\,du\right] & \gamma \le t \le T \end{cases} \tag{7}$$

4. The Optimal Ordering and Pricing Policies

Total cost function is the sum of Ordering Cost (OC), Cost Deterioration (CD), Inventory Carrying Cost (ICC), Interest Charged (IC_1) and Interest Earned (IE_1).
Each cost component is computed as follows:
Ordering Cost, OC is

$$OC = A(0) + A(T) + A(2T) + \ldots + A(n-1)T = A\left[\frac{e^{rH} - 1}{e^{rT} - 1}\right] \tag{8}$$

Cost Deterioration, CD is

$$CD = \sum_{j=0}^{n-1} g e^{rjT}\left[Q - \int_{0}^{T}(a - bpe^{rt})dt\right]$$

where, Q is as given in equation (5). On simplification, we get

$$CD = g\left[a\gamma - aT - \frac{bp}{r}(e^{r\gamma} - e^{rT}) + a\int_{\gamma}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{\gamma}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\left[\frac{e^{rH} - 1}{e^{rT} - 1}\right] \tag{9}$$

Inventory Carrying Cost, ICC is

$$ICC = h\sum_{j=0}^{n-1} g(jT)\left[\int_{0}^{T} I(jT+t)dt\right]$$

$$= hg\left[\frac{a\gamma^2}{2} + \frac{bp}{r^2}[e^{r\gamma}(1 - r\gamma) - 1] + \gamma\left[a\int_{\gamma}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{\gamma}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$
$$\left. + \int_{\gamma}^{T} e^{-\alpha(t-\gamma)^{\beta}}\left[a\int_{t}^{T} e^{\alpha(u-\gamma)^{\beta}}du - bp\int_{t}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}\,du\right]dt\right]\left[\frac{e^{rH} - 1}{e^{rT} - 1}\right] \tag{10}$$

For computing interest charged and earned, there are two possibilities based on the customer's choice. Interest Charges (IC) for unsold items at the initial time or after the permissible delay period M and interest Earned (IE) from the sales revenue during the permissible delay period.
Case (i): Optimum cycle length T is larger than or equal to M i.e., $T \ge M$
Interest Charged in $(0, H)$, IC_1 is

$$IC_1 = I_c \sum_{j=0}^{n-1} g(jT) \left[\int_M^T I(jT+t)dt \right]$$

$$= I_c g \left[\frac{a}{2}(\gamma^2 + M^2 - 2M\gamma) + \frac{bp}{r^2}\left[e^{r\gamma}\left(1 - r(\gamma - M)\right) - e^{rM}\right] + (\gamma - M)\left[a\int_\gamma^T e^{\alpha(u-\gamma)^\beta}du - bp\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du \right] \right.$$

$$\left. + \int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[a\int_t^T e^{\alpha(u-\gamma)^\beta}du - bp\int_t^T e^{ru+\alpha(u-\gamma)^\beta}du \right]dt \right]\left[\frac{e^{rH}-1}{e^{rT}-1}\right] \tag{11}$$

Interest Earned in $(0, H)$, IE_1 is

$$IE_1 = I_e \sum_{j=0}^{n-1} p(jT) \left[\int_0^M (a - bpe^{rt})tdt \right]$$

$$= I_e p \left[\frac{aM^2}{2} - \frac{bp}{r^2}\left[e^{rM}(rM-1)+1\right] \right]\left[\frac{e^{rH}-1}{e^{rT}-1}\right] \tag{12}$$

The total cost over $(0, H)$ is $TC(p,T)$ and is given by

$$TC(p,T) = OC + CD + ICC + IC_1 - IE_1 \tag{13}$$

Substituting equations (8), (9), (10), (11) and (12) in (13), we get

$TC(p,T) =$

$$\left[A + g\left[a\gamma - aT - \frac{bp}{r}(e^{r\gamma} - e^{rT}) + a\int_\gamma^T e^{\alpha(u-\gamma)^\beta}du - bp\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du \right] \right.$$

$$+ hg\left[\frac{a\gamma^2}{2} + \frac{bp}{r^2}\left[e^{r\gamma}(1 - r\gamma) - 1\right] + \gamma\left[a\int_\gamma^T e^{\alpha(u-\gamma)^\beta}du - bp\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du \right] \right.$$

$$\left. + \int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[a\int_t^T e^{\alpha(u-\gamma)^\beta}du - bp\int_t^T e^{ru+\alpha(u-\gamma)^\beta}du \right]dt \right]$$

$$+ I_c g\left[\frac{a}{2}(\gamma^2 + M^2 - 2M\gamma) + \frac{bp}{r^2}\left[e^{r\gamma}\left(1 - r(\gamma - M)\right) - e^{rM}\right] + (\gamma - M)\left[a\int_\gamma^T e^{\alpha(u-\gamma)^\beta}du - bp\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du \right] \right.$$

$$\left. + \int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[a\int_t^T e^{\alpha(u-\gamma)^\beta}du - bp\int_t^T e^{ru+\alpha(u-\gamma)^\beta}du \right]dt \right]$$

$$\left. - I_e p\left[\frac{aM^2}{2} - \frac{bp}{r^2}\left[e^{rM}(rM-1)+1\right] \right] \right]\left[\frac{e^{rH}-1}{e^{rT}-1}\right] \tag{14}$$

The net profit is the difference of gross revenue and total cost.

The gross revenue is $(pe^{rT} - ge^{rT})(a - bpe^{rT})$

Hence, the net profit is $NP(p,T) = (pe^{rT} - ge^{rT})(a - bpe^{rT}) - TC(p,T)$ (15)

where, $TC(p,T)$ is as given in (14)

For obtaining the optimal policies of the model, maximize $NP(p,T)$ with respect to T and p. The conditions for obtaining optimality are

$$\frac{\partial NP(p,T)}{\partial T}=0,\frac{\partial NP(p,T)}{\partial p}=0 \quad and \quad D=\begin{vmatrix}\frac{\partial^2 NP(p,T)}{\partial p^2} & \frac{\partial^2 NP(p,T)}{\partial T\partial p}\\ \frac{\partial^2 NP(p,T)}{\partial T\partial p} & \frac{\partial^2 NP(p,T)}{\partial T^2}\end{vmatrix}<0$$

where D is the determinant of Hessian matrix

$\frac{\partial NP(p,T)}{\partial T}=0$ implies,

$$\begin{aligned}
&(p-g)[are^{rT}-2pbre^{2rT}]\\
&\left\{\left[\frac{e^{rH}-1}{e^{rT}-1}\right]\left[g\left[-a+bpe^{rT}+ae^{\alpha(T-\gamma)^\beta}-bpe^{rT+\alpha(T-\gamma)^\beta}\right]\right.\right.\\
&+hg\left[\gamma\left[ae^{\alpha(T-\gamma)^\beta}-bpe^{rT+\alpha(T-\gamma)^\beta}\right]+\int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[ae^{\alpha(T-\gamma)^\beta}-bpe^{rT+\alpha(T-\gamma)^\beta}\right]dt\right]\\
&+I_c g\left[(\gamma-M)\left[ae^{\alpha(T-\gamma)^\beta}-bpe^{rT+\alpha(T-\gamma)^\beta}\right]+\int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[ae^{\alpha(T-\gamma)^\beta}-bpe^{rT+\alpha(T-\gamma)^\beta}\right]d\,t\right]\\
&+\left[A+g\left[a\gamma-aT-\frac{bp}{r}(e^{r\gamma}-e^{rT})+a\int_\gamma^T e^{\alpha(u-\gamma)^\beta}du-bp\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du\right]\right.\\
&+hg\left[\frac{a\gamma^2}{2}+\frac{bp}{r^2}[e^{r\gamma}(1-r\gamma)-1]+\gamma\left[a\int_\gamma^T e^{\alpha(u-\gamma)^\beta}du-bp\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du\right]\right.\\
&\left.+\int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[a\int_t^T e^{\alpha(u-\gamma)^\beta}du-bp\int_t^T e^{ru+\alpha(u-\gamma)^\beta}du\right]dt\right]\\
&+I_c g\left[\frac{a}{2}(\gamma^2+M^2-2M\gamma)+\frac{bp}{r^2}\left[e^{r\gamma}\big(1-r(\gamma-M)\big)-e^{rM}\right]+(\gamma-M)\left[a\int_\gamma^T e^{\alpha(u-\gamma)^\beta}du-bp\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du\right]\right.\\
&\left.+\int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[a\int_t^T e^{\alpha(u-\gamma)^\beta}du-bp\int_t^T e^{ru+\alpha(u-\gamma)^\beta}du\right]dt\right]\\
&\left.\left.-I_e p\left[\frac{aM^2}{2}-\frac{bp}{r^2}[e^{rM}(rM-1)+1]\right]\right]\left[\frac{e^{rH}-1}{(e^{rT}-1)^2}\right]re^{rT}\right\}=0 \qquad (16)
\end{aligned}$$

$\frac{\partial NP(p,T)}{\partial p}=0$ implies,

$$\begin{aligned}
&e^{rT}(a+bge^{rT}-2pbe^{rT})\\
&-\left\{g\left[-\frac{b}{r}(e^{r\gamma}-e^{rT})-b\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du\right]+hg\left[\frac{b}{r^2}[e^{r\gamma}(1-r\gamma)-1]-b\gamma\left[\int_\gamma^T e^{ru+\alpha(u-\gamma)^\beta}du\right]\right.\right.\\
&\left.-b\int_\gamma^T e^{-\alpha(t-\gamma)^\beta}\left[\int_t^T e^{ru+\alpha(u-\gamma)^\beta}du\right]dt\right]+I_c g\left[\frac{b}{r^2}\left[e^{r\gamma}\big(1-r(\gamma-M)\big)-e^{rM}\right]\right.
\end{aligned}$$

$$-b(\gamma-M)\left[\int_{\gamma}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]-b\int_{\gamma}^{T} e^{-\alpha(t-\gamma)^{\beta}}\left[\int_{t}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]dt\Bigg]$$

$$-I_e\left[\frac{aM^2}{2}-\frac{2bp}{r^2}\left[e^{rM}(rM-1)+1\right]\right]\Bigg\}\left[\frac{e^{rH}-1}{e^{rT}-1}\right]=0 \quad (17)$$

For given values of the parameters and costs, equations (16) and (17) are solved using MATHCAD to get the optimal cycle length $T=T_1$ and selling price $p=p_1$. Substituting the optimal values T_1 and p_1 in equation (14) we get the minimum total cost. Substituting this minimum total cost, T_1 and p_1 in equation (15), we get the maximum profit as

$$NP^*(p_1,T_1)=(p_1e^{rT_1}-ge^{rT_1})(a-bp_1e^{rT_1})$$

$$-\left[A+g\left[a\gamma-aT_1-\frac{bp_1}{r}(e^{r\gamma}-e^{rT_1})+a\int_{\gamma}^{T_1} e^{\alpha(u-\gamma)^{\beta}}du-bp_1\int_{\gamma}^{T_1} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$+hg\left[\frac{a\gamma^2}{2}+\frac{bp_1}{r^2}\left[e^{r\gamma}(1-r\gamma)-1\right]+\gamma\left[a\int_{\gamma}^{T_1} e^{\alpha(u-\gamma)^{\beta}}du-bp_1\int_{\gamma}^{T_1} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$\left.+\int_{\gamma}^{T_1} e^{-\alpha(t-\gamma)^{\beta}}\left[a\int_{t}^{T_1} e^{\alpha(u-\gamma)^{\beta}}du-bp_1\int_{t}^{T_1} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]dt\right]$$

$$+I_cg\left[\frac{a}{2}(\gamma^2+M^2-2M\gamma)+\frac{bp_1}{r^2}\left[e^{r\gamma}\left(1-r(\gamma-M)\right)-e^{rM}\right]+(\gamma-M)\left[a\int_{\gamma}^{T_1} e^{\alpha(u-\gamma)^{\beta}}du-bp_1\int_{\gamma}^{T_1} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$\left.\left.+\int_{\gamma}^{T_1} e^{-\alpha(t-\gamma)^{\beta}}\left[a\int_{t}^{T_1} e^{\alpha(u-\gamma)^{\beta}}du-bp_1\int_{t}^{T_1} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]dt\right]-I_ep_1\left[\frac{aM^2}{2}-\frac{bp_1}{r^2}\left[e^{rM}(rM-1)+1\right]\right]\right]\left[\frac{e^{rH}-1}{e^{rT_1}-1}\right] \quad (18)$$

Case (ii): Cycle Length T is smaller than M i.e., $T<M$

Interest Earned, IE_2 is

$$IE_2=I_e\sum_{j=0}^{n-1}p(j,T)\left\{\int_0^T R(p(t))tdt+R(p(T))[T(M-T)]\right\}$$

$$=pI_e\left\{\int_0^T(a-bpe^{rt})tdt+(a-bpe^{rT})\,[T(M-T)]\right\}\left[\frac{e^{rH}-1}{e^{rT}-1}\right]$$

$$=pI_e\left[\frac{aT^2}{2}-\frac{bp}{r^2}\left[e^{rT}(rT-1)-1\right]+(a-bpe^{rT})[T(M-T)]\right]\left[\frac{e^{rH}-1}{e^{rT}-1}\right] \quad (19)$$

Thus, the total cost over $(0,H)$ is $TC(p,T)$

$$TC(p,T)=OC+CD+ICC-IE_2 \quad (20)$$

Substituting equations (8), (9), (10) and (19) in (20), we get

$$TC(p,T)=\left[A+g\left[a\gamma-aT-\frac{bp}{r}(e^{r\gamma}-e^{rT})+a\int_{\gamma}^{T} e^{\alpha(u-\gamma)^{\beta}}du-bp\int_{\gamma}^{T} e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$+hg\left[\frac{a\gamma^2}{2}+\frac{bp}{r^2}[e^{r\gamma}(1-r\gamma)-1]+\gamma\left[a\int_{\gamma}^{T}e^{\alpha(u-\gamma)^{\beta}}du-bp\int_{\gamma}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$\left.+\int_{\gamma}^{T}e^{-\alpha(t-\gamma)^{\beta}}\left[a\int_{t}^{T}e^{\alpha(u-\gamma)^{\beta}}du-bp\int_{t}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]dt\right]$$

$$-pI_e\left[\frac{aT^2}{2}-\frac{bp}{r^2}[e^{rT}(rT-1)-1]+(a-bpe^{rT})[T(M-T)]\right]\Bigg]\left[\frac{e^{rH}-1}{e^{rT}-1}\right] \tag{21}$$

The net profit is the difference of gross revenue and total cost.

The gross revenue is $(pe^{rT}-ge^{rT})(a-bpe^{rT})$

Hence, the net profit is $NP(p,T)=(pe^{rT}-ge^{rT})(a-bpe^{rT})-TC(p,T)$ (22)

where, $TC(p,T)$ is as given in equation (21)
For obtaining the optimal policies of the model we maximize $NP(p,T)$ with respect to T and p. The conditions for obtaining optimality are

$$\frac{\partial NP(p,T)}{\partial T}=0,\frac{\partial NP(p,T)}{\partial p}=0 \ \ and \ \ D=\begin{vmatrix}\frac{\partial^2NP(p,T)}{\partial p^2} & \frac{\partial^2NP(p,T)}{\partial T\partial p}\\ \frac{\partial^2NP(p,T)}{\partial T\partial p} & \frac{\partial^2NP(p,T)}{\partial T^2}\end{vmatrix}<0$$

where D is the determinant of Hessian matrix

$\frac{\partial NP(p,T)}{\partial T}=0$ implies,

$$(p-g)[are^{rT}-2brpe^{2rT}]-\left\{\left[\frac{e^{rH}-1}{e^{rT}-1}\right]\left[g\left[-a+bpe^{rT}+ae^{\alpha(T-\gamma)^{\beta}}-bpe^{rT+\alpha(T-\gamma)^{\beta}}\right]\right.\right.$$

$$+hg\left[\gamma\left[ae^{\alpha(T-\gamma)^{\beta}}-bpe^{rT+\alpha(T-\gamma)^{\beta}}\right]+\int_{\gamma}^{T}e^{-\alpha(t-\gamma)^{\beta}}\left[ae^{\alpha(T-\gamma)^{\beta}}-bpe^{rT+\alpha(T-\gamma)^{\beta}}\right]dt\right]$$

$$-I_ep[aT-bpTe^{rT}+(a-bpe^{rT})(M-2T)+(MT-T^2)(-bpre^{rT})]$$

$$+\left[A+g\left[a\gamma-aT-\frac{bp}{r}(e^{r\gamma}-e^{rT})+a\int_{\gamma}^{T}e^{\alpha(u-\gamma)^{\beta}}du-bp\int_{\gamma}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$+hg\left[\frac{a\gamma^2}{2}+\frac{bp}{r^2}[e^{r\gamma}(1-r\gamma)-1]+\gamma\left[a\int_{\gamma}^{T}e^{\alpha(u-\gamma)^{\beta}}du-bp\int_{\gamma}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$\left.+\int_{\gamma}^{T}e^{-\alpha(t-\gamma)^{\beta}}\left[a\int_{t}^{T}e^{\alpha(u-\gamma)^{\beta}}du-bp\int_{t}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]dt\right]$$

$$\left.-I_ep\left[\frac{aT^2}{2}-\frac{2bp}{r^2}[e^{rT}(rT-1)]+(a-bpe^{rT})[T(M-T)]\right]\Bigg]\left[\frac{e^{rH}-1}{(e^{rT}-1)^2}\right]re^{rT}\right\}=0 \tag{23}$$

$\frac{\partial NP(p,T)}{\partial p}=0$ implies,

$$e^{rT}(a+bge^{rT}-2pbe^{rT})-\left[g\left[-\frac{b}{r}(e^{r\gamma}-e^{rT})-b\int_{\gamma}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]\right.$$

$$+hg\left[\frac{b}{r^2}[e^{r\gamma}(1-r\gamma)-1]-b\gamma\int_{\gamma}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du-b\int_{\gamma}^{T}e^{-\alpha(t-\gamma)^{\beta}}\left[\int_{t}^{T}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]dt\right]$$

$$-I_e\left[\frac{aT^2}{2}-\frac{2pb}{r^2}[e^{rT}(rT-1)]+(a-2pbe^{rT})[T(M-T)]\right]\Bigg]\left[\frac{e^{rH}-1}{e^{rT}-1}\right]=0 \tag{24}$$

For given values of the parameters and costs, equations (23) and (24) are solved using MATHCAD to get the optimal cycle length $T=T_2$ and selling price $p=p_2$. Substituting the optimal values of T_2 and p_2 in equation (21), we get the minimum total cost. Substituting this minimum total cost, T_2 and p_2 in equation (22), we get the maximum profit as

$$NP^*(p_2,T_2)=(p_2e^{rT_2}-ge^{rT_2})(a-bp_2e^{rT_2})$$

$$-\Bigg[A+g\left[a\gamma-aT_2-\frac{bp_2}{r}(e^{r\gamma}-e^{rT_2})+a\int_{\gamma}^{T_2}e^{\alpha(u-\gamma)^{\beta}}du-bp_2\int_{\gamma}^{T_2}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]$$

$$+hg\Bigg[\frac{a\gamma^2}{2}+\frac{bp_2}{r^2}[e^{r\gamma}(1-r\gamma)-1]+\gamma\left[a\int_{\gamma}^{T_2}e^{\alpha(u-\gamma)^{\beta}}du-bp_2\int_{\gamma}^{T_2}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]$$

$$+\int_{\gamma}^{T_2}e^{-\alpha(t-\gamma)^{\beta}}\left[a\int_{t}^{T_2}e^{\alpha(u-\gamma)^{\beta}}du-bp_2\int_{t}^{T_2}e^{ru+\alpha(u-\gamma)^{\beta}}du\right]dt\Bigg]$$

$$-p_2I_e\left[\frac{aT_2{}^2}{2}-\frac{bp_2}{r^2}[e^{rT_2}(rT_2-1)-1]+(a-bp_2e^{rT_2})[T_2(M-T_2)]\right]\Bigg]\left[\frac{e^{rH}-1}{e^{rT_2}-1}\right] \tag{25}$$

5. Numerical Illustration

The optimal values of selling price (p) and cycle length (T) are obtained by using the equation (16) and (17) or (23) and (24). The optimal values of T are taken as $T=T_1\ if\ T_1\geq M$ and $T=T_2\ if\ T_2<M$.

To illustrate the developed model of Case (i) i.e, if $T_1\geq M$, a numerical example with the following parameter values is considered. The deteriorating parameters $\alpha,\beta\ and\ \gamma$ vary from 0.020 to 0.024, 0.06 to 0.72 and 0.06 to 0.72 respectively. The values of the other parameters and costs are considered as follows: $a=1000\ to 1200$, $b=0.010\ to\ 0.012$ units, $\mathrm{A}=$ Rs. 250.0 to 300.0, $g=Rs.\,0.20\ to\ 0.24=$ Rs. 0.100 to 0.120 $I_c=Rs.\,0.150$ to 0.180, $I_e=Rs.\,0.120$ to 0.144, $M=15\ days=\frac{15}{30}=0.500\ to\ 0.600$, $r=0.010$ to 0.012, $H=12.0$ to 14.4 months.

By substituting the above values in equations (16) and (17) and solving, the optimal values of cycle length T and selling price p are obtained. Substituting the optimal values of cycle length T and selling price p in equations (5) and (15), the optimal values of Order quantity Q and net profit NP are obtained and presented in Table-1.

From Table-1, it is observed that when the parameter 'a' is increasing from 1000 to 1200 units, the optimal ordering quantity 'Q', the cycle length 'T' and the net profit 'NP' are increasing from 1250.845 to 1585.738 units, 1.245 to 1.314 and Rs.1977.152 to Rs.2050.474 respectively and the unit selling price 'p' is decreasing from Rs. 4.275 to Rs. 3.625, when other parameters and costs are fixed.

When the parameter 'b' is increasing from 0.010 to 0.012 units, the optimal ordering quantity 'Q' increasing from 1250.845 to 1250.849, cycle length 'T', selling price 'p' are remains constant at 1.245, Rs.4.275 and the net profit 'NP' is decreasing from Rs.1977.151 to Rs.1977.150 respectively, when other parameters and costs are fixed.

As the deterioration parameter α is increasing from 0.020 to 0.024, the optimal ordering quantity 'Q' and the cycle length 'T' are increasing from 1250.845 to 1375.107 units, 1.245 to 1.365 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. Rs.4.275 to Rs. 4.140 and Rs.1977.152 to Rs.1953.603 respectively, when other parameters and costs are fixed.

Table-1: *Optimal values of Q, NP, T and p for different values of parameters and costs*

For h=0.1, I_c=0.15, I_e=0.12, M=0.5, r=0.01, H=12

a	b	α	β	γ	A	g	Q	T	p	NP
1000	0.01	0.02	0.6	0.6	250	0.2	1250.845	1.245	4.275	1977.152
1050							1328.542	1.259	4.090	1994.236
1100							1410.353	1.275	3.921	2012.050
1150							1496.130	1.294	3.766	2030.755
1200							1585.738	1.314	3.625	2050.474
	0.0105						1250.847	1.245	4.275	1977.151
	0.0110						1250.847	1.245	4.275	1977.151
	0.0115						1250.849	1.245	4.275	1977.150
	0.0120						1250.849	1.245	4.275	1977.150
		0.021					1281.746	1.275	4.239	1971.233
		0.022					1312.768	1.305	4.204	1965.332
		0.023					1343.894	1.335	4.171	1959.454
		0.024					1375.107	1.365	4.140	1953.603
			0.63				1290.509	1.284	4.227	1970.017
			0.66				1332.079	1.325	4.181	1962.661
			0.69				1375.594	1.368	4.135	1955.097
			0.72				1421.082	1.413	4.090	1947.340
				0.63			1252.355	1.247	4.272	1976.975
				0.66			1253.806	1.248	4.269	1976.811
				0.69			1255.199	1.250	4.266	1976.658
				0.72			1256.534	1.252	4.263	1976.517
					262.5		1170.346	1.165	4.489	1984.103
					275.0		1167.304	1.162	4.498	1984.386
					287.5		1161.260	1.156	4.516	1984.953
					300.0		1160.959	1.156	4.517	1984.982
						0.21	1255.235	1.249	4.277	1961.926
						0.22	1259.979	1.254	4.279	1946.543
						0.23	1265.073	1.259	4.280	1930.992
						0.24	1270.516	1.264	4.281	1915.263

For a=1000, b=0.01, α=0.02, β=0.6, γ=0.6, A=250, g=0.2

h	I_c	I_e	M	r	H	Q	T	p	NP
0.105						1252.897	1.247	4.275	1971.381
0.110						1255.002	1.249	4.274	1965.584
0.115						1257.161	1.251	4.274	1959.762
0.120						1259.374	1.253	4.273	1953.913
	0.1575					1251.530	1.245	4.272	1970.434
	0.1650					1252.233	1.246	4.270	1964.532
	0.1725					1253.034	1.247	4.267	1958.606
	0.1800					1254.068	1.248	4.264	1951.804
		0.126				1268.103	1.262	4.228	1975.626
		0.132				1285.617	1.279	4.183	1974.133

h	I_c	I_e	M	r	H	Q	T	p	NP
		0.138				1303.390	1.296	4.139	1972.676
		0.144				1321.422	1.314	4.097	1971.258
			0.525			1282.482	1.276	4.182	1976.588
			0.550			1316.396	1.309	4.091	1976.187
			0.575			1352.685	1.345	4.002	1976.004
			0.600			1391.447	1.383	3.916	1976.099
				0.0105		1242.297	1.236	4.292	1979.547
				0.0110		1242.297	1.236	4.292	1979.547
				0.0115		1233.796	1.228	4.309	1981.935
				0.0120		1233.796	1.228	4.309	1981.935
					12.6	1228.266	1.222	4.330	1974.585
					13.2	1206.405	1.201	4.386	1972.158
					13.8	1185.239	1.180	4.443	1969.864
					14.4	1164.742	1.160	4.501	1967.694

When the parameter β is increasing from 0.60 to 0.72 the optimal ordering quantity 'Q' and the cycle length 'T' are increasing from 1250.845 to 1421.082 units, 1.245 to 1.413 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. 4.275 to Rs. 4.090 and Rs.1977.152 to Rs.1947.340 respectively, when other parameters and costs are fixed.

As the deterioration parameter γ is increasing from 0.60 to 0.72, the optimal ordering quantity 'Q' and the cycle length 'T' are increasing from 1250.845 to 1256.534 units, 1.245 to 1.252 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. 4.275 to Rs. 4.263 and Rs.1977.152 to Rs.1976.517 respectively, when other parameters and costs are fixed.

If the ordering cost 'A' increases from Rs.250 to 300, the optimal ordering quantity 'Q' and the cycle length 'T' are decreasing from 1250.845 to 1160.959 units, 1.245 to 1.156 respectively and the unit selling price 'p' and the net profit 'NP' are increasing from Rs. 4.275 to Rs. 4.517 and Rs.1977.152 to Rs.1984.982 respectively, when other parameters and costs are fixed.

When the unit cost 'g' is increasing from Rs.0.20 to 0.24, the optimal ordering quantity 'Q', the cycle length 'T' and the unit selling price 'p' are increasing from 1250.845 to 1270.516 units, 1.245 to 1.264 and Rs. 4.275 to Rs. 4.281 respectively and the net profit 'NP' is decreasing from Rs.1977.152 to Rs.1915.263 respectively, when other parameters and costs are fixed.

When holding cost 'h' is increasing from Rs.0.100 to 0.120, the optimal ordering quantity 'Q' and the cycle length 'T' are increasing from 1250.845 to 1259.374 units, 1.245 to 1.253 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. 4.275 to Rs. 4.273 and Rs.1977.152 to Rs.1953.913 respectively, when other parameters and costs are fixed.

When interest charged 'I_C' increases from Rs.0.150 to 0.180, the optimal ordering quantity 'Q' and the cycle length 'T' are increasing from 1250.845 to 1254.068 units, 1.245 to 1.248 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. 4.275 to Rs. 4.264 and Rs.1977.152 to Rs.1951.804 respectively, when other parameters and costs are fixed.

If interest charged 'I_e' increases from Rs.0.120 to 0.144, the optimal ordering quantity 'Q' the the cycle length 'T' are increasing from 1250.845 to 1321.422 units, 1.245 to 1.314 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. 4.275 to Rs. 4.097 and Rs.1977.152 to Rs.1971.258 respectively, when other parameters and costs are fixed.

If the permissible delay period 'M' increases from 0.5 months to 0.6 months, the optimal ordering quantity 'Q' and the cycle length 'T' are increasing from 1250.845 to 1391.447 units, 1.245 to 1.383 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. 4.275 to Rs. 3.916 and Rs.1977.152 to Rs.1976.099 respectively, when other parameters and costs are fixed.

The inflation rate 'r' increases from 0.010 to 0.0120 the optimal ordering quantity 'Q' and the cycle length 'T' are decreasing from 1250.845 to 1233.796 units, 1.245 to 1.228 respectively and the unit selling price 'p' and the net profit 'NP' are decreasing from Rs. 4.275 to Rs. 4.309 and Rs.1977.152 to Rs.1981.935 respectively, when other parameters and costs are fixed.

When the time horizon 'H' increases from 12 months to 13.8 then the optimal ordering quantity 'Q', the cycle length 'T' and the net profit 'NP' are decreasing from 1250.845 to 1164.742 units, 1.245 to 1.16 and Rs.1977.152 to Rs.1967.694 respectively and the unit selling price 'p' is increasing from Rs. 4.275 to Rs. 4.501, when other parameters and costs are fixed.

6. Sensitivity Analysis

To study the effect of changes in the model parameters and costs on the optimal values of the order quantity, cycle length, selling price and net profit, the sensitivity analysis is carried by considering $a = 1000$, $b = 0.01$ units, $\alpha = 0.02$, $\beta = 0.60$, $\gamma = 0.60$, $\mathrm{A} = \mathrm{Rs.}\,250$, $g = Rs.\,0.20$, $\mathrm{h} = \mathrm{Rs.}\,0.100$, $I_c = Rs.\,0.150, I_e = Rs.\,0.120$, $M = 0.500, r = 0.01$, $H = 12$ months. Table-2 summarizes these results for variations of -15%, -10%, -5%, 0, 5%, 10%, 15% of the parameters and costs.

As the parameter a increases from -15% to +15%, the optimal order quantity Q is increases from 1044.252 to 1496.13, cycle length 'T' increases from 1.223 to 1.294, selling price 'p' decreases from Rs.4.940 to Rs.3.766 and the net profit increases from Rs.1927.777 to Rs.2030.755.

When the total demand during the cycle period b increases from -15% to +15%, the optimal order quantity 'Q' increases from 1250.843 to 1250.849, cycle length 'T' and selling price 'p' remains constant 1.245 and Rs.4.275 and the net profit 'NP' decreases from Rs.1977.153 to Rs.1977.150.

As the deterioration parameter α increases from -15% to +15%, the optimal order quantity 'Q' increases from 1159.039 to 1343.894, cycle length 'T' increases from 1.155 to 1.335, selling price 'p' decreases from Rs.4.394 to Rs.4.171 and the net profit 'NP' decreases from Rs.1994.984 to Rs.1959.454.

If the parameter β increases from -15% to +15%, the optimal order quantity 'Q' increases from 1165.449 to 1375.594, cycle length 'T' increases from 1.160 to 1.368, selling price 'p' decreases from Rs.4.388 to Rs.4.135 and the net profit 'NP' decreases from Rs.1992.891 to Rs.1955.097

When the deterioration parameter γ increases from -15% to +15%, the optimal order quantity 'Q' increases from 1245.965 to 1255.199, cycle length 'T' increases from 1.238 to 1.250, selling price 'p' decreases from Rs.4.285 to Rs.4.266 and the net profit 'NP' decreases from Rs.1977.753 to Rs.1976.658.

When the ordering cost A increases from -15% to +15%, the optimal order quantity 'Q' decreases from 1636.158 to 1161.260, cycle length 'T' decreases from 1.623 to 1.156, selling price 'p' increases from Rs.3.693 to Rs.4.516 and the net profit 'NP' increases from Rs.1950.119 to Rs.1984.953.

As the unit cost g increases from -15% to +15%, the optimal order quantity 'Q' increases from 1239.817 to 1265.073, cycle length 'T' increases from 1.234 to 1.259, selling price 'p' increases from Rs.4.267 to Rs.4.280 and the net profit 'NP' decreases from Rs.2021.981 to Rs.1930.992.

As the holding cost h increases from -15% to +15%, the optimal order quantity 'Q' increases from 1245.014 to 1257.161, cycle length 'T' increases from 1.239 to 1.251, selling price 'p' decreases from Rs.4.276 to Rs.4.274 and the net profit decreases from Rs.1994.320 to Rs.1959.762.

When the interest charged I_c increases from -15% to +15%, the optimal order quantity 'Q' increases from 1249.630 to 1253.034, cycle length 'T' increases from 1.243 to 1.247, selling price 'p' decreases from Rs.4.281 to Rs.4.267 and the net profit decreases from Rs.1995.489 to Rs.1958.606.

If the interest earned I_e increases from -15% to +15%, the optimal order quantity 'Q' increases from 1200.594 to 1303.390, cycle length 'T' increases from 1.195 to 1.296, selling price 'p' decreases from Rs.4.425 to Rs.4.139 and the net profit 'NP' decreases from Rs.1981.910 to Rs.1972.676.

When the permissible delay period M increases from -15% to +15%, the optimal order quantity 'Q' increases from 1168.658 to 1352.685, cycle length 'T' increases from 1.164 to 1.345, selling price 'p' decreases from Rs.4.562 to Rs.4.002 and the net profit 'NP' decreases from Rs.1979.374 to Rs.1976.004.

If the inflation rate r increases from -15% to +15%, the optimal order quantity 'Q' decreases from 1259.440 to 1233.796, cycle length 'T' decreases from 1.253 to 1.228, selling price 'p' increases from

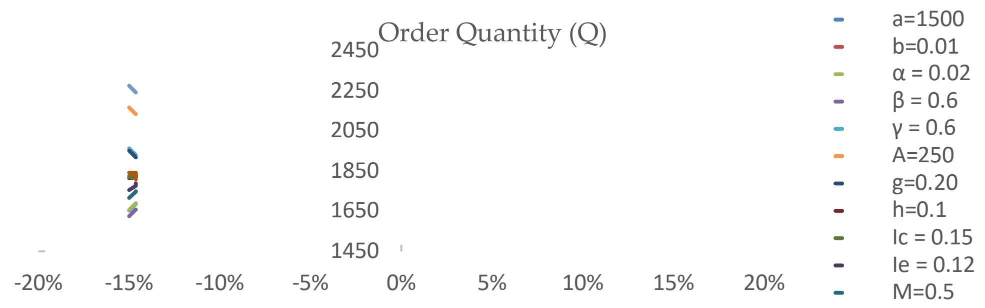
Rs.4.258 to Rs.4.309 and the net profit 'NP' increases from Rs.1974.749 to Rs.1981.935.

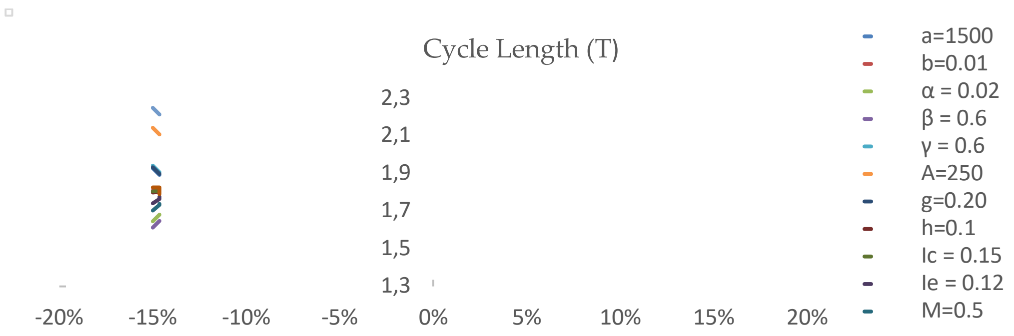
When the time horizon H increases from -15% to +15%, the optimal order quantity 'Q' decreases from 1490.623 to 1185.239, cycle length 'T' decreases from 1.480 to 1.180, selling price 'p' increases from Rs.3.820 to Rs.4.443 and the net profit 'NP' decreases from Rs.2008.083 to Rs.1969.864.

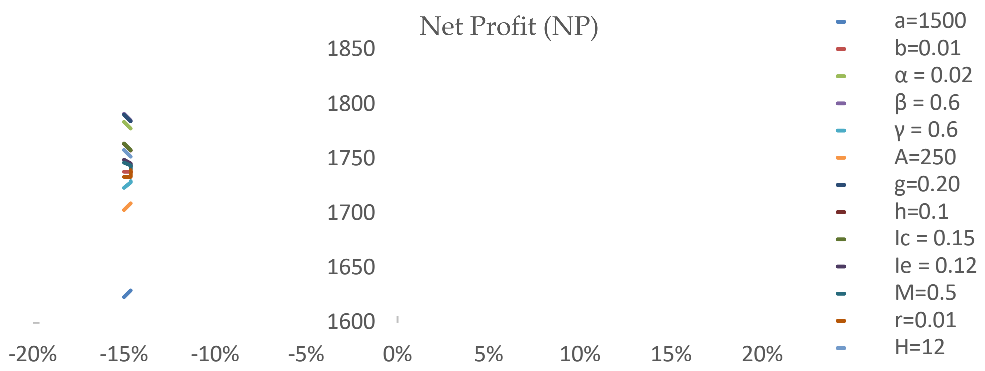
Table-2: *Efect on Optimal Values with Respect to Parameters Variation*

Variation Parameters		Percentage change in parameter						
		-15	-10	-5	0	5	10	15
a	Q	1044.252	1108.494	1177.432	1250.845	1328.542	1410.353	1496.130
	T	1.223	1.226	1.233	1.245	1.259	1.275	1.294
	p	4.940	4.699	4.477	4.275	4.090	3.921	3.766
	NP	1927.777	1944.253	1960.585	1977.152	1994.236	2012.050	2030.755
b	Q	1250.843	1250.843	1250.845	1250.845	1250.847	1250.847	1250.849
	T	1.245	1.245	1.245	1.245	1.245	1.245	1.245
	p	4.275	4.275	4.275	4.275	4.275	4.275	4.275
	NP	1977.153	1977.153	1977.152	1977.152	1977.151	1977.151	1977.150
α	Q	1159.039	1189.474	1220.082	1250.845	1281.746	1312.768	1343.894
	T	1.155	1.185	1.215	4.275	1.275	1.305	1.335
	p	4.394	4.353	4.313	1977.152	4.239	4.204	4.171
	NP	1994.984	1989.032	1983.086	1.245	1971.233	1965.332	1959.454
β	Q	1165.449	1177.053	1213.044	1250.845	1290.509	1332.079	1375.594
	T	1.160	1.172	1.207	1.245	1.284	1.325	1.368
	p	4.388	4.372	4.323	4.275	4.227	4.181	4.135
	NP	1992.891	1990.722	1984.056	1977.152	1970.017	1962.661	1955.097
γ	Q	1245.965	1247.650	1249.277	1250.845	1252.355	1253.806	1255.199
	T	1.238	1.241	1.243	1.245	1.247	1.248	1.250
	p	4.285	4.281	4.278	4.275	4.272	4.269	4.266
	NP	1977.753	1977.540	1977.340	1977.152	1976.975	1976.811	1976.658
A	Q	1636.158	1440.839	1340.631	1250.845	1170.346	1167.304	1161.260
	T	1.623	1.432	1.333	1.245	1.165	1.162	1.156
	p	3.693	3.899	4.078	4.275	4.489	4.498	4.516
	NP	1950.119	1964.813	1970.615	1977.152	1984.103	1984.386	1984.953
g	Q	1239.817	1243.134	1246.810	1250.845	1255.235	1259.979	1265.073
	T	1.234	1.237	1.241	1.245	1.249	1.254	1.259
	p	4.267	4.270	4.273	4.275	4.277	4.279	4.280
	NP	2021.981	2007.169	1992.229	1977.152	1961.926	1946.543	1930.992
h	Q	1245.014	1246.903	1248.847	1250.845	1252.897	1255.002	1257.161
	T	1.239	1.241	1.243	1.245	1.247	1.249	1.251
	p	4.276	4.275	4.275	4.275	4.275	4.274	4.274
	NP	1994.320	1988.621	1982.898	1977.152	1971.381	1965.584	1959.762
I_c	Q	1249.630	1249.910	1250.290	1250.845	1251.530	1252.233	1253.034
	T	1.243	1.244	1.244	1.245	1.245	1.246	1.247
	p	4.281	4.279	4.277	4.275	4.272	4.270	4.267
	NP	1995.489	1989.675	1983.842	1977.152	1970.434	1964.532	1958.606
I_e	Q	1200.594	1217.093	1233.842	1250.845	1268.103	1285.617	1303.390
	T	1.195	1.211	1.228	1.245	1.262	1.279	1.296
	p	4.425	4.373	4.323	4.275	4.228	4.183	4.139
	NP	1981.910	1980.296	1978.709	1977.152	1975.626	1974.133	1972.676

Variation Parameters		Percentage change in parameter						
		-15	-10	-5	0	5	10	15
M	Q	1168.658	1194.025	1221.391	1250.845	1282.482	1316.396	1352.685
	T	1.164	1.189	1.216	1.245	1.276	1.309	1.345
	p	4.562	4.466	4.370	4.275	4.182	4.091	4.002
	NP	1979.374	1978.584	1977.831	1977.152	1976.588	1976.187	1976.004
r	Q	1259.440	1259.440	1250.845	1250.845	1242.297	1242.297	1233.796
	T	1.253	1.253	1.245	1.245	1.236	1.236	1.228
	p	4.258	4.258	4.275	4.275	4.292	4.292	4.309
	NP	1974.749	1974.749	1977.152	1977.152	1979.547	1979.547	1981.935
H	Q	1490.623	1402.823	1323.141	1250.845	1228.266	1206.405	1185.239
	T	1.480	1.394	1.316	1.245	1.222	1.201	1.180
	p	3.820	3.963	4.114	4.275	4.330	4.386	4.443
	NP	2008.083	1995.997	1985.781	1977.152	1974.585	1972.158	1969.864







7. Conclusion

In this paper an EOQ model for deteriorating items with permissible delay in payments having truncated Weibull distribution with inflation is proposed and analyzed. In inventory control, permissible delay in payments has significance influence in obtaining the optimal pricing and ordering policies. The truncated Weibull distribution is one of the most significant life time distributions for items such as food and vegetables markets, market yards and chemical industries, etc., where the deterioration is skewed and having long upper tail. The truncated Weibull distribution includes exponential distribution as a particular case. The sensitivity analysis of the model revealed that the pricing and ordering are highly influenced by the parameters and costs. The model with constraints on warehouse capacity and budget can also be developed with permissible delay in payment and truncated Weibull decay which will be published elsewhere.

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Comparison of $FM/FD/1$ Queuing Performance Using Fuzzy Queuing Model and Intuitionistic Fuzzy Queuing Model with Infinite capacity

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Abstract

Under assorted fuzzy numbers, we portray an FM/FD/1 queuing model with an unrestrained limit. The foremost target of this paper is to compare the efficacy of an FM/FD/1 queuing model based on fuzzy queuing theory and intuitionistic fuzzy queuing theory. Birth (arrival) and death (service) rates are thought to be triangular and triangular intuitionistic fuzzy numbers. The fuzzy consequence of unpredictability modeling is a fuzzy random variable because arbitrary events can only be recognized in an undefined manner. As a consequence, it is essential to interpret the direct correlation between volatility and vagueness. The lining miniature's prosecution dimensions are fuzzified and then examined using arithmetic and logical operations. The evaluation metrics for the fuzzy queuing theory model are furnished as a range of outcomes, meanwhile, the intuitionistic fuzzy queuing theory model has plenty of virtues. An approach is conducted to ascertain quality measures using a methodological approach in which fuzzy values are preserved without being incorporated into crisp values, allowing us to draw scientific conclusions in an uncertain environment. The arithmetical precepts are defined in dealing with various fuzzy numbers to test the model's technical feasibility. A comparison illustration is constituted for each fuzzy number.

Keywords: queuing theory, triangular fuzzy number, triangular intuitionistic fuzzy number, infinite capacity.

1. Introduction

A queue is made up of at least one queue or one or more remodeled offices that carry a system of regulations. To initiate propagation in the queuing hypothesis, the specifications birth rate (birth) and death rate (service) are required. Kaufmann [1] featured an introduction to fuzzy subset theory in 1975. John F Shortle et al [2] envisioned several basic queuing suppositions in 1985. In 1986, Yager [3] proposed a different interpretation of the fuzzy set extension principle. In 1989, Lie et al [4] proposed a fuzzy queuing model. In 1992, Negi et al [5] made an overview of queuing systems. Recently, Lofti A Zadeh [6] depicted fuzzy sets and logic in 1995. Chen [7, 8] posited a parameterized nonlinear optimization strategy to fuzzy queues with the widespread regime in 2005, and an arithmetic programming approach to dealing with equipment interruption with fuzzy parameters in 2006. He evolved FM/FM/1/1/FCFS is a fuzzified exponential time dependent on queuing hypothesis. In 2007, K. Gupta et al [9] published a book that was knowledgeable regarding queuing models. Fuzzy logic with engineering disciplines was proposed by Timothy J. Rose [10] in 2010. In 2012, S. Barak et al [11] published a paper on the cost analysis of fuzzy queuing systems. Srinivasan [12] proposed a fuzzy queuing model based on the DSW algorithm in 2014. Shanmugasundaram et al [13] postulated a DSW computation version on fuzzy multi-server queuing in 2015. Using the DSW algorithm, Mohd Zaki et al [16] likened the queuing model and the fuzzy queuing model. The basic framework is focused on Atanassov's extension principle and α - cut method, and Narayana Moorthy et al [20] used intuitionistic fuzzy numbers as hyperparameters. Arpita Kabiraj et al [21] used intuitionistic notions in a linear programming problem to solve fuzzy linear programming problems. In this analysis, G. Chen et al [22] glanced at optimized and enthalpies techniques in fuzzy M/M/1 queues, using all fuzzy numbers as a covariate. In their paper, S. Hanumantha Rao et al [23] proposed a single semi-Markov queueing system with constraints, encouraging or discouraging arrivals, and a rejigged customer reneging policy. S. Hanumantha Rao et al [24] proposed the membership function of the fuzzy cost function to procure optimistic prognostications for certain key metrics of a customizable 2 different service dedicated server markovian limiting queues with server starts and breakdown over N-policy. The prediction generating function was used by S.S. Sanga et al [25] to generate the stable flow mathematical formulation for predictive distributions and systems assessment processes. R. Sethi et al. [26] used an iterative method for extracting steady queue distributions, making multiple performance indices, and wandering numerical experiments to typify the attitude of the system coefficients as numerous system parameters are updated. F. Ferdowsi [28] envisioned an intuitionistic fuzzy measure to negotiate with uncertainty, in which he used a credibility indicator to integrate a fuzzy model into a crisp model. To investigate the performance of a system, B. R. Kumar et al [29] used estimation theory and defuzzification. A. Tamilarasi [27] researched the intuitionistic fuzzy and queuing model utilizing trapezoidal intuitionistic fuzzy numbers. The fuzzy queues are assessed by transferring fuzzy values into crisp values, as shown in the aforementioned overview. As a consequence, we've proposed a technique for tackling deterministic queues in both fuzzy and intuitionistic fuzzy environments without shifting their nature in this paper. In comparison to previous strategies, this initiative is favorable in that it is eloquent, resilient, and noteworthy. According to the results of the analysis, the fuzzy queuing model's performance measurements are within the spectrum of the intuitionistic fuzzy queuing model's computed performance measures. In the queuing theory, both the specifications, that is, the birth (arrival) times and death (service) times, are geared towards achieving predefined appropriations in the customary lining model. The birth rate and death rate are commonly characterized using terminological terms, such as large, small, incredibly low, and modest, which are better reflected by fuzzy and intuitionistic fuzzy sets. Probably the easiest queue with deterministic service time is the FM/FD/1 queue, which has a multitude of uses in performance measurement, network technologies, and other areas. The main concept is to obtain exact fuzzy values, that is, without attempting to convert them to crisp values, and then apply the queuing performance formulas to two types of participatory cognitive abilities, namely triangular and intuitionistic fuzzy enrolment capacities.

A deterministic queueing model is perhaps the most basic type of queuing problem, as it does not necessitate the use of probability to describe the arrival and service pattern. The set of inputs entering at particular times and the processing times are both fixed. The arrival rate is Poisson in this model, but the service rate is deterministic, i.e., it is consistent. If we can make the service deterministic, that is, suspend evolutionary divergence from the service, we can substantially reduce the virtues of the number in the system as well as the waiting time if we want to optimize the queuing parameters. One solution is to add further servers, and besides, if we can also completely eradicate chance variations, whether, through digitalization or any other means, things will get better remarkably.

Section 2 introduces some underlying concepts and definitions. The insinuations and nomenclature are discussed in Section 3. The suggested queuing model is presented in Section 4. Two numerical precepts are solved in Section 5. The discussion of findings is presented in Section 6. The prototype is checked in Section 7.

2. Preliminaries

The motive of this division is to give some basic definitions, annotations, and outcomes that are used in our subsequent calculations.

Definition 2.1: [14] A fuzzy set $\tilde{A}$ is defined on R, the set of real numbers is called a **fuzzy number** if its membership function $\mu_{\tilde{A}} : R \to [0,1]$ has the following conditions:

(a) $\tilde{A}$ is convex, which means that there exists $x_1, x_2 \in R$ and $\lambda \in [0,1]$, such that $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$

(b) $\tilde{A}$ is normal, which means that there exists an $x \in R$ such that $\mu_{\tilde{A}}(x) = 1$

(c) $\tilde{A}$ is piecewise continuous.

Definition 2.2: [14] A fuzzy number $\tilde{A}$ is defined on R, the set of real numbers is said to be a **triangular fuzzy number (TFN)** if its membership function $\mu_{\tilde{A}} : R \to [0,1]$ which satisfies the following conditions:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-\tilde{a}_1}{\tilde{a}_2-\tilde{a}_1} \text{ for } \tilde{a}_1 \leq x \leq \tilde{a}_2 \\ 1 \text{ for } x = \tilde{a}_2 \\ \frac{\tilde{a}_3-x}{\tilde{a}_3-\tilde{a}_2} \text{ for } \tilde{a}_2 \leq x \leq \tilde{a}_3 \\ 0 \text{ otherwise} \end{cases}$$

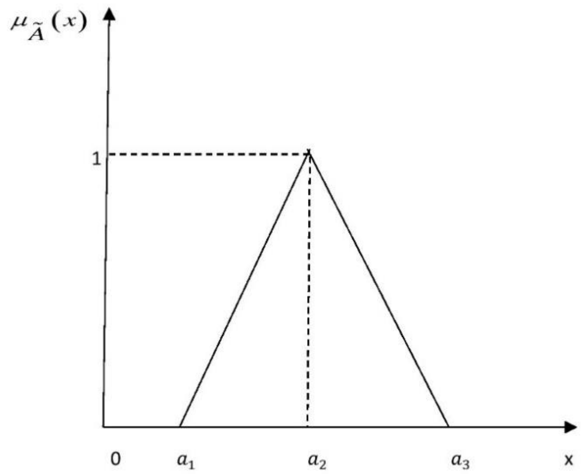

Figure 1: *Triangular fuzzy number*

The triangular fuzzy number is illustrated in Fig 1.

Definition 2.3:[14] Let the two triangular fuzzy numbers be $\tilde{P} \approx (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ and $\tilde{Q} \approx (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)$ and then the **arithmetic operations on TFN** be given as follows:

(A)Addition

$$\tilde{P}+\tilde{Q}\approx\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3\right)+\left(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3\right)$$

$$\tilde{P}+\tilde{Q}\approx\left(\tilde{m}_1,\tilde{\alpha}_1,\tilde{\beta}_1\right)+\left(\tilde{m}_2,\tilde{\alpha}_2,\tilde{\beta}_2\right)$$

$$\tilde{P}+\tilde{Q}\approx\left(\tilde{m}_1+\tilde{m}_2,\max\left\{\tilde{\alpha}_1,\tilde{\alpha}_2\right\},\max\left\{\tilde{\beta}_1,\tilde{\beta}_2\right\}\right) \quad (1)$$

(B)Subtraction

$$\tilde{P}-\tilde{Q}\approx\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3\right)-\left(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3\right)$$

$$\tilde{P}-\tilde{Q}\approx\left(\tilde{m}_1,\tilde{\alpha}_1,\tilde{\beta}_1\right)-\left(\tilde{m}_2,\tilde{\alpha}_2,\tilde{\beta}_2\right)$$

$$\tilde{P}-\tilde{Q}\approx\left(\tilde{m}_1-\tilde{m}_2,\max\left\{\tilde{\alpha}_1,\tilde{\alpha}_2\right\},\max\left\{\tilde{\beta}_1,\tilde{\beta}_2\right\}\right) \quad (2)$$

(C) Multiplication

$$\tilde{P}.\tilde{Q}\approx\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3\right).\left(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3\right)$$

$$\tilde{P}.\tilde{Q}\approx\left(\tilde{m}_1,\tilde{\alpha}_1,\tilde{\beta}_1\right).\left(\tilde{m}_2,\tilde{\alpha}_2,\tilde{\beta}_2\right)$$

$$\tilde{P}.\tilde{Q}\approx\left(\tilde{m}_1.\tilde{m}_2,\max\left\{\tilde{\alpha}_1,\tilde{\alpha}_2\right\},\max\left\{\tilde{\beta}_1,\tilde{\beta}_2\right\}\right) \quad (3)$$

(D) Division

$$\frac{\tilde{P}}{\tilde{Q}}\approx\frac{\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3\right)}{\left(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3\right)}$$

$$\frac{\tilde{P}}{\tilde{Q}}\approx\frac{\left(\tilde{m}_1,\tilde{\alpha}_1,\tilde{\beta}_1\right)}{\left(\tilde{m}_2,\tilde{\alpha}_2,\tilde{\beta}_2\right)}$$

$$\frac{\tilde{P}}{\tilde{Q}}\approx\left(\frac{\tilde{m}_1}{\tilde{m}_2},\max\left\{\tilde{\alpha}_1,\tilde{\alpha}_2\right\},\max\left\{\tilde{\beta}_1,\tilde{\beta}_2\right\}\right) \quad (4)$$

Definition 2.4: For every triangular fuzzy number $\tilde{P}\approx\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3\right)\in F(R)$ **ranking function** $\Re : F(R)\to R$ is defined by graded mean as

$$\Re\left(\tilde{P}\right)=\frac{\left(\tilde{a}_1+4\tilde{a}_2+\tilde{a}_3\right)}{6}$$

For any two TFN $\tilde{P}\approx\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3\right)$ and $\tilde{Q}\approx\left(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3\right)$ We have the following comparisons,

$(a)\tilde{P}\succ\tilde{Q}\Leftrightarrow\Re\left(\tilde{P}\right)>\Re\left(\tilde{Q}\right)$

$(b)\tilde{P}\prec\tilde{Q}\Leftrightarrow\Re\left(\tilde{P}\right)<\Re\left(\tilde{Q}\right)$

$(c)\tilde{P}\approx\tilde{Q}\Leftrightarrow\Re\left(\tilde{P}\right)=\Re\left(\tilde{Q}\right)$

$(d)\tilde{P}-\tilde{Q}\approx 0\Leftrightarrow\Re\left(\tilde{P}\right)-\Re\left(\tilde{Q}\right)=0$

A triangular fuzzy number $\tilde{P}\approx\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3\right)\in F(R)$ is known to be **positive** if $\Re\left(\tilde{P}\right)>0$ and defined by $\tilde{P}\succ 0$.

Definition 2.5:[15] Let a non–empty set be X . An **Intuitionistic fuzzy set (IFS)** $\tilde{A}'$ is defined as $\tilde{A}'=\left\{\left(x,\mu_{\tilde{A}'}(x),\gamma_{\tilde{A}'}(x)/x\in X\right)\right\}$, where $\mu_{\tilde{A}'}:X\to[0,1]$ and $\gamma_{\tilde{A}'}:X\to[0,1]$ denotes the degree of membership and degree of non– membership functions respectively where $x\in X$, for every $x\in X, 0\le\mu_{\tilde{A}'}(x)+\gamma_{\tilde{A}'}(x)\le 1.$

Definition 2.6:[15] An intuitionistic fuzzy set $\tilde{A}'$ described on R, the real numbers are said to be an **Intuitionistic fuzzy number (IFN)** if its membership function $\mu_{\tilde{A}'}:\mathrm{R}\to[0,1]$ and its non –

membership function $\gamma_{\tilde{A}'} : \mathrm{R} \to [0,1]$ should agreeable to the following conditions:

i) $\tilde{A}'$ is normal, which means that there exists an $x \in R$, such that $\mu_{\tilde{A}'}(x) = 1, \gamma_{\tilde{A}'}(x) = 0$

ii) $\tilde{A}'$ is convex for the membership function $\mu_{\tilde{A}'}$, which means that there exists $x_1, x_2 \in R$ and $\lambda \in [0,1]$ such that $\mu_{\tilde{A}'}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}'}(x_1), \mu_{\tilde{A}'}(x_2)\}$.

iii) $\tilde{A}'$ is concave for the non – membership function $\gamma_{\tilde{A}'}$, which means that there exists $x_1, x_2 \in R$ and $\lambda \in [0,1]$ such that $\gamma_{\tilde{A}'}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\gamma_{\tilde{A}'}(x_1), \gamma_{\tilde{A}'}(x_2)\}$.

Definition 2.7:[15] A fuzzy number $\tilde{A}'$ on R is said to be a **triangular intuitionistic fuzzy number (TIFN)** if its membership function $\mu_{\tilde{A}'} : \mathrm{R} \to [0,1]$ and non – membership function $\gamma_{\tilde{A}'} : \mathrm{R} \to [0,1]$ has the following conditions:

$$\mu_{\tilde{A}'}(x) = \begin{cases} \frac{x-\tilde{a}_1}{\tilde{a}_2-\tilde{a}_1} \text{ for } \tilde{a}_1 \leq x \leq \tilde{a}_2 \\ 1 \text{ for } x = \tilde{a}_2 \\ \frac{\tilde{a}_3-x}{\tilde{a}_3-\tilde{a}_2} \text{ for } \tilde{a}_2 \leq x \leq \tilde{a}_3 \\ 0 \text{ otherwise} \end{cases}$$

and

$$\gamma_{\tilde{A}'}(x) = \begin{cases} 1 \text{ for } x < \tilde{a}_1', x > \tilde{a}_3' \\ \frac{\tilde{a}_2-x}{\tilde{a}_2-\tilde{a}_1'} \text{ for } \tilde{a}_1' \leq x \leq \tilde{a}_2 \\ 0 \text{ for } x = \tilde{a}_2 \\ \frac{x-\tilde{a}_2}{\tilde{a}_3-\tilde{a}_2} \text{ for } \tilde{a}_2 \leq x \leq \tilde{a}_3' \end{cases}$$

and is given by $\tilde{A}' = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}_1', \tilde{a}_2, \tilde{a}_3')$ where $\tilde{a}_1' \leq \tilde{a}_1 \leq \tilde{a}_2 \leq \tilde{a}_3 \leq \tilde{a}_3'$.

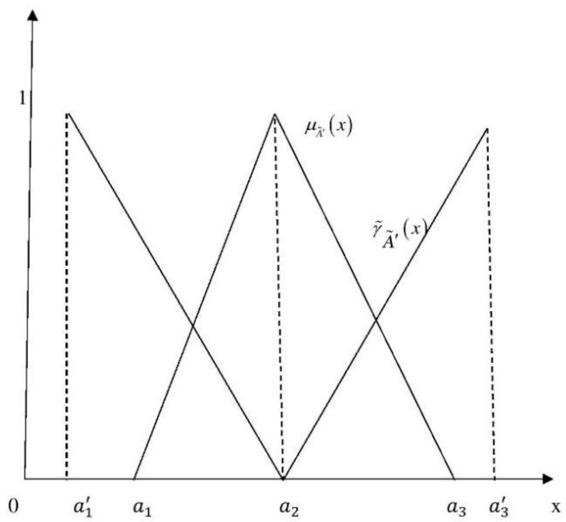

Figure 2: *Intuitionistic triangular fuzzy number*

The intuitionistic triangular fuzzy number is illustrated in Fig 2.

Cases: Let $\tilde{A}' = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}_1', \tilde{a}_2, \tilde{a}_3')$ be a TIFN then the following cases arises.

Case:1 If $\tilde{a}_1' = \tilde{a}_1, \tilde{a}_3' = \tilde{a}_3$ then $\tilde{A}'$ represent a TFN.

Case:2 If $\tilde{a}'_1 = \tilde{a}_1 = \tilde{a}_2 = \tilde{a}'_3 = \tilde{a}_3 = \tilde{m}$ then $\tilde{A}'$ represent a real number $\tilde{m}$. The parametric form of TIFN $\tilde{A}'$ is defined as $\tilde{A}' = (\tilde{\alpha}, \tilde{m}, \tilde{\beta}; \tilde{\alpha}', \tilde{m}, \tilde{\beta}')$ where $\tilde{\alpha}, \tilde{\alpha}'$ & $\tilde{\beta}, \tilde{\beta}'$ represents the left spread and right spread of membership functions and non – membership functions respectively.

Definition 2.8:[15] TIFN $\tilde{A}' \in F(R)$, (where $F(R)$ is the set of all TIFN) can also be represented as a pair $\tilde{A}' = (\tilde{a}, \tilde{\bar{a}}; \tilde{a}', \tilde{\bar{a}}')$ of functions $\tilde{a}(\tilde{r}'), \tilde{\bar{a}}(\tilde{r}'), \tilde{a}'(\tilde{r}')$ & $\tilde{\bar{a}}'(\tilde{r}')$ for $0 \le \tilde{r}' \le 1$ which satisfies the following requirements:

i) $\tilde{a}(\tilde{r}')$ & $\tilde{\bar{a}}'(\tilde{r}')$ is a bounded monotonic increasing left continuous function for membership and non–membership functions respectively.

ii) $\tilde{\bar{a}}(\tilde{r}')$ & $\tilde{a}'(\tilde{r}')$ is a bounded monotonic decreasing left continuous function for membership and non–membership functions respectively.

iii) $\tilde{a}(\tilde{r}') \le \tilde{\bar{a}}(\tilde{r}'), 0 \le \tilde{r}' \le 1.$

iv) $\tilde{a}'(\tilde{r}') \le \tilde{\bar{a}}'(\tilde{r}'), 0 \le \tilde{r}' \le 1.$

Definition 2.9: The extension of fuzzy arithmetic operations of Ming Ma et al [14] to the set of TIFN based upon both location indices and functions of fuzziness indices. The location indices number is taken in the regular arithmetic while the functions of fuzziness indices are assumed to follow the lattice rule which is the least upper bound in the lattice $\tilde{I}'$. For any two arbitrary TIFN $\tilde{P}' \approx (\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1)$ and $\tilde{Q}' \approx (\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2)$ and $* \in \{+, -, \times, \div\}$, then the **arithmetic operations on TIFN** are defined by $\tilde{P}' * \tilde{Q}' = (\tilde{m}_1 * \tilde{m}_2, \tilde{\alpha}_1 \vee \tilde{\alpha}_2, \tilde{\beta}_1 \vee \tilde{\beta}_2; \tilde{m}_1 * \tilde{m}_2, \tilde{\alpha}'_1 \vee \tilde{\alpha}'_2, \tilde{\beta}'_1 \vee \tilde{\beta}'_2)$.

In particular, for any two TIFNs $\tilde{P}' \approx (\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1)$ and $\tilde{Q}' \approx (\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2)$ the arithmetic operations are defined as

$$\tilde{P}' * \tilde{Q}' = (\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1) * (\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2)$$

$$\tilde{P}' * \tilde{Q}' = \left(\tilde{m}_1 * \tilde{m}_2, \max\{\tilde{\alpha}_1, \tilde{\alpha}_2\}, \max\{\tilde{\beta}_1, \tilde{\beta}_2\}; \tilde{m}_1 * \tilde{m}_2, \max\{\tilde{\alpha}'_1, \tilde{\alpha}'_2\}, \max\{\tilde{\beta}'_1, \tilde{\beta}'_2\}\right)$$

$$\tilde{P}' * \tilde{Q}' = (\tilde{m}_1 * \tilde{m}_2, \tilde{\alpha}_1 \vee \tilde{\alpha}_2, \tilde{\beta}_1 \vee \tilde{\beta}_2; \tilde{m}_1 * \tilde{m}_2, \tilde{\alpha}'_1 \vee \tilde{\alpha}'_2, \tilde{\beta}'_1 \vee \tilde{\beta}'_2)$$

In particular, for any two TIFNs $\tilde{P}' \approx (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}'_1, \tilde{a}'_2, \tilde{a}'_3) \approx (\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1)$, $\tilde{Q}' \approx (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{b}'_1, \tilde{b}'_2, \tilde{b}'_3) \approx (\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2)$ we define:

Addition

$$\tilde{P}' + \tilde{Q}' = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}'_1, \tilde{a}_2, \tilde{a}'_3) + (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{b}'_1, \tilde{b}_2, \tilde{b}'_3)$$

$$\tilde{P}' + \tilde{Q}' = (\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1) + (\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2)$$

$$\tilde{P}' + \tilde{Q}' = \left(\tilde{m}_1 + \tilde{m}_2, \max\{\tilde{\alpha}_1, \tilde{\alpha}_2\}, \max\{\tilde{\beta}_1, \tilde{\beta}_2\}; \tilde{m}_1 + \tilde{m}_2, \max\{\tilde{\alpha}'_1, \tilde{\alpha}'_2\}, \max\{\tilde{\beta}'_1, \tilde{\beta}'_2\}\right) \quad (5)$$

Subtraction

$$\tilde{P}' - \tilde{Q}' = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}'_1, \tilde{a}_2, \tilde{a}'_3) - (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{b}'_1, \tilde{b}_2, \tilde{b}'_3)$$

$$\tilde{P}' - \tilde{Q}' = (\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1) - (\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2)$$

$$\tilde{P}' - \tilde{Q}' = \left(\tilde{m}_1 - \tilde{m}_2, \max\{\tilde{\alpha}_1, \tilde{\alpha}_2\}, \max\{\tilde{\beta}_1, \tilde{\beta}_2\}; \tilde{m}_1 - \tilde{m}_2, \max\{\tilde{\alpha}'_1, \tilde{\alpha}'_2\}, \max\{\tilde{\beta}'_1, \tilde{\beta}'_2\}\right) \quad (6)$$

Multiplication

$$\tilde{P}' \times \tilde{Q}' = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}'_1, \tilde{a}_2, \tilde{a}'_3) \times (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{b}'_1, \tilde{b}_2, \tilde{b}'_3)$$

$$\tilde{P}' \times \tilde{Q}' = (\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1) \times (\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2)$$

$$\tilde{P}' \times \tilde{Q}' = \left(\tilde{m}_1 \times \tilde{m}_2, \max\left\{\tilde{\alpha}_1, \tilde{\alpha}_2\right\}, \max\left\{\tilde{\beta}_1, \tilde{\beta}_2\right\}; \tilde{m}_1 \times \tilde{m}_2, \max\left\{\tilde{\alpha}'_1, \tilde{\alpha}'_2\right\}, \max\left\{\tilde{\beta}'_1, \tilde{\beta}'_2\right\}\right) \quad (7)$$

Division

$$\tilde{P}' \div \tilde{Q}' = \left(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}'_1, \tilde{a}_2, \tilde{a}'_3\right) \div \left(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3; \tilde{b}'_1, \tilde{b}_2, \tilde{b}'_3\right)$$

$$\tilde{P}' \div \tilde{Q}' = \left(\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1\right) \div \left(\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2\right)$$

$$\tilde{P}' \div \tilde{Q}' = \left(\tilde{m}_1 \div \tilde{m}_2, \max\left\{\tilde{\alpha}_1, \tilde{\alpha}_2\right\}, \max\left\{\tilde{\beta}_1, \tilde{\beta}_2\right\}; \tilde{m}_1 \div \tilde{m}_2, \max\left\{\tilde{\alpha}'_1, \tilde{\alpha}'_2\right\}, \max\left\{\tilde{\beta}'_1, \tilde{\beta}'_2\right\}\right) \quad (8)$$

Definition 2.10: Consider an arbitrary TIFN $\tilde{A}' = \left(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3; \tilde{a}'_1, \tilde{a}_2, \tilde{a}'_3\right) = \left(\tilde{m}, \tilde{\alpha}, \tilde{\beta}; \tilde{m}, \tilde{\alpha}', \tilde{\beta}'\right)$ and the magnitude of TIFN $\tilde{A}'$ is given by

$$mag\left(\tilde{A}'\right) = \frac{1}{2}\int\left(\tilde{a} + \tilde{\bar{a}} + 2\tilde{m} + \tilde{a}' + \tilde{\bar{a}}'\right)\tilde{f}\left(\tilde{r}'\right)d\tilde{r}'$$

$$mag\left(\tilde{A}'\right) = \frac{1}{2}\int\left(\tilde{\beta} + \tilde{\beta}' + 6\tilde{m} - \tilde{\alpha} - \tilde{\alpha}'\right)\tilde{f}\left(r'\right)d\tilde{r}'$$

In real life scenario, decision-makers select the value of $\tilde{f}\left(\tilde{r}'\right)$ based on their circumstances. Here for our ease, we choose $\tilde{f}\left(\tilde{r}'\right) = \tilde{r}'^2$

$$mag\left(\tilde{A}'\right) = \frac{\left(\tilde{\beta} + \tilde{\beta}' + 6\tilde{m} - \tilde{\alpha} - \tilde{\alpha}'\right)}{6}$$

$$mag\left(\tilde{A}'\right) = \frac{\left(\tilde{a} + \tilde{\bar{a}} + 2\tilde{m} + \tilde{a}' + \tilde{\bar{a}}'\right)}{6}$$

For any two TIFN $\tilde{P}' \approx \left(\tilde{m}_1, \tilde{\alpha}_1, \tilde{\beta}_1; \tilde{m}_1, \tilde{\alpha}'_1, \tilde{\beta}'_1\right)$, $\tilde{Q}' \approx \left(\tilde{m}_2, \tilde{\alpha}_2, \tilde{\beta}_2; \tilde{m}_2, \tilde{\alpha}'_2, \tilde{\beta}'_2\right)$ in $F\left(R\right)$, we define

$(a)\tilde{P}' \geq \tilde{Q}' \Leftrightarrow mag\left(\tilde{P}'\right) \geq mag\left(\tilde{Q}'\right)$

$(b)\tilde{P}' \leq \tilde{Q}' \Leftrightarrow mag\left(\tilde{P}'\right) \leq mag\left(\tilde{Q}'\right)$

$(c)\tilde{P}' \approx \tilde{Q}' \Leftrightarrow mag\left(\tilde{P}'\right) = mag\left(\tilde{Q}'\right)$

3. Suppositions and Diacritical marks

3.1. Suppositions

The following are the accompanying presumptions used in the current model:

i) With only one server, the $\left(FM\ /\ FD\ /\ 1\right):\left(\infty\ /\ FCFS\right)$ queuing model has no bounds.
ii) Service discipline First-Come-First-Served (FCFS)
iii) Arrival times that are widely disseminated.
iv) Fixed deterministic service fetishization.
v) The birth(arrival) rate and death(service) rate are both ambiguous figures.

3.2. Diacritical marks

Here we are using the following notations:

$\tilde{\lambda}, \tilde{\lambda}' \rightarrow$ The mean no. of consumers who arrive in a predetermined period of time.

$\tilde{\mu}, \tilde{\mu}' \rightarrow$ The mean no. of consumers being serviced per unit of time.

$\tilde{\rho} \rightarrow$ Traffic intensity

$\tilde{N}_q, \tilde{N}'_q \rightarrow$ The mean no. of consumers in the line.

$\tilde{N}_s, \tilde{N}'_s \rightarrow$ The mean no. of consumers in the system.

$\tilde{T}_q, \tilde{T}'_q \rightarrow$ The mean sojourn time of the consumers in the queue.

$\tilde{T}_S, \tilde{T}'_S \rightarrow$ The mean sojourn time of the consumers in the system.

$FM \rightarrow$ Fuzzified exponential distribution.

$FD \rightarrow$ Fuzzified Regular service distribution.

$\tilde{P}, \tilde{P}' \rightarrow$ Interarrival rate.

$\tilde{Q}, \tilde{Q}' \rightarrow$ Service rate.

4. Single server deterministic fuzzy queuing model with infinite capacity

We envisage a solo server queuing model governed by the First Come, First Served (FCFS) principle. It's composed as $(FM / FD / 1):(\infty / FCFS)$ in Kendall's notation. Fuzzified exponential distribution with arrival rate is denoted by FM, and fuzzified stable (consistent) dispersion with service rate is denoted by FD. This is a stochastic process, and the state vector is the collection $\{0,1,2,...\}$ in which the value implies the number of customers in the system, which encompasses any enterprise currently in the establishment. Because it is unbounded in size, there is no limit to the number of customers it can hold. Let $\tilde{\lambda}$ and $\tilde{\lambda}'$ be the fuzzy and intuitionistic fuzzy arrival rates respectively. Let $\tilde{\mu}$ and $\tilde{\mu}'$ be the fuzzy and intuitionistic fuzzy service rates respectively. At the steady-state, the FCFS discipline is upheld but the capacity is unlimited.

The following are the fabrication characteristics of the above model:

i) The anticipated number of customers in the system is given as

$$\tilde{N}_S = \tilde{\rho} + \frac{\tilde{\rho}^2}{2(1-\tilde{\rho})}, \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}} \tag{9}$$

ii) The anticipated number of customers in the queue is given as

$$\tilde{N}_q = \frac{\tilde{\rho}^2}{2(1-\tilde{\rho})} \tag{10}$$

iii) The anticipated waiting time of customers in the queue is given as

$$\tilde{T}_q = \frac{\tilde{\rho}}{2(1-\tilde{\rho})\tilde{\mu}} \tag{11}$$

iv) The anticipated waiting time of customers in the system is given as

$$\tilde{T}_S = \frac{1}{\tilde{\mu}} + \frac{\tilde{\rho}}{2(1-\tilde{\rho})\tilde{\mu}} \tag{12}$$

5. Solo server deterministic fuzzy queuing model with unlimited capability

The plaza has a ginormous market and over 10 different manufacturers. Contemplate a shopping centre with a parking slot facility on one floor, with 12 access points and 12 exit ramps for vehicles that are free of charge. Envision two vehicles arriving at the parking spot every minute. Under this scenario, we enumerate the examples and solve them. Interpret the entry rate and the departure rate as both TFNs and TIFNs symbolized by $\tilde{\lambda}$, $\tilde{\lambda}'$ and $\tilde{\mu}$, $\tilde{\mu}'$ respectively. We postulate the system's limit, is infinity.

5.1. Single server deterministic fuzzy queuing model with infinite capacity

Let $\tilde{\lambda} = (1, 2, 3)$ is the arrival rate and $\tilde{\mu} = (11, 12, 13)$ is the service rate of the queuing model. Determine the TFN in the form of $(\tilde{m}, \tilde{\alpha}, \tilde{\beta})$ as $\tilde{\lambda} = (2, 1, 1)$ and $\tilde{\mu} = (12, 1, 1)$. To determine the values of a number of customers and their sojourn time in the queue as well as a system using suitable formulas among (9), (10), (11), & (12). It is necessary to use the appropriate arithmetic operations described in (1), (2), (3), and (4) for add, sub, multiply, and divide, respectively.

The metrics of performance are calculated and tabulated in Table 1.

Table 1: *Performance measures using TFN*

Quantifiable metrics using TFN	
$\tilde{N}_q$	$(-0.9834, 0.0166, 1.0166)$
$\tilde{N}_S$	$(-0.8168, 0.1832, 1.1832)$
$\tilde{T}_q$	$(-0.9917, 0.0083, 1.0083)$
$\tilde{T}_S$	$(-0.9084, 0.0916, 1.0916)$

5.2. Single server deterministic intuitionistic fuzzy queuing model with infinite capacity

Let $\tilde{\lambda}' = (1.5, 2, 2.5; 1, 2, 3)$ is the arrival rate and $\tilde{\mu}' = (11.5, 12, 12.5; 11, 12, 13)$ is the service rate of the queuing model. Determine the TIFN in the form of $(\tilde{m}, \tilde{\alpha}, \tilde{\beta}; \tilde{m}, \tilde{\alpha}', \tilde{\beta}')$ as $\tilde{\lambda}' = (2, 0.5, 0.5; 2, 1, 1)$ and $\tilde{\mu}' = (12, 0.5, 0.5; 12, 1, 1)$. To determine the values of a number of customers and their sojourn time in the queue as well as a system using suitable formulas among (9), (10), (11), & (12). It is necessary to use the appropriate arithmetic operations described in (5), (6), (7), and (8) for add, sub, multiply, and divide, respectively.

The metrics of performance are calculated and tabulated in Table 2.

Table 2: *Performance measures using TIFN*

Quantifiable metrics using TIFN	
$\tilde{N}'_q$	$(-0.4834, 0.0166, 0.5166; -0.9834, 0.0166, 1.0166)$
$\tilde{N}'_S$	$(-0.3168, 0.1832, 0.6832; -0.8168, 0.1832, 1.1832)$
$\tilde{T}'_q$	$(-0.4917, 0.0083, 0.5083; -0.9917, 0.0083, 1.0083)$
$\tilde{T}'_S$	$(-0.4084, 0.0916, 0.5916; -0.9084, 0.0916, 1.0916)$

The following figures depict the visualizations of Tables 1 and 2.

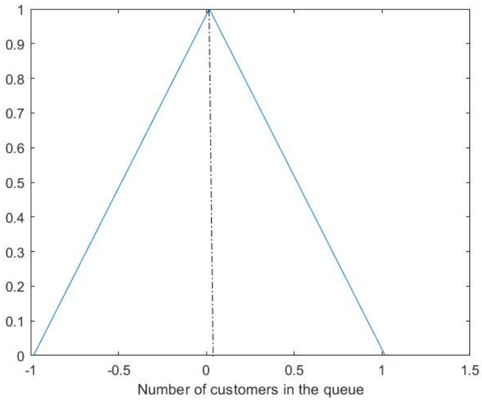

Figure 3: *The number of customers in the queue* $\tilde{N}_q$

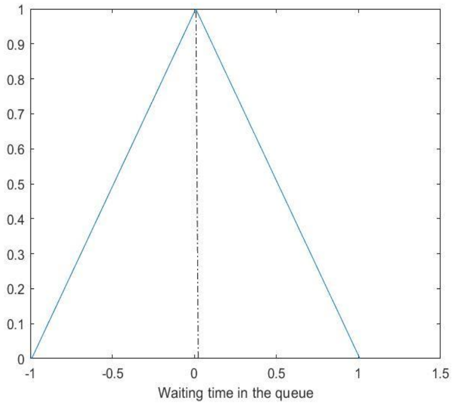

Figure 4: *The waiting time of the customers in the queue* $\tilde{T}_q$

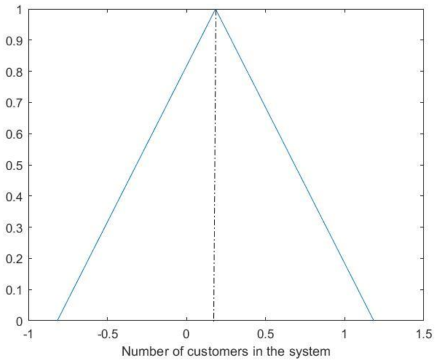

Figure 5: *The number of customers in the system* $\tilde{N}_s$

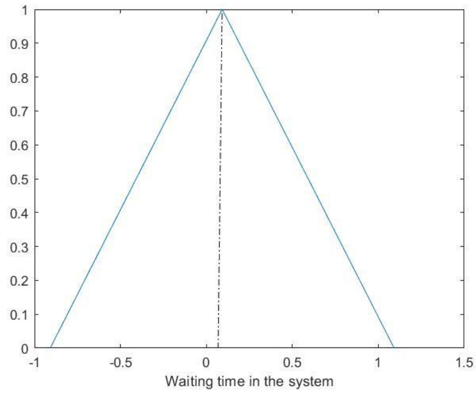

Figure 6: *The waiting time of customers in the system* $\tilde{T}_s$

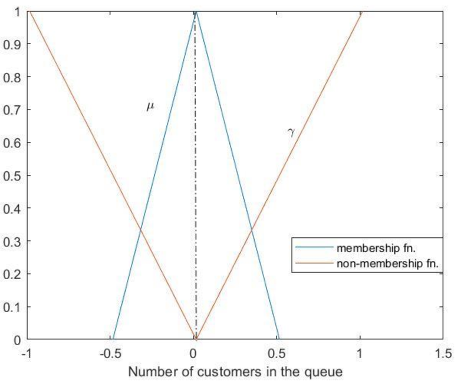

Figure 7: *The membership* $(\tilde{\mu})$ *and the non-membership functions* $(\tilde{\gamma})$ *of the number of customers in the queue* $\tilde{N}'_q$

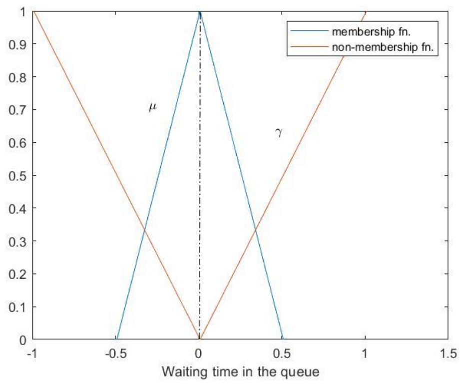

Figure 8: *The membership* $(\tilde{\mu})$ *and the non-membership* $(\tilde{\gamma})$ *functions of the waiting time of consumers in the queue* $\tilde{T}'_q$

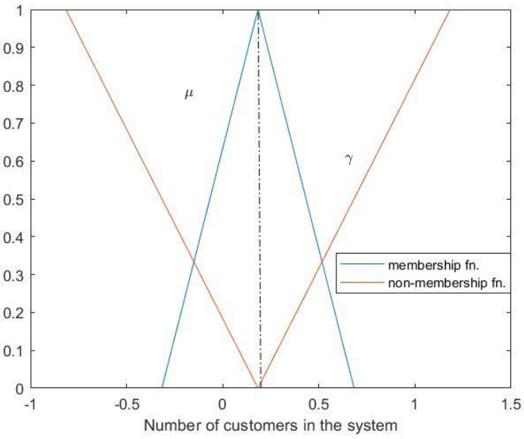

Figure 9: *The membership $(\tilde{\mu})$ and the non-membership $(\tilde{\gamma})$ functions of the number of consumers in the system* $\tilde{N}'_S$

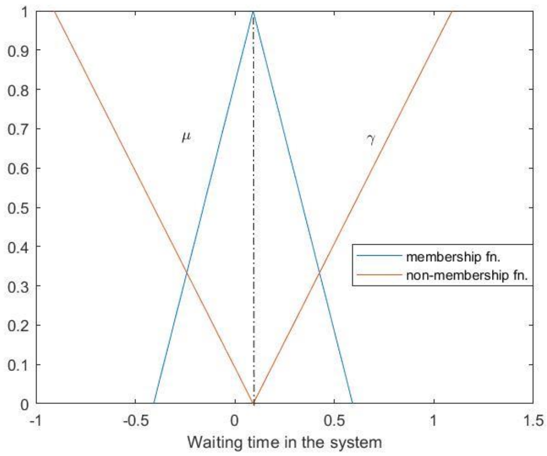

Figure 10: *The membership $(\tilde{\mu})$ and the non-membership $(\tilde{\gamma})$ functions of the waiting time of consumers in the system* $\tilde{T}'_s$

6. Results and Discussions

Tables 1-2 provide the results, which show different assessments for a multitude of membership functions (TFN and TIFN).

i) The mean value of $\tilde{N}_q = 0.0166$ and the left and right stretched values are -0.9834 and 1.0166 respectively emphasizing that the queue length of consumers is closely between -0.9834 and 1.0166. Its most assured value is 0.0166.

ii) The mean value of $\tilde{N}_S = 0.1832$ and the left and right stretched values are -0.8168 and 1.1832 respectively emphasizing that the system length of consumers is closely between -0.8168 and 1.1832. Its most assured value is 0.1832.

iii) The mean value of $\tilde{T}_q = 0.0083$ and the left and right stretched values are -0.9917 and 1.0083 respectively emphasizing that the sojourn time of consumers in the queue is closely between -0.9917 and 1.0083. Its most assured value is 0.0083.

iv) The mean value of $\tilde{T}_S = 0.0916$ and the left and right stretched values are -0.9084 and 1.0916 respectively emphasizing that the sojourn time of consumers in the system is closely between -0.9084 and 1.0916. Its most assured value is 0.0916.

v) The mean value of $\tilde{N}'_q = 0.0166$ and the left and right fuzziness of membership $(\tilde{\mu})$ functions are -0.4834 and 0.5166 respectively and the left and right fuzziness of non-membership $(\tilde{\gamma})$ functions are -0.9834 and 1.0166 respectively. Its most assured value is 0.0166.

vi) The mean value of $\tilde{N}'_S = 0.1832$ and the left and right fuzziness of membership $(\tilde{\mu})$ functions are -0.3168 and 0.6832 respectively and the left and right fuzziness of non-membership $(\tilde{\gamma})$ functions are -0.8168 and 1.1832 respectively. Its most assured value is 0.1832.

vii) The mean value of $\tilde{T}'_q = 0.0083$ and the left and right fuzziness of membership $(\tilde{\mu})$ functions are -0.4917 and 0.5083 respectively and the left and right fuzziness of non-membership $(\tilde{\gamma})$ functions are -0.9917 and 1.0083 respectively. Its most assured value is 0.0083.

viii) The mean value of $\tilde{T}'_S = 0.0916$ and the left and right fuzziness of membership $(\tilde{\mu})$ functions are -0.4084 and 0.5916 respectively and the left and right fuzziness of non-membership $(\tilde{\gamma})$ functions are -0.9084 and 1.0916 respectively. Its most assured value is 0.0916.

The findings demonstrate that the exhibition measures $\tilde{N}_q, \tilde{N}_S, \tilde{T}_q, \tilde{T}_S, \tilde{N}'_q, \tilde{N}'_S, \tilde{T}'_q$ & $\tilde{T}'_S$ for both the fuzzy queuing theory and intuitionistic fuzzy queuing models were metabolized and tested in this study. Because the intuitionistic fuzzy set supposition is more efficaciously customizable, the intuitionistic fuzzy queuing model is significantly more effective and efficient in measuring the exhibition of the deterministic FM/FD/1 queuing model framework. The intuitionistic fuzzy queuing model produces more comprehensive data, which is immensely beneficial when characterizing a model framework. As a result, this study concludes that intuitionistic fuzzy lining is one of the options for registering exposition parameters because the data obtained from the implementation is much better to recognize and perceive.

7. Conclusion

In this manuscript, IFS is shown as a quite crucial asset to fuzzy set theory when interacting with ostensible implementation in single server deterministic queuing models with infinite capacity. We validated the system using comparison sorting rubrics such as the extrapolated length of the customer line and the system for both classifications of arrivals without changing the composition of the queues from fuzzy to crisp. In addition, fuzzy values and intuitionistic fuzzy values are used to ascertain the assumptive sojourn time of customers in the line and throughout the system. Another cause to use the suggested technique indicator is that it delivers viable paths to beliefs in the queuing utilizing multiple forms of membership functions (TFN and TIFN) while retaining exactness within the shuttered crisp interval.

The birth rate and death rate are fuzzy allegiances, so we make a comparison between the fuzzy and intuitionistic fuzzy set hypothesis. System length, queue length, system sojourn time, queue sojourn time, and other execution proportions are interestingly ambiguous. The proposed method's effectiveness is supported by mathematical precepts. It should be acknowledged that by raising the number of variables, the accomplishment of the queuing model can be enhanced. Entrepreneurs, retail outlets, and dealerships can use the proposed model to precisely determine the best queueing system execution.

The prediction model is used to reach scientific claims, and the fuzzy and intuitionistic fuzzy queue with infinite capacity is explained in greater detail. The envisaged queuing system's authenticity and conciseness are appraised using the TFN and TIFN mathematical manifestations. A numerical model demonstrates the design flexibility of the preferred methodology. The intuitionistic fuzzy queuing model is certainly better and more favorable in appraising dimensions of queuing models because the intuitionistic fuzzy theory is more configurable. As an outcome, intuitionistic fuzzy queuing is one of the healthiest modes of computing performance standards, according to this study, because the evidence found from the application is easier to spot and explore. The paper can be extended from various angles. One is to consider birth and death rates as random variables, or fuzzy random variables. Consider neutrosophic fuzzy numbers as an additional aspect to protrude this paper.

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ANALYSIS OF A TWO-STATE PARALLEL SERVERS RETRIAL QUEUEING MODEL WITH BATCH DEPARTURES

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Abstract

This paper deals with the transient state behavior of an M/M/1 retrial queueing model contains two parallel servers with departures occur in batches. At the arrival epoch, if all servers are busy then customers join the retrial group. Whereas, if the customers find any of one server is free then they join the free server and start its service immediately. Here, we assume that primary customers arrive according to Poisson process. The retrial customers also follow the same fashion. Service time follows an exponential distribution. Explicit time dependent probabilities of exact number of arrivals and exact number of departures when both servers are free or when one server is busy or when both servers are busy are obtained by solving the difference differential equation recursively. Some important verification and conversion of two-state model into single state are also discussed. Some of the existing results in the form of special cases have been deduced.

Keywords: Retrial, Queueing, Arrivals, Departures, Batch

1. Introduction

In recent years, computer networks and data communication systems are the fastest growing technologies, which have led to significant development in applications such as advance in internet, audio data traffic, video data traffic, etc. Recently there have been significant contributions to retrial queueing system in which arriving customer who finds the server busy upon arrival is require leaving the service area and repeating his demand after some time. Between trials, a blocked customer who remains in a retrial group is said to be in orbit. Retrial queue have applications in telephone switching systems, telecommunication networks and computers are competing to gain service from a central processing unit. Moreover, retrial queues are also used as mathematical models of several computer systems: packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks etc. There are enough of literatures available on retrial queues. We referred some of the work like Artalejo and Corral [1], Falin and Templeton [2] and Artalejo [3] etc.

In many queueing systems it is assumed that customers arrive singly at a service facility and depart singly from the service facility. However, this assumption is violated in many other real word situations. Letters arriving at a post office, ships arriving at a port in convoy, people going to a theatre and so on are some examples of queuing in which customers do not arrive and depart singly but in bulk or groups. The size of an arriving group and departing group may be a random variable or a fixed number. Mathematically as well as practically the cases where the size of an arriving group and departing group is a random variable, are more common, and also more difficult to handle.

One can note that the batch arrival queue may not always be given the name 'batch' but instead of this many authors chose to use the term 'bulk'. Predominantly, this reflects two leading strands of applications, where 'bulk' often gives a connotation of transportation settings whereas 'batch' frequently implies applications in communications.

Queueing situations in which arrivals occur singly, but service is in bulk are considered in this research. Bulk service queues have potential applications in many areas e.g. in loading and unloading of cargoes at a seaport, in traffic signal systems, in computer networks where jobs are processed in batches, manufacturing/ production systems, cinema halls, in transportation processes involving buses, airplanes, trains, ships, elevators etc. Bailey [4] introduced the concept of bulk service and the same was later studied by a number of parishioners. Juan [5] obtained a numerical method for the single server bulk service queueing system with variable capacity. Janssen and Leeuwaarden [6] presented an analytic rather than a numerical framework for dealing with discrete time bulk service queue. Goswami et al. [7] analysed a discrete time single server infinite buffer bulk service queues. In this research, the inter-arrival time of successive arrivals and service times of batches are assumed to be independent and geometrically distributed. Al-khedhairi and Tadj [8] investigated the queueing process of a bulk service queueing system under Bernoulli schedule.

The classical transient results for the M/M/1, M/M/c and M/G/1 queue provide little insight into the behavior of a queueing system through a fixed operation time t. The function $P_n(t)$ gives the distribution for the number in the system at time t, but practically provides no information on how the system has regulated up until time t. The question seems to be answered by Pegden and Rosenshine [9]. The analysis of their paper based on M/M/1 queueing model in which the state of the system is given by (i, j), where i is the number of arrivals and j is the number of departures until time t. Kalra and Singla [10] investigated the performance analysis of a two-state retrial queueing model with batch departures. In this paper, they obtained time dependent probabilities of exact number of arrivals in the system and exact number of departures from the system when only one server is free or busy. Garg and Kumar [11] studied a single server retrial queue with impatient customers and obtained time-dependent probabilities of number of exact arrivals and number of exact departures from the orbit.

This research studies a time dependent retrial queueing model by obtaining the explicit probabilities of the exact number of arrivals in the system and the exact number of departures from the system by a given time t wherein the departures occur from the orbit in batches of variable size.

The rest of this paper is organized as follows: Section 2 gives a relatively formal description of the queueing model. In Section 3, we defined the two-dimensional state model and derived the difference-differential equations. The time dependent solution for the model is obtained in section 4. Section 5 presents the some useful performance measures of the system and Section 6 discussed some special cases. The last section ends with a suitable conclusion.

2. Model Description

2.1. Assumption and Notation

The two parallel servers retrial queueing system is considered wherein departures take place in batches of variable size whenever these occur from the orbit. The primary calls follow a Poisson distribution with rate λ. If the server is busy at the arrival time, then the arriving call joins the orbit, whereas if the server is free then the service of arriving call gets started. The behavior of customers in orbit is same as in the main model, i.e. every customer in orbit produces a Poisson flow of repeated calls with rate θ. If a batch of repeated calls finds the server free, it is served and leaves the system after service otherwise, if the server is occupied at that time then the system state does not change. Arrivals occur one by one and departures occur from the orbit in batches of variable size with rate μ. The input flow of primary calls, intervals between repeated trials and service times are mutually independent. For distribution of arrivals, service times and retrials, we make use of the following assumptions and notations:

1) The repeated calls for each server follow a Poisson distribution with parameter θ.
2) In this model the departures occur from the orbit is treated as bulk departures whose capacity is determined afresh before each service which is equal to newly determined capacity of the server or units present in the orbit, whichever is less. In this case capacity of the server is a random variable. The size of the batch is determined at beginning of the each service. The probability that the server can serve a batch of γ units is b_γ so that $\sum_{\gamma=1}^{K} b_\gamma = 1$, where K is the maximum capacity of the server.
3) The Service times for each call depart in batches of variable size and follow an exponential distribution with parameter μ.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by

$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$, Re (s) > 0

The Laplace inverse of

$\frac{Q(p)}{P(p)}$ is $\sum_{k=1}^{n}\sum_{l=1}^{m_k} \frac{t^{m_k-l} e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \frac{Q(p)}{P(p)}$ $(p-a_k)^{m_k}$ ∀ p= a_k, $a_i \neq a_k$ for i ≠ k.

where,

$P(p) = (p-a_1)^{m_1} (p-a_2)^{m_2} \dots\dots\dots (p-a_n)^{m_n}$

$Q(p)$ is a polynomial of degree < m_1+m_2+m_3 +…………m_n-1.

If $L^{-1}\{p(s)\} = P(t)$ and $L^{-1}\{q(s)\} = Q(t)$, then

$L^{-1}\{p(s)\, q(s)\} = \int_0^t P(\mathrm{u})Q(\mathrm{t}-\mathrm{u})\mathrm{du} = P * Q$, where $P * Q$ is the convolution of P and Q.

3. The Two-Dimensional State Model

3.1. Definitions

$P_{i,j,0}(t)$= Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is idle.
$P_{i,j,k}(t)$= Probability that there are exactly i arrivals in the system and j departures from the system by time t when k servers are busy. $k = 1,2$.
$P_{i,j}(t)$= Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t) \qquad \forall i, j; \quad i \geq j.$$

Also

$$P_{i,j,1}(t) = 0, i \leq j;\; P_{i,j,0}(t) = 0, i < j.$$

Initially

$$P_{0,0,0}(0) = 1;\; P_{i,j,0}(0) = 0 \;\&\; P_{i,j,k}(0) = 0;\;\; \forall i, j \neq 0.\;\; k = 1,2.$$

3.2. The difference – differential equations governing the system are

$$\frac{d}{dt}P_{i,i,0}(t) = -\lambda\, P_{i,i,0}(t) + \mu\, \sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right) P_{i,i-\gamma,1}(t) \qquad i \geq 0, i \geq \mathrm{K} \qquad (1)$$

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu\, \sum_{\gamma=1}^{K} b_\gamma\, P_{i,j-\gamma,1}(t) \qquad i > j, i > 0; j \geq \mathrm{K} \qquad (2)$$

$$\frac{d}{dt}P_{1,0,1}(t) = -(\lambda+\mu)P_{1,0,1}(t) + \lambda P_{0,0,0}(t) \qquad (3)$$

$$\frac{d}{dt}P_{2,0,2}(t) = -(\lambda+\mu)P_{2,0,2}(t) + \lambda P_{1,0,1}(t) \qquad (4)$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda + \mu + (i-j-1)\theta)P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + (i-j)\theta P_{i,j,0}(t) + 2\mu P_{i,j-1,2}(t) \qquad i > 1, i > j \geq 0 \qquad (5)$$

$$\frac{d}{dt}P_{i,j,2}(t) = -(\lambda+2\mu)P_{i,j,2}(t) + \lambda P_{i-1,j,1}(t) + \lambda\left(1-\delta_{i-2,j}\right)P_{i-1,j,2}(t) + (i-j-1)\theta P_{i,j,1}(t) \qquad i > 2, i > j \geq 0 \qquad (6)$$

$$\text{where } \delta_{i-2,j} = \begin{cases} 1, & when\; i-2 = j \\ 0, & otherwise \end{cases}$$

Using the Laplace transformation $\bar{f}$ (s) of $f(t)$ which is given by

$\bar{f}$ (s) = $\int_0^\infty e^{-st} f(t)$ dt, Re (s) > 0

in the equations (1) - (6) along with the initial conditions, the following equations are obtained:

$$\left.\begin{aligned} (s+\lambda)\bar{P}_{0,0,0}(s) &= P_{0,0,0}(0) \\ (s+\lambda)\bar{P}_{i,i,0}(s) &= \mu \sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)\bar{P}_{i,i-\gamma,1}(s) \end{aligned}\right\} \qquad i > 0, i \geq \mathrm{K} \qquad (7)$$

$$(s+\lambda+(i-j)\theta)\bar{P}_{i,j,0}(s) = \mu \sum_{\gamma=1}^{K} b_\gamma\, \bar{P}_{i,j-\gamma,1}(s) \qquad i > j, i > 0,\; j \geq \mathrm{K} \qquad (8)$$

$$(s+\lambda+\mu)\bar{P}_{1,0,1}(s) = \lambda\, \bar{P}_{0,0,0}(s) \qquad (9)$$

$$(s+\lambda+\mu)\bar{P}_{2,0,2}(s) = \lambda\, \bar{P}_{1,0,1}(s) \qquad (10)$$

$$(s+\lambda+\mu+(i-j-1)\theta)\bar{P}_{i,j,1}(s) = \lambda\, \bar{P}_{i-1,j,0}(s) + (i-j)\theta\bar{P}_{i,j,0}(s) + 2\mu\bar{P}_{i,j,-1,2}(s) \qquad i > 1, i > j \geq 0 \qquad (11)$$

$$(s+\lambda+2\mu)\bar{P}_{i,j,2}(s) = \lambda\, \bar{P}_{i-1,j,1}(s) + \lambda\left(1-\delta_{i-1,j}\right)\bar{P}_{i-1,j,2}(s) + (i-j-1)\theta\bar{P}_{i,j,1}(s) \qquad i > 2, i > j \geq 0 \qquad (12)$$

3.3. Solution of the Problem

Solving equations (7) to (12) recursively, the following results are obtained

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} \qquad (13)$$

$$\bar{P}_{1,1,0}(s) = \frac{\lambda\mu}{(s+\lambda)^2\,(s+\lambda+\mu)} \qquad (14)$$

$$\bar{P}_{i,i,0}(s) = \frac{1}{s+\lambda}\mu \sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)\bar{P}_{i,i-\gamma,1}(s) \qquad i > 1 \qquad (15)$$

$$\bar{P}_{i,2,0}(s) = \frac{\mu}{(s+\lambda+\mu+(i-2)\theta)}\left[b_1\bar{P}_{i,1,1}(s) + b_2\bar{P}_{i,0,1}(s)\right] \qquad i > 2 \qquad (16)$$

$$\bar{P}_{1,0,1}(s) = \left(\frac{1}{s+\lambda}\right)\left(\frac{\lambda}{s+\lambda+\mu}\right) \tag{17}$$

$$\bar{P}_{2,1,1}(s) = \frac{\lambda}{(s+\lambda+\mu)}\bar{P}_{1,1,0}(s) + 2\mu\frac{\lambda}{(s+\lambda+2\mu)(s+\lambda+\mu)}\bar{P}_{1,0,1} \tag{18}$$

$$\bar{P}_{i,1,1}(s) = \frac{2\mu}{(s+\lambda+\mu+(i-2)\theta)}\frac{\lambda^{i-1}}{(s+\lambda+2\mu)^{i-1}}\bar{P}_{1,0,1}(s) \qquad i > 2 \tag{19}$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{(s+\lambda+\mu)}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{(s+\lambda+\mu)}\bar{P}_{i,i-1,0}(s) + \frac{2\mu}{(s+\lambda+\mu)}\bar{P}_{i,i-2,2}(s) \qquad i > 2 \tag{20}$$

$$\bar{P}_{i,0,2}(s) = \frac{\lambda^{i-1}}{(s+\lambda+2\mu)^{i-1}}\bar{P}_{1,0,1}(s) \qquad i > 1 \tag{21}$$

$$\bar{P}_{i,j,2}(s) = \left(\sum_{k=1}^{i-j}\left(\frac{\lambda}{s+\lambda+2\mu}\right)^{i-j-k}\eta'_k(s)\bar{P}_{j+k,j,1}(s)\right) \qquad i \geq j+2, j \geq 1 \tag{22}$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 & for\ k = 1 \\ \left(1+\frac{(k-1)\theta}{s+\lambda+2\mu}\right) & for\ k = 2\ to\ i-j-1 \\ \frac{(k-1)\theta}{s+\lambda+2\mu} & for\ k = i-j \end{cases}$$

$$\bar{P}_{i,j,1}(s) = \frac{\lambda}{(s+\lambda+\mu+(i-j-1)\theta)}\bar{P}_{i-1,j,0}(s) + \frac{(i-j)\theta}{(s+\lambda+\mu+(i-j-1)\theta)}\bar{P}_{i,j,0}(s)$$
$$\frac{2\mu}{(s+\lambda+\mu+(i-j-1)\theta)}\left(\sum_{k=0}^{i-j}\left(\frac{\lambda}{s+\lambda+2\mu}\right)^{i-j-k}\eta'_k(s)\bar{P}_{j+k,j-1,1}(s)\right)$$
$$i \geq j+2, j \geq 2 \tag{23}$$

$$where\ \eta'_k(s) = \begin{cases} 1 & for\ k = 0 \\ \left(1+\frac{k\theta}{s+\lambda+2\mu}\right) & for\ k = 1\ to\ i-j-1 \\ \frac{k\theta}{s+\lambda+2\mu} & for\ k = i-j \end{cases}$$

$$\bar{P}_{i,j,0}(s) = \frac{1}{(s+\lambda+(i-j)\theta)}\left(\mu\sum_{\gamma=1}^{K}b_\gamma\right)\bar{P}_{i,j-\gamma,1}(s) \qquad i > j \geq 3 \tag{24}$$

Using the Inverse Laplace transformation

$\frac{Q(p)}{P(p)}$ is $\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{t^{m_k-l}e^{a_k t}}{(m_k-l)!(l-1)!}\times\frac{d^{l-1}}{dp^{l-1}}\left(\frac{Q(p)}{P(p)}\right)(p-a_k)^{m_k}$ $\quad \forall\, p = a_k,\ a_i \neq a_k$ for $i \neq k$.

where

$P(p) = (p-a_1)^{m_1}(p-a_2)^{m_2}\ldots\ldots\ldots.(p-a_n)^{m_n}$

$Q(p)$ is a polynomial of degree $< m_1+m_2+m_3+\ldots\ldots\ldots.m_n - 1$.

If L^{-1} {f(s)} = F(t) and L^{-1} {g(s)} = G(t), then

L^{-1} {f(s) g(s)} = $\int_0^t F(u)G(t-u)du$ = F * G, F * G is called the convolution of F and G.

and

The Laplace inverse of $\bar{N}^{a,b,c}_{n_1,n_2,n_3}(s) = \frac{1}{(s+a)^{n_1}(s+b)^{n_2}(s+c)^{n_3}}$ is

$$N^{a,b,c}_{n_1,n_2,n_3}(t) = \sum_{l=1}^{n_3}\sum_{m=1}^{l}\frac{e^{-at}\ t^{n_3-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_2+g_2)\right)}{(n_3-l)!(m-1)!\,(b-a)^{n_2+m-1}(c-a)^{n_1+l-m}}$$

$$+\sum_{l=1}^{n_2}\sum_{m=1}^{l}\frac{e^{-bt}\ t^{n_2-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_2-l)!\,(m-1)!\ (a-b)^{n_3+m-1}(c-b)^{n_1+l-m}}$$

$$+\sum_{l=1}^{n_1}\sum_{m=1}^{l}\frac{e^{-ct}\ t^{n_1-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(n_2+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_1-l)!\,(m-1)!\ (a-c)^{n_3+m-1}(b-c)^{n_2+l-m}}$$

in equations (13) to (24), the following probabilities are

$$P_{0,0,0}(t) = e^{-\lambda t} \tag{25}$$

$$P_{1,1,0}(t) = \lambda\mu(te^{-\lambda t})e^{-(\lambda+\mu)t} \tag{26}$$

$$P_{i,i,0}(t) = \left\{\mu\sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)e^{-\lambda t}\right\} * P_{1,i-\gamma,1}(t) \qquad i>1 \tag{27}$$

$$P_{i,2,0}(t) = \mu b_1 e^{-(\lambda+\mu+(i-2)\theta)t} * P_{i,1,1}(t) + \mu b_2 e^{-(\lambda+\mu+(i-2)\theta)t} * P_{i,0,1}(t) \qquad i>2 \tag{28}$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right) \tag{29}$$

$$P_{2,1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{1,1,0}(t) + 2\lambda\mu e^{-(\lambda+\mu)t}\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right) * P_{1,0,1}(t) \tag{30}$$

$$P_{i,1,1}(t) = \left[2\mu\lambda^{i-1}e^{-(\lambda+\mu+(i-2)\theta)t}\left\{\frac{1}{(2\mu)^{i-1}} - e^{-2\mu t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-r}}\right\}\right] * P_{1,0,1}(t) \qquad i>2 \tag{31}$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t) + 2\mu e^{-(\lambda+\mu)t} * P_{i,i-2,2}(t) \qquad i>2 \tag{32}$$

$$P_{i,0,2}(t) = \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!}e^{-(\lambda+2\mu)t}\right) * P_{1,0,1}(t) \qquad i>1 \tag{33}$$

$$P_{i,j,2}(t) = \left(\lambda^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-(\lambda+2\mu)t}\right) * P_{j+1,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}(\mathrm{k}-1)\theta\frac{t^{i-j-k}}{(i-j-k)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \left((\mathrm{i}-\mathrm{j}-1)\theta e^{-(\lambda+2\mu)t}\right) * P_{i,j,1}(t) \qquad i\geq j+2, j\geq 1 \tag{34}$$

$$P_{i,j,1}(t) = \lambda e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i,j,0}(t) + 2\mu\lambda^{i-j}e^{-(\lambda+\mu+(i-j-1)\theta)t}\left\{\frac{1}{(2\mu)^{i-j}} - e^{-2\mu t}\sum_{r=1}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(2\mu)^{i-j-k}} - e^{-2\mu t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k-r}}\right\} * P_{j+k,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}(\mathrm{k}\theta)\left\{\frac{1}{(2\mu)^{i-j-k+1}} - e^{-2\mu t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k+1-r}}\right\} * P_{j+k,j-1,1}(t) + 2\mu(i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t}\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right) * P_{i,j-1,1}(t) \qquad i\geq j+2, j\geq 2 \tag{35}$$

$$P_{i,j,0}(t) = \left(\mu \sum_{\gamma=1}^{K} b_\gamma e^{-(\lambda+\mu+(i-j)\theta)t}\right) * P_{i,j-\gamma,1}(t) \qquad i > j \geq 3 \tag{36}$$

4. Measures of Effectiveness

4.1. The Laplace transform of the probability $P_{i.}(t)$ that exactly i units arrive by time t is :

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}} ;\text{i} > 0 \tag{37}$$

And its Inverse Laplace transform is

$$P_{i.}(t) = \frac{e^{-\lambda t} (\lambda t)^i}{i!} \tag{38}$$

The basic assumption on primary arrivals is that it forms a Poisson process and above analysis of abstract solution also verifies the same.

4.2. The probability that exactly j customers have been served by time t. $P_{.j}(t)$ in terms of $P_{i,j}(t)$ is given by:

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

4.3. From the abstract solution of our model, we verified that the sum of all possible probabilities is one i.e. taking summation over i and j on equations (15)-(31) and adding, we get

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} \{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s) + \bar{P}_{i,j,2}(s)\} = \frac{1}{s}$$

Taking inverse Laplace transformation, we get

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} \{P_{i,j,0}(t) + P_{i,j,1}(t) + +P_{i,j,2}(t)\} = 1,$$

which is a verification of our results.

4.4. Converting two-state model into single state model:

To convert two-dimensional state model into a single state model probability $Q_{n,k}(t)$ is defined as under:

$Q_{n,k}(t)$ = Probability that there are n customers in the orbit at time t and the servers are free or busy according as $k = 0,1,2$.

The probability of exactly n customers in the system at time t in terms of $P_{i,j,0}(t)$ and $P_{i,j,k}(t)$:

When the server is free, it is defined by probability $Q_{n,0}(t)$

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures)

When k servers are busy, it is defined by probability $Q_{n,k}(t)$

$$Q_{n,k}(t) = \sum_{j=0}^{\infty} P_{j+n+k,j,k}(t) \qquad (k = 1,2)$$

where k defines the number of servers.

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures – k)

Using the above definitions from the equations (1) to (6) the set of equations in statistical equilibrium are:

$$\lambda Q_{0,0} = \mu\left[\sum_{\gamma=1}^{K}\left(\sum_{l=\gamma}^{K} b_l\right)\right]Q_{\gamma,1} \qquad (39)$$

$$(\lambda + n\theta)Q_{n,0} = \mu\left(\sum_{l=\gamma}^{K} b_l\right)Q_{n+\gamma,1} \qquad n > 0 \qquad (40)$$

$$(\lambda + n\theta + \mu)Q_{n,1} = \lambda Q_{n,} + (n+1)\theta Q_{n+1,0} + 2\mu Q_{n,2,0} \qquad n \geq 0 \qquad (41)$$

$$(\lambda + 2\mu)Q_{n,2} = \lambda Q_{n,1} + (n+1)\theta Q_{n+1,1} + \lambda Q_{n-1,2} \qquad n \geq 0 \qquad (42)$$

4.5. Special Case:

1. When the units are served singly and considering$K = 1,\ b_1 = 1,\ b_2 = b_3 = b_4 = \cdots = b_K = 0$ in equations (25) to (36), then the probabilities coincide with the results of Singla and Kalra [12].

$$P_{0,0,0}(t) = e^{-\lambda t} \qquad (43)$$

$$P_{1,1,0}(t) = \lambda\mu(te^{-\lambda t})e^{-(\lambda+\mu)t} \qquad (44)$$

$$P_{i,i,0}(t) = \lambda\mu e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i-1,i-1,0}(t) + \mu\theta e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i,i-1,0}(t) + 2\mu^2 e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i,i-2,2}(t) \qquad i > 1 \qquad (45)$$

$$P_{i,2,0}(t) = 2\mu^2 e^{-(\lambda+(i-2)\theta)t}\left(\frac{1}{(\mu+(i-2)\theta)} - \frac{e^{-(\mu+(i-2)\theta)t}}{(\mu+(i-2)\theta)}\right) * P_{i,0,2}(t) \qquad i \geq 3 \qquad (46)$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) \qquad (47)$$

$$P_{2,1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{1,1,0}(t) + 2\lambda\mu e^{-(\lambda+\mu)t}\left(\frac{1}{2\mu} - \frac{e^{-2\mu t}}{2\mu}\right) * P_{1,0,1}(t) \qquad (48)$$

$$P_{i,1,1}(t) = \left[2\mu\lambda^{i-1}e^{-(\lambda+\mu+(i-2)\theta)t}\left\{\frac{1}{(2\mu)^{i-1}} - e^{-2\mu t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-r}}\right\}\right] * P_{1,0,1}(t) \qquad i > 2 \qquad (49)$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t) + 2\mu e^{-(\lambda+\mu)t} * P_{i,i-2,2}(t) \qquad i > 2 \qquad (50)$$

$$P_{i,0,2}(t) = \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!}e^{-(\lambda+2\mu)t}\right) * P_{1,0,1}(t) \qquad i > 1 \qquad (51)$$

$$P_{i,j,2}(t) = \left(\lambda^{i-j-1}\frac{t^{i-j-2}}{(i-j-2)!}e^{-(\lambda+2\mu)t}\right) * P_{j+1,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}\frac{t^{i-j-k-1}}{(i-j-k-1)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \sum_{k=2}^{i-j-1}\left(\lambda^{i-j-k}(\mathrm{k}-1)\theta\frac{t^{i-j-k}}{(i-j-k)!}e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t) + \left((\mathrm{i}-\mathrm{j}-1)\theta e^{-(\lambda+2\mu)t}\right) * P_{i,j,1}(t) \qquad i \geq j+2, j \geq 1 \qquad (52)$$

$$P_{i,j,1}(t) = \lambda e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i,j,0}(t) + 2\mu\lambda^{i-j}e^{-(\lambda+\mu+(i-j-1)\theta)t}\left\{\frac{1}{(2\mu)^{i-j}} - e^{-2\mu t}\sum_{r=1}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(t) + 2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\left\{\frac{1}{(2\mu)^{i-j-k}} - e^{-2\mu t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k-r}}\right\} * P_{j+k,j-1,1}(t) +$$

$$2\mu e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\,(k\theta)\left\{\frac{1}{(2\mu)^{i-j-k+1}}-e^{-2\mu t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-j-k+1-r}}\right\}*$$

$$P_{j+k,j-1,1}(t)+2\mu(i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t}\left(\frac{1}{2\mu}-\frac{e^{-2\mu t}}{2\mu}\right)*P_{i,j-1,1}(t)$$

$$i\geq j+2, j\geq 2 \qquad (53)$$

$$P_{i,j,0}(t)=\lambda\mu e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu+(i-j)\theta}-\frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right)*P_{i-1,j-1,0}(t)+$$

$$\mu(i-j+1)\theta e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu+(i-j)\theta}-\frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right)*P_{i,j-1,0}(t)$$

$$+2\mu^2\lambda^{i-j+1}\left[\sum_{l=1}^{i-j+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\;t^{(i-j+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((i-j+1)-l)!\,(m-1)!\;(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}}\right.$$

$$\left.-\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)}(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)}(\mu-(i-j)\theta)}\right]*P_{j-1,j-2,1}(t)+2\mu^2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}$$

$$\left[\sum_{l=1}^{(i-j+1)-k}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\;t^{((i-j+1)-k)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{(((i-j+1)-k)-l)!(m-1)!\,(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}}-\right.$$

$$\left.\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)-k}(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)-k}(\mu-(i-j)\theta)}\right]*P_{j+k-1,j-2,1}(t)+2\mu^2\sum_{k=1}^{i-j}\lambda^{(i-j+1)-k}\,(k\theta)$$

$$\left[\sum_{l=1}^{((i-j+1)-k)+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\;t^{(((i-j+1)-k)+1)-l}(-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{((((i-j+1)-k)+1)-l)!(m-1)!\,(\mu)^m(2\mu-(i-j)\theta)^{1+l-m}}-\right.$$

$$\left.\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{((i-j+1)-k)+1}(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{((i-j+1)-k)+1}(\mu-(i-j)\theta)}\right]*P_{j+k-1,j-2,1}(t)+2\mu^2\,(i-j+$$

$$1)\theta\left[\frac{e^{-(\lambda+(i-j)\theta)t}}{(\mu)(2\mu-(i-j)\theta)}-\frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)(\mu-(i-j)\theta)}+\frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)(\mu-(i-j)\theta)}\right]*P_{i,j-2,1}(t)$$

$$i>j\geq 3 \qquad (54)$$

2. Letting $K=1,\ b_1=1,\ b_2=b_3=b_4=\cdots=b_K=0$ and $\mu=1$ in (39) to (42), then the following equations are:

$$(\lambda+n\theta)Q_{n,0}=Q_{n,1} \qquad n\geq 0 \qquad (55)$$

$$(\lambda+n\theta+1)Q_{n,1}=\lambda Q_{n,0}+(n+1)\theta Q_{n+1,0}+2Q_{n,2} \qquad n\geq 0 \qquad (56)$$

$$(\lambda+2)Q_{n,2}=\lambda Q_{n,1}+(n+1)\theta Q_{n+1,1}+\lambda Q_{n-1,2} \qquad n\geq 0 \qquad (57)$$

which coincide with the results (2.1) – (2.3) of Falin and Templeton [2].

6. Conclusion

In this study, a two retrial queueing system with bulk departures having two identical parallel servers is investigated. Bulk queueing systems are common in real-life situations such as elevators, loading and unloading cargoes, giant wheel, chemical manufacturing process, communication networks and tourism etc. Transient probabilities of exact number of arrivals and departures are found by solving difference differential equations recursively when no, one or both servers are busy. Further, some particular cases of interest are discussed along with special cases. From two-dimensional state queueing model, factors are well understood and quantified.

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The Transmuted Weibull Frechet Distribution: Properties and Applications

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Abstract

The behaviour of everyday real life processes played a greater role in distribution theory. Thus, this article proposes a transmuted Weibull Frechet (TWFr) distribution for modeling real life datasets. Of most important, the statistical properties of the TWFr distribution such as the hazard, survival functions, order statistic, quantile, odd, cumulative functions were derived and examined. A simulation study to examine the performance of the TWFr distribution was also conducted. A glass fiber data and breaking stress of carbon data real life application were used to showcase the performance of the proposed model. The results showed that the TWFr distribution competes favourably well with other types of continuous distributions in the Frechet family of distributions.

Keywords: Frechét distribution, Hazard rate function, Order statistics, Transmutation, Weibull distribution.

1. Introduction

Modeling the distributions of real life processes poses greater challenges despite the numerous distributions that have been proposed in literature. However, there is a growing interest in developing newer classes of classical univariate distributions for modeling variety of data sets that arise from our daily scenarios. Thus, it becomes necessary to model these processes either by compounding one or more distributions to address these complex situations.

The Weibull distribution proposed by a famous statistician called Weibull in 1951 [27] has a wide range of applications in modeling failure time processes, lifetime processes, mechanical and electrical systems. More so, the Weibull distribution has been found to be better for modeling the minimum of large number of independent positive random variables in extreme value theory. On the other hand, the Frechet distribution is a special case of the Weibull distribution used to model extreme value scenarios like earthquakes, horse racing, floods, rainfall, wind speed, queues in supermarkets and sea waves (see [2]). The Frechet distribution has been widely used to model extreme value scenarios because of its stochastic phenomena. However, the Frechet distribution is used for modeling maximum of a large number of independent random variables from a particular class of distributions ([1]). Hence, because of its usefulness, improving the flexibility of the Frechet distribution becomes necessary by adding a transmuted parameter that can reflect the true characteristics of the data set(see [13]).

Several statistical distributions have been proposed in literature. For example, [2] proposed the Weibull Frechet distribution, [12] proposed the generalized odd Weibull generated family of distributions. Recently, [26] proposed the gamma extended Frechet distribution. [17] proposed the beta Frechet distribution. [16] proposed the exponentiated Frechet distribution. [14] proposed Kumaraswamy Frechet distribution. The generalized transmuted Frechet (GTFr) distribution was proposed in [18]. [14] estimated the Frechet type 11 parameters and [23] proposed a new generalization of the Frechet distribution. In this study, the transmuted Weibull Frechet distribution is introduced.

A random variable X with a scale parameter α, and β, τ, ν the shape parameters has a Weibull Frechet distribution if the cumulative density function is given as

$$G(x) = 1 - exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big). \tag{1}$$

The corresponding probability density function is given as

$$\begin{aligned} g(x) =& \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\Big[-\nu(\frac{\alpha}{x})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{-\nu-1} \\ &\times exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big). \end{aligned} \tag{2}$$

However, the pdf and cdf of a random variable X can be transmuted with a transmutation parameter $|\lambda| \leq 1$ as

$$f(x;\lambda) = g(x)\Big[1 + \lambda - 2\lambda G(x)\Big], \tag{3}$$

and

$$F(x;\lambda) = \Big(1 + \lambda\Big)G(x) - \lambda\Big[G(x)\Big]^2 \tag{4}$$

with $G(x)$ and $g(x)$ as the cdf and pdf of the baseline/parent distribution respectively.

This article is organized as follows: The introduction was given in section 1, Section 2 is the formulation of the transmuted Weibull Frechet distribution. Section 3 discussed the maximum likelihood of model parameters. In Section 4, we derived some properties of the TWFr distribution. Section 5 is the simulation study and real life application to validate the proposed model and Section 6 is the conclusion.

2. The Transmuted Weibull Frechet Distribution

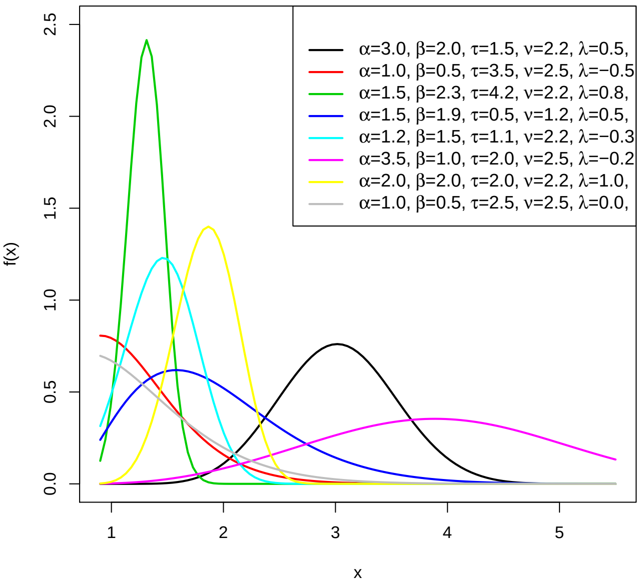
Let X be random variable. Then, the pdf of the TWFr is defined as

$$\begin{aligned} f_{TWFr}(x) =& \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\Big[-\nu(\frac{\alpha}{x})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{-\nu-1} \\ &\times exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big) \\ &\times \Big[1 - \lambda + 2\lambda exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big)\Big]. \end{aligned} \tag{5}$$

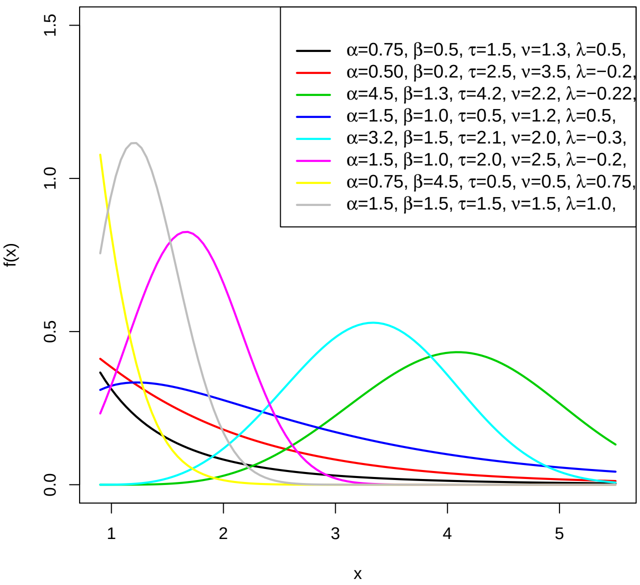
The corresponding cdf is given as

$$\begin{aligned} F_{TWFr}(x) =& \Big(1 + \lambda\Big)\Big(1 - exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big)\Big) \\ &- \lambda\Big[1 - exp\Big(-\tau\Big\{exp\Big[(\frac{\alpha}{x})^{\beta}\Big] - 1\Big\}^{-\nu}\Big)\Big]^2. \end{aligned} \tag{6}$$

Figure 1 shows the plot of the pdf for TWFr distribution with different parameters values. The plot of the TWFr distribution shows that it could be increasing, decreasing and skewed to the right and left depending on the values of the parameters.



(a)



(b)

Figure 1 *The plots of the TWFr pdf for some parameter values*

The reliability or survival function (rf) of the random variable X is given as

$$\begin{aligned} rf_{TWFr}(x) =& exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\left(1-\lambda\right) \\ &+\lambda\left(exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right)^{2}. \end{aligned} \tag{7}$$

Its hazard rate function (hrf) is given as

$$
\begin{aligned}
hr_{TWFr}(x) =& \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\left[-\nu(\frac{\alpha}{x})^{\beta}\right]\left\{1-exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{-\nu-1}\\
&\times exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\\
&\times\left[1-\lambda+2\lambda exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right]\\
&\times\left\{exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\left(1-\lambda\right)+\lambda\left(exp\left(-\tau\left\{\right.\right.\right.\right.\\
&\left.\left.\left.\left. exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right)^{2}\right\}^{-1}.
\end{aligned}
\tag{8}
$$

Figure 2 shows the plot for the hazard rate function of the TWFr distribution. The plot shows that the TWFr model is decreasing and bathtub depending on the values of the associated parameters.

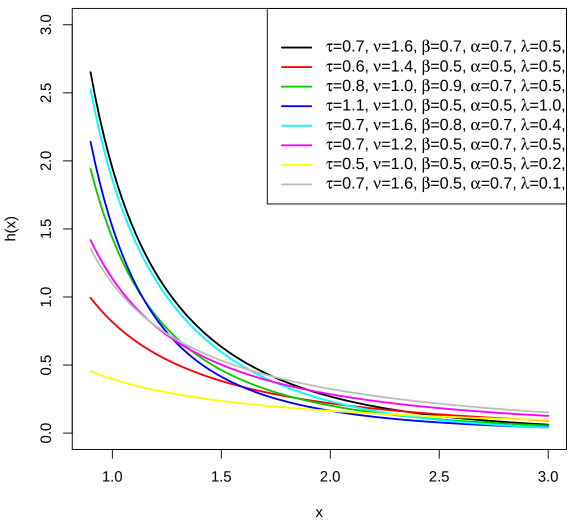


Figure 2 *The plots of the TWFr hrf for some parameter values*

The cumulative hazard rate function (chrf) of the TWFr model is given as

$$
\begin{aligned}
chrf_{TWFr}(x) =& -\ln\left\{exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\left(1-\lambda\right)\right.\\
&\left.+\lambda\left(exp\left(-\tau\left\{exp\left[(\frac{\alpha}{x})^{\beta}\right]-1\right\}^{-\nu}\right)\right)^{2}\right\}.
\end{aligned}
\tag{9}
$$

3. Parameter Estimation of the Transmuted Weibull Frechet Distribution

Let X be random variable with TWFr distribution function. Then, the log-likelihood ℓ of the distribution for parameter vector $(\lambda, \beta, \tau, \nu, \alpha)^T$ is given as,

$$
\ell = n\log(\tau\nu\beta\alpha^{\beta}) + s + m + z + p \tag{10}
$$

$$\frac{\partial \ell}{\partial \lambda} = p'_{\lambda} = 0, \tag{11}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{k'_{\alpha}}{k} - s'_{\alpha} + m'_{\alpha} - z'_{\alpha} + p'_{\alpha} = 0, \tag{12}$$

$$\frac{\partial \ell}{\partial \tau} = \frac{n}{\tau} - z'_{\tau} + p'_{\tau} = 0, \tag{13}$$

$$\frac{\partial \ell}{\partial \nu} = \frac{n}{\nu} - s'_{\nu} + m'_{\nu} - z'_{\nu} + p'_{\nu} = 0, \tag{14}$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \frac{k'_{\beta}}{k} - s'_{\beta} + m'_{\beta} - z'_{\beta} + p'_{\beta} = 0. \tag{15}$$

where $'$ denotes partial derivative and subscript the respective parameter and

$$k = \alpha^{\beta}; s = \sum_{i=1}^{n} \left[-\nu (\frac{\alpha}{x})^{\beta} \right]; m = \sum_{i=1}^{n} \log \left\{ 1 - exp \left[-(\frac{\alpha}{x})^{\beta} \right] \right\}^{-\nu-1};$$

$$z = \sum_{i=1}^{n} \left(-\tau \left\{ exp \left[(\frac{\alpha}{x})^{\beta} \right] - 1 \right\}^{-\nu} \right);$$

$$p = \sum_{i=1}^{n} \log \left[1 - \lambda + 2\lambda exp \left(-\tau \left\{ exp \left[(\frac{\alpha}{x})^{\beta} \right] - 1 \right\}^{-\nu} \right) \right].$$

The Equations (11), (12), (13), (14) and (15) are nonlinear and can not be easily obtained in closed form. Thus, the solutions to the parameter vector are obtained numerically using the Newton-Raphson algorithm and with various statistical and mathematical softwares like R, Mathematical, Maple and Matlab.

4. Some Statistical Properties of the Transmuted Weibull Frechet Distribution

This section investigates some statistical properties of the TWFr distribution. These include, quantile and random number generation and order statistics.

4.1. Quantile Function and Median

Let X be a random variable such that $X \sim TWFr(\alpha, \lambda, \beta, a, b)$. Then, the quantile function of X for $u \in (0, 1)$ is real solution of the following equation given as

$$Q(u) = F^{-1}(x). \tag{16}$$

Thus,

$$x_u = \alpha \left[\log \left\{ 1 + \left[(-\tau^{-1}) \log(1 - \phi(u)) \right]^{-\frac{1}{\nu}} \right\} \right]^{-\frac{1}{\beta}} \tag{17}$$

$$0 < u < 1$$

where

$$\phi(u) = \begin{cases} \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}, & \text{if } \lambda > 0, \\ u, & \text{otherwise } \lambda = 0. \end{cases}$$

By setting $u = 0.5$ in Equation (17), we have the median (M) of X as

$$M = \alpha\Bigg[\log\Bigg\{1 + \Bigg[(-\tau^{-1})\log(1-\phi(0.5))\Bigg]^{-\frac{1}{\nu}}\Bigg\}\Bigg]^{-\frac{1}{\beta}}, \tag{18}$$

with

$$\phi(u) = \begin{cases} \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-2\lambda}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-2\lambda}}{2\lambda}, & \text{if } \lambda > 0, \\ 0.5, & \text{otherwise } \lambda = 0. \end{cases}$$

However, the 25^{th} and 75^{th} percentile for the random variable X is obtained as

$$Q_1 = \alpha\Bigg[\log\Bigg\{1 + \Bigg[(-\tau^{-1})\log(1-\phi(0.25))\Bigg]^{-\frac{1}{\nu}}\Bigg\}\Bigg]^{-\frac{1}{\beta}}, \tag{19}$$

with

$$\phi(u) = \begin{cases} \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-\lambda}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-\lambda}}{2\lambda}, & \text{if } \lambda > 0, \\ 0.25, & \text{otherwise } \lambda = 0. \end{cases}$$

$$Q_3 = \alpha\Bigg[\log\Bigg\{1 + \Bigg[(-\tau^{-1})\log(1-\phi(0.75))\Bigg]^{-\frac{1}{\nu}}\Bigg\}\Bigg]^{-\frac{1}{\beta}}, \tag{20}$$

with

$$\phi(u) = \begin{cases} \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-3\lambda}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-3\lambda}}{2\lambda}, & \text{if } \lambda > 0, \\ 0.75, & \text{otherwise } \lambda = 0. \end{cases}$$

4.2. Reversed Hazard Function and Odds Functions

The reversed hazard function (rhf) of the TWFr distribution is given as

$$\begin{aligned} rhf_{TWFr}(x) =& \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\Bigg[-\nu(\frac{\alpha}{x})^{\beta}\Bigg]\Bigg\{1-exp\Bigg[-(\frac{\alpha}{x})^{\beta}\Bigg]\Bigg\}^{-\nu-1} \\ &\times exp\Bigg(-\tau\nu\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}\Bigg) \\ &\times \Bigg[1-\lambda+2\lambda exp\Bigg(-\tau\nu\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}\Bigg)\Bigg] \\ &\times \Bigg\{1+exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg(\lambda-1\Bigg) \\ &-\lambda\Bigg(exp\Bigg(-2\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg]-1\Bigg\}^{-\nu}\Bigg)\Bigg)\Bigg\}^{-1}. \end{aligned} \tag{21}$$

The Odd function that corresponds to the TWFr distribution is given as

$$\begin{aligned} O_{TWFr}(x) =& 1 + \Bigg\{exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg)\Bigg(1-\lambda\Bigg) \\ & + \lambda\Bigg(exp\Bigg(-2\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg)\Bigg)\Bigg\}^{-1}. \end{aligned} \tag{22}$$

4.3. Distribution of the Order Statistics

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the $f_{TWFr}(x)$ distribution and $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$be the corresponding order statistics, Then, probability density function of the kth order statistics X_k, say $f_k(x)$ is given as

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!}\Bigg[F_{TWFr}(x)\Bigg]^{k-1} f_{TWFr}(x)\Bigg[1 - F_{TWFr}(x)\Bigg]^{n-k}. \tag{23}$$

$$-\infty < x < \infty.$$

On substituting into Equation (23), we have

$$\begin{aligned} f_k(x) =& \frac{n!}{(k-1)!(n-k)!} \\ & \times \tau\nu\beta\alpha^{\beta}x^{-\beta-1}exp\Bigg[-\nu(\frac{\alpha}{x})^{\beta}\Bigg]\Bigg\{1 - exp\Bigg[-(\frac{\alpha}{x})^{\beta}\Bigg]\Bigg\}^{-\nu-1} \\ & \times exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg) \\ & \times \Bigg[1 - \lambda + 2\lambda exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg)\Bigg] \\ \times & \Bigg[\Bigg(1+\lambda\Bigg)\Bigg(1 - exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg)\Bigg) \\ & - \lambda\Bigg[1 - exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg)\Bigg]^2\Bigg]^{k-1} \\ & \times \Bigg[1 - \Bigg(1+\lambda\Bigg)\Bigg(1 - exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg)\Bigg) \\ & - \lambda\Bigg[1 - exp\Bigg(-\tau\Bigg\{exp\Bigg[(\frac{\alpha}{x})^{\beta}\Bigg] - 1\Bigg\}^{-\nu}\Bigg)\Bigg]^2\Bigg]^{n-k}. \end{aligned} \tag{24}$$

The minimum and maximum order statistics are obtained when $k = 1$ and $k = n$ respectively.

4.4. Simulation Study

A simulation is performed to examine the flexibility and efficiency of the TWFr distribution. Tables 1 and 2 show the simulation results for different values of parameters. The simulation was performed as follows:

- Data were generated using

$$x_u = \alpha\Bigg[\log\Bigg\{1 + \Bigg[(-\tau^{-1})\log(1-\phi(u))\Bigg]^{-\frac{1}{\nu}}\Bigg\}\Bigg]^{-\frac{1}{\beta}} \quad 0 < u < 1.$$

with

$$\phi(u) = \begin{cases} \frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}, & \text{if } \lambda < 0, \\ \frac{(1+\lambda)+\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}, & \text{if } \lambda > 0, \\ u, & \text{otherwise } \lambda = 0. \end{cases}$$

- The values of the parameters were purportedly set as $\alpha = 1.5$, $\beta = 0.5$,$\tau = 1.0$, $\nu = 1.0$ and $\lambda = 0.5$, $\alpha = 1.5$, $\beta = 0.5$,$\tau = 1.0$, $\nu = 1.0$ and $\lambda = -0.5$.
- The sample sizes were taken as $n = 10, 50, 150, 350, 400$ and 500.
- Each sample size was replicated 5000 times.

In the simulation study, we investigated the mean estimates (MEs), variance, biases and means squared errors (MSEs) of the MLEs.
The bias was calculated as (for $S = \alpha, \lambda, \beta, \tau, \nu$)

$$\hat{B}ias_S = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right).$$

Also, the MSE was obtained as

$$\hat{M}SE_S = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right)^2.$$

In Tables 1 and 2, the results of the Monte Carlo study show that the MSEs decay towards zero as the sample size increases which corroborates with the first-order asymptotic theory. The mean estimates of the TWFr distribution parameter estimates tend to the true parameter values as the sample size increases which also corroborates the fact that the asymptotic normal distribution provides an adequate approximation of the estimates.

5. Real-Life Applications

A breaking stress of carbon fibers and glass fiber real life datasets were used to examine the performance and flexibility of the model based on its test statistic. Several criteria were used to determine the distribution of the best fit: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC).

Table 1 *Simulation results for mean estimates, biases and root mean squared errors of $\hat{\tau}, \hat{\upsilon}, \hat{\alpha}, \hat{\lambda} > 0$ and $\hat{\beta}$ for the TWFr distribution.*

n	Parameter	ME	Bias	Variance	MSE
10	$\hat{\alpha}$	1.4627	-0.0373	0.1504	0.1518
	$\hat{\beta}$	0.6653	0.1653	0.0738	0.1011
	$\hat{\tau}$	1.0968	0.0968	0.1034	0.1128
	$\hat{\upsilon}$	0.9871	-0.0129	0.2505	0.2507
	$\hat{\lambda}$	0.3617	-0.1383	0.1493	0.1684
50	$\hat{\alpha}$	1.5383	0.0383	0.0618	0.0632
	$\hat{\beta}$	0.5326	0.0326	0.0167	0.0177
	$\hat{\tau}$	1.0924	0.0924	0.0485	0.0571
	$\hat{\upsilon}$	1.0037	0.0037	0.0924	0.0924
	$\hat{\lambda}$	0.4170	-0.0830	0.0690	0.0759
150	$\hat{\alpha}$	1.5540	0.0540	0.0279	0.0308
	$\hat{\beta}$	0.5068	0.0068	0.0050	0.0050
	$\hat{\tau}$	1.0529	0.0529	0.0240	0.0268
	$\hat{\upsilon}$	1.0012	0.0012	0.0322	0.0322
	$\hat{\lambda}$	0.4742	-0.0258	0.0417	0.0423
350	$\hat{\alpha}$	1.5528	0.0528	0.0142	0.0170
	$\hat{\beta}$	0.4983	-0.0017	0.0019	0.0019
	$\hat{\tau}$	1.0367	0.0367	0.0116	0.0129
	$\hat{\upsilon}$	1.0063	0.0063	0.0137	0.0137
	$\hat{\lambda}$	0.4960	-0.0040	0.0234	0.0234
400	$\hat{\alpha}$	1.5539	0.0539	0.0134	0.0163
	$\hat{\beta}$	0.4975	-0.0025	0.0016	0.0016
	$\hat{\tau}$	1.0334	0.0334	0.0102	0.0113
	$\hat{\upsilon}$	1.0072	0.0072	0.0115	0.0116
	$\hat{\lambda}$	0.5001	0.0001	0.0205	0.0205
500	$\hat{\alpha}$	1.5500	0.0500	0.0104	0.0129
	$\hat{\beta}$	0.4968	-0.0032	0.0013	0.0013
	$\hat{\tau}$	1.0276	0.0276	0.0079	0.0087
	$\hat{\upsilon}$	1.0071	0.0071	0.0091	0.0091
	$\hat{\lambda}$	0.5058	0.0058	0.0176	0.0176

The density functions considered include (for $x > 0$)

- Weibull Frechet: $f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}exp\Big[-b(\frac{\alpha}{x})^{\beta}\Big]\Big\{1-exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{-b-1}$ $\times\, exp\bigg(-a\Big\{exp[(\frac{\alpha}{x})^{\beta}]-1\Big\}^{-b}\bigg);$
- Exponentiated Frechect: $f(x) = \lambda\beta\alpha^{\beta}x^{-\beta-1}exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\{1-exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{\lambda-1};$
- Kumaraswamy Frechet: $f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}exp\Big[-a(\frac{\alpha}{x})^{\beta}\Big]\Big\{1-exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{b-1};$
- Beta Frechet: $f(x) = \frac{\beta\alpha^{\beta}x^{-\beta-1}}{B(a,b)}exp\Big[-a(\frac{\alpha}{x})^{\beta}\Big]\Big\{1-exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{b-1};$
- Gamma Extended Frechet: $f(x) = \frac{a\beta\alpha^{\beta}x^{-\beta-1}}{\Gamma(b)}exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\{1-exp\Big[-(\frac{\alpha}{x})^{\beta}\Big]\Big\}^{a-1}\Big(-log\{1-$

Table 2 *Simulation results for mean estimates, biases and root mean squared errors of $\hat{\tau}, \hat{\nu}, \hat{\alpha}, \hat{\lambda} < 0$ and $\hat{\beta}$ for the TWFr distribution.*

n	Parameter	ME	Bias	Variance	MSE
10	$\hat{\alpha}$	1.5887	0.0887	0.1859	0.1937
	$\hat{\beta}$	0.6626	0.1626	0.0532	0.0796
	$\hat{\tau}$	0.9814	-0.0186	0.1996	0.1999
	$\hat{\nu}$	0.9641	-0.0359	0.1531	0.1544
	$\hat{\lambda}$	-0.5970	-0.0970	0.7110	0.7204
50	$\hat{\alpha}$	1.5604	0.0604	0.0658	0.0694
	$\hat{\beta}$	0.5448	0.0448	0.0164	0.0184
	$\hat{\tau}$	0.9843	-0.0157	0.0538	0.0540
	$\hat{\nu}$	1.0009	0.0009	0.0683	0.0683
	$\hat{\lambda}$	-0.4541	0.0459	0.0995	0.1016
150	$\hat{\alpha}$	1.5587	0.0587	0.0284	0.0318
	$\hat{\beta}$	0.5117	0.0117	0.0060	0.0062
	$\hat{\tau}$	1.0013	0.0013	0.0224	0.0224
	$\hat{\nu}$	1.0134	0.0134	0.0272	0.0274
	$\hat{\lambda}$	-0.4521	0.0479	0.0363	0.0386
350	$\hat{\alpha}$	1.5561	0.0561	0.0146	0.0178
	$\hat{\beta}$	0.5010	0.0010	0.0024	0.0024
	$\hat{\tau}$	1.0100	0.0100	0.0100	0.0101
	$\hat{\nu}$	1.0158	0.0158	0.0114	0.0116
	$\hat{\lambda}$	-0.4636	0.0364	0.0168	0.0181
400	$\hat{\alpha}$	1.5582	0.0582	0.0130	0.0164
	$\hat{\beta}$	0.4998	-0.0002	0.0021	0.0021
	$\hat{\tau}$	1.0110	0.0110	0.0089	0.0091
	$\hat{\nu}$	1.0166	0.0166	0.0100	0.0103
	$\hat{\lambda}$	-0.4642	0.0358	0.0160	0.0173
500	$\hat{\alpha}$	1.5563	0.0563	0.0102	0.0134
	$\hat{\beta}$	0.4991	-0.0009	0.0017	0.0017
	$\hat{\tau}$	1.0109	0.0109	0.0074	0.0075
	$\hat{\nu}$	1.0159	0.0159	0.0078	0.0081
	$\hat{\lambda}$	-0.4650	0.0350	0.0129	0.0141

$exp\left[-(\frac{\alpha}{x})^{\beta}\right]\Big\}^{a}\Big)^{b-1}$;

- Transmuted Frechet: $f(x) = \beta\alpha^{\beta}x^{-\beta-1}exp\left[-(\frac{\alpha}{x})^{\beta}\right]\left\{1+\lambda-2\lambda exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}$;
- Frechet: $f(x) = \lambda\alpha^{\lambda}x^{-\lambda-1}exp\left[-(\frac{\alpha}{x})^{\lambda}\right]$;
- Alpha Power Inverse Weibull: $f(x) = \frac{\log(\alpha)}{(\alpha-1)}\lambda\beta exp(-\lambda x^{-\beta})\alpha^{exp(-\lambda x^{-\beta})}$;
- Transmuted Rayleigh: $f(x) = \frac{x}{\alpha^2}exp\left(-\frac{x^2}{2\alpha^2}\right)\left(1-\beta+2\beta exp\left(-\frac{x^2}{2\alpha^2}\right)\right)$.

5.1. Breaking Stress of Carbon fibres

The first data consist of 100 breaking stress of carbon fibres as used in [19] and [5]. It consists of 100 observations taken on breaking stress of carbon fibers (in Gba). Table 3 shows the test statistics. The dataset are as follow:

2.81, 2.77, 2.17, 2.83, 1.92, 1.41, ,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 3.68, 2.97, 1.36,3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19 2.82, 2.05, 3.65.

Table 3 *Performance rating of the TWFr distribution with breaking stress of carbon fibres dataset*

Model	Parameter MLEs(Std. Errors)	AIC	CAIC	BIC	HQIC	-2ℓ
Transmuted Weibull Frechét	$\hat{\tau} = 86.6226(41.4818)$ $\hat{\nu} = 0.4167(0.2361)$ $\hat{\alpha} = 3.7977(1.3096)$ $\hat{\lambda} = 0.6301(0.1704)$ $\hat{\beta} = 0.4463(0.1260)$	**292.0566**	**292.4776**	**302.4773**	**296.2740**	**282.0566**
Weibull Frechét	$\hat{\alpha} = 0.6942(0.363)$ $\hat{\beta} = 0.6178(0.284)$ $\hat{a} = 0.0947(0.456)$ $\hat{b} = 3.5178(2.942)$	294.6000	295.0211	305.0207	298.8174	286.6000
Exponentiated Frechect	$\hat{\alpha} = 69.1489(57.349)$ $\hat{\beta} = 0.5019(0.0800)$ $\hat{\lambda} = 145.3275(122.824)$	295.7000	295.8237	300.9103	297.8087	291.7000
Kumaraswamy Frechét	$\hat{\alpha} = 2.0556(0.0710)$ $\hat{\beta} = 0.4654(0.0070)$ $\hat{a} = 6.2815(0.0630)$ $\hat{b} = 224.1800(0.1640)$	297.1000	297.5211	307.5207	301.3174	289.1000
Beta Frechét	$\hat{\alpha} = 1.6097(2.4980)$ $\hat{\beta} = 0.4046(0.1080)$ $\hat{a} = 22.0143(21.432)$ $\hat{b} = 29.7617(17.4790)$	311.1000	311.5211	321.5207	315.3174	303.1000
Gamma Extended Frechét	$\hat{\alpha} = 1.3692(1.3692)$ $\hat{\beta} = 0.4776(0.1330)$ $\hat{a} = 27.6452(14.1360)$ $\hat{b} = 17.4581(14.8180)$	312.0000	312.4211	322.4207	316.2174	304.0000
Alpha Power Inverted Weibull	$\hat{\alpha} = 109.8227(75.5562)$ $\hat{\beta} = 1.1138(0.2018)$ $\hat{\lambda} = 2.2803(0.1420)$	328.4842	328.7342	336.2997	331.6473	322.4842
Transmuted Frechét	$\hat{\alpha} = 1.9315(0.0971)$ $\hat{\beta} = 1.7435(0.0760)$ $\hat{\lambda} = 0.0819(0.1980)$	350.5000	350.7500	358.3155	353.6631	344.5
Frechét	$\hat{\alpha} = 1.8705(0.1120)$ $\hat{\lambda} = 1.7766(0.113)$	348.3000	348.4237	353.5103	350.4087	344.3000

Figures 3 and 4 show the empirical pdf and cdf for the breaking stress of carbon for the TWFr model.

5.2. Glass fibres data

The second data consist of 1.5 cm strengths of glass fibres obtained at the UK National Physical Laboratory. The data were used to compare the performance of the *TWFr* distribution as used in [25], [11], [3], [15], [24], [21], [5], [6], [4], [7], [22], [8], [9], [10], [20] and [28]. The observations are as follows:

1.53, 1.54, 1.55, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.61, 1.58, 1.59, 1.60, 1.61,0.55, 0.74,1.50, 1.50, 1.55, 1.52, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67,.

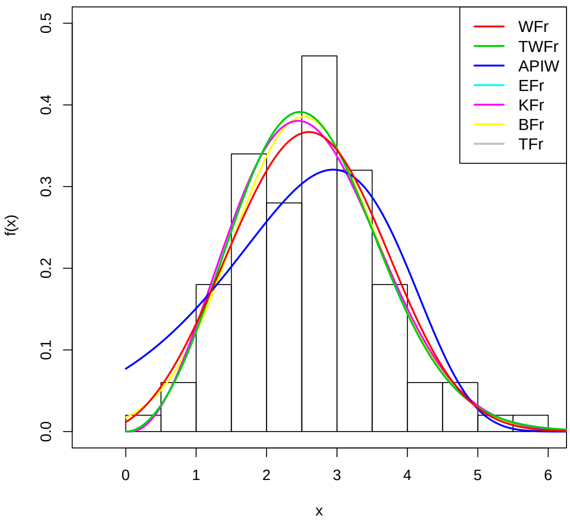


Figure 3 *The empirical cdfs of the TWFr density for the breaking stress of carbon*

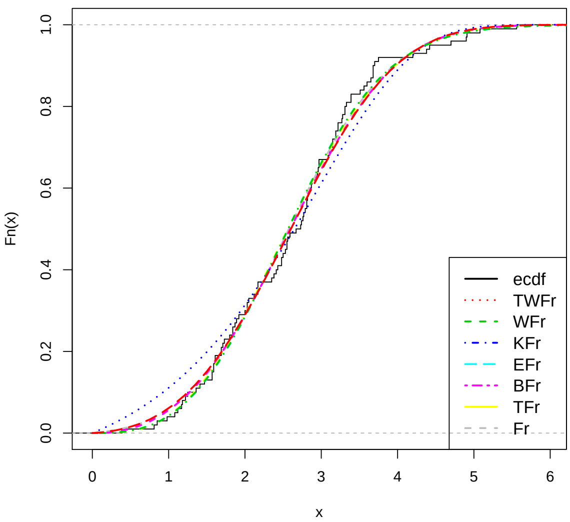


Figure 4 *The empirical pdfs of the TWFr density for the breaking stress of carbon*

The descriptive statistics of the glass fibers dataset are showed in Table 4. Table 5 shows the measure of comparison for the various distribution under consideration.

Table 4 *Descriptive statistics for the glass fibres dataset to 2 decimal points*

Mean	Median	Mode	St.D	IQR	Variance	Skewness	Kurtosis	$25^{th}P.$	$75^{th}P.$
1.51	1.59	1.61	0.32	0.31	0.11	-0.81	0.80	1.38	1.69

Figures 5 and 6 show the empirical pdf and cdf for the glass fiber data for the TWFr model.

Table 5 *Performance rating of the TWFr distribution with glass fibres dataset*

Model	Parameter MLEs(Std. Errors)	AIC	CAIC	BIC	HQIC	-2ℓ
Transmuted Weibull Frechét	$\hat{\tau} = 0.5739(0.1877)$ $\hat{\nu} = 3.9560(2.2162)$ $\hat{\lambda} = 0.0300(0.0135)$ $\hat{\alpha} = 5.8185(4.2645)$ $\hat{\beta} = 1.2345(0.1734)$	**35.7214**	**36.7740**	**46.4370**	**39.9359**	**25.7214**
Weibull Frechét	$\hat{\alpha} = 0.3865(0.7990)$ $\hat{\beta} = 0.2436(0.2850)$ $\hat{a} = 1.4762(4.782)$ $\hat{b} = 16.8561(20.4850)$	39.0000	39.6896	47.5725	42.3716	31.0000
Beta Frechét	$\hat{\alpha} = 2.0518(0.9886)$ $\hat{\beta} = 0.6466(0.1630)$ $\hat{a} = 15.0756(12.057)$ $\hat{b} = 36.9397(22.649)$	68.6261	69.3157	77.1986	71.9977	69.300
Gamma Extended Frechét	$\hat{\alpha} = 1.6625(0.9520)$ $\hat{\beta} = 0.7421(0.197)$ $\hat{a} = 32.1120(17.3970)$ $\hat{b} = 13.2688(9.967)$	69.6237	70.3016	78.1098	72.9007	61.4503
Alpha Power Inverted Weibull	$\hat{\alpha} = 61.10(48.14)$ $\hat{\beta} = 0.78(0.16)$ $\hat{\lambda} = 3.80(0.30)$	82.5800	82.9900	89.0100	85.1100	76.5848
Frechét	$\hat{\alpha} = 1.2640(0.0589)$ $\hat{\lambda} = 2.8879(0.2340)$	97.7105	97.9045	102.0078	99.3560	93.6980
Transmuted Frechét	$\hat{\alpha} = 1.3068(0.034)$ $\hat{\beta} = 2.7898(0.1648)$ $\hat{\lambda} = 0.1298(0.2080)$	100.1009	100.5078	106.4897	102.5908	94.0893
Transmuted Rayleigh	$\hat{\alpha} = 1.0895(1.1e-08)$ $\hat{\beta} = 1.0e-10(1.7e-12)$	103.5818	103.7820	107.8680	105.2676	99.5818
Alpha Power Inverted Exponential	$\hat{\alpha} = 83.4497(79.2814)$ $\hat{\lambda} = 0.3137(0.0774)$	196.3253	196.5253	200.6116	198.0111	191.4580

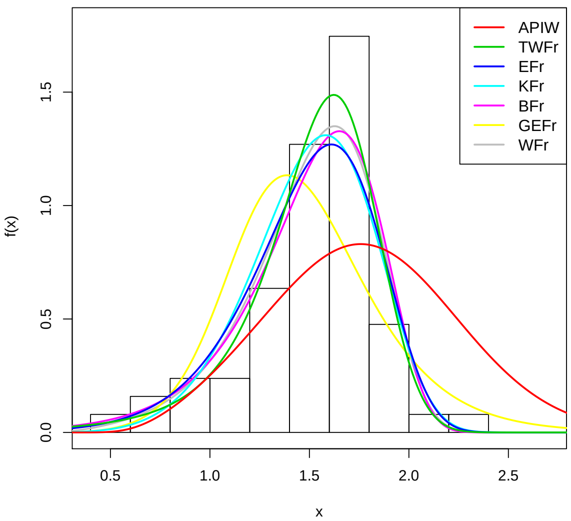


Figure 5 *The empirical cdfs of the TWFr density for the glass fiber data*

5.3. Discussion

The performance of a model is determined by the value that corresponds to the lowest Akaike Information Criteria (AIC) or the highest Log-likelihood value is regarded as the *best* model. In

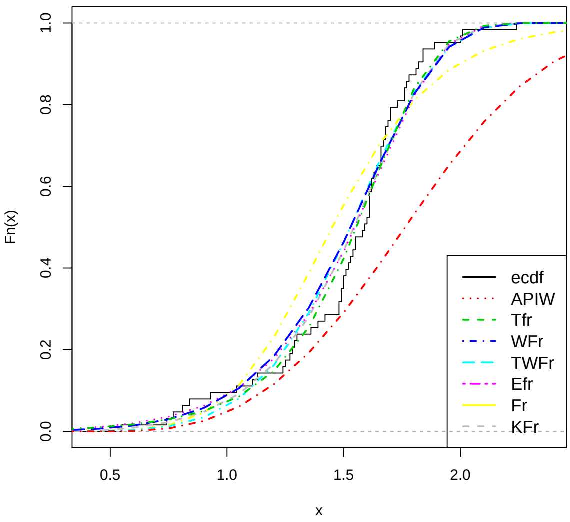


Figure 6 *The empirical pdfs of the TWFr density for the glass fiber data*

the two real life cases considered, the *TWFr* distribution has the lowest AIC values.

6. Conclusion

The Transmuted Weibull Frechet distribution has been successfully derived. Its expressions for the basic statistical properties which include the order statistics distribution, cumulative hazard function, reversed hazard function, quantile, median, hazard function, odds function have been successfully established. The shape of the distribution could be increasing (depending on the value of the parameters). An application of real life data shows that the TWFr distribution is a better competitor for some other families of distributions.

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AN IMPROVED DIFFERENCE CUM – EXPONENTIAL RATIO TYPE ESTIMATOR IN RANKED SET SAMPLING

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Abstract

Ranked set sampling is an approach to data collection originally combines simple random sampling with the field investigator's professional knowledge and judgment to pick places to collect samples. Alternatively, field screening measurements can replace professional judgment when appropriate and analysis that continues to stimulate substantial methodological research. The use of ranked set sampling increases the chance that the collected samples will yield representative measurements. This results in better estimates of the mean as well as improved performance of many statistical procedures. Moreover, ranked set sampling can be more cost-efficient than simple random sampling because fewer samples need to be collected and measured. The use of professional judgment in the process of selecting sampling locations is a powerful incentive to use ranked set sampling. This paper is devoted to the study, we introduce an approach to the mean estimators in ranked set sampling. The amount of information carried by the auxiliary variable is measured with the on populations and samples and to use this information in the estimator, the basic ratio and the generalized exponential ratio estimators are as an improved form of a difference cum exponential ratio type estimator under the ranked set sampling in order to estimate the population mean $\overline{Y}$ *of study variate Y using single auxiliary variable X. The expressions for the mean squared error of propose estimator under ranked set sampling is derived and theoretical comparisons are made with competing estimators. We show that the proposed estimator has a lower mean square error than the existing estimators. In addition, these theoretical results are supported with the aid of some real data sets using R studio. Therefore, Under RSS architecture, a better difference cum exponential ratio type estimator has been suggested. The estimator's mathematical form has been developed, and its efficiency requirements have been developed in relation to various already-existing estimators from the literature. By imputing various values for the constants used in the creation of our proposed estimator, we also provide several specific situations of our estimator.*

Keywords: Ranked Set Sampling; Exponential Ratio Type Estimator; Ratio Estimator, Mean Square Error (MSE), Efficiency, R studio.

1. Introduction

It is well known that the information of the auxiliary variable is commonly used in order to increase efficiency and precision in sample surveys. It has also a role in the related methods of estimation, such as ratio, product, and regression. If the correlation between the study variable (Y) and the auxiliary variable (X) is highly positive, the ratio method of estimation is used. If not, the product method of estimation is employed effectively provided that this correlation is highly negative. In recent years, there have been many articles on estimators for the population mean in the Sampling Theory Literature, such as unbiased estimators in general form for estimating the finite population mean in stratified random sampling [1], a generalized ratio estimator is proposed by using some robust measures with single auxiliary variable [2 and 3], an efficient families of ratio-type estimators to estimate finite population mean using known correlation coefficient between study variable and auxiliary variable by [5 aand 6], Estimation of rare and clustered population mean using stratified adaptive cluster sampling and using auxiliary character in stratified random sampling [7 and 8]. The estimation of population mean using auxiliary attribute under ranked set sampling (RSS) [9, 10 and 11]. The problem of exponential estimator for estimating the population mean considered under RSS using attribute, two phase sampling by [12, 13, 14, and 15].

In addition to the Simple Random Sampling (SRS) method, RSS, which may be considered as a controlled random sampling design, was first introduced to estimate the pasture yield by [16]. The RSS procedure involves randomly drawing n sets of n units each from the population for which the mean is to be estimated. It is assumed that the units in each set can be ranked visually. From the first set of n units, the lowest unit ranked is measured. From the second set of n units, the second lowest unit ranked is measured. This process continues until the n^{th} ranked unit is measured. The gain in efficiency by a computation involving five distributions illustrated by [16]. As a simple introduction to the concept of RSS, when X is a random variable with a density function $F(x)$ and $(x_1,x_2,...,x_n)$ are the unobserved values from n units, we may then rank them by visual inspection or based on a concomitant variable. RSS involves selecting one unit among every ranked set consisting of m units for quantification.

The RSS method can be briefly described step by step as follows:

Step 1: Randomly select m^2 units from the target population.

Step 2: Allocate the m^2 selected units as randomly as possible into m sets, each of size m.

Step 3: Without knowing any values of the variable of interest, rank the units within each set with respect to variable of interest. This may be based on personal professional judgment or done with concomitant variable correlated with the variable of interest.

Step 4: Choose a sample for actual quantification by including the smallest ranked unit in the first set, the second smallest ranked unit in the second set and this process continues in this way until the largest ranked unit is selected from the last set.

Step 5: Repeat Steps 1 through 4 for n cycles to obtain a sample of size mn for actual quantification. [17]

When it is ranked on the auxiliary variable, let $y_{(i)}, x_{(i)}$ denote an i^{th} judgment ordering in the i^{th} set for the study variable and the i^{th} order statistic in the i^{th} set for the auxiliary variable, respectively.

In the remaining part of this article, the estimators for the population mean under RSS are mentioned in Section 2, the adapted estimator from the SRS to RSS is given in Section 3, theoretical and numerical comparisons of the adapted estimator are performed with the existing adapted estimators in literature in Sections 4 and 5, respectively.

2. Estimators in literature

The estimator of the population ratio using the RSS as defined by [19].

$$t_{Rss} = \frac{\bar{y}_{[n]}}{\bar{x}_{[n]}} \tag{2.1}$$

Where $\bar{y}_{[n]} = \frac{1}{n}\sum_{i=1}^{n} y(i)$ and $\bar{x}_{[n]} = \frac{1}{n}\sum_{i=1}^{n} x(i)$. Note that the estimator in (2.1) can also be used for the population total and mean. Then, the estimator for the population mean can be written as follows:

$$\bar{y}_{rRss} = \frac{\bar{y}_{[n]}}{\bar{x}_{[n]}}\bar{X} \tag{2.2}$$

Where it is assumed that the population mean $\bar{X}$ of the auxiliary variable x is known and the MSE equation of the estimator in (2.2) can be given by

$$MSE(\bar{y}_{rRss}) \cong \frac{1}{mr}\left(S_y^2 - 2RS_{yx} + R^2S_x^2\right) - \frac{1}{m^2r}\left(\sum_{i=1}^{m}\tau_{y(i)}^2 - 2R\sum_{i=1}^{m}\tau_{yx(i)} + R^2\sum_{i=1}^{m}\tau_{x(i)}^2\right) \tag{2.3}$$

Where, $R = \frac{\bar{Y}}{\bar{X}}$, S_x^2 is the population variance of the auxiliary variable, S_y^2 is the population variance of the study variable, S_{yx} is the population covariance between the auxiliary and study variables, $\tau_{x(i)} = (\mu_{x(i)} - \bar{X})$, $\tau_{y(i)} = (\mu_{y(i)} - \bar{Y})$, and $\tau_{yx(i)} = (\mu_{y(i)} - \bar{Y})(\mu_{x(i)} - \bar{X})$. Here, $\bar{Y}$ is the population mean of the study variable. Note that the values of $\mu_{x(i)}$ and $\mu_{y(i)}$ depend on the order statistics from some specific distributions and these values can be found in [19]. We would like to remind that the values of $\mu_{x(i)}$ and $\mu_{y(i)}$ can be taken to be same in the absence of judgment error if the variables have the same distribution (see the appendix of [20]

The following estimator by adapting [21] to the RSS proposed by [22]:

$$\bar{y}_{kRss} = \frac{k\bar{y}_{[n]}}{\bar{x}_{[n]}}\bar{X} \tag{2.4}$$

Where k is a constant.

The MSE of the estimator in (2.4) is given by

$$MSE(\bar{y}_{rRss}) \cong \frac{1}{mr}\left(k^{*2}S_y^2 - 2Rk^*S_{yx} + R^2S_x^2\right) + \bar{Y}^2(k^*-1)^2 - \frac{1}{m^2r}\left(\sum_{i=1}^{m}k^{*2}\tau_{y(i)}^2 - 2Rk^*\sum_{i=1}^{m}\tau_{yx(i)} + R^2\sum_{i=1}^{m}\tau_{x(i)}^2\right) \tag{2.5}$$

where $k^* = \frac{1+\gamma\rho C_xC_y - W_{yx(i)}}{1+\gamma C_y^2 - W_{y[i]}^2}$, Here, $W_{yx(i)} = \frac{1}{m^2r\bar{X}\bar{Y}}\sum_{i=1}^{m}\tau_{yx(i)}$ and $W_{y[i]}^2 = \frac{1}{m^2r\bar{Y}^2}\sum_{i=1}^{m}\tau_{y(i)}^2$, $\gamma = \frac{1}{mr}$, C_x and C_y are the population coefficients of variation of the auxiliary and study variables, respectively, ρ is the population correlation between the auxiliary and the study variables.

3. Proposed estimator

An improved difference cum-exponential ratio type is defined for estimating $\bar{Y}$ as following [18 and 21]

$$\bar{Y}_{RK} = \{t_1\bar{y}_{[n]} + t_2(\bar{X} - \bar{x}_{[n]})\}\left\{\exp\left[\frac{\bar{X} - \bar{x}_{[n]}}{\bar{X} + \bar{x}_{[n]}}\right]\right\} \tag{3.1}$$

To obtain the MSE of $\bar{Y}_{RK}$, write

$\bar{y}_{(n)} = \bar{Y}(1 + \epsilon_0)$, and $\bar{x}_{(n)} = \bar{X}(1 + \epsilon_1)$,

Such that $E(\epsilon_0) = E(\epsilon_1) = 0$,

and $E(\epsilon_0)^2 = V\left(\frac{\bar{y}_{(n)}}{\bar{Y}^2}\right) = \frac{1}{mr}\frac{1}{\bar{Y}^2}\left[S_y^2 - \frac{1}{m}\sum t_{y(i)}^2\right] = \left[\theta C_y^2 - w_{y(i)}^2\right]$,

$E(\epsilon_1)^2 = V\left(\frac{\bar{x}_{(n)}}{\bar{X}^2}\right) = \frac{1}{mr}\frac{1}{\bar{Y}^2}\left[S_x^2 - \frac{1}{m}\sum t_{y(i)}^2\right] = \left[\theta C_x^2 - w_{x(i)}^2\right]$,

$E(\epsilon_0\epsilon_1) = \frac{1}{mr}\frac{1}{\bar{Y}\bar{X}}\left[S_{yx} - \frac{1}{m}\sum t_{y(i)}^2\right] = \left[\theta C_{yx} - w_{yx(i)}\right]$,

Where $W_{x[i]}^2 = \frac{1}{m^2r\bar{X}^2}\sum_{i=1}^{m}\tau_{x(i)}^2$

Expressing (1.1) in terms of e's,

$$\bar{Y}_{RK} = \{t_1\bar{Y}(1 + \epsilon_0) + t_2(\bar{X} - \bar{X}(1 + \epsilon_1))\}\left\{\exp\left[\frac{\bar{X} - \bar{X}(1 + \epsilon_1)}{\bar{X} + \bar{X}(1 + \epsilon_1)}\right]\right\}$$

$$\bar{Y}_{RK} = \{t_1\bar{Y} + t_1\bar{Y}\,\epsilon_0 - t_2\bar{X}\,\epsilon_1\}\left\{\exp\left(-\frac{\epsilon_1}{2}\right)\left[1 + \frac{\epsilon_1}{2}\right]^{-1}\right\} \tag{3.2}$$

Expanding the right hand side of (1.2) and retaining terms up to the second power of e's,

$$\bar{Y}_{RK} = \{t_1\bar{Y} + t_1\bar{Y} \in_0 - t_2\bar{X} \in_1\}\left\{\exp\left(-\frac{\in_1}{2}\right)\left[1+\frac{\in_1}{2}+\frac{{\in_1}^2}{4}\right]\right\}$$

$$\bar{Y}_{RK} = \{t_1\bar{Y} + t_1\bar{Y} \in_0 - t_2\bar{X} \in_1\}\left\{1-\frac{\in_1}{2}+\frac{{\in_1}^2}{4}+\frac{{\in_1}^2}{8}\right\} \quad (3.3)$$

From (3.3),

$$\bar{Y}_{RK} - \bar{Y} = \bar{Y}\left\{(t_1-1) + t_1 \in_0 - \frac{t_1 \in_1}{2} - t_2 R \in_1 - \frac{t_1 \in_0 \in_1}{2} + \frac{t_2 R {\in_1}^2}{2} + \frac{3t_1 {\in_1}^2}{8}\right\} \quad (3.4)$$

Squaring (3.4) and then taking expectation of both sides, the MSE of the estimator $\bar{Y}_{RK}$ is

$$MSE(\bar{Y}_{RK}) = \bar{Y}^2\{{t_1}^2\varphi_1 - t_1\varphi_2 + {t_2}^2R^2\varphi_3 - t_1t_2R\varphi_4\} \quad (3.5)$$

Where,

$\varphi_1 = \gamma\{C_y^2 + C_x^2 - 2C_{yx}\} - \{w_{y[i]}^2 + w_{x[i]}^2 - 2w_{yx[i]}\}$

$\varphi_2 = \gamma\{C_x^2\} - \{w_{x[i]}^2\}$

$\varphi_3 = \gamma\left\{\frac{3}{4}C_x^2 - C_{yx}\right\} + \left\{\frac{3}{4}w_{x[i]}^2 - w_{yx[i]}\right\}$

$\varphi_4 = \gamma\{C_x^2 - C_{yx}\} + \{w_{x[i]}^2 - w_{yx[i]}\}$

Obtain the optimum t_1 and t_2 to minimize $MSE(\bar{Y}_{RK})$. Differentiate $MSE(\bar{Y}_{RK})$ with respect to t_1 and t_2 and equating the derivatives to zero, optimum values of t_1 and t_2 is given by

$$t_{1opt} = \frac{2\varphi_2\varphi_3}{4\varphi_1\varphi_3 - {\varphi_4}^2}$$

$$t_{2opt} = \frac{\varphi_2\varphi_4}{4\varphi_1\varphi_3 - {\varphi_4}^2}$$

Substituting the value of t_{1opt} and t_{2opt} in (3.5), we get the minimum value of $MSE(\bar{Y}_{RK})$ as

$$MSE_{min}(\bar{Y}_{RK}) = \bar{Y}^2\{{t_1}^2\varphi_1 - t_1\varphi_2 - {t_2}^2R^2\varphi_3\} \quad (3.6)$$

4. Efficiency

In this section, the performances of the proposed estimator have been demonstrated over the traditional ratio estimator in the RSS and the estimator of [23] respectively, as follows:

$$MSE(\bar{y}_{rRss}) - MSE_{min}(\bar{Y}_{RK}) > 0$$
$$\{(1-{t_1}^2)\varphi_1 + t_1\varphi_2 + {t_2}^2R^2\varphi_3\} > 0 \quad (4.1)$$

$$MSE(\bar{y}_{kRss}) - MSE_{min}(\bar{Y}_{RK}) > 0$$
$$\{(k^*-1)^2 + (1-{t_1}^2)\varphi_1 + t_1\varphi_2 + {t_2}^2R^2\varphi_3\} > 0 \quad (4.2)$$

Table 1: *Some members of exponential ratio type estimator in ranked set sampling*

Estimator	**t1**	**t2**
$\bar{Y}_{RK1} = \{\bar{y}_{[n]} + (\bar{X} - \bar{x}_{[n]})\}\left\{\exp\left[\frac{\bar{X} - \bar{x}_{[n]}}{\bar{X} + \bar{x}_{[n]}}\right]\right\}$	1	1
$\bar{Y}_{RK2} = \{(\bar{X} - \bar{x}_{[n]})\}\left\{\exp\left[\frac{\bar{X} - \bar{x}_{[n]}}{\bar{X} + \bar{x}_{[n]}}\right]\right\}$	0	1
$\bar{Y}_{RK3} = \{\bar{y}_{[n]}\}\left\{\exp\left[\frac{\bar{X} - \bar{x}_{[n]}}{\bar{X} + \bar{x}_{[n]}}\right]\right\}$	1	0

5. Numerical example

To observe performances of the estimators, we use some real-life populations. The descriptions of these populations are given below:

Population I {source: [24]}

Y: Acceleration of automobiles
X: Engine horsepower of automobiles
Objective: To estimate population mean of Acceleration of automobiles.
The summary statistics are given below:

$$N = 392, n = 30, m = 10, r = 3, \mu_x = 104.4694, \mu_y = 15.5413, S_y = 2.7589, S_x = 38.4912,$$
$$C_x = 0.3684, C_y = 0.1775, C_{xy} = -0.0451, \beta_{2(x)} = 0.6541, \beta_{1(x)} = 1.079, \rho_{xy} = 0.9091$$

Population II {source: [25]}
Y: Body Mass Index (BMI) of Crohn's disease patients
X: Weight of Crohn's disease patients
Objective: To estimate population mean of Body Mass Index (BMI) of Crohn's disease patients.
The summary statistics are given below:

$$N = 117, n = 20, m = 5, r = 4, \mu_x = 69.0256, \mu_y = 26.0624, S_y = 4.9888, S_x = 14.2438,$$
$$C_x = 0.2063, C_y = 0.1914, C_{xy} = 0.0325, \beta_{2(x)} = 0.7746, \beta_{1(x)} = 0.6571, \rho_{xy} = 0.8222$$

Population III {source: [26]}
Y: Body Mass Index (BMI)
X: Thigh Circumference
Objective: To estimate population mean of Body Mass Index (BMI).
The summary statistics are given below:

$$\mathrm{N} = 36, \mathrm{n} = 8, \mathrm{m} = 4, \mathrm{r} = 2, \mu_x = 49.3806, \mu_y = 25.678, S_y = 3.8198, S_x = 3.7599,$$
$$C_x = 0.0761, C_y = 0.1488, C_{xy} = 0.0066, \beta_{2(x)} = -0.6159, \beta_{1(x)} = -0.0607, \rho_{xy} = 0.9848$$

Percent Relative Efficiencies (PREs) of our proposed estimators along with competitor estimators from literature have been presented in Table 2, 3 and 4 for different real-life populations.

Table 2: *PRE of Estimators for Population I*

	$\overline{y}_{rRSS}$	$\overline{y}_{\kappa RSS}$	$\overline{Y}_{RK1}$	$\overline{Y}_{RK2}$	$\overline{Y}_{RK3}$	$\overline{Y}_{RK}$
$\overline{y}_{rRSS}$	100					
$\overline{y}_{\kappa RSS}$	212.19	100				
$\overline{Y}_{RK1}$	245.19	231.72	100			
$\overline{Y}_{RK2}$	241.45	210.47	98.81	100		
$\overline{Y}_{RK3}$	238.97	189.37	90.84	93.08	100	
$\overline{Y}_{RK}$	***361.74***	***275.18***	***249.18***	***245.15***	***213.49***	***100***

Table 2, revealed the percent relative efficiencies (PRE) of estimators for population I. It is observed that the proposed difference cum exponential ratio type estimator in ranked set sampling $\overline{Y}_{RK}$ proved to be the best estimator in the sense of having highest percent relative efficiency than usual unbiased estimators$\overline{Y}_{rRSS}$, $\overline{Y}_{kRSS}$ for the population I. The generalized form of proposed difference cum exponential ratio type estimator $\overline{Y}_{RK}$ is 361.74% more efficient than the existing estimator $\overline{y}_{rRSS}$ and 275.18% more efficient than $\overline{Y}_{kRSS}$.

Moreover, the special cases of our proposed generalized estimator $\overline{Y}_{RK1}$, $\overline{Y}_{RK2}$ and $\overline{Y}_{RK3}$ are also proved to be more efficient than existing estimators. These results suggest using proposed difference cum exponential ratio type estimator to estimate population mean of Acceleration of automobiles.

Table 3: *PRE of Estimators for Population II*

	$\overline{y}_{rRSS}$	$\overline{y}_{\kappa RSS}$	$\bar{Y}_{RK1}$	$\bar{Y}_{RK2}$	$\bar{Y}_{RK3}$	$\bar{Y}_{RK}$
$\overline{y}_{rRSS}$	100					
$\overline{y}_{\kappa RSS}$	204.74	100				
$\bar{Y}_{RK1}$	238.48	238.29	100			
$\bar{Y}_{RK2}$	237.37	204.28	93.92	100		
$\bar{Y}_{RK3}$	221.49	174.28	89.32	82.74	100	
$\bar{Y}_{RK}$	***352.86***	***252.48***	***248.82***	***229.23***	***190.48***	***100***

Table 3, showed the percent relative efficiencies (PRE) of estimators for population II. It is observed that the proposed difference cum exponential ratio type estimator in ranked set sampling $\bar{Y}_{RK}$ also proved to be the best estimator in the sense of having highest percent relative efficiency than usual unbiased estimators $\bar{Y}_{rRSS}$, $\bar{Y}_{kRSS}$ for the population II. The generalized form of proposed difference cum exponential ratio type estimator $\bar{Y}_{RK}$ is 352.86% more efficient than the existing estimator $\overline{y}_{rRSS}$ and 252.48% more efficient than $\bar{Y}_{kRSS}$. Moreover, the special cases of our proposed generalized estimator $\bar{Y}_{RK1}$, $\bar{Y}_{RK2}$ and $\bar{Y}_{RK3}$ are also proved to be more efficient than existing estimators. These results suggest using proposed difference cum exponential ratio type estimator to estimate population mean of Body Mass Index (BMI) of Crohn's disease patients.

Table 4: *PRE of Estimators for Population III*

	$\overline{y}_{rRSS}$	$\overline{y}_{\kappa RSS}$	$\bar{Y}_{RK1}$	$\bar{Y}_{RK2}$	$\bar{Y}_{RK3}$	$\bar{Y}_{RK}$
$\overline{y}_{rRSS}$	100					
$\overline{y}_{\kappa RSS}$	238.48	100				
$\bar{Y}_{RK1}$	275.28	264.82	100			
$\bar{Y}_{RK2}$	263.82	249.27	98.47	100		
$\bar{Y}_{RK3}$	239.83	237.42	97.38	98.37	100	
$\bar{Y}_{RK}$	***384.27***	***283.38***	***259.37***	***278.38***	***239.57***	***100***

Table 4, showed the percent relative efficiencies (PRE) of estimators for population III. It is observed that the proposed difference cum exponential ratio type estimator in ranked set sampling $\bar{Y}_{RK}$ also proved to be the best estimator in the sense of having highest percent relative efficiency than usual unbiased estimators$\bar{Y}_{rRSS}$, $\bar{Y}_{kRSS}$ for the population III. The generalized form of proposed difference cum exponential ratio type estimator $\bar{Y}_{RK}$ is 384.27% more efficient than the existing estimator $\overline{y}_{rRSS}$ and 283.38% more efficient than $\bar{Y}_{kRSS}$. Moreover, the special cases of our proposed generalized estimator $\bar{Y}_{RK1}$, $\bar{Y}_{RK2}$ and $\bar{Y}_{RK3}$ are also proved to be more efficient than existing estimators. These results suggest using proposed difference cum exponential ratio type estimator to estimate population mean of Body Mass Index (BMI).

6. Conclusion

In this article, an improved difference cum exponential ratio type estimator has been proposed under RSS design. The mathematical form of the estimator has been derived and its condition of efficiencies has been formulated with respect to some existing estimators from literature. Further, we present some special cases of our proposed estimator by imputing different values of constants utilized in the formation of proposed estimator. For comparing the efficiencies of proposed estimator with some existing estimators, we utilized some real-life populations for estimating population mean of Acceleration of automobiles, population mean of Body Mass Index (BMI) of Crohn's disease patients and population mean of Body Mass Index (BMI). The result from these populations shows that our proposed estimator and its special cases perform efficiently as compare to existing estimators. We also observe that efficiency of proposed estimator and its special cases increases when the correlation between study and auxiliary variable increases. Therefore, it is recommended to use proposed estimator for estimating population mean when correlation between study and auxiliary variable is strong positive.

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Bayesian Analysis of Type II Generalized Topp–Leone Accelerated Failure Time Models Using R and Stan

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Abstract

With a Bayesian framework, the current study intends to fit the Type II generalized Topp–Leone-G (TIIGTL-G) model as an accelerated failure time (AFT) model to censored survival data. In this paper, we have obtained and analysed three AFT models using Type II Generalized Topp-Leone (TIIGTL) distribution as generator and considering Weibull, Exponential, and Log-Logistic as a baseline distribution. The fitting of these models to the censored survival data is done with the help of R and STAN. A comparison of these two models is conducted, and the best model is chosen using the Bayesian model evaluation criteria LOOIC and WAIC.

Keywords: Type II generalized Topp–Leone G Model, Bayesian Survival Modelling, Censored data, Leave one out information criteria, STAN

1. Introduction

[1] proposed the Type II generalised Topp-Leone-G (TIIGTL-G) family of distributions, which uses the Topp-Leone random variable as a generator, and investigated its mathematical properties and how they were used to fit lifetime data. Research analysts are evaluating lifetime data and issues with modelling the survival process using the extended form of the standard distribution in the survival study. It has been shown that the Bayesian paradigm is instrumental in analyzing survival models in many real-world contexts. [2] set up and analysed Topp-Leone exponential distribution, Topp-Leone exponentiated exponential distribution and Topp-Leone exponentiated extension distribution using Bayesian approach. Also, [3] fitted the Weibull, Topp-Leone-Weibull (TL-W), and Generalized Topp-Leone-Weibull (GTL-W) survival models as accelerated failure time models using Bayesian approach and have shown that TL-W AFT model is the most suitable model for fitting a censored data (tumor data). Recently, [4] analysed and compared three accelerated failure time models—Weibull, log-normal, and log-logistic under Bayesian framework.

In this article, We have fitted a censored survival data using TIIGTL-G model as an accelerated failure time (AFT) model. The aforementioned models were fitted using the full Bayesian inference-supporting probabilistic programming language STAN [5] in R. The programming language Stan is used to define statistical models, and in Bayesian analysis, it is most frequently employed as an Hamiltonian Monte Carlo (HMC) sampler [6, 7]. STAN primarily uses the No-U-Turn sampler (NUTS) [8] to obtain posterior simulation for Bayesian analysis. Thus, we have also evaluated and selected the best model using Leave-One-Out information criteria (LOOIC) and Watanabe-Akaike information criteria or widely applicable information criteria (WAIC) for the diet data. Using a fitted Bayesian model and the log-likelihood assessed at the posterior simulations of the parameter values, LOO and WAIC are two methods for evaluating the precision of pointwise out-of-sample predictions [9]. Thus, in this article, we have conducted a Bayesian analysis of TIIGTL-Weibull AFT, TIIGTL-Exponential AFT, and TIIGTL-Log-logistic AFT models by presenting summaries of the posterior densities in both numerical and graphical form by using R and Stan.

2. Type II Generalized Topp–Leone–G (TIIGTL-G) family

Let a continuous random variable T with baseline cdf and pdf $G(t,\phi)$ and $g(t,\phi)$ respectively with parameter vector ϕ. The cumulative distribution function (cdf), probability density function (pdf), survival function, and hazard function of the TIIGTL-G family are respectively given by

$$F_T(t,c,d,\phi) = 1 - (1 - G(t,\phi)^{2d})^c \tag{1}$$

$$f_T(t,c,d,\phi) = 2cdg(t,\phi)[G(t,\phi)]^{2d-1}(1 - G(t,\phi)^{2d})^c \tag{2}$$

$$S_T(t,c,d,\phi) = 1 - F_T(t,c,d,\phi) = (1 - G(t,\phi)^{2d})^c \tag{3}$$

$$h_T(t,c,d,\phi) = f_T(t,c,d,\phi)/S_T(t,c,d,\phi) \tag{4}$$

Thus, the random variable T with the pdf given in Equation 2 will be denoted as $T \sim TIIGTL - G(c,d,\phi)$ where c,d are two shape parameters and ϕ is parameter vector of baseline distribution. Also, random number generation from the survival model is accomplished by equating $F(t)$ and v, where V has $Uniform(0,1)$ distribution. Thus,

$$F(t) = v \tag{5}$$

$$1 - (1 - G(t)^{2d})^c = v \tag{6}$$

then we have,

$$G(t) = [1 - (1 - v)^{1/c}]^{1/2d} \tag{7}$$

For any baseline cdf G(t), this is the TIIGTL-G model's general expression for producing random numbers.

3. Accelerated Failure Time (AFT) models

It has been noted in statistical literature that many models have been created for assessing survival data or life time data. The Cox Proportional Hazard (PH) model is the most well-liked of them all. When examining survival data, the Accelerated Failure Time (AFT) model can be thought of as a good substitute for the Cox PH model [10]. AFT models are parametric models that take into account the linear regression of the logarithm of the survival time T on a variety of covariates. They are used to investigate the impact of a covariate on how quickly or slowly the survival process advances [3]. According to the AFT model, covariates and failure time have a direct relationship [11]. If number of covariates $x_1, x_2, ..., x_p$ have an impact on survival time T then we can write the AFT model as:

$$log(T) = \beta_0 + \sum_{k=1}^{L} \beta_k x_k + \sigma e = \mathbf{x}'\beta + \sigma\epsilon \tag{8}$$

where $\beta_k, k = 1, 2, ..., L$ are the coefficients of regression, σ is a scale parameter such that $\sigma > 0$ and ϵ is the random error with a specified probability distribution.

3.1. Weibull AFT model

Let survival time T follows Weibull distribution with scale and shape parameter λ and α respectively. Then the probability density function, cumulative distribution function, and survival function of Weibull distribution are provided as follows [12]:

$$g(t|\alpha,\lambda) = (\alpha/\lambda)(t/\lambda)^{\alpha-1}exp(-(t/\lambda)^{\alpha}) \tag{9}$$

$$G(t|\alpha,\lambda) = 1 - exp(-(t/\lambda)^{\alpha}) \tag{10}$$

$$S(t|\alpha,\lambda) = exp(-(t/\lambda)^{\alpha}) \tag{11}$$

hence, we can write $T \sim Weibull(\alpha,\lambda)$. Now, Let a random variable ϵ has a standard extreme value distribution with density function $g(e) = exp(e - exp(e))$ and survival function $S(e) = exp(-exp(e))$ substituting $e = (logt - \mathbf{x}'\beta)/\sigma$ from the Equation 8 in the extreme value distribution and then the Weibull AFT model is obtained and we can write it as $T \sim Weibull(1/\sigma, exp(\mathbf{x}'\beta))$.

3.1.1 TIIGTL-W AFT model

The Type two generalized Topp-Leone-Weibull (TIIGTL-W) AFT model is obtained by considering weibull AFT model as the baseline model G and substituting it in the TIIGTL-G model. Thus, the cdf, pdf, survival function, and hazard function of the TIIGTL-W AFT model are respectively given by

$$F(t|\Omega,\mathbf{x}) = 1 - (1 - G(t)^{2d})^{c} \tag{12}$$

$$f(t|\Omega,\mathbf{x}) = 2abg(t)[G(t)]^{2d-1}(1 - G(t)^{2d})^{c} \tag{13}$$

$$S(t|\Omega,\mathbf{x}) = (1 - G(t)^{2d})^{c} \tag{14}$$

$$h(t|\Omega,\mathbf{x}) = f(t|\Omega,x)/S(t|\Omega,x) \tag{15}$$

Where $t > 0$, g(t) and G(t) are the pdf and cdf of Weibull AFT model. $\Omega = (c,d,\sigma,\beta)$,c,d and α are shape parameters and λ is scale parameter. Also $\sigma = 1/\alpha, \lambda = exp(\mathbf{x}'\beta)$ from the AFT model and we have $T \sim TIIGTL - W(c,d,1/\sigma,exp(\mathbf{x}'\beta))$. Now, for random number generation from TIIGTL-W we proceed as follows, Let $V \sim Uniform(0,1)$. Then from Equation 7 we have

$$G(t) = [1 - (1 - v)^{1/c}]^{1/2d} \tag{16}$$

$$1 - exp(-(t/\lambda)^{\alpha}) = [1 - (1 - v)^{1/c}]^{1/2d} \tag{17}$$

then we get,

$$t = exp(\mathbf{x}'\beta)[-log(1 - (1 - (1 - v)^{1/c})^{1/2d})]^{\sigma} \tag{18}$$

This is the TIIGTL-W AFT model's general expression for producing random numbers, where $\lambda = exp(\mathbf{x}'\beta)$ and $\sigma = 1/\alpha$.

3.2. Exponential AFT model

Let survival time T follows Exponential distribution with inversescale or rate parameter $\lambda > 0$ Then the probability density function, cumulative distribution function, and survival function of Exponential distribution are provided as follows [12]:

$$g(t|\alpha,\lambda) = 1 - exp(-\lambda t) \tag{19}$$

$$G(t|\alpha,\lambda) = \lambda exp(-\lambda t) \tag{20}$$

$$S(t|\alpha,\lambda) = exp(-\lambda t) \tag{21}$$

hence, we can write $T \sim Exp(\lambda)$. Now, Let a random variable ϵ has a standard extreme value distribution with density function $g(e) = exp(e - exp(e))$ and survival function $S(e) = exp(-exp(e))$. Considering $\sigma = 1$ substituting $e = (logt - \mathbf{x}'\beta)$ from the Equation 8 in the extreme value distribution and then the Exponential AFT model is obtained and we can write it as $T \sim Exp(exp(-\mathbf{x}'\beta))$.

3.2.1 TIIGTL-E AFT model

The Type two generalized Topp-Leone-Exponential (TIIGTL-E) AFT model is obtained by considering exponential AFT model as the baseline model G and substituting it in the TIIGTL-G model. Thus, the cdf, pdf, survival function, and hazard function of the TIIGTL-E AFT model are respectively given by

$$F(t|\Omega, \mathbf{x}) = 1 - (1 - G(t)^{2d})^c \tag{22}$$

$$f(t|\Omega, \mathbf{x}) = 2cdg(t)[G(t)]^{2d-1}(1 - G(t)^{2d})^c \tag{23}$$

$$S(t|\Omega, \mathbf{x}) = (1 - G(t)^{2d})^c \tag{24}$$

$$h(t|\Omega, \mathbf{x}) = f(t|\Omega, x)/S(t|\Omega, x) \tag{25}$$

Where $t > 0$, g(t) and G(t) are the pdf and cdf of Exponential AFT model. $\Omega = (c, d, \beta)$,c,d are shape parameters and λ is inversescale parameter. Also $\lambda = exp(-\mathbf{x}'\beta)$ from the AFT model and we have $T \sim TIIGTL - E(c, d, exp(-\mathbf{x}'\beta))$. Now, for random number generation from TIIGTL-E we proceed as follows, Let $V \sim Uniform(0, 1)$. Then from Equation 7 we have

$$G(t) = [1 - (1 - v)^{1/c}]^{1/2d} \tag{26}$$

$$1 - exp(-\lambda t) = [1 - (1 - v)^{1/c}]^{1/2d} \tag{27}$$

then we get,

$$t = (-exp(\mathbf{x}'\beta))log[1 - (1 - (1 - v)^{1/c})^{1/2d}] \tag{28}$$

This is the TIIGTL-E AFT model's general expression for producing random numbers, where $\lambda = exp(\mathbf{x}'\beta)$.

3.3. Log Logistic AFT model

Let survival time T follows Log Logistic distribution with scale and shape parameter λ and α respectively. Then the probability density function, cumulative distribution function, and survival function of Log Logistic distribution are provided as follows [12]:

$$g(t|\alpha, \lambda) = (\alpha/\lambda)(t/\lambda)^{\alpha-1}(1 + (t/\lambda)^{\alpha})^{-2} \tag{29}$$

$$G(t|\alpha, \lambda) = 1 - (1 + (t/\lambda)^{\alpha})^{-1} \tag{30}$$

$$S(t|\alpha, \lambda) = (1 + (t/\lambda)^{\alpha})^{-1} \tag{31}$$

hence, we can write $T \sim LL(\alpha, \lambda)$. Now, Let a random variable ϵ has a standard logistic value distribution with density function $g(e) = exp(e)(1 - exp(e))^{-2}$ and survival function $S(e) = (1 - exp(e))^{-1}$ substituting $e = (logt - \mathbf{x}'\beta)/\sigma$ from the Equation 8 in the extreme value distribution and then the Log Logistic AFT model is obtained and we can write it as $T \sim LL(1/\sigma, exp(\mathbf{x}'\beta))$.

3.3.1 TIIGTL-LL AFT model

The Type two generalized Topp-Leone-log-logistic (TIIGTL-LL) AFT model is obtained by considering log-logistic AFT model as the baseline model G and substituting it in the TIIGTL-G model. Thus, the cdf, pdf, survival function, and hazard function of the TIIGTL-W AFT model are respectively given by

$$F(t|\Omega, \mathbf{x}) = 1 - (1 - G(t)^{2d})^c \tag{32}$$

$$f(t|\Omega, \mathbf{x}) = 2cdg(t)[G(t)]^{2d-1}(1 - G(t)^{2d})^c \tag{33}$$

$$S(t|\Omega, \mathbf{x}) = (1 - G(t)^{2d})^c \tag{34}$$

$$h(t|\Omega, \mathbf{x}) = f(t|\Omega, x)/S(t|\Omega, x) \tag{35}$$

Where $t > 0$, g(t) and G(t) are the pdf and cdf of Log-logistic AFT model. $\Omega = (c, d, \sigma, \beta)$,c,d and α are shape parameters and λ is scale parameter. Also $\sigma = 1/\alpha, \lambda = exp(\mathbf{x}'\beta)$ from the AFT model and we have $T \sim TIIGTL - LL(c, d, 1/\sigma, exp(\mathbf{x}'\beta))$. Now, for random number generation from TIIGTL-LL we proceed as follows, Let $V \sim Uniform(0, 1)$. Then from Equation 7 we have

$$G(t) = [1 - (1 - v)^{1/c}]^{1/2d} \tag{36}$$

$$1 - (1 + (t/\lambda)^{\alpha})^{-1} = [1 - (1 - v)^{1/c}]^{1/2d} \tag{37}$$

then we get,

$$t = exp(\mathbf{x}'\beta)[(1 - (1 - (1 - v)^{1/c})^{1/2d})^{-1} - 1]^{\sigma} \tag{38}$$

This is the TIIGTL-LL AFT model's general expression for producing random numbers, where $\lambda = exp(\mathbf{x}'\beta)$ and $\sigma = 1/\alpha$.

4. Diet Data

90 homogenous rats of the same species, age, and environmental conditions were separated into three groups and fed with low, saturated, and unsaturated fat diets, respectively, as reported by [13]. Each rat's foot pad received an identical dosage of tumour cells. 200 days of observation of the rats revealed the growth of a tumour as the event. Several of the rats got tumours, but several did not. Survival time is defined as the amount of time without a tumour or the amount of time before one develops one. The survival times of the tumor-free animals are marked with stars and treated as censored. As a result, the data is correctly suppressed, as shown in the Table 1. The primary objective of this study is to compare the three diets' tumor-preventing capacities in rats.

Table 1: *Tumor-free duration (days) of 90 rats on three different diets (∗ indicates censored)*

Low Fat (30 rats)	**Saturated Fat** (30 rats)	**Unsaturated Fat** (30 rats)
140 87 200∗	124 96 81	112 63 66
177 56 200∗	58 142 133	68 63 94
50 66 200∗	56 86 165	84 77 101
65 73 200∗	68 75 170∗	109 91 105
86 119 200∗	79 117 200∗	153 91 108
153 140∗ 200∗	89 98 200∗	143 66 112
181 200∗ 200∗	107 105 200∗	60 70 115
191 200∗ 200∗	86 126 200∗	70 77 126
77 200∗ 200∗	142 43 200∗	98 63 161
84 200∗ 200∗	110 46 200∗	164 66 178

4.1. Data Structure for computation in R

We have produced the data in a listed form necessary for fitting Bayesian models to the data using *stan* function.

```
y = survival times (Tumor-free time in days)
y <- c(140,177,50,65,86,153,181,191,77,84,87,56,66,73,119,140,200,200,
200,200,200,200,200,200,200,200,200,200,200,200,124,58,56,68,79,89,107,
86,142,110,96,142,86,75,117,98,105,126,43,46,81,133,165,170,200,200,200,
200,200,200,112,68,84,109,153,143,60,70,98,164,63,63,77,91,91,66,70,77,
63,66,66,94,101,105,108, 112,115,126,161,178 )
event=1 if tumor is developed or zero if it is censored
event <- c(rep(1,15),rep(0,15),rep(1,23),rep(0,7),rep(1,30))
Low-Fat is considered as reference category
x1 = 1 if saturated fat is applied and 0 otherwise
x1 <- c(rep(0,30),rep(1,30),rep(0,30))
```

```
x2 = 1 if unsaturated fat is applied and 0 otherwise
x2 <- c(rep(0,30),rep(0,30),rep(1,30))
x = cbind(1,x1,x2)
N = nrow(x)
M = ncol(x)
datt = list(y=y, event=event,x=x,N=N,M=M)
```

5. Bayesian Analysis

In Bayesian analysis, following Bayes Theorem, we look for the exact parameter distributions known as the posterior distribution by fusing the prior distribution of parameter with the data or likelihood. We must define a prior distribution for the model's parameters and likelihood of the data before building the Bayesian regression model.

5.1. Likelihood

Following the [14] , the joint likelihood function for right censored data is given as

$$L = \prod_{i=1}^{n} h(t_i)^{\gamma_i} S(t_i) \tag{39}$$

Also as an alternative to the likelihood, the log-likelihood can be written as

$$logL = \sum_{i=1}^{n} (\gamma_i (logh(t_i) + logS(t_i))) \tag{40}$$

here γ_i is an indicator variable such that γ = 0 if the observed value is censored and γ = 1 if the observed value is failed (recorded). In equation 39 we can sustitute the hazard function $h(t_i)$ and survival function $S(t_i)$ of TIIGTL-W AFT , TIIGTL-E AFT and TIIGTL-LL AFT models in order to get the likelihood of TIIGTL-W AFT , TIIGTL-E AFT and TIIGTL-LL AFT survival models respectively.

5.2. Prior

A prior distribution must be specified for the model's parameters in order to build a Bayesian regression model. Two prior types—the student t prior and the normal prior, are used by the researchers in the remaining sections of this work. Student t distribution is used for the priors of shape and scale parameters and Normal distribution is used as a prior for the regression coefficients. These priors are weekly informative priors and are discussed briefly by [3].

5.3. Posterior

The Bayes Theorem can be used to determine the joint posterior distribution of parameter $\Omega = (c, d, \sigma, \beta) = (c, d, \sigma, \beta_0, \beta_1, ..., \beta_p)$ given data as

$$P(\Omega|t, X) \propto L(\Omega|t, X)P(\Omega) \tag{41}$$

$$P(\Omega|t, X) \propto L(\Omega|t, X)P(c)P(d)P(\sigma)P(\beta) \tag{42}$$

Here parameters are assumed to be independent and X is the matrix of covariates. Hence we can obtain the joint posterior distribution of TIIGTL-W AFT Model, TIIGTL-W AFT Model and TIIGTL-LL AFT Model by sustituting the likelihood and priors of corresponding models in equation 42. Because it is challenging to determine the marginal distributions of the parameters and the normalised joint posterior distribution analytically, the estimates and other relevant results are obtained using the Markov chain Monte Carlo (MCMC) simulation technique.

5.4. Implementation using Stan

The package **rstan** is necessary to run STAN code in R. For the Bayesian modeling, there are several blocks in Stan such as Data and Transformed data block, Parameter, and Transformed parameter block, Generated quantities block, etc. Following are the stan codes containing all these blocks for all three models discussed in this article.

5.4.1 Stan code for TIIGTL-W AFT model

```
stancode_ttgtlw = "
functions{
// defines the log survival
vector log_S (vector t,real shape1,real shape2,
real shape3,vector scale){
vector[num_elements(t)] log_S ;
for (i in 1:num_elements(t)){
log_S[i] = log(((1-((weibull_cdf(t[i],shape3,
scale[i]))^(2*shape2)))^(shape1)));
}
return log_S;
}
//defines the log hazard
vector log_h (vector t,real shape1,real shape2,
real shape3,vector scale){
vector[num_elements(t)] log_h ;
vector[num_elements(t)] ls ;
ls = log_S(t,shape1,shape2,shape3,scale) ;
for (i in 1:num_elements(t)){
log_h[i] = (log(2)+log(shape1)+log(shape2)+
weibull_lpdf(t[i]|shape3,scale[i])+
(((2*shape2)-1)*weibull_lcdf(t[i]|shape3,scale[i]))+
((shape1-1)*(log(1-(weibull_cdf(t[i],shape3,
scale[i]))^(2*shape2))))) - ls[i];
}
return log_h;
}
//defines the log-likelihood for right censored data
real surv_ttgtlw_lpdf(vector t,vector d,real shape1,
real shape2,real shape3,vector scale){
vector[num_elements(t)] log_lik;
real prob;
log_lik = d .* log_h(t,shape1,shape2,shape3,scale)+
log_S(t,shape1,shape2,shape3,scale);
prob = sum(log_lik);
return prob;
}
}
//data block
data{
int N; // number of observations
vector <lower=0> [N] y;// observed times
vector <lower=0,upper=1> [N] event;//censoring(1=obs.,
// 0=cens.)
int M; // number of covariates
```

```
matrix[N,M] x;//model matrix (N rows, M columns)
}
//parameters block
parameters{
vector [M] beta;//coef.in the linear predictor
real<lower=0> shape1;// shape parameter
real<lower=0> shape2;// shape parameter
real<lower=0> sigma;//scale parameter sigma=1/shape3
}
// transformed parameters block
transformed parameters{
vector[N] linpred;
vector[N] mu;
linpred = x*beta; //linear predictor
for (i in 1:N){
mu[i] = exp(linpred[i]);
}
}
// model block
model{
shape1 ~ student_t(5,0,10) T[0, ];//prior for shape1
shape2 ~ student_t(5,0,10) T[0, ];//prior for shape2
sigma ~ student_t(2,0,10) T[0, ];//prior for sigma
beta ~ normal(0,10);//prior for reg. coefficients
y ~ surv_ttgtlw(event,shape1,shape2,1/sigma,mu);
//model for the data
}
// generated quantities block
generated quantities{
vector[N] y_rep;//posterior predictive value
vector[N] log_lik;//log-likelihood
{ for(n in 1:N){
log_lik[n] = ((log(2)+log(shape1)+log(shape2)+
weibull_lpdf(y[n]|1/sigma,exp(x[n,]*beta))+
(((2*shape2)-1)*weibull_lcdf(y[n]|1/sigma,
exp(x[n,]*beta))))+((shape1-1)*
(log(1-(weibull_cdf(y[n],1/sigma,exp(x[n,]*beta)))^
(2*shape2))))-(log(((1-((weibull_cdf(y[n],1/sigma,
exp(x[n,]*beta)))^(2*shape2)))^(shape1))))*event[n])+
(log(((1-((weibull_cdf(y[n],1/sigma,exp(x[n,]*beta)))^(2*shape2)))
^(shape1))));}
}
{real u;
u=uniform_rng(0,1);
for (n in 1:N){
y_rep[n] = (exp(x[n,]*beta))*(-log((1-(1-((1-u)^(1/shape1)))
^(1/(2*shape2)))^(sigma)));}
}
}
"
```

5.4.2 Stan code for TIIGTL-E AFT model

```
stancode_ttgtle = "
```

```
functions{
// defines the log survival
vector log_S (vector t,real shape1,real shape2,vector inversescale){
vector[num_elements(t)] log_S ;
for (i in 1:num_elements(t)){
log_S[i] = log(((1-((exponential_cdf(t[i],
inversescale[i]))^(2*shape2)))^(shape1)));
}
return log_S;
}
//defines the log hazard
vector log_h (vector t,real shape1,real shape2,vector inversescale){
vector[num_elements(t)] log_h ;
vector[num_elements(t)] ls ;
ls = log_S(t,shape1,shape2,inversescale) ;
for (i in 1:num_elements(t)){
log_h[i] = (log(2)+log(shape1)+log(shape2)+
exponential_lpdf(t[i]|inversescale[i])+
(((2*shape2)-1)*exponential_lcdf
(t[i]|inversescale[i]))+
((shape1-1)*(log(1-(exponential_cdf
(t[i],inversescale[i]))^(2*shape2))))) - ls[i];
}
return log_h;
}
//defines the log-likelihood for right censored data
real surv_ttgtle_lpdf(vector t,vector d,real shape1,
real shape2,vector inversescale){
vector[num_elements(t)] log_lik;
real prob;
log_lik = d .* log_h(t,shape1,shape2,inversescale)+
log_S(t,shape1,shape2,inversescale);
prob = sum(log_lik);
return prob;
}
}
//data block
data{
int N; // number of observations
vector <lower=0> [N] y;// observed times
vector <lower=0,upper=1> [N] event;//censoring(1=obs.,
// 0=cens.)
int M; // number of covariates
matrix[N,M] x;//model matrix (N rows, M columns)
}
//parameters block
parameters{
vector [M] beta;//coef.in the linear predictor
real<lower=0> shape1;// shape parameter
real<lower=0> shape2;// shape parameter
}
// transformed parameters block
transformed parameters{
```

```
vector[N] linpred;
vector[N] mu;
linpred = -x*beta; //linear predictor
for (i in 1:N){
mu[i] = exp(linpred[i]);
}
}
// model block
model{
shape1 ~ student_t(5,0,10) T[0,];//prior for shape1
shape2 ~ student_t(5,0,10) T[0,];//prior for shape2
beta ~ normal(0,10);//prior for reg. coefficients
y ~ surv_ttgtle(event,shape1,shape2,mu);
//model for the data
}
// generated quantities block
generated quantities{
vector[N] y_rep;//posterior predictive value
vector[N] log_lik;//log-likelihood
{ for(n in 1:N){
log_lik[n] = ((log(2)+log(shape1)+log(shape2)+
exponential_lpdf(y[n]|exp(-(x[n,]*beta)))+
(((2*shape2)-1)*exponential_lcdf(y[n]|exp(-(x[n,]*
beta)))))+((shape1-1)*
(log(1-(exponential_cdf(y[n],exp(-(x[n,]*
beta))))^
(2*shape2))))-(log(((1-((exponential_cdf(y[n],
exp(-(x[n,]*beta))))^(2*shape2)))^(shape1))))*event[n])+
(log(((1-((exponential_cdf(y[n],
exp(-(x[n,]*beta))))^(2*shape2)))^(shape1))));}
}
{real u;
u=uniform_rng(0,1);
for (n in 1:N){
y_rep[n] = (exp(x[n,]*beta))*(-log(1-(1-((1-u)^(1/shape1)))
^(1/(2*shape2))));}
}
}
"
```

5.4.3 Stan code for TIIGTL-LL AFT model

```
stancode_ttgtlll = "
functions{
// defines the log survival
vector log_S (vector t,real shape1,real shape2,
real shape3,vector scale){
vector[num_elements(t)] log_S ;
for (i in 1:num_elements(t)){
log_S[i] = log((1-(((1+(t[i]/scale[i])^(-shape3))^(-1))^(2*shape2)))
^(shape1));
}
return log_S;
}
```

```
//defines the log hazard
vector log_h (vector t,real shape1,real shape2,
real shape3,vector scale){
vector[num_elements(t)] log_h ;
vector[num_elements(t)] ls ;
ls = log_S(t,shape1,shape2,shape3,scale) ;
for (i in 1:num_elements(t)){
log_h[i] = (log(2)+log(shape1)+log(shape2)+
(log(shape3)-(shape3)*log(scale[i])+(shape3-1)*
log(t[i])-2*log(1+(t[i]/scale[i])^(shape3)))+
(((2*shape2)-1)*log(((1+(t[i]/scale[i])^(-shape3))^(-1)))+
((shape1-1)*(log(1-((1+(t[i]/scale[i])^(-shape3))^(-1))^(2*shape2))))))
-ls[i];
}
return log_h;
}
//defines the log-likelihood for right censored data
real surv_ttgtlll_lpdf(vector t,vector d,real shape1,
real shape2,real shape3,vector scale){
vector[num_elements(t)] log_lik;
real prob;
log_lik = d .* log_h(t,shape1,shape2,shape3,scale)+
log_S(t,shape1,shape2,shape3,scale);
prob = sum(log_lik);
return prob;
}
}
//data block
data{
int N; // number of observations
vector <lower=0> [N] y;// observed times
vector <lower=0,upper=1> [N] event;//censoring(1=obs.,
// 0=cens.)
int M; // number of covariates
matrix[N,M] x;//model matrix (N rows, M columns)
}
//parameters block
parameters{
vector [M] beta;//coef.in the linear predictor
real<lower=0> shape1;// shape parameter
real<lower=0> shape2;// shape parameter
real<lower=0> sigma;//scale parameter sigma=1/shape3
}
// transformed parameters block
transformed parameters{
vector[N] linpred;
vector[N] mu;
linpred = x*beta; //linear predictor
for (i in 1:N){
mu[i] = exp(linpred[i]);
}
}
// model block
```

```
model{
shape1 ~ student_t(5,0,10) T[0, ];//prior for shape1
shape2 ~ student_t(5,0,10) T[0, ];//prior for shape2
sigma ~ student_t(2,0,10) T[0, ];//prior for sigma
beta ~ normal(0,10);//prior for reg. coefficients
y ~ surv_ttgtlll(event,shape1,shape2,1/sigma,mu);
//model for the data
}
// generated quantities block
generated quantities{
vector[N] y_rep;//posterior predictive value
vector[N] log_lik;//log-likelihood
{ for(n in 1:N){
log_lik[n] = ((log(2)+log(shape1)+log(shape2)+(log(1/sigma)-
(1/sigma)*(x[n,]*beta)+((1/sigma)-1)*log(y[n])-
2*log(1+(y[n]/exp(x[n,]*beta))^(1/sigma)))+
(((2*shape2)-1)*log(((1+(y[n]/exp(x[n,]*beta))^(-1/sigma))^(-1))))+
((shape1-1)*(log(1-(((1+(y[n]/exp(x[n,]*beta))^(-1/sigma))^(-1))^
(2*shape2))))))-(log(((1-((((1+(y[n]/exp(x[n,]*beta))^(-1/sigma))
^(-1))^(2*shape2)))^(shape1))))*event[n]))+
(log((1-(((((1+(y[n]/exp(x[n,]*beta))^(-1/sigma))^(-1)))^(2*shape2)))
^(shape1))));}
}
{real u;
u=uniform_rng(0,1);
for (n in 1:N){
y_rep[n] = (exp(x[n,]*beta))*((((1-(1-((1-u)^(1/shape1)))^(1/(2*shape2)))
^(-1))-1)^(sigma));}
}
}
"
```

5.5. Model fitting with Stan

The function *stan* from the package **rstan** is used for the fitting of all three models based on TIIGTL-G family. All relevant codes for the numeric as well as graphical summary are attached in upcoming sub sections.

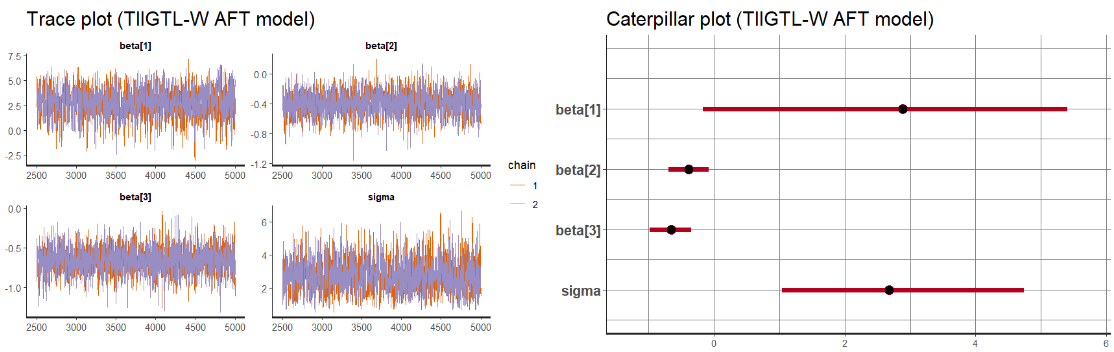
5.5.1 Fitting of TIIGTL-W AFT model

```
require(survival)
betaw = solve(crossprod(x),crossprod(x,log(y)))
betaw = c(betaw)
TTGTLWAFT <- stan(model_code = stancode_ttgtlw,data=datt,
init=list(list(beta=betaw),list(beta=betaw)),iter=5000,chains=2)
```

Output and graphics Summarization: Table 2 contains the results obtained after fitting the TIIGTL-W AFT model to the diet data set. The coeffcients beta[2] of saturated fat (x1) and beta[3] of unsaturated fat (x2) are negative which indicate that both x1 and x2 expedite the tumor development process, consequently, survival time (time to develop a tumor) will be shorter. From the summary results and from the caterpillar plot (Figure 1b), it is seen that the 95% credible intervals do not contain a value of zero for the coefficients of the diets, so the coefficients are statistically significant. Additionally, we can see the posterior estimates (mean and se_mean), the standard deviation (sd), and the credible interval. Also we can observe the n_eff (rough estimate of the effective sample size), and the Rhat, also known as the potential scale reduction factor [15], which calculates the Markov chain's convergence to the target distribution. According to [15] the allowable range of n_eff is greater than 100 and Rhat values less than 1.1. We can observe Rhat values for all parameters of the TIIGTL-W AFT model is less than 1.1, this indicates that the Monte Carlo error is tolerable, the Markov chains reach to the target distribution, and the effective sample size is appropriate.. Trace plots are also attached (Figure 1a) as indicator of convergence of MCMC algorithm. Using the **Bayesplot** package, posterior predictive density (PPD) charts are used to visually evaluate the model. Posterior predictive density (Figure 2a) graphs shows that the TIIGTL-W AFT model is consistent with the current data.

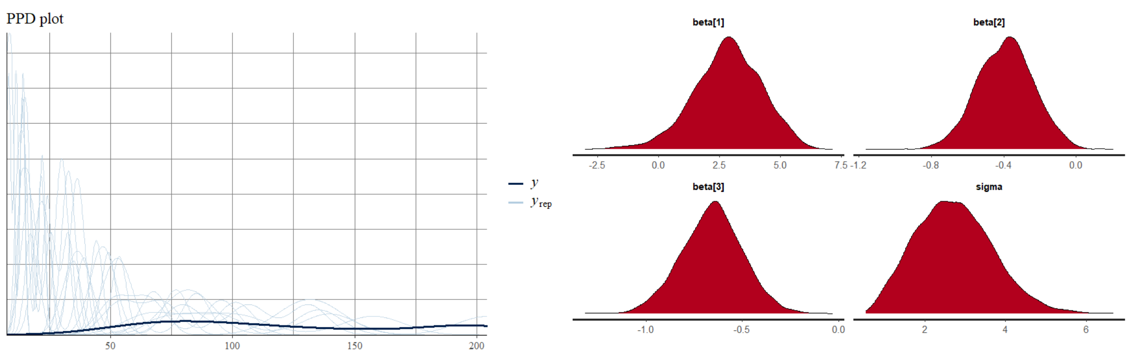
Table 2: *Summary of Posterior estimates of TIIGTL-W AFT model parameters*

parametrs	mean	se_mean	sd	2.5%	50%	97.5%	n_eff	Rhat
beta[1]	2.825	0.037	1.416	-0.168	0.398	5.402	1440	1.002
beta[2]	-0.390	0.003	0.157	-0.695	-0.648	-0.086	2353	1.000
beta[3]	-0.658	0.004	0.161	-0.980	-0.930	-0.345	2049	1.000
shape1	9.569	0.154	8.365	0.524	0.936	31.259	2953	1.001
shape2	13.829	0.253	10.081	2.275	2.945	40.073	1586	1.001
sigma	2.726	0.025	0.956	1.044	1.241	4.739	1494	1.002



(a) (b)

Figure 1: *(a) Traceplot for TIIGTL-W AFT model, In two separate runs, two chains were displayed; combining the two chains successfully indicates that MCMC algorithm has converged to the target joint posterior distribution. (b) Caterpillar plot for TIIGTL-W AFT model*



(a) (b)

Figure 2: *(a) Posterior predictive density (PPD) plot of the TIIGTL-W AFT model to check model convergence. The TIIGTL-W AFT model's posterior predictive density adequately fits the data, according to the PPD plot (b) Posterior density plot for TIIGTL-W AFT model*

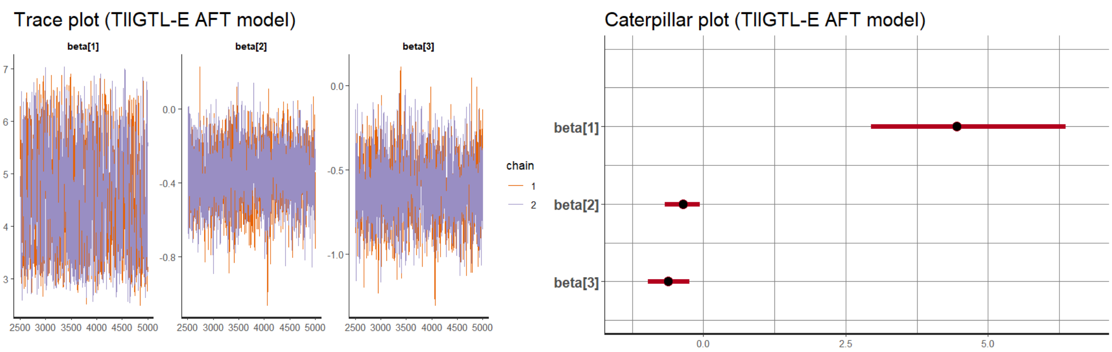
5.5.2 Fitting of TIIGTL-E AFT model

```
TTGTLEAFT <- stan(model_code = stancode_ttgtle,data=datt,
init=list(list(beta=betae),list(beta=betae)),iter=5000,chains=2)
```

Output and graphics Summarization: From Table 3 we can observe that the coeffcients beta[2] of saturated fat (x1) and beta[3] of unsaturated fat (x2) are negative and the Rhat of the TIIGTL-E AFT model parameters are less than 1.1, which shows Markov chain converges to the target distribution. Also, n_eff is greater than 100. From the caterpillar plot (Figure 3b), it is seen that the 95% credible intervals do not contain a value of zero for the coefficients of the diets, so the coefficients are statistically significant. The PPD plot (Figure 4a) of the TIIGTL-E AFT model indicates that the posterior predictive density matched the data well.

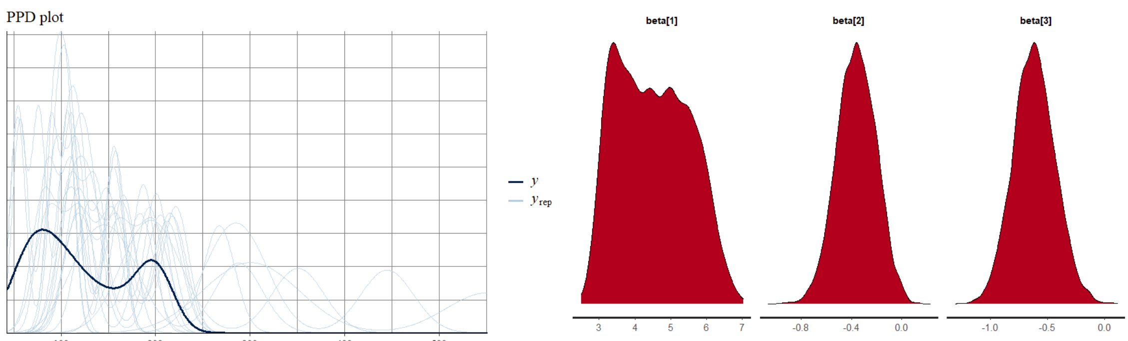
Table 3: *Summary of Posterior estimates of TIIGTL-E AFT model parameters*

parametrs	mean	se_mean	sd	2.5%	50%	97.5%	n_eff	Rhat
beta[1]	4.507	0.034	1.000	2.950	3.063	6.360	889	1.001
beta[2]	-0.362	0.004	0.158	-0.671	-0.619	-0.059	1672	1.001
beta[3]	-0.615	0.005	0.185	-0.968	-0.914	-0.245	1231	1.001
shape1	3.206	0.131	5.022	0.168	0.192	18.503	1473	1.000
shape2	4.755	0.163	5.236	1.211	1.296	19.390	1036	1.002



(a) (b)

Figure 3: *(a) Traceplot for TIIGTL-E AFT model parameters (b) Caterpillar plot for the TIIGTL-E AFT model*



(a) (b)

Figure 4: *(a) The posterior predictive density (PPD) plot of the TIIGTL-E AFT model (b) Posterior density plot TIIGTL-E AFT model parameters*

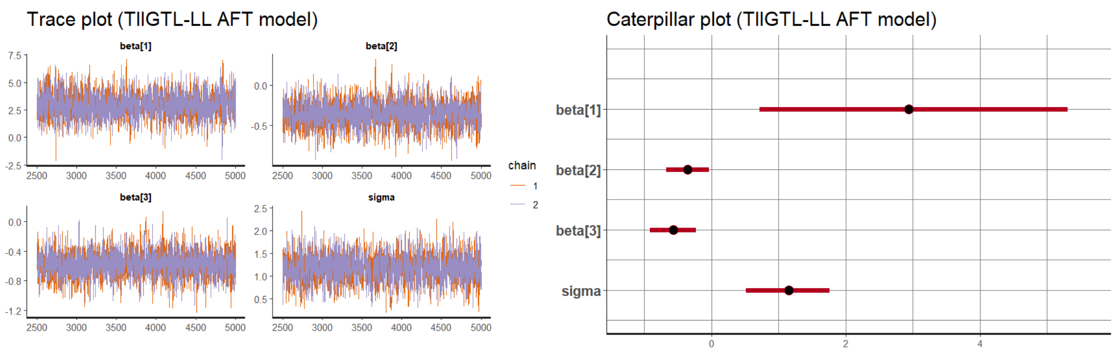
5.5.3 Fitting of TIIGTL-LL AFT model

```
TTGTLLLAFT <- stan(model_code = stancode_ttgtlll,,data=datt,
init=list(list(beta=betall),list(beta=betall)),iter=5000,chains=2)
```

Output and Graphics Summarization: From Table 4 we can observe that the coeffcients beta[2] of saturated fat (x1) and beta[3] of unsaturated fat (x2) are negative and the Rhat of the TIIGTL-LL AFT model parameters are less than 1.1, which shows Markov chain converges to the target distribution. Also, n_eff is greater than 100. From the caterpillar plot (Figure 5b), it is seen that the 95% credible intervals do not contain a value of zero for the coefficients of the diets, so the coefficients are statistically significant. The PPD plot (Figure 6a) of the TIIGTL-LL AFT model indicates that the posterior predictive density matched the data well.

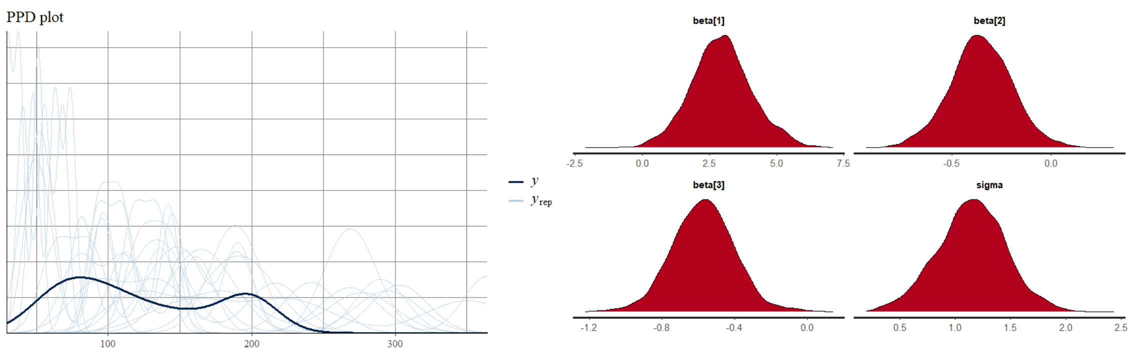
Table 4: *Summary of Posterior estimates of TIIGTL-LL AFT model parameters*

parametrs	mean	se_mean	sd	2.5%	50%	97.5%	n_eff	Rhat
beta[1]	2.951	0.025	1.141	0.713	1.108	5.304	2017	1.000
beta[2]	-0.354	0.004	0.157	-0.674	-0.618	-0.039	1987	1.000
beta[3]	-0.575	0.004	0.172	-0.913	-0.850	-0.234	1651	1.001
shape1	11.323	0.176	8.929	1.188	1.829	34.794	2566	1.000
shape2	11.555	0.167	8.733	1.568	2.175	33.203	2738	1.000
sigma	1.145	0.007	0.309	0.514	0.616	1.758	1713	1.000



(a) (b)

Figure 5: *(a) Traceplot of TIIGTL-LL AFT model parameters (b) Caterpillar plot for TIIGTL-LL AFT model*



(a) (b)

Figure 6: *(a) The posterior predictive density (PPD) plot of the TIIGTL-LL AFT model (b) Posterior density plot for TIIGTL-LL AFT model*

5.6. Bayesian model Comparison

We take into account model evaluation and selection standards such as Watanabe Akaike Information Criteria (WAIC) and Leave One Out cross-validation Information Criteria (LOOIC) ([16] ,[17]) in order to compare the fitted models. In R, **loo** package [17] is used to obtain LOOIC and WAIC by using the log-likelihood evaluated at the posterior simulations of the parameters after fitting the model through STAN. The lower value of these selection strategies, however, denotes a better model fit.

Table 5: *LOOIC and WAIC values for all models.*

Model	LOOIC	WAIC
TIIGTL-E AFT	1026.4	1026.3
TIIGTL-W AFT	1024.5	1024.5
TIIGTL-LL AFT	1015.0	982.8

From Table 5, we can see that the LOOIC and WAIC value of the TIIGTL-LL AFT model is lowest among the three, which shows in comparison to other models for diet data, the TIIGTL-LL AFT model is a superior survival model.

5.7. Conclusion

In a Bayesian framework, the Weibull, Exponential, and Log-Logistic Accelerated Failure Time models for the diet data are fitted using the Type II Generalized Topp–Leone distribution. Diet coefficients for each model have statistical significance. The posterior predictive density (PPD) plots for the TIIGTL-W AFT, TIIGTL-E AFT, and TIIGTL-LL AFT models were used to calculate the posterior predictive check. The replicated data sets are derived from the same model as the original data set, and all are sufficient models for projecting the future value, as seen in the PPD plot where the data y and replicated data set y_{rep} exhibit the same behaviour and share a similar appearance. TIIGTL-LL AFT model fits the censored diet data better than the other models, according to comparisons of posterior predictive density plots, LOOIC and WAIC.

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Reliability and Performance Analysis of a Complex Manufacturing System with Inspection facility using Copula Methodology

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Abstract

This paper deals with the assessment of various reliability factors of a real-life manufacturing system having inspection facility. This multistate manufacturing system have five workstations those are connected in series configuration as: W_1, W_2, W_3, W_4, W_5. Workstations W_2 and W_4 has the configuration 2-out-of -3: G and 1-out-of-3: F. Due to failure of the any of the workstation, whole manufacturing system can completely fail. Apart from this machine failure can also make system down. To avoid sudden failure in the system pre-emptive maintenance strategy has been adopted. This is a corrective maintenance action before a failure occurs and scheduled during off days. Risk analysis is done because of fault of W_5 workstation in material quality inspection. Probability distributions like exponential time distribution is followed by all failures and general time distribution by all repairs. To study the probabilistic behavior of the system in different possible transition states, Markov process have been used. Supplementary variable technique and copula method of finding joint probability distribution have been used to obtained various reliability features such as steady state behavior of the system, reliability function, availability, Mean time to failure, sensitivity analysis and profit analysis.

Keywords: Reliability analysis, Mean time to failure, Availability, Sensitivity analysis, Risk analysis

1. Introduction

Nowadays, Due to the globalization of the market and business, a lot of problems related to manufacturing industries like delays in product delivery, machine failure, cancellation of demand, etc. are encountered by the industries daily. Therefore, reliability and availability analysis are important for the performance analysis of discrete manufacturing systems. A lot of work has been done to discuss reliability measures of manufacturing systems using different approaches [1, 3, 5, 6 and 7]. Here, a concept of making a methodical approach to analyze a failure-free system for a manufacturing industry is developed for a practical period [9].

The objective of this work is to assess the performance and risk analysis of a manufacturing system under different operating conditions. This multistate manufacturing system contains five workstations that are connected in the series configuration as *W_1, W_2, W_3, W_4, and W_5*. Workstation W_1 consists of the raw material supplied by a merchant or vendor for making finished goods. At workstation W_2 material provided by workstation W_1 is transformed into welded usable form and small and big components are used to make the final product. Later on, these welded components are sent to workstation W_3 for the dye or tint process. Workstation W_4 plays a vital role in this complex manufacturing system as the finished product assembles here only by connecting and arranging equipped components in logical order received from workstation W_3. At last, before

sending the finished product to suppliers in the market, it goes through the inspection process at workstation W_5 for quality inspection. On the workstations W_2 and W_4, three machines are involved in performing the same task connected parallelly. These workstations follow 2-out -of-3:G and 1-out-of-3:F conformations, which means that for the fully operational stage of the system in which it can achieve the required target, it is essential that at least two machines of workstation W_2 and W_4 are in working condition otherwise in the opposite case the system fails [2]. Along with this, Machine failure is also considered which may be major or minor. To get maximum reliability, two groups of repairmen are involved in repairing the system according to their knowledge and skills. Here, a joint probability distribution is obtained using copula methodology when both groups are involved in repairing the system at the same time [4 and 8]. After getting finished goods, product inspection is done by workstation W_5. Here, any fault or ignorance in the inspection of the product can take the system into a risk state that can cause system failure. For example, if a technical fault is there in final assembled product due to wrong assembly or material use which can result in a failure after a certain period of use of the product or in certain climatic conditions, also the product have been not tested for that period or under that climatic conditions. The transition state diagram and state specification of the considered system are shown in figure-1 and table-1 respectively. The figure-1 shows positive transition intensities, and the transition probabilities for time Δ are proportional to the intensities, the remaining transition probabilities for time Δ are equal to o (Δ).

2. Notations

$P_0(t)$	:	Denotes Probability at time t when the system is in initial state S_0
Pi (k, t)	:	Denotes the probability of system getting in breakdown state because of failure of the i[th] workstation at time t, also elapsed repair time considered in between k and k+Δ, where i= 1, 2, 3, 4, 5, M, QR, and k$\in$ [0, +∞)
W_1/W_2/W_3/ W_4/W_5	:	Workstation 1 to Workstation 5
K	:	Elapsed repair time, where k$\in$ [0, +∞)
ψ_{1W} / ψ_{1A}	:	The Showing Failure rate of the one machine of the Workstation 2/ 4.
ψ_M	:	Machine failure rate
$P_{1W}(t)$	:	Showing the probability of 2-out -of-3: G state of workstation 2 i.e. system is in fully operational mode even after one machine of workstation 2 is failed and rest two are in working condition
$P_{1A}(t)$	:	Showing the probability of 2-out -of-3: G state of workstation 4 i.e. system is in fully operational mode even after one machine of workstation 4 is failed and rest two are in working condition
ψ_i	:	The General failure rate of i[th] workstation, where i= 1, 2, 3, M, 4, 5.

$\phi_i(k)$	:	Showing repair rate of i[th] workstation between the time interval (k, k+Δ), where i= 1, 2, 3, 4, 5, M, QR, and k∈ [0, +∞)
γ_Q	:	Showing risk rate or the factor which indicates the level of risk taken into consideration which can lead the system into the risk stage
Pi, W (k, t)	:	Shows the probability of the failed state of the system due to failure of the i[th] workstation from the state S_2 when one machine of workstation W_2 is not working. Elapsed repair time for the i[th] subsystem lies between (k, k+Δ), where i=1, 3, 4 and k∈ [0, +∞)
Pi, A (k, t)	:	Shows the probability of the failed state of the system due to failure of the i[th] workstation from the state S_2 when one machine of workstation 4 is not working. Elapsed repair time for the i[th] subsystem lies between (k, k+Δ), where i= 1, 2, 3 and k∈ [0, +∞).
$P_{QR}(q, t)$	:	Shows the probability at time t when the system is in the risk due to ignorance in inspection at the workstation 5
K_1, K_2	:	Profit and service cost per unit time respectively

Also, consider $u_1 = e^r$ and $u_2 = \phi_M(r)$, according to Gumbel- Hougaard copula methodology the joint probability distribution is given by

$$\phi_M = \exp\left[r^\theta + \left[\log \phi_M(r)\right]^\theta\right]^{1/\theta}.$$

2.1. Assumptions

For reliability analysis of this manufacturing system, the following assumptions are taken into consideration.

- All the workstations are fully operational at t=0.
- Failures follow exponential time distribution and are statistically independent while repairs follow arbitrary time distributions.
- Repaired workstations are assumed like in good working conditions.
- Workstations 2 and 4 follow 2-out-of-3: G and 1-out-of-3: F conformation.
- It is also considered that the manufacturing system can fail due to any mechanical failure that may be major or minor or both at the same time. Here joint probability distribution is used to solve these failures using the copula methodology [6]. The whole system can also fail due to machine failures that may be either major or minor or both.

2.2. State specification

Table -1 shows the state specification of the transition diagram-1

Table 1: *State specification table*

States	Description	System State
S0	The system is in the fully operational stage.	G
S1	The system is in the failed state due to the failure of the workstation W_1.	F_R
S2	The system is in a working state when workstation W_2 is in 2-out-of-3: G configuration.	G
S3	Due to the failure of workstation W_3, the whole system is in the failed state.	F_R
S4	The system is in a working state when workstation W_4 is in 2-out-of-3: G configuration.	G
S5	Due to machine failure, the whole system is in a breakdown condition.	F_R
S6	System is working at high risk due to negligence of the workstation W_5.	R_S
S7	The system is in inoperable condition from the risk state i.e. S_6 due to ignorance in the inspection.	F_R
S8	The system is in inoperable condition from the state S_2 as workstation W_1 is unable to work due to some vendor issues.	F_R
S9	The system is in inoperable condition from the state S_2 due to the failure of workstation W_4.	F_R
S10	The system is in the failed state from the state S_2 because workstation W_2 follows 1-out-of-3: F configuration.	F_R
S11	The system is inoperable due to the not functioning of workstation W_3.	F_R
S12	The system is in inoperable condition from the state S4 due to the failure of workstation W_1.	F_R
S13	The system is inoperable from the state S_4 due to the failure of the workstation W_2.	F_R
S14	The system is inoperable from the state S_4 due to the failure of the workstation W_3.	F_R
S15	The system is inoperable from the state S_4 due to the failure of the workstation W_4.	F_R

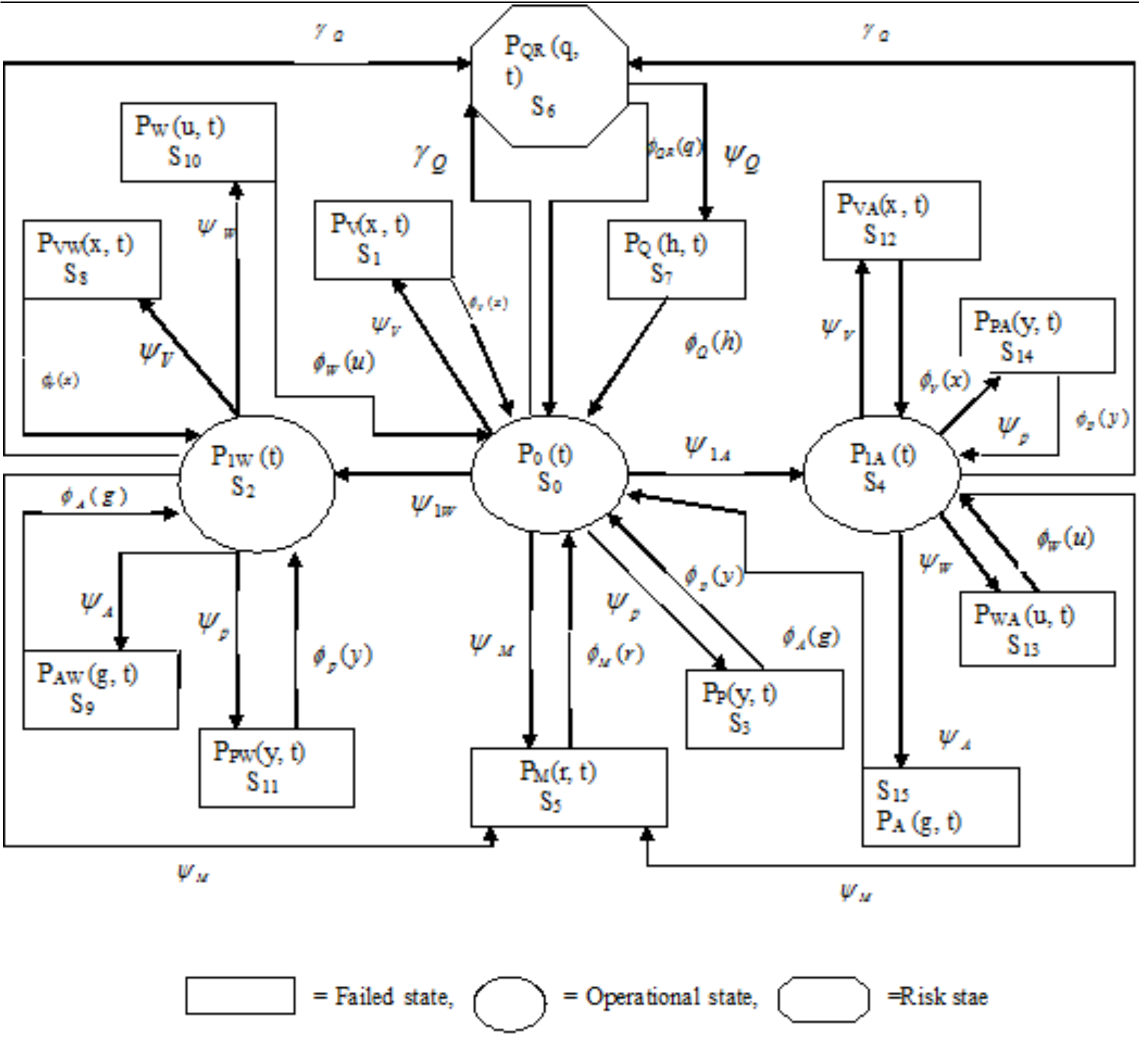

Figure 1: *Transition State diagram*

3. Formulation of the mathematical model

Following integro- differential equations which satisfying the model are obtained after probabilistic considerations and limiting process:

$$\left[\frac{d}{dt}+\psi_{1A}+\psi_{1W}+\psi_{V}+\psi_{M}+\psi_{p}+\gamma_{Q}\right]P_0(t)=\int_0^\infty \phi_V(x)P_V(x,t)dx+\int_0^\infty \phi_p(y)Pp(y,t)dy+$$
$$\int_0^\infty \phi_M(r)P_M(r,t)dg+\int_0^\infty \phi_Q(q)P_Q(q,t)dq+\int_0^\infty \phi_A(g)P_A(g,t)dg+$$
$$\int_0^\infty \phi_W(u)P_W(u,t)du+\int_0^\infty \phi_Q(h)P_Q(h,t)dh \tag{1}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\phi_V(x)\right]P_V(x,t)=0 \tag{2}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\phi_p(y)\right]P_p(y,t)=0 \tag{3}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\exp\left[r^{\theta}+\left\{\left[\log\phi_M(r)\right]\right\}^{\theta}\right]^{1/\theta}\right]P_M(r,t)=0 \tag{4}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial q}+\psi_Q+\phi_{QR}(q)\right]P_{QR}(q,t)=0 \tag{5}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial h}+\phi_Q(h)\right]P_Q(h,t)=0 \tag{6}$$

$$\left[\frac{d}{dt}+\psi_A+\psi_W+\psi_V+\psi_M+\psi_p+\gamma_Q\right]P_{1w}(t)=\int_0^\infty\phi_V(x)P_{VW}(x,t)dx+$$
$$+\int_0^\infty\phi_A(g)P_{AW}(g,t)dg+\int_0^\infty\phi_p(y)P_{PW}(y,t)dy+\psi_{1W}P_0(t) \tag{7}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\phi_V(x)\right]P_{VW}(x,t)=0 \tag{8}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial g}+\phi_A(g)\right]P_{AW}(g,t)=0 \tag{9}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\phi_p(y)\right]P_{PW}(y,t)=0 \tag{10}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+\phi_W(u)\right]P_W(u,t)=0 \tag{11}$$

$$\left[\frac{d}{dt}+\psi_A+\psi_W+\psi_V+\psi_M+\psi_p+\gamma_Q\right]P_{1A}(t)=\int_0^\infty\phi_V(x)P_{VA}(x,t)dx+$$
$$\int_0^\infty\phi_p(y)P_{PA}(y,t)dy+\int_0^\infty\phi_W(u)P_{WA}(u,t)du+\psi_{1A}P_0(t) \tag{12}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\phi_V(x)\right]P_{VA}(x,t)=0 \tag{13}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\phi_p(y)\right]P_{PA}(y,t)=0 \tag{14}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial u}+\phi_W(u)\right]P_{WA}(u,t)=0 \tag{15}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial g}+\phi_A(g)\right]P_A(g,t)=0 \tag{16}$$

3.1 Boundary conditions

$$P_V(0,t)=\psi_V P_0(t) \tag{17}$$

$$P_p(0,t)=\psi_p P_0(t) \tag{18}$$

$$P_M(0,t)=\psi_M P_0(t) \tag{19}$$

$$P_{QR}(0,t) = \gamma_Q \left[P_0(t) + P_A(t) + P_W(t)\right] \tag{20}$$

$$P_Q(0,t) = \psi_Q P_{QR}(t) \tag{21}$$

$$P_{VW}(0,t) = \psi_V P_W(t) \tag{22}$$

$$P_{AW}(0,t) = \psi_A P_W(t) \tag{23}$$

$$P_{PW}(0,t) = \psi_p P_W(t) \tag{24}$$

$$P_W(0,t) = \psi_W P_W(t) \tag{25}$$

$$P_{VA}(0,t) = \psi_V P_A(t) \tag{26}$$

$$P_{pA}(0,t) = \psi_p P_p(t) \tag{27}$$

$$P_{WA}(0,t) = \psi_W P_A(t) \tag{28}$$

$$P_A(0,t) = \psi_A P_A(t) \tag{29}$$

3.2 Initial Condition

$P_0(0) = 1$, otherwise zero. (30)

After solving equations (1) to (16), using initial and boundary conditions by taking Laplace transform, one can obtain following up and down probabilities of the system.

$$\overline{P_0}(s) = \frac{1}{K(s)} \tag{31}$$

$$\overline{P_{1w}}(s) = \frac{B(s)}{K(s)} \tag{32}$$

$$\overline{P_{1A}}(s) = \frac{A(s)}{K(s)} \tag{33}$$

$$\overline{P_v}(s) = \frac{\psi_v}{K(s)} J_v(s) \tag{34}$$

$$\overline{P_P}(s) = \frac{\psi_P}{K(s)} J_P(s) \tag{35}$$

$$\overline{P_M}(s) = \frac{\psi_M}{K(s)} J_M(s) \tag{36}$$

$$\overline{P_{QR}}(s) = \frac{\gamma_Q}{K(s)} \left[1 + B(s) + A(s)\right] J_{QR}(s) \tag{37}$$

$$\overline{P_Q}(s) = \frac{\gamma_Q \psi_Q}{K(s)} \left[1 + A(s) + B(s)\right] J_{QR}(s) J_Q(s) \tag{38}$$

$$\overline{P_{VW}}(s) = \frac{\psi_V\, B(s)}{K(s)} J_V(s) \tag{39}$$

$$\overline{P_{PW}}(s)=\frac{\psi_P\ B(s)}{K(s)}J_P(s) \tag{40}$$

$$\overline{P_{AW}}(s)=\frac{\psi_A\ B(s)}{K(s)}J_A(s) \tag{41}$$

$$\overline{P_{W}}(s)=\frac{\psi_W\ B(s)}{K(s)}J_W(s) \tag{42}$$

$$\overline{P_{VA}}(s)=\frac{\psi_V\ A(s)}{K(s)}J_V(s) \tag{43}$$

$$\overline{P_{PA}}(s)=\frac{\psi_P\ A(s)}{K(s)}J_P(s) \tag{44}$$

$$\overline{P_{WA}}(s)=\frac{\psi_W\ A(s)}{K(s)}J_W(s) \tag{45}$$

$$\overline{P_{A}}(s)=\frac{\psi_A\ A(s)}{K(s)}J_A(s) \tag{46}$$

where,

$$\begin{aligned}K(s)=&\ s+\psi_V+\psi_M+\psi_P+\psi_{1A}+\psi_{1W}+\gamma_Q-\psi_V\overline{S}_V(s)-\psi_P\overline{S}_P(s)-\psi_M\overline{S}_M(s)-\\&\gamma_Q\overline{S}_{QR}(s)-\left[\gamma_Q\overline{S}_{QR}(s)+\psi_A\overline{S}_A(s)\right]\frac{\psi_{1A}}{C_1}-\left[\gamma_Q\overline{S}_{QR}(s)+\psi_W\overline{S}_W(s)\right]\frac{\psi_{1w}}{c_2}-\\&\psi_Q\gamma_Q\overline{S}_Q(s)\left[1+\frac{\psi_{1W}}{C_1}+\frac{\psi_{1A}}{C_2}\right]J_{QR}(s)\end{aligned} \tag{47}$$

$$J_i(s)=\frac{1-\overline{S_v(s)}}{s},\ \text{for}\ i=V,P,M,QR,Q,A,W \tag{48}$$

$$J_{QR}(s)=\frac{1-\overline{S}_{QR}(s)}{s+\psi_Q} \tag{49}$$

$$A(s)=\frac{\psi_{1A}}{C_1} \tag{50}$$

$$B(s)=\frac{\psi_{1W}}{C_2}$$

(51)

$$C_1=s+\psi_V+\psi_M+\psi_P+\psi_A+\psi_W+\gamma_Q-\psi_V\overline{S}_V(s)-\psi_P\overline{S}_P(s)-\psi_W\overline{S}_W(s) \tag{52}$$

$$C_2=s+\psi_V+\psi_M+\psi_P+\psi_A+\psi_W+\gamma_Q-\psi_V\overline{S}_V(s)-\psi_P\overline{S}_P(s)-\psi_A\overline{S}_A(s) \tag{53}$$

$$\overline{S}_i(j)=\int_0^\infty\phi_i(j)\exp[-s_j-\int_0^i\phi_i(j)dj]dj,\ for\ i=V,P,M,QR,Q,A,W\ \&\ j=x,y,r,q,h,g,u. \tag{54}$$

$$\phi_M=\exp\left[r^\theta+\left[\log\phi_M(r)\right]^\theta\right]^{1/\theta} \tag{55}$$

Also,

$$\overline{P}_{up}(s)+\overline{P}_{down}(s)=\frac{1}{s} \tag{56}$$

To study the steady-state behavior of the system using Abel's lemma we have

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) = f(\text{say})$$

$$\overline{Pup}(s) = P_0(s) + P_{1W}(s) + P_{1A}(s) \tag{57}$$

$$P_0 = \frac{1}{K^{'}(0)} \tag{58}$$

$$P_{1W} = \frac{B(0)}{K^{'}(0)} \tag{59}$$

$$P_{1A} = \frac{A(0)}{K^{'}(0)} \tag{60}$$

where,

$$K^{'}(0) = \left[\frac{d}{ds}K(s)\right]_{s=0} \tag{61}$$

$T_i = -\overline{S}_i(0)$ =Mean time to repair the i^th^ failure (62)

$$A(0) = \frac{\psi_{1A}}{\psi_M + \psi_A + \gamma_Q} \tag{63}$$

$$B(0) = \frac{\psi_{1W}}{\psi_M + \psi_A + \gamma_Q} \tag{64}$$

$$\underline{Lim} J_{QR}(s) = \frac{1}{\psi_Q} M_{QR} \quad \text{(As s tends to 0)} \tag{65}$$

$$\mathrm{S}_{\phi_i}(j) = \frac{\phi_i}{j + \phi_i} \tag{66}$$

For a non-repairable system, the Laplace transform of the reliability when all repair rates of the system are zero, then from equation (31), we have

$$\overline{\mathrm{R}}(s) = \frac{1}{s + \psi_v + \psi_P + \psi_{1A} + \psi_{1W} + \psi_M + \gamma_Q}$$

where R(s) is the Laplace transform of the reliability function.

The reliability of the transit system is obtained as:

$$R(t) = \exp[-(\psi_V + \psi_M + \psi_P + \psi_{1A} + \psi_{1W} + \gamma_Q) * t] \tag{67}$$

The mean time to failure of the system is given by,

$$\text{MTTF} = Lim\overline{R}(s) = \int_0^\infty R(t)dt$$

$$MTTF = \frac{1}{\psi_V + \psi_M + \psi_P + \psi_{1A} + \psi_{1W} + \gamma_Q} \tag{68}$$

Availability of the system is given by,

$$\overline{Pup(s)} = P_0(s) + P_{1W}(s) + P_{1A}(s)$$

$$\overline{P}up(s) = \frac{1}{K(s)}\left[1 + B(s) + A(s)\right]$$

$$\overline{P}up(s) = \frac{1 + \dfrac{.016}{s + .055}}{s + .054}$$

Taking inverse Laplace transforms, we have

$$Pup(t) = -16.e^{(-.05500000000t)} + 17.e^{(-.0540000000t)} \tag{69}$$

Sensitivity analysis is performed for monitoring changes in reliability and MTTF of the system with respect to workstations W1, W3, and risk factor γ_Q.

we obtain

$$\frac{\partial R(t)}{\partial \psi_V} = -te^{(-(\psi_v + \psi_P + \psi_{1A} + \psi_{1W} + \psi_M + \gamma_Q)^* t)} \tag{70}$$

Also, we can get $\dfrac{\partial R(t)}{\partial \psi_P}$ and $\dfrac{\partial R(t)}{\partial \gamma_Q}$.

$$\frac{\partial}{\partial \psi_V} MTTF = -\frac{1}{(\psi_v + \psi_P + \psi_{1A} + \psi_{1W} + \psi_M + \gamma_Q)^2} \tag{71}$$

Also, $\partial MTTF / \partial \psi_P$ and $\partial MTTF / \partial \gamma_Q$.

The profit function of the considered manufacturing system is given by

$$G(t) = K_1 \cdot \int_0^t P_{up}(t)dt - K_2 t$$

where, K1 and K2 are revenue and repair costs per unit time, respectively.
Also

$$G(t) = K_1 \int_0^t e^{(-(\psi_v + \psi_P + \psi_{1A} + \psi_{1W} + \psi_M + \gamma_Q)t)} + \left[\frac{\psi_{1A} + \psi_{1W}}{\psi_{1A} + \psi_{1W} - \psi_W - \psi_A}\right] e^{(-(\psi_v + \psi_P + \psi_{1A} + \psi_{1W} + \psi_M + \gamma_Q)t)} - \left[\frac{\psi_{1A} + \psi_{1W}}{\psi_{1A} + \psi_{1W} - \psi_W - \psi_A}\right] e^{(-(\psi_v + \psi_P + \psi_A + \psi_W + \psi_M + \gamma_Q)t)} dt - K_2 t \tag{72}$$

4. Results and discussion

To check more concrete behavior of the system, Numerical computation of reliability, availability, and profit function is done concerning time by keeping other parameter fixed and also MTTF of the system for different failure rates.

Figure 2 shows the movement of reliability with respect to time. It reveals that due to ignorance of the workstation W_5 in inspection, the reliability decreases with the passage of time. Figure 3 shows a rapid decrease in MTTF with an increment in Workstation W_1, W_3, and machine failure rate. It is also observed that in some instances MTTF is almost the same with respect to these three failure rates. Also, as the risk rate increases, the MTTF of the system decreases smoothly shown in figure 4. Figure 5 gives an idea about the availability of the system that decreases constantly as time increases.

Sensitivity analysis of system reliability is done for different workstations failure rates as shown in figures 6, 7, 8, and 9. Here we observe that the system has almost same sensitivity for W_1 workstation failure and risk rate, although machine failure and workstation W_2 come next in magnitude.

Finally, Figure 10 shows that the cost of the system increases in general with time.

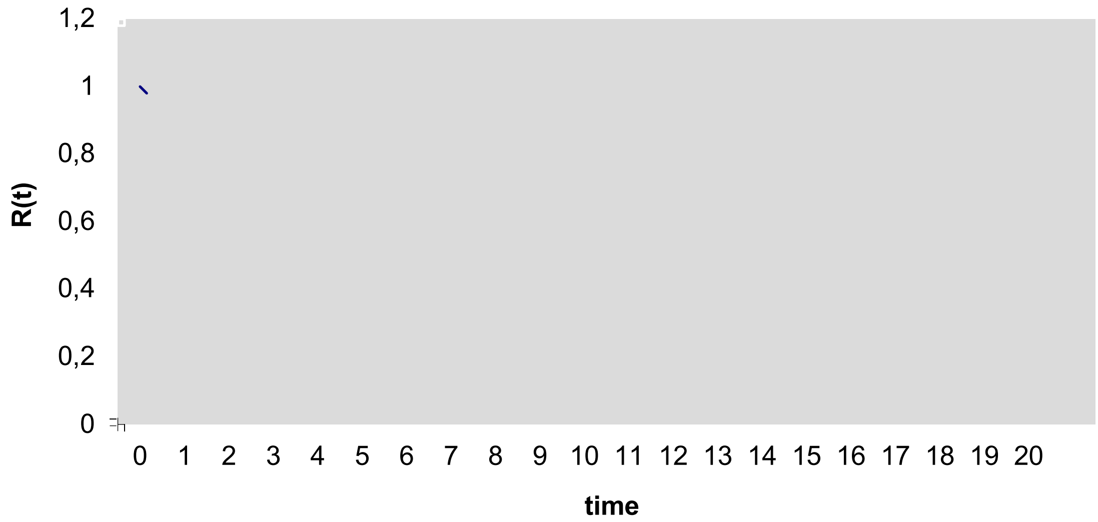

Figure 2: *Reliability Vs Time*

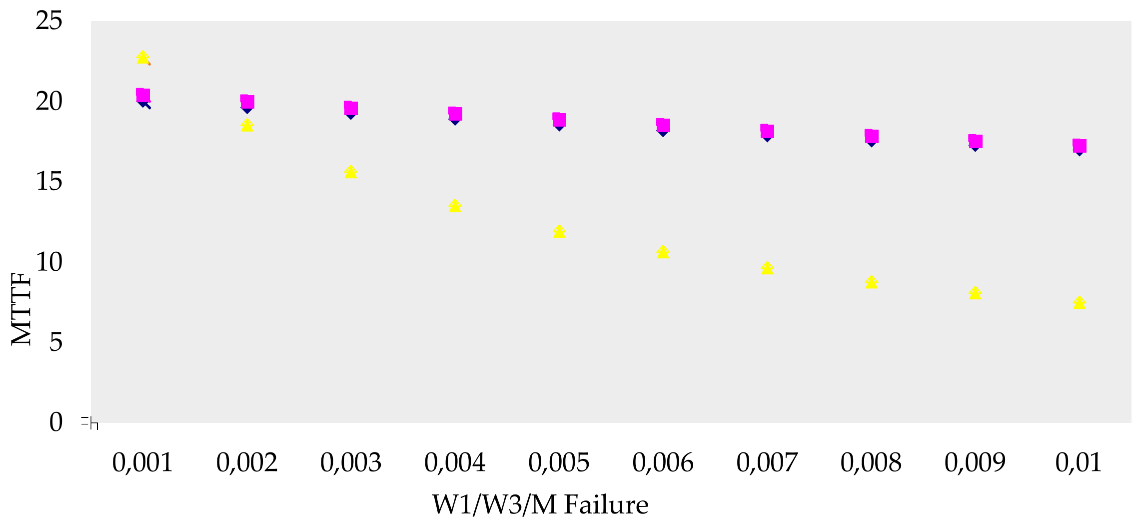

Figure 3: *MTTF Vs workstation W_1, workstation W_3 and Machine failure*

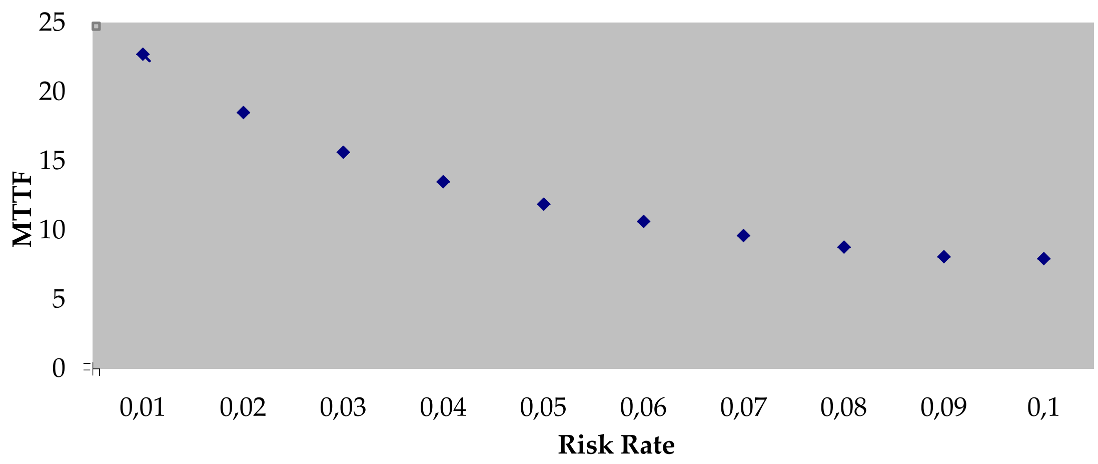

Figure 4: *MTTF Vs Risk Rate*

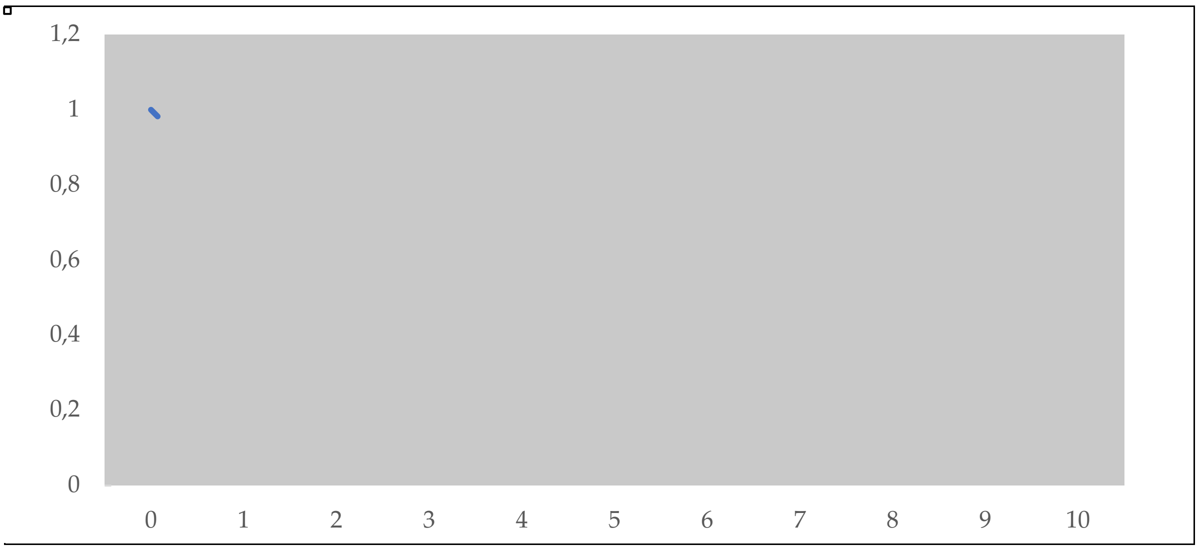

Figure 5: *Availability Vs Time*

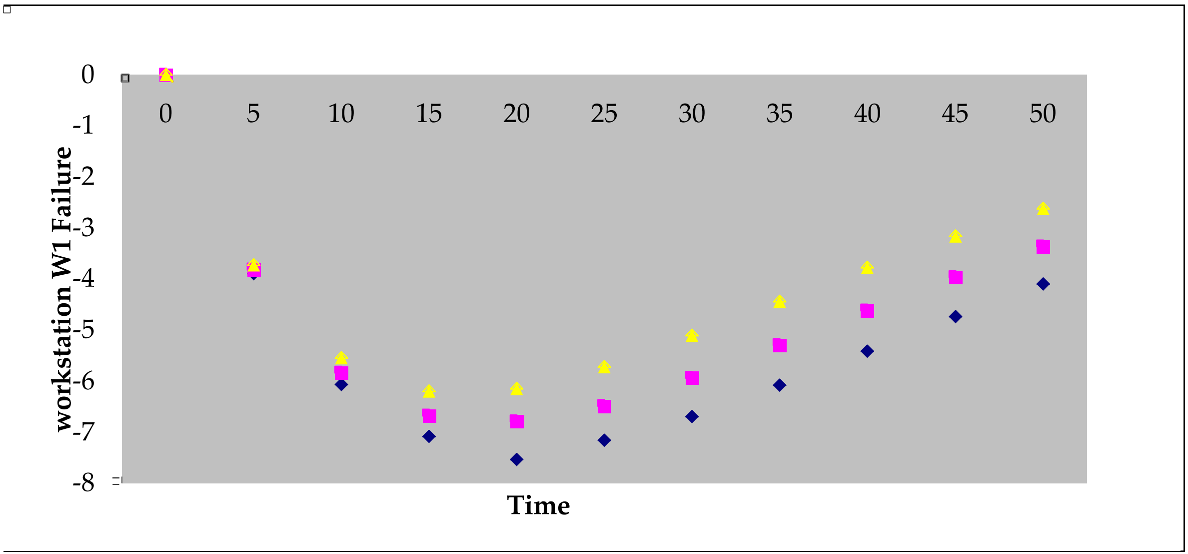

Figure 6: *Sensitivity of system reliability with respect to workstation W_1 failure.*

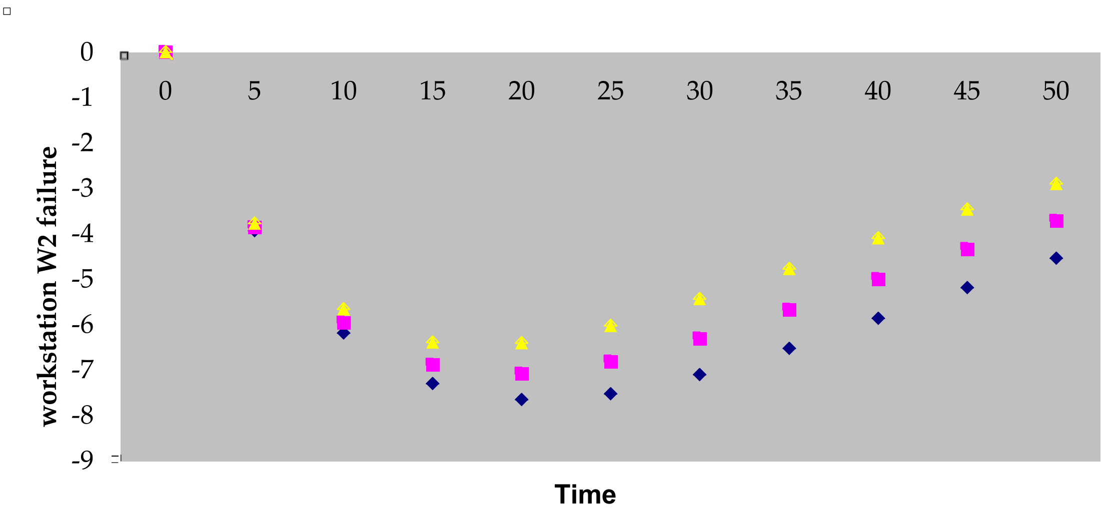

Figure 7: *Sensitivity of system reliability with respect to workstation W_2 failure.*

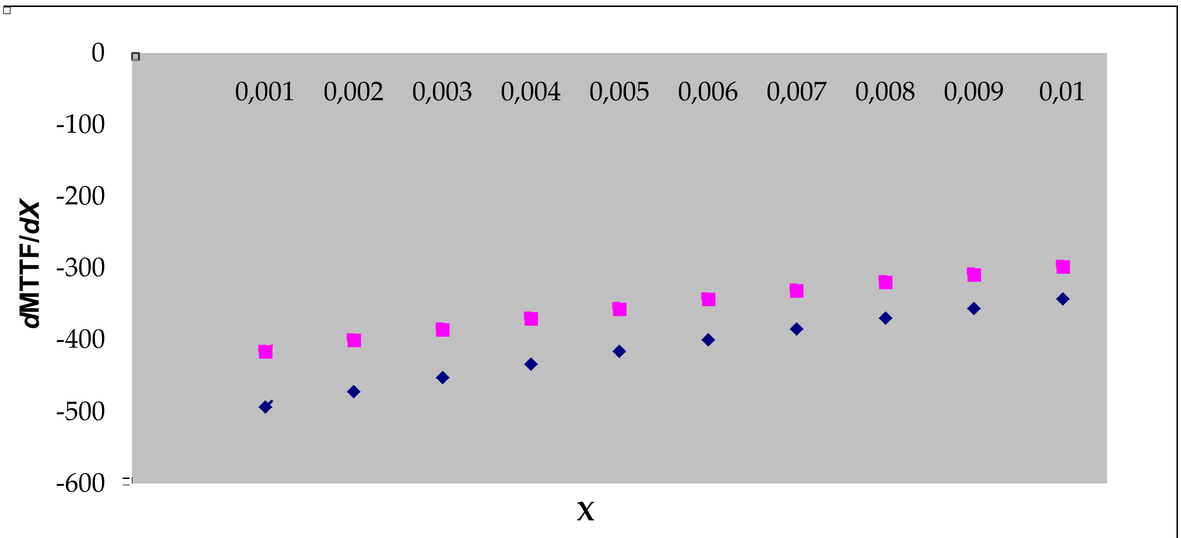

Figure 8: *Sensitivity for MTTF with respect to X1=ψ_V , X2=ψ_P. (X=X1=X2)($\psi_V = \psi_P$=.001, .002 ….01)*

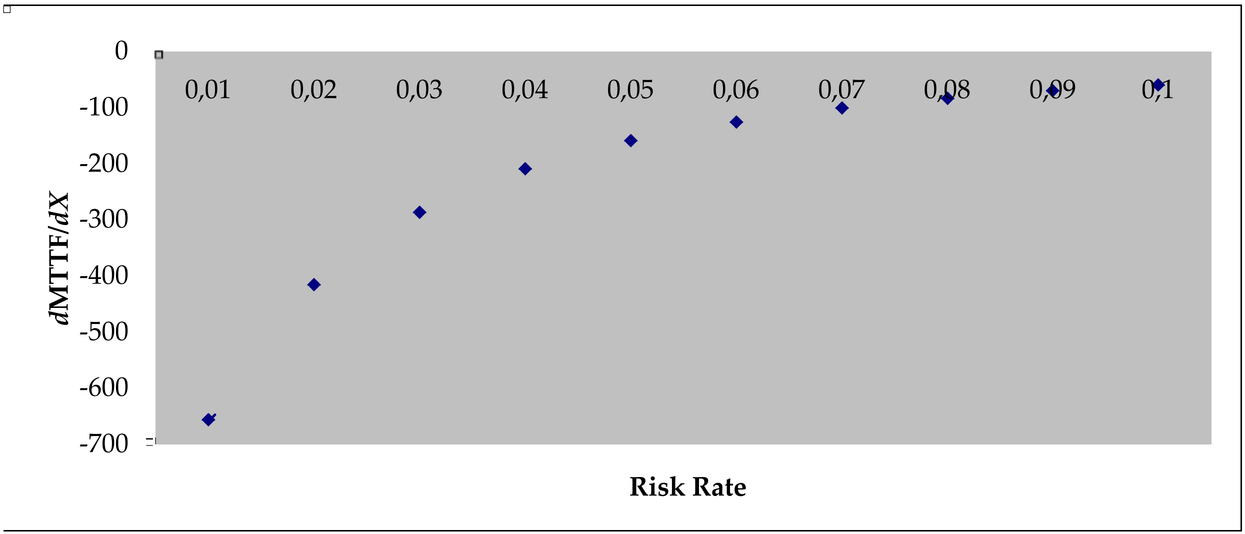

Figure 9: *Sensitivity for MTTF with respect to X=* γ_Q *(*γ_Q*=.01,. 02,…,.1)*

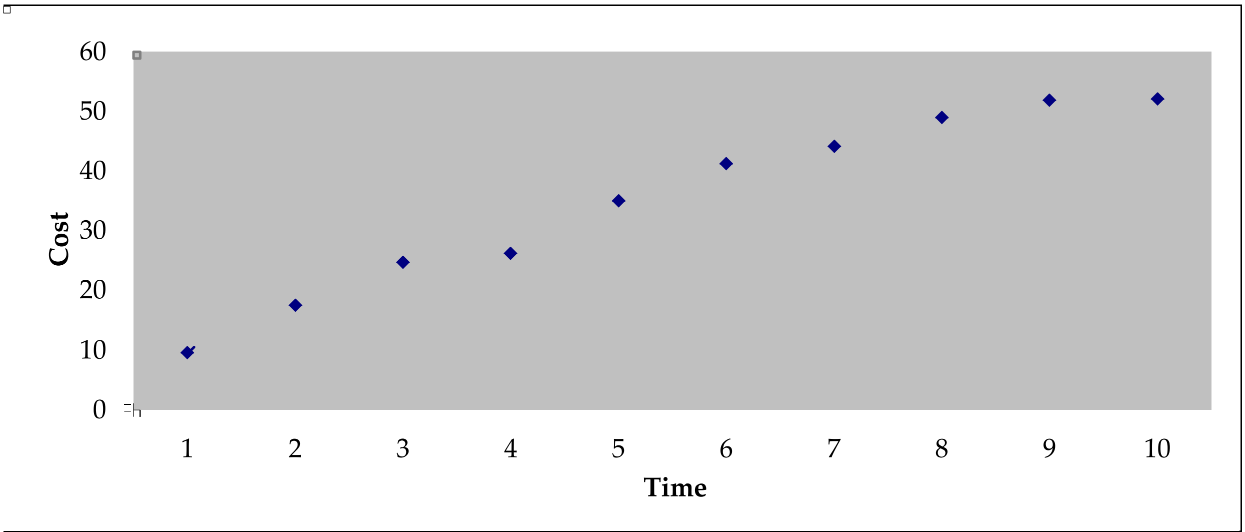

Figure 10: *Cost Vs Time*

5. Conclusion

In this work, the operational behavior of a k-out of-n configuration system is discussed including risk factor using mathematical modelling technique. Also, a comparative analysis of reliability, availability, MTTF, risk, sensitivity, and profit function are done with time for different workstations. The proposed technique has an advantage of analyzing reliability of a complex manufacturing system in a more flexible way.

The study may help a manufacturing industry in:

a. Handling resources and suppliers
b. Planning of production strategies and maintenance policies
c. Decision making.

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On the Minimum of Exponential and Teissier Distributions

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Abstract

In reliability theory minimum of two random variables has a significant meaning, and models with increasing failure rates play a vital role. Motivated by these facts, in this article, a two-parameter lifetime distribution with an increasing failure rate is constructed by considering the method of a minimum of two independent random variables following the exponential and Teissier distributions and studied in detail. Several exciting features, such as moments, quantiles, Bonferroni and Lorenz curves, entropies, stress–strength reliability, moments of a residual lifetime, and order statistics, are derived for the proposed distribution. For the estimation purpose, eight different techniques have been used, including maximum likelihood, ordinary least square, weighted least square, Cramer-von Mises, maximum product spacing, Anderson-Darling, right-tailed Anderson-Darling, and bootstrapping (parametric and nonparametric). The performance of these estimators is compared using three real datasets. The exact Fisher information matrix elements are derived, and confidence intervals based on the information matrix and bootstrapping techniques are constructed. A simulation study is carried out to see the efficiency of the maximum likelihood in terms of mean square error and bias. Negative log-likelihood, Akaike information criteria, Bayesian information criteria, Consistent Akaike information criteria, and Hannan-Quinn information criteria are the goodness-of-fit statistics employed. Furthermore, other nonparametric test statistics such as Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises are used for model selection. Moreover, three real datasets related to epidemiology, seismology, and reliability are modeled and compared with exponential, exponentiated exponential, Lindley, exponentiated Lindley, Rayleigh, exponentiated Rayleigh, Gompertz, exponentiated Gompertz, Weibull, and exponentiated Weibull distributions to demonstrate how the suggested model performs in practice. And it is observed that the proposed distribution provides a better fit among all considered models, according to most of the test statistics. The proposed lifetime distribution is unimodal and capable of modeling positive datasets with an increasing failure rate which contains Gompertz one-parameter model as a particular case. It is a simple model with only two parameters resulting from expressions for different characteristics that are analytically tractable. So, it is expected that it will be helpful in various disciplines where such types of data exist, such as reliability, lifetime modeling, and survival analysis.

Keywords: Probability distribution, Moments, Information Matrix, Maximum likelihood estimator, Bootstrap, Simulations.

1. Introduction

To model, the frequency of mortality associated with aging alone, Teissier [1] developed an increasing failure rate distribution, known as the Teissier distribution (TD). Muth [2] pointed out that the TD has a heavier tail than some classical distributions like the gamma, Weibull, and log-normal distributions. The TD was used by Rinne [3] to model the lifetime of a real dataset related to used motor cars. Leemis and McQueston [4] established a univariate distributional relationship, in which this model was reconsidered and renamed the "Muth distribution." Some statistical features of Muth distribution were thoroughly investigated by Jodra et al. [5]. Irshad et al. [6, 7] studied inference and some other extensions of the Muth distribution. Saroj et al. [12] introduced inverse muth distribution.

Recently, exponentiated Teissier distribution (ETD) has been given by Sharma et al. [11]. Eghwerido [12], and Poonia and Aazad [13] applied the alpha power transform (APT) technique of Mahdavi and Kundu [14] on TD and ETD, respectively. The exponential distribution (ED) is a well-known classical distribution with some distinguishing features such as a constant hazard rate and memorylessness. Related references on exponential distributions can be found in the literature, for example, see Gupta and Kundu [15], Nadarajah and Haghighi [16] and Mahdavi and Kundu [14]. Gompertz proposed that human mortality increases exponentially with age. Makeham extended Gompertz's suggestion of competing risks by adding one and two parameters to the standard Gompertz distribution known as Gompertz Makeham-I (GMD-I) and Gompertz Makeham-II (GMD-II) distributions, respectively. Chapter 10 of Marshall and Olkin [17] provides a comprehensive review of the Gompertz and all extensions made by Makeham. According to chapter 10 of Marshall and Olkin [17], GMD-I has three cases. The cdf of the second case of GMD-I is given as

$$F(x;\theta,\beta,\xi) = 1 - e^{-\xi(e^{\theta x}+\beta\theta x-1)}\ \theta > 0, -1 \leq \beta < 0,\ \xi > 0. \tag{1}$$

However, TD is a particular case of the second case of GMD-I when $\xi = 1$ and $\beta = -1$. The case of $\beta = -1$ is under communication. Recently, many lifetime distributions has been developed in reliability theory, Deepthy and Sebastian [9] developed Burr III Modified Weibull Distribution, Manoharan and Kavya [13] extended Lomax distribution to construct a reliability model. Some probable scenarios that arise in real-life applications because of the distribution of the minimum of two random variables are fascinating, see chapters 5 and 17 of Marshall and Olkin [17].

Suppose X_1 and X_2 are two independent random variables follow Teissier and exponential distribution with parameters θ and $\theta\lambda$ respectively. The cumulative density functions (cdfs) of X_1 and X_2 are given by (for $x > 0,\ \theta > 0,\ \lambda \geq 0$)

$$F_{X_1}(x;\theta) = 1 - e^{\theta x - e^{\theta x}+1},\ \ F_{X_2}(x;\theta,\lambda) = 1 - e^{-\theta\lambda x}, \tag{2}$$

respectively. Suppose X=Minimum$\{X_1, X_2\}$. The cdf, pdf, survival function, hazard rate function (hrf), cumulative hazard rate and reversed hazard rate of METD are given by (for $x > 0,\ \theta > 0,\ \lambda > 0$)

$$F(x;\theta,\lambda) = 1 - e^{1+\theta x - e^{\theta x} - \theta\lambda x}, \tag{3}$$

$$f(x;\theta,\lambda) = \theta\left(\lambda - 1 + e^{\theta x}\right) e^{1+\theta x - e^{\theta x} - \theta\lambda x}, \tag{4}$$

$$S(x;\theta,\lambda) = e^{1+\theta x - e^{\theta x} - \theta\lambda x},\ h(x;\theta,\lambda) = \theta\left(\lambda - 1 + e^{\theta x}\right), \tag{5}$$

$$H(x;\theta,\lambda) = (e^{\theta x} + \theta\lambda x - \theta x - 1),\ r(x;\theta,\lambda) = \frac{\theta\left(\lambda - 1 + e^{\theta x}\right) e^{\theta x+1}}{e^{\theta\lambda x + e^{\theta x}} - e^{\theta x+1}}. \tag{6}$$

Interestingly, the pdf of METD can be obtained from second case of GMD-I as a special case by substituting $\xi = 1$ and $\beta = \lambda - 1$ in Eqn.(1), in this case $0 \leq \lambda < 1$. Unfortunately, this case has not received much attention in the literature. However, the METD model work for $\lambda \geq 1$ also. It should be noted that Gompertz's one-parameter distribution is a particular case of METD when $\lambda = 1$ in METD.

The rest of the article is organized as follows. In Section 2, some statistical properties of the METD have been derived. Section 3 deals with the estimation of the parameters of METD. In Section 4, simulation is carried out. In section 5 three applications are presented to show that the proposed distribution can be used quite effectively in analyzing the real-life datasets. Finally, section 6 provides some conclusions.

2. Statistical Properties

In this section some basic features of the proposed distribustions such as shape analysis of pdf and hrf of METD, moments, quantiles, Bonferroni and Lorenz Curve, Renyi entropy, stress-strength reliability (ssr), moments of residual life function and order statistics are studied.

2.1. Shape of pdf and hrf

The pdf of METD is log-concave as

$$(\log f(x))'' = \frac{-\theta^2 e^{\theta x}\left[(e^{\theta x}-1)^2 + 2\lambda(e^{\theta x}-1) + (\lambda-1/2)^2 + 3/4\right]}{(e^{\theta x}+\lambda-1)^2}, \tag{7}$$

is negative for all $x > 0,\ \theta > 0,\ \lambda > 0$. Moreover, $\lim_{x\to 0^+} f(x) = \theta\lambda$ and $\lim_{x\to\infty} f(x) = 0$. The pdf is decreasing for $\lambda \geq 1$ and having a unique mode at $\frac{1}{\theta}\log\left(\frac{1}{2}\left(3-2\lambda+\sqrt{5-4\lambda}\right)\right)$ for $0 \leq \lambda < 1$, see Fig.1(a). Also $(h(x))' = \theta^2 e^{\theta x} > 0 \implies$ hrf is exponentially increasing.

2.2. Moment generating function and moments

By using $u = e^{\theta x}$, the moment generating function (mgf) of METD can be written as

$$M_X(t) = e\left[E^0_{\lambda-\frac{t}{\theta}-1}(1) - (1-\lambda)E^0_{\lambda-\frac{t}{\theta}}(1)\right], \tag{8}$$

where

$$E^l_s(z) = \frac{1}{\Gamma(l+1)}\int_1^\infty (\log u)^l e^{-zu}u^{-s}\,ds, \tag{9}$$

$l > -1$, $s \in \mathbb{R}$ and $\Gamma(.)$ is the gamma function. Moreover, the rth derivative of $M_X(t)$ at $t = 0$ also the rth moment about origin, can be given as

$$E(X^r) = M_X^{(r)}(0) = e\theta^{-r}\Gamma(r+1)\left(E^r_{\lambda-1}(1) - (1-\lambda)E^r_\lambda(1)\right). \tag{10}$$

Using the moments, mean, variance, skewness and excess kurtosis of the METD can be calculated and shown in Fig.1.

2.3. Quantile function

By inverting the cdf of METD, the quantile function of METD(θ, λ) can be expressed as

$$\xi_p = \begin{cases} \frac{1}{\theta}\log\left[(\lambda-1)W_{-1}\left(\frac{e^{\frac{1}{\lambda-1}}(1-p)^{\frac{1}{1-\lambda}}}{\lambda-1}\right)\right] & \text{if } \lambda < 1 \\ \frac{1}{\theta}\log\left[(\lambda-1)W\left(\frac{e^{\frac{1}{\lambda-1}}(1-p)^{\frac{1}{1-\lambda}}}{\lambda-1}\right)\right] & \text{if } \lambda > 1 \end{cases},\quad \frac{1}{\theta}\log\left[1-\log(1-p)\right] \text{ if } \lambda = 1 \tag{11}$$

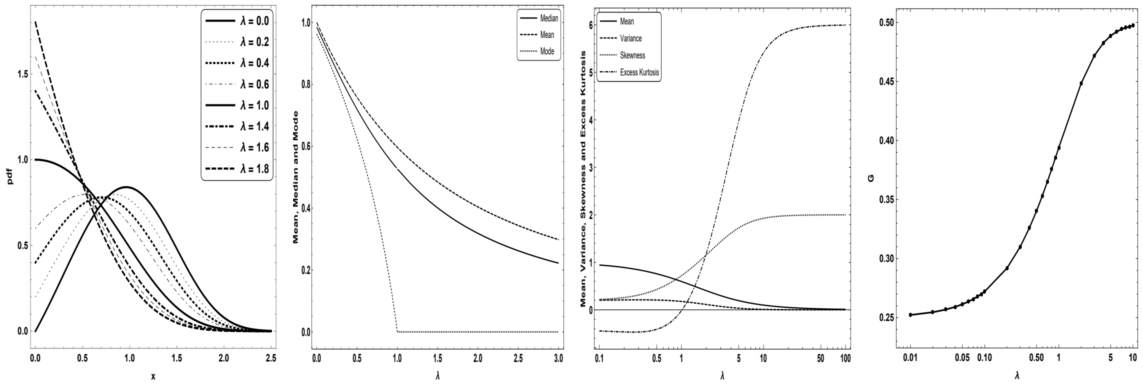
where $p \in (0, 1)$, $W(.)$ represent the Lambert-W function(see Jodra [18]) and $W_{-1}(.)$ is the negative branch of the Lambert-W function, $p = 0.25, 0.50, 0.75$ corresponds to the first, second(median) and third quartiles.

2.4. Bonferroni and Lorenz Curve

The Bonferroni curve, Lorenz curve and Gini coefficient are defined as $B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(t)\,dt$, $L(p) = \frac{1}{\mu}\int_0^p F^{-1}(t)\,dt$, and $G = 1 - 2\int_0^1 L(p)\,dp$, respectively, where $0 < p < 1$, $\mu = \int_0^1 F^{-1}(t)\,dt$ and $F^{-1}(.)$ is the quantile function. Lorenz curve for METD is given as

$$L(p) = 1 - \frac{J\left(\theta, \xi_p, 1, \lambda-1, 1\right) - (1-\lambda)J\left(\theta, \xi_p, 1, \lambda, 1\right)}{J(\theta, 0, 1, \lambda-1, 1) - (1-\lambda)J(\theta, 0, 1, \lambda, 1)}, \tag{12}$$

where $J(\theta, t, r, s, z) = \frac{1}{\Gamma(r+1)}\int_{e^{\theta t}}^\infty (\log u)^r e^{-zu}u^{-s}\,du$.



(a) **(b)** **(c)** **(d)**

Figure 1: *Plots of (a) Probability density function, (b) Mean, median, and mode, and (c) Mean, variance, skewness, and excess kurtosis, (d) Gini coefficient, for the METD when $\theta = 1$.*

2.5. Renyi entropy

The Renyi entropy of a non-negative continuous random variable X with pdf $f(x)$ is a measure of variation and is defined as $I_R(\gamma) = \frac{1}{1-\gamma} \log \int_0^\infty f(x)^\gamma dx$, where $\gamma > 0$ and $\gamma \neq 1$. If $X \sim$METD(θ, λ) then the Renyi entropy of X is given by (using $u = e^x$)

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left[e^\gamma \theta^{\gamma-1} H(\gamma, \gamma(\lambda-1)+1, \lambda-1, \gamma) \right], \tag{13}$$

where $H(z, s, c, p) = \int_1^\infty u^{-s} e^{-zu} (c+u)^p \, du,\ c > -1$.

2.6. Stress strength reliability

Suppose X_1 and X_2 are two random variables from the METD family such that $X_1 \sim$ METD(θ_1, λ_1) and $X_2 \sim$ METD(θ_2, λ_2). The ssr of METD is specified as

$$R = P(X_1 > X_2) = \int_0^\infty f_{X_1}(x) F_{X_2}(x) \, dx. \tag{14}$$

using $u = e^{\theta x}$, the expression of R can be written as

$$R = 1 - e^2 \left(Q\left(\theta, \theta\lambda_2 - \theta + \lambda_1 - 1\right) + (\lambda_1 - 1)\, Q\left(\theta, \theta\left(\lambda_2 - 1\right) + \lambda_1\right) \right), \tag{15}$$

where $Q(\theta, s) = \int_1^\infty u^{-s} e^{-(u+u^\theta)} \, du$ and $\theta = \theta_2/\theta_1$.

2.7. Moments of residual life function

The rth moment of residual life of a random variable X with pdf $f(x)$ and cdf $F(x)$ is given by

$$m_r(t) = E[(X-t)^r | X > t] = \frac{1}{1-F(t)} \int_t^\infty (x-t)^r f(x) \, dx, \tag{16}$$

where $r = 1, 2, 3, \ldots$.
The mean and variance of residual life may be expressed as $m(t) = m_1^{'}(t) - t$, and $V(t) = m_2^{'}(t) - (m_1^{'}(t))^2$ where $m_r^{'}(t) = \frac{1}{1-F(t)} \int_t^\infty x^r f(x) \, dx$. By using $u = e^{\theta x}$ and defining $J(\theta, t, r, s, z) = \frac{1}{\Gamma(r+1)} \int_{e^{\theta t}}^\infty (\log u)^r e^{-zu} u^{-s} \, du$, the numerator of $m_r^{'}(t)$ can be written as

$$\int_t^\infty x^r f(x) \, dx = e\Gamma(r+1)\theta^{-r} \left[J(\theta, t, r, \lambda-1, 1) + (\lambda-1) J(\theta, t, r, \lambda, 1) \right]. \tag{17}$$

2.8. Order statistics

Assume that $X_1, X_2, ..., X_n$ is a random sample of size n drawn from a population with the pdf $f(x)$ and the related order statistics $X_{(1)}, X_{(2)}, ..., X_{(n)}$. The pdf of rth order statistics of METD is given as

$$f_{X(r)}(x) = \frac{\theta}{B(r, n-r+1)} \left(\lambda - 1 + e^{\theta x}\right) (1-\phi(x))^{r-1} (\phi(x))^{n-r+1}, \tag{18}$$

where $\phi(x) = e^{1-\theta\lambda x+\theta x-e^{\theta x}}$, $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $x > 0$, $\theta > 0, \beta > 0, r \leq n$. The mth moment of rth order statistics $X_{(r)}$ is derived as

$$E[X_{(r)}^m] = \frac{\theta^{-m}\Gamma(m+1)}{B(r, n-r+1)} \sum_{i=0}^{n-r} \sum_{j=0}^{i+r-1} (-1)^{i+j} e^{j+1} \binom{i+r-1}{j} \binom{n-r}{i} \Big[E_{q_j}^m(j+1) + (\lambda-1)E_{q_j+1}^m(j+1)\Big], \tag{19}$$

where $E_s^l(z)$ given in Eqn. (9) and $q_j = (j+1)(\lambda-1)$.

3. Estimation of Parameters

For the estimation purpose, eight different techniques have been used, including maximum likelihood (MLE), ordinary least square (OLS), weighted least square (WLS), Cramer-von Mises (CVM), maximum product spacing (MPS), Anderson-Darling (AD), right-tailed Anderson-Darling (RTAD), and bootstrapping (parametric and nonparametric). For the OLS and MPS techniques, see Swain et al. [19], and Cheng and Amin [20], respectively. A general theory of various estimation techniques can be found in Sharma et al. [11] and Dey et al. [**?**]. For parametric and nonparametric bootstrap estimation, 1000 samples were generated according to the algorithm given by Kharazmi et al. [21]. The discussed estimation procedures are applied to two real datasets, and the results are shown in section 5.

3.1. Method of the Maximum likelihood

The maximum likelihood (MLE) approach is the most extensively used approach for parameter estimation. This approach has numerous flexible properties, including consistency, asymptotic efficiency, and invariance. Suppose $x_1, x_2, ..., x_n$ be a sample of size n from the METD. For the vector of parameters $\Theta = (\theta, \lambda)$, the log-likelihood function of METD is given as

$$l(\Theta) = n\log\theta + \sum_{i=1}^{n} \left[1 - e^{\theta x_i} + \theta(1-\lambda)x_i\right] + \sum_{i=1}^{n} \log\left[\lambda - 1 + e^{\theta x_i}\right]. \tag{20}$$

The MLEs of the parameters have been calculated numerically using Mathematica 12.3. After differentiation of the log-likelihood function with respect to the parameters θ and λ, the components of the score vector $U(\Theta)$ can be expressed as

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{x_i e^{\theta x_i}}{\lambda - 1 + e^{\theta x_i}} + \sum_{i=1}^{n} x_i(1-\lambda-e^{\theta x_i}),\ U_\lambda = \sum_{i=1}^{n} \frac{1}{\lambda - 1 + e^{\theta x_i}} - \theta\sum_{i=1}^{n} x_i. \tag{21}$$

The MLEs of the parameters can be obtained by setting these equations to zero and solving them. Another advantage of the MLE approach is that it is useful to construct approximated confidence interval (ACI) of the parameters, see Lawless [**?**]. The exact 2×2 information matrix $I(\Theta)$ required for interval estimate of METD parameters defined as $I(\Theta) = \begin{pmatrix} I_{\theta,\theta} & I_{\theta,\lambda} \\ I_{\lambda,\theta} & I_{\lambda,\lambda} \end{pmatrix}$. The members of the $I(\Theta|x)$ for the METD are given as

$$I_{\theta,\theta} = -E\Big(\frac{\partial^2 l(\Theta|x)}{\partial\theta^2}\Big) = \theta^{-2}\big[2e\big\{(\lambda-1)(E_{\lambda-1}^2(1) - J_{\lambda-1}^2(\lambda-1)) + E_{\lambda-2}^2(1)\big\} + 1\big], \tag{22}$$

RT&A, No 4 (71)
Volume 17, December 2022

$$I_{\theta,\lambda} = I_{\lambda,\theta} = -E\Big(\frac{\partial^2 l(\Theta|x)}{\partial\theta\partial\lambda}\Big) = \frac{1}{\theta}\Big[e(E^1_{\lambda-1}(1) + (\lambda-1)E^1_{\lambda}(1) + J^1_{\lambda-1}(\lambda-1))\Big], \tag{23}$$

and

$$I_{\lambda,\lambda} = -E\Big(\frac{\partial^2 l(\Theta|x)}{\partial\lambda^2}\Big) = eJ^0_{\lambda}(\lambda-1), \tag{24}$$

where the integral $E^l_s(z)$ is defined in Eqn.(9) and the integral $J^r_s(k) = \frac{1}{\Gamma(r+1)}\int_1^\infty \frac{e^{-u}u^{-s}\log^r(u)}{k+u}\,du$. The multivariate normal distribution $N_2(0, I(\hat{\Theta})^{-1})$ may be used to generate confidence intervals for model parameters under usual regularity conditions.

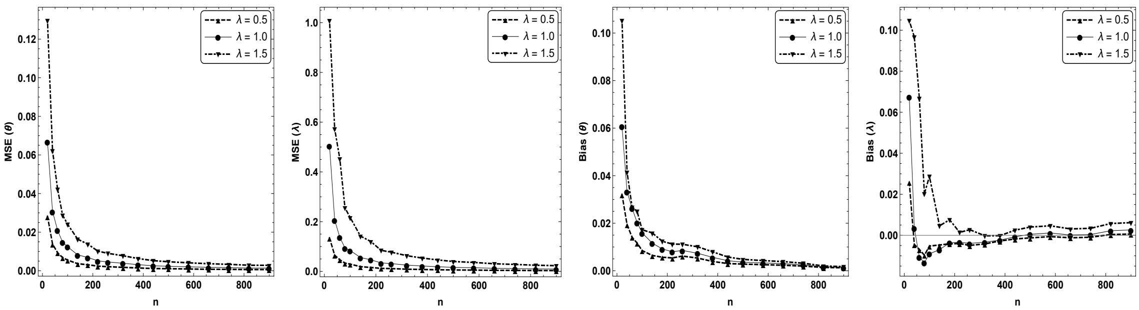


Figure 2: *The MSE and bias of the parameters for the simulated samples.*

4. Simulation

To investigate the performance of the MLEs for the METD parameters, several simulations are explored for different sample sizes. The samples are generated from Eqn. (11), using the inverse cdf technique. The parameters values are taken as $\theta = 1$ and $\lambda = 0.5, 1.0$ and 1.5 and the sample sizes are selected as $n = 20, 40, 60, 80, 100, 140, 180, 220, 260, 320, 380, 440, 500, 580, 660, 740, 820$ and 900. Each sample size is repeated 1000 times, biases and mean squared errors are calculated. Fig.2 displays the results of the simulation. The trends in Fig.2 reveal that as the sample size increases, the MSEs and bias of the MLEs decay toward zero, as expected by first-order asymptotic theory.

5. Data Analysis

Three real datasets are being used to demonstrate the practical significance of the METD model. The first dataset was downloaded from the webportal of the World Health Organization (https://covid19.who.int/) on October 12, 2021, which denotes the daily number of deaths in South Africa due to the novel coronavirus from May 11, 2020, to June 28, 2020. Currently, the first dataset is slightly modified by the WHO. The second dataset is related to seismology and is taken from the Wolfram data repository, which indicates the earthquake waiting times in days. The third dataset is taken from Murthy et al. [25], which is about aircraft windshield data and demonstrates the service times of windshields that had not failed at the time of observation.

- Dataset I
  8, 12, 1, 13, 19, 9, 14, 3, 22, 26, 27, 30, 28, 10, 22, 52, 43, 28, 25, 34, 32, 40, 22, 50, 37, 56, 60, 44, 46, 82, 82, 48, 74, 70, 69, 57, 88, 57, 49, 63, 94, 46, 53, 61, 111, 103, 87, 48, 73.

- Dataset II
  840, 1901, 40, 139, 246, 157, 695, 1336, 780, 1617, 145, 294, 335, 203, 638, 44, 562, 1354, 436, 937, 33, 721, 454, 30, 735, 121, 76, 36, 384, 38, 150, 710, 667, 129, 365, 280, 46, 40, 9, 92, 434, 402, 556, 209, 82, 736, 194, 99, 599, 220, 584, 759, 304, 83, 887, 319, 375, 832, 263, 460, 567, 328.

- Dataset III
  0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

Table 1 give a brief summary of the datasets and fitted METD. The datasets are starting from non-zero and right-skewed. Therefore, METD may be the right choice to model these datasets. Parametric and non-parametric bootstrap techniques also have been applied for estimation purpose by adopting the methodology of Efron and LePage [22]. According to the described methodology in the estimation section, the exact information matrix for the datasets I, II and III are given as: $\begin{pmatrix} 14517.2 & 77.4383 \\ 77.4383 & 0.7927 \end{pmatrix}$, $\begin{pmatrix} 3.0789\times 10^{6} & 484.359 \\ 484.359 & 0.0799 \end{pmatrix}$ and $\begin{pmatrix} 30.5202 & 3.54375 \\ 3.54375 & 0.787313 \end{pmatrix}$, respectively. Inverse of the exact information matrix for the datasets I, II and III are given as: $\begin{pmatrix} 0.0001 & -0.0140 \\ -0.0140 & 2.6341 \end{pmatrix}$, $\begin{pmatrix} 6.8421\times 10^{-6} & -0.0414 \\ -0.0414 & 263.358 \end{pmatrix}$ and $\begin{pmatrix} 0.0686361 & -0.308936 \\ -0.308936 & 2.66068 \end{pmatrix}$, respectively. The interval estimates of the parameters based on the expected information matrix are given as: (i) For the dataset I, $\hat{\theta} \in (0.0134, 0.0201)$, $\hat{\lambda} \in (0.0253, 0.9342)$. (ii) For the dataset II, $\hat{\theta} \in (0, 0.0012)$, $\hat{\lambda} \in (0, 7.2586)$ and (iii) For the dataset III, $\hat{\theta} \in (0.3018, 0.4312)$, $\hat{\lambda} \in (0.0812, 0.8867)$. The point and interval estimates of the parameters based on the parametric bootstrap are given as: (i) For the dataset I, $\hat{\theta} = 0.0170, \hat{\lambda} = 0.4903, \hat{\theta} \in (0.0137, 0.0206), \hat{\lambda} \in (0.1577, 1.0143)$. (ii) For the dataset II, $\hat{\theta} = 0.0007, \hat{\lambda} = 3.1266, \hat{\theta} \in (0.0003, 0.0012), \hat{\lambda} \in (1.4779, 6.6621)$ and (iii) For the dataset III, $\hat{\theta} = 0.3709, \hat{\lambda} = 0.4771, \hat{\theta} \in (0.3096, 0.4411), \hat{\lambda} \in (0.1274, 0.9156)$. The point and interval estimates of the parameters based on the non-parametric bootstrap are given as: (i) For the dataset I, $\hat{\theta} = 0.0169, \hat{\lambda} = 0.4885, \hat{\theta} \in (0.0141, 0.0201), \hat{\lambda} \in (0.1315, 0.9469)$. (ii) For the dataset II, $\hat{\theta} = 0.0007, \hat{\lambda} = 3.1244, \hat{\theta} \in (0.0003, 0.0012), \hat{\lambda} \in (1.4249, 6.0172)$ and (iii) For the dataset III, $\hat{\theta} = 0.3696, \hat{\lambda} = 0.4913, \hat{\theta} \in (0.3115, 0.4367), \hat{\lambda} \in (0.1812, 0.9456)$ All the seven different estimation

Table 1: *Summary of the datasets and fitted METD.*

	Size	Min	$Q_{.25}$	Median	$Q_{.75}$	Max	Mean	Skew	Ex-Ku
Data I	49	1	25	46	61	111	45.46	0.42	-0.52
METD	-	0	23.99	43.26	64.22	∞	45.47	0.41	-0.41
Data II	62	9	129	328	667	1901	437.21	1.49	2.52
METD	-	0	136.87	323.22	624.12	∞	437.20	1.48	2.57
Data III	63	0.04	1.09	2.06	2.81	5.14	2.08	0.43	-0.26
METD	-	0	1.09	1.98	2.94	∞	2.08	0.41	-0.41

approaches, as given in the estimation section, have been applied to estimate the parameters of METD for the both datasets, and results are shown in Table 2 with different test statistics and ranking based on Kolmogorov-Smirnov (KS) test. From Table 2, it may be concluded that according to the KS test, the maximum likelihood (ML) and CVM are the best estimator among considered estimation procedures for dataset I and II respectively, whereas MPS is the worst techniques among all considered methods for the both datasets. For third dataset CVM is most effective estimator and AD is not much efficient among all estimators.

The METD is compared with the following distributions: exponential distribution (ED), exponentiated exponential distribution (EED) of Kundu and Gupta [15], Teissier distribution (TD) of Teissier [1], exponentiated Teissier distribution (ETD) of Sharma et al. [11], Rayleigh distribution (RD), exponentiated Rayleigh distribution (ERD), Lindley distribution, exponentiated Lindley distribution, Gompertz Distribution (GOD), exponentiated Gompertz Distribution (EGOD) of El-Gohary et al. [24], Weibull distribution (WD), and exponentiated Weibull distribution (EWD) of Mudholkar and Srivastava [23]. Several goodness-of-fit (gof) statistics are used for model selection, including negative log-likelihood (NLL), Akaike (AIC), Bayesian (BIC), and Consistent

Table 2: *Estimation of parameters by different techniques and various test statistics with ranking(r).*

Method	$\hat{\theta}$	$\hat{\lambda}$	NLL	OLSS	WLSS	CVMS	MPSS	ADS	RTADS	KS	p-Value(KS)	r
Dataset I												
ML	0.0168	0.4797	227.402	0.0175	6.0462	0.0178	-4.0207	0.1138	0.0556	0.0475	0.9996	1
OLS	0.0162	0.5409	227.453	0.0159	5.0371	0.0190	-4.0177	0.1182	0.0548	0.0507	0.9989	4
WLS	0.0160	0.5599	227.496	0.0161	4.9512	0.0202	-4.0172	0.1262	0.0593	0.0518	0.9984	6
CVM	0.0168	0.4873	227.403	0.0172	5.9974	0.0177	-4.0206	0.1136	0.0556	0.0498	0.9991	3
MPS	0.0157	0.5926	227.587	0.0171	5.1453	0.0228	-4.0168	0.1450	0.0706	0.0551	0.9964	7
AD	0.0166	0.5003	227.408	0.0166	5.5836	0.0178	-4.0195	0.1122	0.0536	0.0489	0.9993	2
RTAD	0.0165	0.5208	227.421	0.0161	5.3265	0.0182	-4.0187	0.1137	0.0530	0.0513	0.9987	5
Dataset II												
ML	0.0006	3.2178	438.650	0.0343	17.6435	0.0335	-4.5439	0.2896	0.1375	0.0610	0.9642	6
OLS	0.0005	3.8710	438.714	0.0331	17.7131	0.0338	-4.5406	0.2960	0.1444	0.0582	0.9764	2
WLS	0.0005	3.8398	438.690	0.0338	17.4445	0.0341	-4.5409	0.2905	0.1404	0.0596	0.9708	4
CVM	0.0006	2.9352	438.662	0.0340	17.9474	0.0329	-4.5459	0.2940	0.1378	**0.0577**	**0.9784**	**1**
MPS	0.0002	8.7240	438.942	0.0364	18.2424	0.0386	-4.5389	0.3158	0.1631	0.0612	0.9631	7
AD	0.0006	3.4156	438.657	0.0339	17.5170	0.0336	-4.5426	0.2889	0.1376	0.0600	0.9687	5
RTAD	0.0006	3.1760	438.651	0.0338	17.6555	0.0330	-4.5439	0.2900	0.1370	0.0588	0.9742	3
Dataset III												
ML	0.3665	0.4840	98.1613	0.0347	16.0498	0.0379	-4.6152	0.2623	0.1334	0.0681	0.9127	4
OLS	0.3799	0.4184	98.2516	0.0322	16.8590	0.0325	-4.6204	0.2722	0.1394	0.0637	0.9458	2
WLS	0.3682	0.4751	98.1627	0.0341	16.0314	0.0370	-4.6156	0.2612	0.1329	0.0673	0.9188	3
CVM	0.3888	0.3811	98.4149	0.0332	18.4910	0.0315	-4.6256	0.3026	0.1577	**0.0625**	**0.9534**	**1**
MPS	0.3480	0.5702	98.3114	0.0442	18.2744	0.0514	-4.6127	0.3125	0.1647	0.0709	0.8870	6
AD	0.3660	0.3162	99.2066	0.1814	67.5575	0.1848	-4.6290	1.0657	0.5165	0.0959	0.5740	7
RTAD	0.3695	0.4734	98.1657	0.0340	16.0688	0.0367	-4.6160	0.2611	0.1325	0.0687	0.9074	5

Akaike (CAIC), and Hannan-Quinn (HQIC) information criteria. Other robust test statistics, including Kolmogorov-Smirnov (KS), Anderson-Darling (AD), and Cramer-von Mises (CVM), as well as the p-value of KS, are examined for model selection in addition to these information metrics. The model with the lowest test statistics value and the highest p-value was chosen as the best model among all competitors. Table 3 shows all the relevant gof statistics of the fitted distributions for both datasets. For dataset I, according to AD, CVM, and KS tests, METD achieved the first rank, whereas according to AIC, BIC, CAIC, and HQIC, METD achieved the second rank, and ETD achieved the first rank among all competitor models. According to NLL, EGOD, ETD, and EWD have better ranks than METD. Therefore, as METD is a simple model in comparison with exponentiated models and has a significant p-value, it is concluded that METD may be a good choice to model the first dataset.
METD ranked first in NLL, AD, CVM, and KS tests for dataset II, while METD ranked second in AIC, BIC, CAIC, and HQIC tests, and ED ranked first among all competitors models. ED has no significant p-value. Therefore, it is concluded that METD provides a reasonable fit for the second dataset in comparison with all competitor distributions. For third dataset, METD is consistently achieved best rank according to all gof test statistics. Only it is second under NLL and KS test. According to the AIC and KS test, the top four models are selected for both datasets. Histograms of datasets with pdfs of distributions, are displayed in Fig. 3. Once again, these plots confirm the conclusion that the METD is an appropriate model for these datasets.

6. Conclusion

The METD is a two-parameter distribution proposed in this study. The hazard rate function of the METD is exponentially increasing and the probability density function is unimodal ($0 \leq \lambda < 1$) and decreasing ($\lambda \geq 1$). Two different datasets of different characteristics (unimodal and decreasing) are provided to show the practical significance of the present distribution. Furthermore, the proposed distribution can be used as an alternative to some well-known distributions such as exponential, Lindley, Rayleigh, Gompertz, Weibull, and their exponentiated models, and it is expected that it will provide a better fit for similar datasets than the models discussed in this paper. The METD demonstrated in this study shows its ability to model

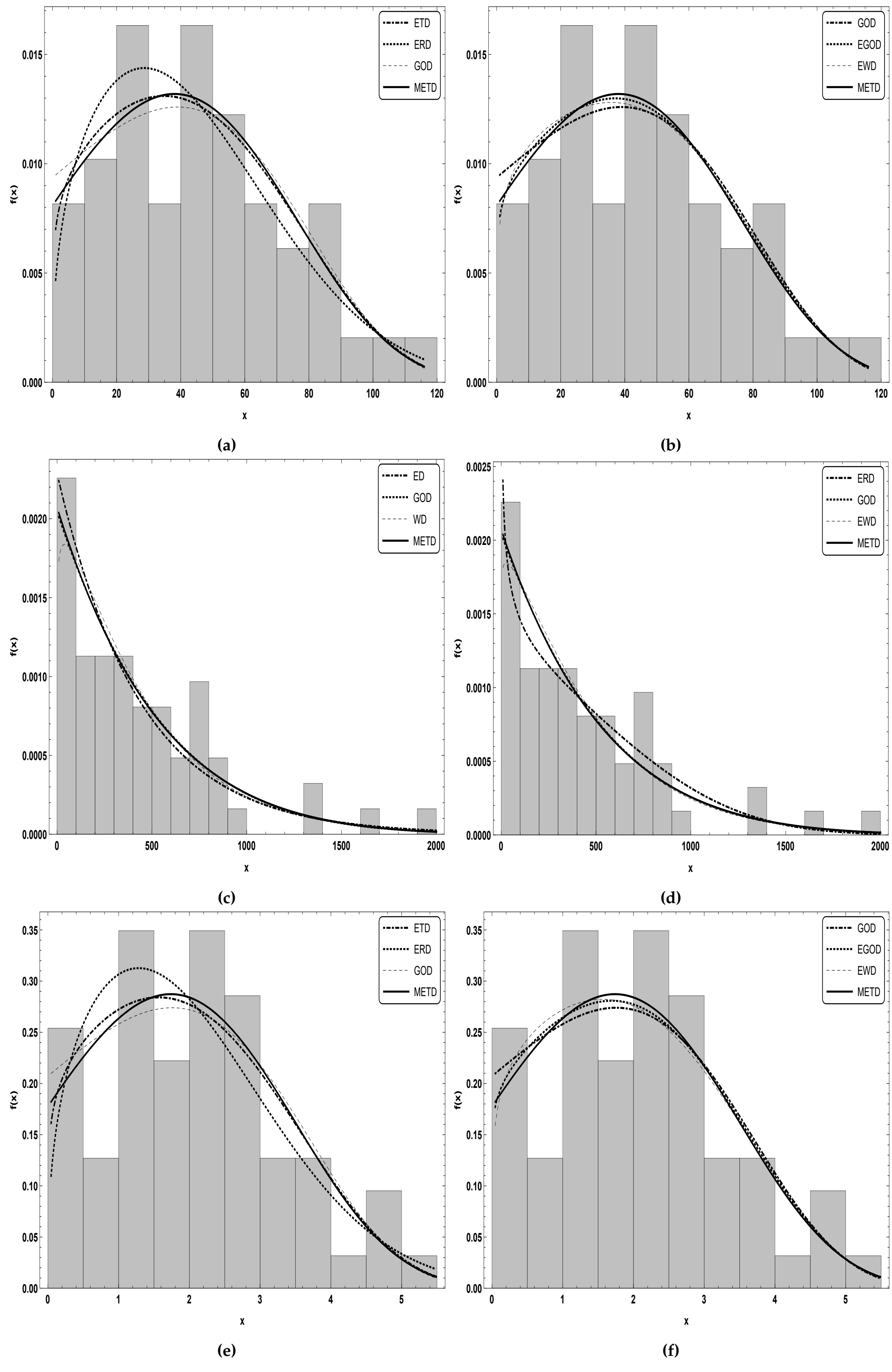


Figure 3: *Histogram with some better fitted pdfs for dataset I (a),(b), dataset II (c),(d) and dataset III (e),(f) where better models according to AIC in first column and better models according to KS test in second column.*

Table 3: *Various goodness-of-fit statistics, the number of parameters (NP) with respective ranking r of all fitted distributions for the first and second datasets.*

Distribution	NP	NLL	r	AIC	r	BIC	r	CAIC	r	HQIC	r	AD	r	CVM	r	KS	PV(KS)	r
Dataset I																		
ED	1	236.035	13	474.070	13	475.962	13	476.962	13	474.788	13	2.8483	12	0.5423	13	0.1999	0.0344	13
EED	2	231.116	11	466.232	11	470.015	11	472.015	11	467.667	11	0.6772	10	0.1161	11	0.1207	0.4377	11
TD	1	233.764	12	469.528	12	471.42	12	472.42	12	470.246	12	3.0822	13	0.3962	12	0.1780	0.0787	12
ETD	2	227.391	2	458.782	1	462.566	1	464.566	1	460.218	1	0.1155	2	0.0180	2	0.0532	0.9977	5
RD	1	230.394	10	462.787	9	464.679	6	465.679	5	463.505	9	0.8837	11	0.0876	8	0.1026	0.6419	8
ERD	2	228.079	6	460.157	4	463.941	4	465.941	6	461.593	4	0.2162	6	0.0378	6	0.0838	0.8529	6
LD	1	230.188	9	462.376	8	464.268	5	465.268	4	463.094	8	0.5572	9	0.0998	10	0.1155	0.4933	10
ELD	2	230.165	8	464.330	10	468.114	10	470.114	10	465.765	10	0.5232	8	0.0911	9	0.1132	0.5191	9
GOD	2	227.437	5	458.875	3	462.659	3	464.659	3	460.310	3	0.1311	5	0.0204	5	0.0506	0.9989	4
EGOD	3	227.383	1	460.766	5	466.442	8	469.442	8	462.919	5	0.1163	3	0.0181	3	0.0479	0.9995	2
WD	2	228.807	7	461.614	7	465.397	7	467.397	7	463.049	7	0.2932	7	0.0442	7	0.0874	0.8158	7
EWD	3	227.401	3	460.803	6	466.478	9	469.478	9	462.956	6	0.1204	4	0.0187	4	0.0484	0.9994	3
METD	**2**	**227.402**	**4**	**458.805**	**2**	**462.588**	**2**	**464.588**	**2**	**460.240**	**2**	**0.1138**	**1**	**0.0178**	**1**	**0.0475**	**0.9996**	**1**
Dataset II																		
ED	1	438.986	7	879.971	1	882.098	1	883.098	1	880.806	1	0.3666	7	0.0534	7	0.0744	0.8562	8
EED	2	438.748	5	881.496	5	885.750	5	887.750	5	883.166	5	0.3574	6	0.0486	6	0.0738	0.8627	7
TD	1	487.937	13	977.874	13	980.001	13	981.001	13	978.709	13	32.2274	13	3.8607	13	0.3786	2.67×10^{-8}	13
ETD	2	442.409	10	888.818	9	893.073	9	895.073	9	890.489	9	1.2232	10	0.1945	10	0.1106	0.4035	10
RD	1	464.709	12	931.417	12	933.544	12	934.544	12	932.252	12	13.7411	12	1.4724	12	0.2574	0.0004	12
ERD	2	439.801	8	883.602	8	887.857	7	889.857	7	885.273	7	0.4831	8	0.0589	8	0.0658	0.9347	3
LD	1	446.268	11	894.536	11	896.663	11	897.663	10	895.371	11	3.6521	11	0.3507	11	0.1594	0.0765	11
ELD	2	438.755	6	881.510	6	885.764	6	887.764	6	883.180	6	0.3317	4	0.0421	4	0.0684	0.9138	5
GOD	2	438.656	2	881.312	3	885.566	3	887.566	3	882.982	3	0.2905	2	0.0337	2	0.0614	0.9622	2
EGOD	3	441.424	9	888.847	10	895.229	10	898.229	11	891.353	10	0.8585	9	0.1241	9	0.0848	0.7306	9
WD	2	438.703	4	881.406	4	885.660	4	887.660	4	883.076	4	0.3406	5	0.0440	5	0.0700	0.9006	6
EWD	3	438.692	3	883.384	7	889.766	8	892.766	8	885.890	8	0.3230	3	0.0403	3	0.0669	0.9261	4
METD	**2**	**438.650**	**1**	**881.301**	**2**	**885.555**	**2**	**887.555**	**2**	**882.971**	**2**	**0.2896**	**1**	**0.0335**	**1**	**0.0610**	**0.9641**	**1**
Dataset III																		
ED	1	109.299	13	220.597	13	222.740	13	223.740	13	221.440	13	3.8816	12	0.7789	13	0.2077	0.0074	13
EED	2	103.547	10	211.093	10	215.380	11	217.380	11	212.779	11	1.3151	10	0.2329	10	0.1437	0.1339	10
TD	1	106.974	12	215.947	12	218.090	12	219.090	12	216.790	12	3.9012	13	0.4427	12	0.1705	0.0453	12
ETD	2	98.288	4	200.577	3	204.863	3	206.863	3	202.263	3	0.3049	4	0.0464	3	0.0761	0.8311	5
RD	1	102.492	9	206.984	8	209.127	8	210.127	5	207.827	8	1.2469	9	0.0841	6	0.0958	0.5754	6
ERD	2	99.198	6	202.397	4	206.683	4	208.683	4	204.083	4	0.5116	6	0.0903	7	0.1067	0.4383	7
LD	1	104.578	11	211.156	11	213.299	10	214.299	10	211.999	10	2.1351	11	0.4159	11	0.1564	0.0821	11
ELD	2	101.888	8	207.776	9	212.063	9	214.063	9	209.462	9	0.9654	8	0.1682	9	0.1300	0.2170	9
GOD	2	98.276	3	200.553	2	204.840	2	206.840	2	202.239	2	0.3033	3	0.0466	4	0.0679	0.9143	1
EGOD	3	98.231	2	202.463	5	208.893	5	211.893	7	204.992	5	0.2890	2	0.0428	2	0.0694	0.9009	3
WD	2	100.318	7	204.635	7	208.922	6	210.922	6	206.321	7	0.6425	7	0.0929	8	0.1086	0.4164	8
EWD	3	98.327	5	202.654	6	209.084	7	212.084	8	205.183	6	0.3105	5	0.0473	5	0.0760	0.8325	4
METD	**2**	**98.161**	**1**	**200.323**	**1**	**204.609**	**1**	**206.609**	**1**	**202.008**	**1**	**0.2623**	**1**	**0.0379**	**1**	**0.0681**	**0.9127**	**2**

Covid-19, earthquake waiting times and service times data appropriately.

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On consistency of Bayesian parameter estimators for a class of ergodic Markov models

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Abstract

The consistency of the Bayesian estimation of a parameter is shown for a class of ergodic discrete Markov chains. J.L. Doob's method was used, offered earlier for the i.i.d. situation. The result may be useful in the reliability theory for models with unknown parameters, in the risk management in financial mathematics, and in other applications.

Keywords: Bayesian estimator; consistency; ergodic Markov chain

MSC2020: 62F12; 62F15; 62M05

1. Introduction

Parameter estimation plays a significant and in some cases possibly even a crucial role in quite a few applications such as the reliability theory for models with unknown parameters, see [4, chapter 3], in the Extreme Values theory for Markov processes, in the risk management in financial mathematics, et al. In the asymptotic sense, one of the basic desirable properties of any estimator in the long run is its consistency, weak or strong, as it shows that the estimation is close to the "true" parameter if the classical setting is accepted. Similarly, in the Bayesian setting consistency means literally the same – convergence to the sample value of the parameter, even though there is no such thing as a "true parameter value" because it is to be sampled from the prior distribution. Also, as it is well-known, Bayesian estimators often work well in the classical setting, too, assuming some fictitious prior distribution for the parameter is chosen.

In this paper the problem of strong consistency is tackled for a certain class of Markov models in the Bayesian setting, and, as was already mentioned, in the classical situation with a fixed nonrandom "true" parameter value. Assume that there is a family of distributions $\{\mathbb{P}^\theta\}$ parameterised by some variable $\theta \in \Theta$, where $\Theta \subset R^m$ is a given parametric space. Any estimator is a measurable function of the observations, or, a bit more generally, a mapping from the space of outcomes Ω, say, to the space $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ which is Borel measurable with respect to the sigma-algebra of the observations $\mathcal{F}^X$; here $\mathcal{B}(\mathbb{R}^m))$ is the Borel sigma-algebra in $\mathbb{R}^m$.

In the Bayesian setting it is assumed that there is some prior distribution for θ on the set Θ; the latter is usually a topological space, and in this paper, it will be assumed that Θ is a domain in $\mathbb{R}^m$ which is not necessarily bounded. S.N. Bernstein and R. von Mises were the first to establish consistency and the first steps towards the asymptotic normality of the Bayesian estimator for some particular i.i.d. cases, see [1, Chapter IV, p.271], [18, pp. 188-192]. The general theory about asymptotic normality was developed later by Le Cam [13] and Ibragimov and Khasmisnky [5]; for more recent results see, for example, [12], [15]. Another direction related to the problem was asymptotic singularity of measures for large observation samples based on martingale theory and developed in [6, 7, 8, 14, 17], et al. Naturally asymptotic normality requires more restrictive assumptions. On the other hand, "just" consistency may often be used for constructions of more efficient estimations by certain modifications. Also, in a situation where the conditions for asymptotic normality are not met, it may be even more desirable to know whether the applied estimator is consistent. Hence, it makes sense to separate the studies of sufficiency conditions for both properties, asymptotic normality and consistency.

In this paper the approach offered for the i.i.d. observations in [3] is used, adjusted for a class of markovian models. An important point in [3] was a Strong Law of Large Numbers for the sample distribution functions (d.f. in what follows) as the number of observations tends to infinity. Also essential was an assumption that theoretical d.f. are different for different parameters. In this paper discrete densities on a finite or countable state space are used. This restriction looks not crucial and likely may be relaxed. At the level of ideas, the most close to this study is the paper [17], where the earlier basic results from [6, 7, 8] are applied precisely to the problem of parameter estimators' consistency. However, formally conditions in for this property [17] and in what follows are different. Also, in a way, this paper is based on a more simple background than that in [6, 7, 8, 17].

The paper consists of this Introduction, The setting, Auxiliary lemmata, Main result (theorem 4), and Proof of theorem 4.

2. The setting

Let$\{X_t\}$ be a homogeneous Markov chain (MC) in discrete time $T = \{0, 1, \ldots\}$ with a finite or countable (denumerable) state space $\mathcal{X} \subset \mathbb{R}^1$ (it will be clear in what follows why it is convenient to work on $\mathbb{R}^1$: although it is not a restriction, but it may be desirable that the elements of the state space are linearly ordered). The transition probabilities are denoted by $p_{ij}(s,t) = \mathbb{P}(X_t = j|X_s = i)) = \mathbb{P}(X_{t-s} = j|X_0 = i)) = p_{ij}(t-s)$ for $s \le t$, and let $\mathcal{P}(t) = (p_{ij}(t))$ be the transition probability matrix over time t; furthermore, they will all depend on a parameter θ. The notion of ergodicity of a MC is not uniquely determined in the literature; in the present paper we understand it as follows.

Definition 1. *A homogeneous MC $(X_n, n = 0, 1, \ldots)$ is called ergodic if there exists a limiting invariant probability measure μ which does not depend on the initial distribution – say, μ_0 – and to which there is a convergence in total variation for each μ_0:*

$$\lim_{t\to\infty} \|p_{\mu_0,\cdot}(t) - \mu_{\cdot}\|_{TV} = 0, \tag{1}$$

where $p_{\mu_0,j}(t) = \mathbb{P}_{\mu_0}(X_t = j)$. Recall that the total variation metric, or distance is given by the formula

$$\|\mu - \nu\|_{TV} := 2 \sup_{A \in \mathcal{F}(\mathcal{X})} (\mu(A) - \nu(A)).$$

As it was said, the transition probabilities depend on a parameter and the problem under consideration is estimation of this parameter given observations on the time interval $[1, n]$ where $n \to \infty$. It is assumed that $\theta \in \Theta \subset \mathbb{R}^m$; Θ is a domain, not necessarily bounded. Naturally, a stationary measure, generally speaking, also depends on θ: denote it from now on by $\mu^\theta(dx)$ and note that under the assumption of convergence (1) it is necessarily unique. We will need the extended process $Y_n = (X_n, X_{n+1})$ which is also a MC on the state space $\mathcal{X} \times \mathcal{X}$. The symbol $\mu^\theta(dx, dx')$ will denote the stationary measure for the MC (Y_n); it is easy to see that such an invariant measure does exist. Assume that the functions $p^\theta(\cdot, \cdot)$ are Borel measurable with respect to the variable θ. Then due to the ergodicity (see (1)) the invariant probabilities are also Borel measurable in θ. Following Doob's approach, *suppose* that a (weak) Law of Large Numbers (LLN) holds true for the MC (X_n) with respect to the corresponding measure $\mathbb{P}^\theta$, for each θ . In this case, LLN is also valid for the MC (Y_n), where $Y_n := (X_n, X_{n+1})$. It is easy to see that these two conditions – LLN for the MC (X_n) and for the MC (Y_n) – are equivalent. Hence, the following assumption will be accepted in what follows.

Assumption 2. *It is assumed that for each $\theta \in \Theta$ and any measurable A, B a convergence holds true,*

$$\left| \frac{1}{T} \sum_{s=0}^{T-1} 1(X_s \in A, X_{s+1} \in B) - \mu^\theta(A \times B) \right| \overset{\mathbb{P}^\theta}{\to} 0, \quad T \to \infty.$$

This assumption is equivalent also to the condition

$$\left|\frac{1}{T}\sum_{s=0}^{T-1} g(Y_s) - \int g(y)\mu^\theta(dy)\right| \stackrel{\mathbb{P}^\theta}{\to} 0, \quad T \to \infty,$$

for any bounded measurable function $g(y)$, where $y = (x, x')$.

Let us collect the comments made earlier in the form of a proposition.

Proposition 3. *Under the assumptions made above the following statements hold:*

1. *If (X_n) is a homogeneous MC then Y_n is also a homogeneous MC.*
2. *If the MC (X_n) is ergodic then the MC (Y_n) is also ergodic, and vice versa.*

Note that, as usual, all sigma-algebras in the text are regarded as completed with respect to the corresponding probability measures.

3. Main result

The Bayesian setting assumes that the parameter θ is random; let it have a prior probability distribution $\mathbb{Q}$ on Θ. Recall that here Θ is a domain in $\mathbb{R}^m$, not necessarily bounded. It is assumed that

$$\mathbb{E}\theta < \infty. \tag{2}$$

Any estimator of the parameter given observations is represented by some Borel measurable function $\hat\theta_n = \hat\theta_n(X_1, \ldots, X_n)$. As it is well-known (cf., for example, [2, chapter 19]), there exists a Borel measurable function ϕ_n such that the Bayesian estimator reads,

$$\mathbb{E}(\theta|X_1, \ldots, X_n) \stackrel{\mathbb{P}^\theta\text{-a.s.}}{=} \phi_n(X_1, \ldots, X_n).$$

So, the statistic $\hat\theta_n := \phi_n(X_1, \ldots, X_n) = \mathbb{E}(\theta|X_1, \ldots, X_n)$ is necessarily $(\mathcal{F}^X_N, \mathcal{B}(\mathbb{R}^m))$-measurable; hence, also $(\mathcal{F}^X_\infty, \mathcal{B}(\mathbb{R}^m))$-measurable, and ϕ_n is measurable with respect to the pair of σ-algebras $(\mathcal{B}(\mathcal{X})^n, \mathcal{B}(\mathbb{R}^m)), \forall n \in \mathbb{N}$, where $\mathcal{B}(\mathcal{X})$ is the set of all subsets of the state space $\mathcal{X}$, that is, $\mathcal{B}(\mathcal{X}) = 2^{\mathcal{X}}$. Recall that a pointwise limit of measurable functions is also measurable.

Theorem 4. *Let the following conditions be satisfied:*

1. *Transition probability matrices of the MC (X_n) for different values of θ are different, that is, for any $\theta \neq \theta'$ there exist i, j such that $p^\theta_{ij} \neq p^{\theta'}_{ij}$.*
2. *Let MC (Y_n) be ergodic for each θ under the measure $\mathbb{P}^\theta$ in the sense of the definition (1), and let the (weak) LLN hold for the process Y for each θ in the sense of the assumption (2). Then there is a convergence*

$$\hat\theta_n \to \theta, \quad n \to \infty, \quad \mathbb{P} - a.s. \tag{3}$$

Here, as usual in the Bayesian setting,

$$\mathbb{P}(d\theta, d\omega) = \mathbb{Q}(d\theta)\mathbb{P}^\theta(d\omega).$$

Remark 5. *Recall that similar results under different conditions were established in [17, theorems 1-2]. Formally, those conditions in [17] may be applicable, or not applicable in our situation because the assumption of the absolute continuity for the projection measures on the sigma-algebra $\mathcal{F}^X_n$ for any two values of the parameter is not assumed, see [17, Theorem 1, condition (C)] and [17, Theorem 2, condition (b)]. In markovian examples in [7, §13] a similar condition to [17, Theorem 1, condition (C)] was assumed as well, see theorem 22, condition (b). In the present paper such a condition is neither assumed, nor it follows from the other assumptions. Intuitively, the lack of continuity should only help consistency; nevertheless, even if so, it apparently does require some calculus. In any case, the proof of the theorem 4 in what follows does not distinguish between the cases tackled in [17] and the cases not covered by this cited paper.*

Remark 6. *As in the setting of Doob in [3], this result may also be used in the classical setting where θ is not random and there exists a unique "true" parameter value. For that, an artificial prior density should be introduced on Θ which must be everywhere positive. Then, as in [3], the analogous assertion will hold true about an almost sure convergence of the artificial Bayesian estimator under the product measure on $\Theta \times \mathcal{X}^\infty$.*

In particular, what is usually highlighted about Bernstein and von Mises theorem is that if the measure $\mathbb{Q}$ has a densty $q(\theta)$ which is everywhere positive, then convergence of the Bayesian estimator towards θ will take place almost everywhere in Θ with respect to the Lebesgue measure. Actually, it suffices for this property that the measure $\mathbb{Q}$ were absolutely continuous with respect to the latter. However, in either case there is no way to know for which particular values of θ this convergence is valid and for which maybe not; it may only be claimed that the set of "bad" values of θ with no convergence has measure zero.

4. Auxiliary results

Let us define the sample distribution function

$$F_N(x,x') := \frac{1}{N}\sum_{t=0}^{N-1} 1(X_t \le x, X_{t+1} \le x').$$

Denote by $\mathcal{S} = \{F(x,x'), x, x' \in \mathbb{R}\}$ the space of all functions of two variables (x,x') with the following properties:

1. $0 \le F(x,x') \le 1$ for each $x, x' \in \mathbb{R}$.
2. If $x \le z$, $x' \le z'$, then $F(x,x') \le F(z,z')$ (monotonicity).
3. For each $x, x' \in R$
$$\lim_{z\downarrow x, z'\downarrow x'} F(z,z') = F(x,x').$$
4. For each $x, x' \in \mathbb{R}$ there exists a limit
$$\lim_{z\uparrow x, z'\uparrow x'} F(z,z') =: F(x,x')_-.$$
(NB: Actually, the latter notation will not be used in what follows; it is just an analogue of the one-dimensional property of "làg" – possessing "limites à gauche" – for the one-dimensional case. Respectively, the property 3 is the analogue of the "càd" – being "continue à droite" for a function of one variable.)
5. $$\lim_{z\uparrow +\infty, z'\uparrow +\infty} F(z,z') = 1.$$
6. $$\lim_{z\downarrow -\infty, z'\downarrow -\infty} F(z,z') = 0.$$

In fact, in the situation under the consideration we deal with some proper subset of all distribution functions of two variables, because all corresponding measures on $\mathbb{R}^2$ have atoms in our setting. However, all we need is that this more general space of distribution functions with a certain metric is a Polish space, and this will be guaranteed by proposition 16 in what follows.

Denote by $\Sigma(\mathcal{S})$ the sigma-algebra on $\mathcal{S}$ generated by all finite cylinders, i.e.,

$$\Sigma(\mathcal{S}) := \sigma(F \in \mathcal{S} : F(x_1,x_1') \le a_1, \ldots F(x_n,x_n') \le a_n))$$

for any $(x_1,x_1'), \ldots, (x_n,x_n') \in \mathbb{R}^2$ and $a_1, \ldots, a_n \in [0,1]$.

Note that the distribution function of any two-dimensional random vector belongs to the space $\mathcal{S}$, and all sample d.f. F_N belong to this space, too.

Lemma 7. *The function $F_N : \Omega \mapsto \mathbb{R}$ is a measurable map with respect to the corresponding pair of sigma-algebras $(\mathcal{F}_N^X; \Sigma(S))$.*

Proof. The proof is elementary and is shown here only for the convenience of the reader. Indeed, for any couple (x, x') the mapping

$$F_N(x, x') := \frac{1}{N}\sum_{t=0}^{N-1} 1(X_t \le x, X_{t+1} \le x')$$

is measurable as a function of ω, as a finite sum of random variables (indicators), which may be expressed by the relation

$$(\omega : \ F_N(x, x') \le a) \in \mathcal{F}_N^X$$

for any $a \in R$. Then, for any finite sets of $(x_1, x_1'), \ldots, (x_n, x_n')$ and $a_1, \ldots, a_n$ we have,

$$(\omega : \ F_N(x_1, x_1') \le a_1, \ldots F_N(x_n, x_n') \le a_n)) \in \mathcal{F}_N^X,$$

by the definition of what is a sigma-algebra. Therefore, $F_N(\cdot, \cdot)$ as a function of ω is, indeed, $(\mathcal{F}_N^X; \Sigma(S))$-measurable, as required. ∎

In the next lemma it is assumed that the distribution of $Y_0 = (X_0, X_1)$ is invariant. In this case its distribution function is denoted by $\hat{F}^\theta(x, x')$; recall that due to ergodicity it is unique. It may be presented by the formula

$$\hat{F}^\theta(x, x') = \hat{F}^\theta(x) p^\theta_{x,x'}, \tag{4}$$

where, in turn, $\hat{F}^\theta(x)$ is the (unique) invariant distribution function of the MC X_n with respect to the probability measure P^θ, which is simultaneously the limiting distribution function for the (X_n).

Lemma 8. *Under the assumption that all transition probabilities p^θ_{ij}, $i, j \in \mathcal{X}$ are Borel measurable in θ, the invariant distribution function $\hat{F}^\theta(x, x')$ is Borel measurable in θ for each pair (x, x').*

Proof. Indeed, invariant probabilities $p^\theta_{inv}(i)$, $i \in \mathcal{X}$ are measurable in θ as limits of measurable n-step transition probabilities. So, the "double" invariant probabilities $p^\theta_{inv}(i)p^\theta(ij)$, $i, j \in \mathcal{X}$ also have the same property. Hence, the theoretical d.f.

$$\mathbb{P}^\theta_{x_0}(X_n \le x, X_{n+1} \le x') = \sum_{i \le x} p^\theta_{x_0,i}(n) \sum_{j \le x'} p^\theta_{i,j}$$

is clearly Borel measurable in θ, too. So is its limit at $n \to \infty$ which equals $\hat{F}^\theta(x, x')$, as required. ∎

Let us recall the Lèvy–Doob theorem on convergence of conditional expectations.

Proposition 9. *(see, e.g., [11, Theorem 4.3.10]) Let $E|\xi| < \infty$ and let $\mathcal{F}_n, n = 0, 1, \ldots$ be an increasing sequence of σ-algebras, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, and let $\mathcal{F}_\infty$ be the minimal σ-algebra which contans all $\mathcal{F}_n$, that is, $\mathcal{F}_\infty = \bigvee\limits_n \mathcal{F}_n$ (that is the minimal sigma-algebra generated by all $\mathcal{F}_n$). Then*

$$\lim_{n\to\infty} \mathbb{E}(\xi|\mathcal{F}_n) = \mathbb{E}(\xi|\mathcal{F}_\infty), \quad a.s.$$

and

$$\lim_{n\to\infty} \mathbb{E}|\mathbb{E}(\xi|\mathcal{F}_\infty) - \mathbb{E}(\xi|\mathcal{F}_n)| = 0.$$

In our setting due to the proposition 9 we have,

$$\lim_{n\to\infty} \mathbb{E}(\theta|X_1, \ldots, X_n) = \lim_{n\to\infty} \phi_n(X) = \mathbb{E}(\theta|\mathcal{F}^X_\infty) \quad \text{a.s.}$$

This implies that the limit in the left hand side in the latter double equality is $\mathcal{F}^X_\infty$-measurable.

Lemma 10. *Assume that the transition probability matrices are different for different parameter values, that is, $\theta \neq \theta'$ implies that there exist i, j such that $p^{\theta}_{ij} \neq p^{\theta}_{ij}$. Then the mapping $\theta \mapsto \hat{F}^{\theta}(j, j'), j, j' \in \mathcal{X}$ is one-to-one. Moreover, the mapping*

$$G: \theta \mapsto \hat{F}^{\theta}(x, x'), \quad x, x' \in \mathbb{R}, \tag{5}$$

is also one-to-one.

Proof. The proof follows from the formula (4). Indeed, if for $\theta \neq \theta'$ the one-dimensional invariant distribution functions $\hat{F}_{\theta}(\cdot)$ are different, then two-dimensional are different, too. If for some pair $\theta \neq \theta'$ the one-dimensional d.f. coincide, $\hat{F}^{\theta}(\cdot) = \hat{F}^{\theta'}(\cdot)$, then the two-dimensional one are yet different due to the formula (4) and by virtue of the distinguishability assumption of transition probabilities for different parameter values. The same property for the mapping G follows straightforwardly. ■

Further, due to the assumed LLN the following convergence of relative frequencies holds,

$$\frac{1}{n}\sum_{t=0}^{n-1} 1(X_t \leq j) \stackrel{\mathbb{P}^{\theta}}{\to} \hat{F}^{\theta}(j) = \mathbb{E}^{\theta}_{inv} 1(X_0 \leq j), \quad n \to \infty,$$

where $\mathbb{E}^{\theta}_{inv}$ is expectation with respect to the corresponding invariant measure. A similar convergence holds true for two-dimensional relative frequencies,

$$\frac{1}{n}\sum_{t=0}^{n-1} 1(X_t \leq j, X_{t+1} \leq j') \stackrel{P^{\theta}}{\to} \hat{F}^{\theta}(j, j') = \mathbb{E}^{\theta}_{inv} 1(X_0 \leq j, X_1 \leq j'), \quad n \to \infty.$$

Since two-dimensional invariant d.f. $\hat{F}^{\theta}(\cdot, \cdot)$ are different for any two different parameter values, the value θ is uniquely determined by the infinite trajectory of observations $X = (X_n, n = 1, \ldots)$. In other words, the mapping $\theta \mapsto \hat{F}^{\theta}(\cdot, \cdot)$ is one-to-one. This mapping is measurable due to the LLN and because the limit of measurable mappings is also measurable. Moreover, as it follows from proposition 13 (see below; it is not linked to this lemma), the inverse mapping is also measurable.

Let us recall some further definitions; it is necessary because one of them is not standard in most of mathematics areas (see definition 12 in what follows).

Definition 11. *Borel measurable sets in a Polish (& more generally, in any topological) space $\mathbb{X}$ are the sets of the minimal σ-algebra $\mathcal{B}(\mathbb{X})$ of subsets in $\mathbb{X}$ which contains all open subsets in $\mathbb{X}$.*

Definition 12. *Let X, Y be Borel measurable sets in Polish spaces $\mathbb{X}, \mathbb{Y}$, respectively. The mapping $f : X \to Y$ is called:*

1. *Borel iff its graph $\Gamma_f = \{(x, y) : x \in X, f(x) = y\}$ is a Borel set in the space $\mathbb{X} \times \mathbb{Y}$;*
2. *B-measurable iff the image of any Borel set from the space Y under the inverse mapping f^{-1} is a Borel set in $\mathbb{X}$.*

Note that the "usual" definition of a Borel function in the majority of areas of mathematics coincides with 10.2.

The next result may be found in [10, Theorem 2.4.3] (we only state the part of this theorem which will be used in what follows).

Proposition 13 ([10, Theorem 2.4.3])**.** *Let X, Y be Borel sets in Polish spaces and $f : X \to Y$ be some mapping. Then:*

1. *If f is Borel measurable then the images of all Borel sets from Y under the inverse mapping f^{-1} are also Borel, so that the mapping f is B-measurable;*
2. *Vice versa, if f is B-measurable then it is a Borel function.*

Corollary 14. *If the mapping f is Borel measurable and one-to-one, then its inverse f^{-1} is B-measurable and, hence, Borel one in the sense of definition 12.*

In order to apply proposition 13 in the proof of our main result in the next section, let us show that both proposition and its corollary 14 are applicable to the mapping G (see (5)).

Lemma 15. *Under the assumptions of theorem 4 the mapping G^{-1} is Borel and B-measurable.*

Proof. Firstly, the mapping $G : \theta \mapsto \hat{F}^{\theta}(\cdot,\cdot)$ is B-measurable in the sense of the definition 12. Indeed, the element $\hat{F}^{\theta}(\cdot,\cdot)$ is a limit in probability $\mathbb{P}^{\theta}$ of the sequence of functions $\mathbb{E}^{\theta}F_N(\cdot,\cdot)$, which are all B-measurable in θ; therefore, so is their limit.

Secondly, according to lemma 10, the mapping G is one-to one; hence, so is its inverse is G^{-1}. The claim of lemma 15 now follows from corollary 14. ■

Further, it is desirable that the parametric space Θ and the space of invariant distribution functions were complete and separable metric spaces. It is trivial with $\Theta \subset \mathbb{R}^m$ with the Euclidean metric; for the space of "double" distribution functions a suitable matric should be chosen which is, of course, not unique. To each distribution function there correspond a probability distribution on $\mathbb{R}^2$. Let us accept that the distance between two distribution functions is defined as a distance between their corresponding measures. Let us choose Prokhorov's metric $d_p(\nu_1,\nu_2)$ for them: if α-neighbourhood of a set $A \subset \mathbb{R}^2$ is denoted by

$$A_\alpha := \{\bar{a} := (a_1,a_2) \in \mathbb{R}^2 : \ d(\bar{a},A) < \alpha\}, \text{ if } A \neq \emptyset, \ \emptyset_\alpha = \emptyset \quad \forall \alpha > 0,$$

then the distance between probability measures ν_1, ν_2 on $\mathbb{R}^2$ is defined by the formula

$$d_p(\nu_1,\nu_2) := \inf\{\alpha > 0 : \nu_2(A) \le \nu_1(A_\alpha) + \alpha \ \& \ \nu_1(A) \le \nu_2(A_\alpha) + \alpha, \ \forall A \in \mathcal{B}(S)\}.$$

The same formula provides the distance between two distribution functions, namely, as a distance $d_p(\cdot,\cdot)$ between the corresponding measures on $\mathbb{R}^2$.

Proposition 16 ([16, Lemma 1.4])**.** *Let a metric space be complete and separable. Then the space of probability measures on it with the Prokhorov metric is also complete and separable.*

5. Proof of theorem 4

Proof. By virtue of Lèvy–Doob's theorem (see proposition 9) we have,

$$\hat{\theta}_n \equiv \mathbb{E}(\theta|X_1,\ldots,X_n) = \mathbb{E}(\theta|\mathcal{F}^X_n) \to \mathbb{E}(\theta|\mathcal{F}^X_\infty) =: \hat{\theta}_\infty, \quad n \to \infty \quad P\text{-a.s.}. \tag{6}$$

Due to its definition, the random variable $\hat{\theta}_\infty$ is $\mathcal{F}^X_\infty$-measurable; being a conditional expectation, it is a Bayesian estimator of θ constructed upon the infinite sequence of observations $X_1, X_2, \ldots$ For the proof of the theorem, it suffices to establish the equality

$$\hat{\theta}_\infty \overset{\mathbb{P}-\text{a.s.}}{=} \theta. \tag{7}$$

The basis for thsi equality is the empirical fact that θ is uniquely deterined by the infinite sequence of observations due to the assumed LLN and because of the one-to-one correspondence between θ and the invariant distribution of the pair (X_0, X_1). Let us provide more rigorous considerations related, in particular, to the measurability.

By virtue of the LLN assumptions, we have

$$\mathcal{F}^X_{N-1} \ni F_N(x) = \frac{1}{N}\sum_{i=0}^{N-1} I(X_i \le x) \overset{\mathbb{P}^\theta}{\to} \hat{F}^\theta(x) = \mathbb{E}^\theta_{inv} I(X_0 \le x),$$

and also

$$\mathcal{F}^X_N \ni F_N(x,x') = \frac{1}{N}\sum_{i=0}^{N-1} I(X_i \le x, X_{i+1} \le x') \overset{\mathbb{P}^\theta}{\to} \hat{F}^\theta(x,x') = \mathbb{E}^\theta_{inv} I(X_0 \le x, X_1 \le x').$$

The random variable $F_N(x,x')$ is $(\mathcal{F}_N^X, \mathcal{B}(\mathbb{R}^2))$-measurable for any pair $(x,x') \in \mathbb{R}^2$. According to lemma 7, the mapping $F_N(\cdot,\cdot)$ is $(\mathcal{F}_N^X, \Sigma(\mathcal{S}))$-measurable, hence, it is a random variable in the space of distribution functions.

Now, according to lemma 10 the following equality holds true,

$$\theta \overset{\mathbb{P}-\text{a.s.}}{=} G^{-1}(\underbrace{\hat{F}^\theta(\cdot,\cdot)}_{\in \mathcal{F}_\infty^X}).$$

By virtue of lemma 15, the mapping G^{-1} is Borel and B-measurable. Therefore,

$$G^{-1}(\hat{F}^\theta(\cdot,\cdot)) \in \mathcal{F}_\infty^X.$$

Therefore, by virtue of (6),

$$\hat{\theta}_n \to \mathbb{E}(\theta|\mathcal{F}_\infty^X) \overset{\mathbb{P}-\text{a.s.}}{=} \mathbb{E}(G^{-1}(\hat{F}^\theta(\cdot,\cdot))|\mathcal{F}_\infty^X) \overset{\mathbb{P}-\text{a.s.}}{=} G^{-1}(\hat{F}^\theta(\cdot,\cdot)) \overset{\mathbb{P}-\text{a.s.}}{=} \theta.$$

This means that (7) holds true. implies the desired convergence (3). Theorem 4 is proved. ■

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On the Degree of Mutual Dependence of Three Events

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"...one of the most important problems
in the philosophy of natural sciences is
... to make precise premises which would
make it possible to regard any given
real events as independent."

A. N. Kolmogorov,
Foundations of the Theory of Probability

Abstract

We define degree of mutual dependence of three events in a probability space by using Boltzmann-Shannon entropy function of an appropriate variable distribution produced by these events and depending on four parameters varying, in general, within of a polytope. It turns out that the entropy function attains its absolute maximum exactly when the three events are mutually independent and its absolute minimum at some vertices of the polytope where the events are "maximally" dependent. By composing the entropy function with an appropriate linear function we obtain a continuous "degree of mutual dependence" function with the same domain and the interval $[0, 1]$ *as a target. It attains value* 0 *when the events are mutually independent (the entropy is maximal) and value* 1 *when they are "maximally" dependent (the entropy is minimal). A link is available for downloading a Java code which evaluates the degree of mutual dependence of three events in the classical case of a sample space with equally likely outcomes.*

Keywords: entropy; average information; degree of dependence; probability space; probability distribution; experiment in a sample space; linear system; affine isomorphism; classification space.

1. Introduction

In our papers [6] and [7]) we introduce and study a measure of dependence of two events in a probability space, based on the fundamental notion of Boltzmann-Shannon entropy. The present work is written as a natural conceptual continuation of the above papers for the case of three events A_1, A_2, A_3. By analogy, we consider the joint experiment $\mathfrak{J}_3$ of the corresponding three binary trials, whose probability distribution gives rise to the entropy function that, in turn, measures the mutual dependence of these events.

In accord with [6, 4.1], any one of the three pairs of events A_i, A_j, $1 \leq i < j \leq 3$, produces a joint experiment $\mathfrak{J}_{ij}$ whose probability distribution satisfies the linear system (3). Since the partition $\mathfrak{J}_3$ of the sample space is finer than each partition $\mathfrak{J}_{ij}$, its probability distribution $(\xi_1, \ldots, \xi_8)$ satisfies the linear system (5). After fixing the probabilities $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of the components of *Yule's triple* $A = (A_1, A_2, A_3)$, the general solution of the last system depends on four parameters $\theta = (\theta_0, \ldots, \theta_3)$ chosen among ξ_k's. Taking into account that $\xi_k(\theta)$'s are probabilities, we obtain that θ varies within a subset $I_7(\alpha)$ of $\mathbb{R}^4$, which is described in Theorem 1. In case $\alpha \in (0, 1)^3$ the set $I_7(\alpha)$ is a polytope, see [2, Ch. 12]. Since the system of linear inequalities (9) which define the polytope $I_7(\alpha)$ is minimal (Lemma 2), we can apply the machinery from the previous citation in order to use the corresponding properties of this polytope.

The 7-tuples (α, θ) vary within a polytope $I_7 \subset \mathbb{R}^7$ which is the inverse image of the 7-dimensional simplex Δ_7 via the affine isomorphism (7). The projection $p(\alpha, \theta) = \alpha$ produces the fibre bundle $(I_7, p, [0,1]^3)$ with fibre $p^{-1}(\alpha) = C_7(\alpha)$ where $C_7(\alpha) = \{\alpha\} \times I_7(\alpha)$, for the definition see [5, Part I, 2, 1.1]. This fibre bundle is used for classification of all equivalence classes of Yule's triples with given α and θ, cf. [6, Theorem 1]. An isomorphic fibre bundle can be used for classification of all probability distributions produced by the above equivalence classes of Yule's triples. The general patterns of these two fibre bundles are described in terms of very elementary algebraic geometry at the end of Subsection 4.2 where also classification Theorem 2 is formulated.

Corollary 1, (ii), yields that $0 < \xi_k(\theta) < 1$, $k = 1, \ldots, 8$, if and only if $\theta \in \mathring{I}_7(\alpha)$. In particular, $\mathring{I}_7(\alpha)$ is the natural domain of the entropy function $E_\alpha(\theta)$ of the probability distribution $(\xi_k(\theta))_{k=1}^8$, defined in (11).

In Lemma 4 we prove that $E_\alpha(\theta)$ is a strictly concave function that can be extended in a unique way as continuous at the polytope $I_7(\alpha)$. Moreover, its continuous extension $\hat{E}_\alpha$ is also a strictly concave function. In Corollary 2 we show that all permutations of the members of Yule's triple $A = (A_1, A_2, A_3)$ have the same entropy.

Subsection 5.2 is devoted to finding the set of critical points of the entropy function $E_\alpha(\theta)$. It turns out that this set is not empty: The special point $\theta^{(\alpha)} \in \mathring{I}_7(\alpha)$ defined by the formulae (10) is critical, see Lemma 6.

Since the Hessian of $E_\alpha(\theta)$ is a negative definite quadratic form everywhere in its domain $\mathring{I}_7(\alpha)$, we obtain that the set of local maximums of the entropy function $E_\alpha(\theta)$ coincides with the set of its critical points, see Lemma 7.

In accord with Weierstrass theorem, the extended entropy function $\hat{E}_\alpha(\theta)$ attains an absolute maximum and an absolute minimum in its compact domain $I_7(\alpha)$. Theorems 3 and 4 make this statement more precise. The former asserts that $\hat{E}_\alpha(\theta)$ has a unique absolute maximum at the point $\theta^{(\alpha)}$. The latter uses the structure of the frontier of the polytope $I_7(\alpha)$, described, for example, in [2, Chapter 12, 12.1], and shows that $\hat{E}_\alpha(\theta)$ attains its absolute minimum only at some of its vertices. We note here an analogy with the simplex method.

Subsection 6.1 contains two statements that motivate the use the extended entropy function $\hat{E}_\alpha(\theta)$ for measuring the power of mutual relations among three events. In Lemma 8 we show that the components of a Yule's triple are mutually independent if and only if the corresponding θ coincides with $\theta^{(\alpha)}$. In other words, we observe mutual independence exactly when $\hat{E}_\alpha(\theta)$ attains its absolute maximum, which is in keeping conformity with our intuition. In the case of sample space with equally likely outcomes, Lemma 9 establishes the set-theoretic relations among the components of a Yule's triple when the corresponding θ lies on any one of the 3-faces of the polytope $I_7(\alpha)$. Intuitively, the "maximally" tight-fitting is observed at the vertices some of which are points of absolute minimum of $\hat{E}_\alpha(\theta)$.

Let $A = (A_1, A_2, A_3)$ be a Yule's triple with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 = \Pr(A_1)$, $\alpha_2 = \Pr(A_2)$, $\alpha_3 = \Pr(A_3)$. In the final Subsection 6.2 we compose the extended entropy function $\hat{E}_\alpha(\theta)$ with a linear function and define a function $e_\alpha \colon I_7(\alpha) \to [0,1]$, whose value at any $\theta \in I_7(\alpha)$ corresponding to A is said to be degree of dependence of the events A_1, A_2, A_3. Note that $e_\alpha(\theta^{(\alpha)}) = 0$ (the events A_1, A_2, A_3 are mutually independent) and $e_\alpha(\theta_1) = 1$ for any vertex θ_1 where $\hat{E}_\alpha(\theta)$ attains its absolute minimum (the events A_1, A_2, A_3 are maximally dependent).

2. Definitions and Notation

Let $(\Omega, \mathcal{A}, \Pr)$ be a probability space with set of outcomes Ω, σ-algebra $\mathcal{A}$, and probability function Pr. In this paper we are using only the structure of Boolean algebra on $\mathcal{A}$.

We introduce the following notation:

Given events A_1, A_2, A_3 from $\mathcal{A}$, we set $A = (A_1, A_2, A_3) \in \mathcal{A}^3$;

R is the range of the probability function $\Pr\colon \mathcal{A} \to \mathbb{R}$;
Given $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we set $\alpha = (\alpha_1, \alpha_2, \alpha_3)$;
Given $\theta_0, \theta_1, \theta_2, \theta_3 \in \mathbb{R}$, we set $\theta = (\theta_0, \theta_1, \theta_2, \theta_3)$;
$I(\alpha_i, \alpha_j) = [\max(0, \alpha_i + \alpha_j - 1), \min(\alpha_i, \alpha_j)]$, $1 \leq i < j \leq 3$, see [6, 4.1];
$I^{(\alpha_i, \alpha_j)} = [\max(0, \alpha_i - \alpha_j), \min(\alpha_i, 1 - \alpha_j)]$, $1 \leq i < j \leq 3$;
$[(\alpha)]$ is the fiber of the surjective map

$$\mathcal{A}^3 \to R^3, (A_1, A_2, A_3) \mapsto (\Pr(A_1), \Pr(A_2), \Pr(A_3)),$$

over $\alpha \in R^3$;
$[(\alpha_i, \alpha_j)]$ is the fiber of the surjective map

$$\mathcal{A}^2 \to R^2, (A_i, A_j) \mapsto (\Pr(A_i), \Pr(A_j)),$$

over $(\alpha_i, \alpha_j) \in R^2$, $1 \leq i < j \leq 3$;

$$\theta_0^{(A)} = \Pr(A_1 \cap A_2 \cap A_3), \theta_1^{(A)} = \Pr(A_1^c \cap A_2 \cap A_3),$$

$$\theta_2^{(A)} = \Pr(A_1 \cap A_2^c \cap A_3), \theta_3^{(A)} = \Pr(A_1 \cap A_2 \cap A_3^c), A \in \mathcal{A}^3;$$

$$\theta^{(A)} = (\theta_0^{(A)}, \theta_1^{(A)}, \theta_2^{(A)}, \theta_3^{(A)});$$

$[(\alpha, \theta)]$ is the fiber of the map $[(\alpha)] \to R^4, A \mapsto \theta^{(A)}$, over any $\theta \in R^4$, and $R^{(\alpha)}$ is its range.

We note that the fibers $[(\alpha)]$ for $(\alpha) \in R^3$ form a partition of $\mathcal{A}^3$ and the fibers $[(\alpha, \theta)]$ for $\theta \in R^{(\alpha)}$ form a partition of $[(\alpha)]$.

The members of the fiber $[(\alpha)]$ are said to be *Yule's triples of type* (α). The members of the fiber $[(\alpha, \theta)]$ are called *Yule's triples of type* (α, θ).

3. Methods

In this paper we are using fundamentals of:
- Linear algebra,
- Affine geometry,
- Polytope theory,
- Fibre bundles,
- Real algebraic geometry.

4. Classification of Yule's Triples and Their Probability Distributions

4.1. The Probability Distribution of a Yule's Triple

Any ordered triple $A = (A_1, A_2, A_3) \in \mathcal{A}^3$ produces three experiments of the form

$$\mathfrak{I}_{ij} = (A_i \cap A_j) \cup (A_i \cap A_j^c) \cup (A_i^c \cap A_j) \cup (A_i^c \cap A_j^c), 1 \leq i < j \leq 3,$$

and the experiment

$$\mathfrak{I}_3 = (A_1 \cap A_2 \cap A_3) \cup (A_1^c \cap A_2 \cap A_3) \cup (A_1 \cap A_2^c \cap A_3) \cup (A_1 \cap A_2 \cap A_3^c) \cup$$

$$(A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c) \cup (A_1^c \cap A_2^c \cap A_3) \cup (A_1^c \cap A_2^c \cap A_3^c)$$

(cf. [8, I,§5]). We introduce the following notation:

$$\xi_1^{(A_i, A_j)} = \Pr(A_i \cap A_j), \xi_2^{(A_i, A_j)} = \Pr(A_i \cap A_j^c),$$

$$\xi_3^{(A_i,A_j)} = \Pr(A_i^c \cap A_j), \xi_4^{(A_i,A_j)} = \Pr(A_i^c \cap A_j^c), 1 \leq i < j \leq 3.$$

Moreover, we set

$$\begin{aligned}
\xi_1^{(A)} &= \Pr(A_1 \cap A_2^c \cap A_3^c), \xi_2^{(A)} = \Pr(A_1^c \cap A_2 \cap A_3^c),\\
\xi_3^{(A)} &= \Pr(A_1^c \cap A_2^c \cap A_3), \xi_4^{(A)} = \Pr(A_1^c \cap A_2^c \cap A_3^c),\\
\xi_5^{(A)} &= \Pr(A_1 \cap A_2 \cap A_3), \xi_6^{(A)} = \Pr(A_1^c \cap A_2 \cap A_3),\\
\xi_7^{(A)} &= \Pr(A_1 \cap A_2^c \cap A_3), \xi_8^{(A)} = \Pr(A_1 \cap A_2 \cap A_3^c).
\end{aligned} \tag{1}$$

The above probabilities satisfy the following identities:

$$\begin{aligned}
&\xi_5^{(A)} + \xi_8^{(A)} = \xi_1^{(A_1,A_2)}, \xi_1^{(A)} + \xi_7^{(A)} = \xi_2^{(A_1,A_2)},\\
&\xi_2^{(A)} + \xi_6^{(A)} = \xi_3^{(A_1,A_2)}, \xi_3^{(A)} + \xi_4^{(A)} = \xi_4^{(A_1,A_2)},\\
&\xi_5^{(A)} + \xi_7^{(A)} = \xi_1^{(A_1,A_3)}, \xi_1^{(A)} + \xi_8^{(A)} = \xi_2^{(A_1,A_3)},\\
&\xi_3^{(A)} + \xi_6^{(A)} = \xi_3^{(A_1,A_3)}, \xi_2^{(A)} + \xi_4^{(A)} = \xi_4^{(A_1,A_3)},\\
&\xi_5^{(A)} + \xi_6^{(A)} = \xi_1^{(A_2,A_3)}, \xi_2^{(A)} + \xi_8^{(A)} = \xi_2^{(A_2,A_3)},\\
&\xi_3^{(A)} + \xi_7^{(A)} = \xi_3^{(A_2,A_3)}, \xi_1^{(A)} + \xi_4^{(A)} = \xi_4^{(A_2,A_3)}.
\end{aligned} \tag{2}$$

For any $1 \leq i < j \leq 3$ and any $(A_i, A_j) \in [(\alpha_i, \alpha_j)]$, the probability distribution

$$(\xi_1^{(i,j)}, \xi_2^{(i,j)}, \xi_3^{(i,j)}, \xi_4^{(i,j)}) = (\xi_1^{(A_i,A_j)}, \xi_2^{(A_i,A_j)}, \xi_3^{(A_i,A_j)}, \xi_4^{(A_i,A_j)})$$

satisfies the linear system

$$\left|\begin{array}{ccccccccc}
\xi_1^{(i,j)} & + & \xi_2^{(i,j)} & & & & & = & \alpha_i\\
 & & & & \xi_3^{(i,j)} & + & \xi_4^{(i,j)} & = & 1-\alpha_i\\
\xi_1^{(i,j)} & & & + & \xi_3^{(i,j)} & & & = & \alpha_j\\
 & & \xi_2^{(i,j)} & & & + & \xi_4^{(i,j)} & = & 1-\alpha_j.
\end{array}\right. \tag{3}$$

The identities (2) and the linear systems (3) yield that for any ordered triple $A \in [\alpha]$, the probability distribution

$$(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8) = (\xi_1^{(A)}, \xi_2^{(A)}, \xi_3^{(A)}, \xi_4^{(A)}, \xi_5^{(A)}, \xi_6^{(A)}, \xi_7^{(A)}, \xi_8^{(A)}) \tag{4}$$

satisfies the linear system

$$\left|\begin{array}{ccccccccccccccccc}
\xi_1 & & & & & & & + & \xi_5 & & & + & \xi_7 & + & \xi_8 & = & \alpha_1\\
 & & \xi_2 & + & \xi_3 & + & \xi_4 & & & + & \xi_6 & & & & & = & 1-\alpha_1\\
 & & \xi_2 & & & & & + & \xi_5 & + & \xi_6 & & & + & \xi_8 & = & \alpha_2\\
\xi_1 & & & + & \xi_3 & + & \xi_4 & & & & & + & \xi_7 & & & = & 1-\alpha_2\\
 & & & & \xi_3 & & & + & \xi_5 & + & \xi_6 & + & \xi_7 & & & = & \alpha_3\\
\xi_1 & + & \xi_2 & & & + & \xi_4 & & & & & & & + & \xi_8 & = & 1-\alpha_3.
\end{array}\right. \tag{5}$$

Let us denote for short $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8)$ and let H_7 be the affine hyperplane in $\mathbb{R}^8$ with equation $\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 + \xi_7 + \xi_8 = 1$. For any $\alpha \in \mathbb{R}^3$ the solutions of (5) depend on four parameters, say $\theta_0 = \xi_5$, $\theta_1 = \xi_6$, $\theta_2 = \xi_7$, $\theta_3 = \xi_8$, and for any triple $\alpha \in \mathbb{R}^3$ form a 4-dimensional affine space ℓ_α in H_7 with parametric representation

$$\ell_\alpha\colon \left\{ \begin{array}{lclcccccccccccccc} \xi_1 & = & & & \alpha_1 & & & & & - & \theta_0 & & & - & \theta_2 & - & \theta_3 \\ \xi_2 & = & & & & & \alpha_2 & & & - & \theta_0 & - & \theta_1 & & & - & \theta_3 \\ \xi_3 & = & & & & & & & \alpha_3 & - & \theta_0 & - & \theta_1 & - & \theta_2 & & \\ \xi_4 & = & 1 & - & \alpha_1 & - & \alpha_2 & - & \alpha_3 & + & 2\theta_0 & + & \theta_1 & + & \theta_2 & + & \theta_3 \\ \xi_5 & = & & & & & & & & & \theta_0 & & & & & & \\ \xi_6 & = & & & & & & & & & & & \theta_1 & & & & \\ \xi_7 & = & & & & & & & & & & & & & \theta_2 & & \\ \xi_8 & = & & & & & & & & & & & & & & & \theta_3 \end{array} \right. \tag{6}$$

The map

$$\iota_7\colon \mathbb{R}^7 \to H_7, (\alpha, \theta) \mapsto \xi, \tag{7}$$

defined by formulae (6) is an affine isomorphism with inverse affine isomorphism

$$\chi_7\colon H_7 \to \mathbb{R}^7, \xi \mapsto (\xi_1 + \xi_5 + \xi_7 + \xi_8, \xi_2 + \xi_5 + \xi_6 + \xi_8, \xi_3 + \xi_5 + \xi_6 + \xi_7, \xi_5, \xi_6, \xi_7, \xi_8). \tag{8}$$

The symmetric group S_3 acts on $\mathbb{R}^7$ by the rule $\sigma(\alpha, \theta) = (\sigma\alpha; \sigma\theta)$, where $\sigma\alpha = (\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)})$ and $\sigma\theta = (\theta_0, \theta_{\sigma^{-1}(1)}, \theta_{\sigma^{-1}(2)}, \theta_{\sigma^{-1}(3)})$, $\sigma \in S_3$. When necessary, we write σ_α and σ_θ in order to distinguish the actions of σ on α's and θ's, respectively.

On the other hand, we transport the action of S_3 on the set $\{6, 7, 8\}$ via the bijection $1 \mapsto 6, 2 \mapsto 7, 3 \mapsto 8$ and define an action of S_3 on the hyperplane H_7 by the formula

$$\sigma\xi = (\xi_{\sigma^{-1}(1)}, \xi_{\sigma^{-1}(2)}, \xi_{\sigma^{-1}(3)}, \xi_4, \xi_5, \xi_{\sigma^{-1}(6)}, \xi_{\sigma^{-1}(7)}, \xi_{\sigma^{-1}(8)}).$$

Lemma 1. The affine isomorphism ι_7 is also an isomorphism of S_3-sets: $\iota_7(\sigma(\alpha, \theta)) = \sigma\iota_7(\alpha, \theta)$.

Proof. We check the statement for a set of generators of S_3: For $\sigma = (12)$ we have

$$\xi_1((12)(\alpha, \theta)) = \xi_2(\alpha, \theta), \xi_2((12)(\alpha, \theta)) = \xi_1(\alpha, \theta),$$

$$\xi_6((12)(\alpha, \theta)) = \xi_7(\alpha, \theta), \xi_7((12)(\alpha, \theta)) = \xi_6(\alpha, \theta).$$

For $\sigma = (23)$ we have

$$\xi_2((23)(\alpha, \theta)) = \xi_3(\alpha, \theta), \xi_3((23)(\alpha, \theta)) = \xi_2(\alpha, \theta),$$

$$\xi_7((23)(\alpha, \theta)) = \xi_8(\alpha, \theta), \xi_8((23)(\alpha, \theta)) = \xi_7(\alpha, \theta).$$

■

4.2. The Geometric Classification

After fixing the coordinates α_1, α_2, and α_3, the isomorphism ι_7 from (7) maps the 4-dimensional affine space $\zeta_\alpha = \{\alpha\} \times \mathbb{R}^4$ onto the 4-dimensional affine space ℓ_α in H_7. We denote by $\iota_7^{(\alpha)}$ the (affine) restriction of ι_7 on ζ_α, so $\iota_7^{(\alpha)}\colon \zeta_\alpha \to \ell_\alpha$.

The trace of the 8-dimensional cube $\{\xi \in \mathbb{R}^8 | 0 \leq \xi_k \leq 1, k = 1, \ldots, 8\}$ onto the hyperplane H_7 is the 7-dimensional simplex Δ_7 defined in H_7 by the inequalities $\xi_1 \geq 0, \ldots, \xi_8 \geq 0$. The inverse image $T_7 = \iota_7^{-1}(\Delta_7)$ via the affine isomorphism ι_7 is the convex polyhedron in $\mathbb{R}^7$ with non-empty interior, defined by the system of inequalities

$$T_7\colon \left\{ \begin{array}{ccccccccc} \theta_0 & & & + & \theta_2 & + & \theta_3 & \leq & \alpha_1 \\ \theta_0 & + & \theta_1 & & & + & \theta_3 & \leq & \alpha_2 \\ \theta_0 & + & \theta_1 & + & \theta_2 & & & \leq & \alpha_3 \\ 2\theta_0 & + & \theta_1 & + & \theta_2 & + & \theta_3 & \geq & \alpha_1 + \alpha_2 + \alpha_3 - 1. \\ \theta_0 & & & & & & & \geq & 0 \\ & & \theta_1 & & & & & \geq & 0 \\ & & & & \theta_2 & & & \geq & 0 \\ & & & & & & \theta_3 & \geq & 0 \end{array} \right. \tag{9}$$

The form (8) of the inverse isomorphism χ_7 yields that $T_7 \subset [0,1]^7$. In particular, T_7 is a polytope. Note that we are using the terminology about polytopes introduced in [2, Ch. 12].

For any $\alpha \in \mathbb{R}^3$ we set $C_7(\alpha) = \zeta_\alpha \cap T_7$, so $C_7(\alpha) = \{\alpha\} \times I_7(\alpha)$, where $I_7(\alpha) \subset \mathbb{R}^4$ and $\mathbb{R}^4$ is furnished with coordinates θ. The subset $I_7(\alpha)$ is defined in $\mathbb{R}^4$ via the system (9) with fixed α. Hence $I_7(\alpha)$ is a convex bounded polyhedron in $\mathbb{R}^4$. We also set $D_7(\alpha) = \iota_7(C_7(\alpha))$. Since $\iota_7(\zeta_\alpha) = \ell_\alpha$, we obtain that $D_7(\alpha) = \ell_\alpha \cap \Delta_7$.

We consider T_7, $\zeta_\alpha \simeq \mathbb{R}^4$, $C_7(\alpha)$, $I_7(\alpha)$, ℓ_α, Δ_7, and $D_7(\alpha)$ as topological subspaces of the corresponding ambient linear spaces, with topology induced by their standard topology. Moreover, for each subset A of a topological space X we denote by $\mathring{A}$ its interior with respect to X. We note that $\mathring{A}$ is the largest open set contained in A, see [3, § 1, $n^o 6$].

Lemma 2. The minimal number of half-spaces in $\mathbb{R}^4$, whose intersection is the polyhedron $I_7(\alpha)$ is 8.

Proof. We can not omit any one of the inequalities in (9) formed by the free variables $\xi_5 = \theta_0$, $\xi_6 = \theta_1$, $\xi_7 = \theta_2$, and $\xi_8 = \theta_3$. It turns out that the general solution of the linear system (5) can also be written in terms of the free variables ξ_1, ξ_2, ξ_3, and ξ_4. In particular, neither of the inequalities $\xi_1 \geq 0$, $\xi_2 \geq 0$, $\xi_3 \geq 0$, and $\xi_4 \geq 0$, that define the polytope T_7 can be omitted, too. ∎

We define the point $\theta^{(\alpha)} \in \mathbb{R}^4$ by the formulae

$$\theta_0^{(\alpha)} = \alpha_1\alpha_2\alpha_3, \theta_1^{(\alpha)} = (1-\alpha_1)\alpha_2\alpha_3, \theta_2^{(\alpha)} = \alpha_1(1-\alpha_2)\alpha_3, \theta_3^{(\alpha)} = \alpha_1\alpha_2(1-\alpha_3). \tag{10}$$

Lemma 3. If $\alpha \in [0,1]^3$, then $\theta^{(\alpha)} \in I_7(\alpha)$ and the following three statements are equivalent:
(i) One has $\alpha \in (0,1)^3$.
(ii) One has $\theta^{(\alpha)} \in \mathring{I}_7(\alpha)$.
(iii) One has $\mathring{I}_7(\alpha) \neq \emptyset$.

Proof. The equalities $\theta_1 + \theta_3 + \theta_4 - \alpha_1 = -\alpha_1(1-\alpha_2)(1-\alpha_3)$, $\theta_1 + \theta_2 + \theta_4 - \alpha_2 = -\alpha_2(1-\alpha_1)(1-\alpha_3)$, $\theta_1 + \theta_2 + \theta_3 - \alpha_3 = -\alpha_3(1-\alpha_1)(1-\alpha_2)$, and $2\theta_1 + \theta_2 + \theta_3 + \theta_4 - \alpha_1 - \alpha_2 - \alpha_3 + 1 = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)$ yield that the system (9) is satisfied if $\alpha \in [0,1]^3$. If, in addition, $\alpha \in (0,1)^3$, then (9) with strict inequalities holds. Thus, the implication (i) $\Longrightarrow$ (ii) is also proved.

(ii) $\Longrightarrow$ (iii) This is trivial.

(iii) $\Longrightarrow$ (i) Let $\theta \in \mathring{I}_7(\alpha)$. Then $\xi_k(\theta) > 0$, $k = 1,\ldots,8$, their sum is 1, and satisfy the linear system (5). Therefore $\alpha \in (0,1)^3$. ∎

Theorem 1. (i) One has

$$I_7(\alpha) = \begin{cases} (0,0,0,0) & \text{if at least two of } \alpha_i's \text{ are } 0 \\ \{0\} \times I(\alpha_2,\alpha_3) \times \{0\} \times \{0\} & \text{if } \alpha_1 = 0, \alpha_2 > 0, \alpha_3 > 0 \\ \{0\} \times \{0\} \times I(\alpha_1,\alpha_3) \times \{0\} & \text{if } \alpha_2 = 0, \alpha_1 > 0, \alpha_3 > 0 \\ \{0\} \times \{0\} \times \{0\} \times I(\alpha_1,\alpha_2) & \text{if } \alpha_3 = 0, \alpha_1 > 0, \alpha_2 > 0 \\ \{\alpha_3\} \times \{0\} \times \{0\} \times \{1-\alpha_3\} & \text{if } \alpha_1 = 1, \alpha_2 = 1, \alpha_3 > 0 \\ \{\alpha_2\} \times \{0\} \times \{1-\alpha_2\} \times \{0\} & \text{if } \alpha_1 = 1, \alpha_3 = 1, \alpha_2 > 0 \\ \{\alpha_1\} \times \{1-\alpha_1\} \times \{0\} \times \{0\} & \text{if } \alpha_2 = 1, \alpha_3 = 1, \alpha_1 > 0 \\ \{(\alpha_2-\theta_3, 0, \alpha_3-\alpha_2+\theta_3, \theta_3) | \theta_3 \in I^{(\alpha_2,\alpha_3)}\} & \text{if } \alpha_1 = 1, \alpha_2 > 0, \alpha_3 > 0 \\ \{(\alpha_3-\theta_1, \theta_1, 0, \alpha_1-\alpha_3+\theta_1) | \theta_1 \in I^{(\alpha_3,\alpha_1)}\} & \text{if } \alpha_2 = 1, \alpha_1 > 0, \alpha_3 > 0 \\ \{(\alpha_1-\theta_2, \alpha_2-\alpha_1+\theta_2, \theta_2, 0) | \theta_2 \in I^{(\alpha_1,\alpha_2)}\} & \text{if } \alpha_3 = 1, \alpha_1 > 0, \alpha_2 > 0 \end{cases}$$

and $I_7(\alpha)$ is a polytope in $\mathbb{R}^4$ if $\alpha \in (0,1)^3$.

(ii) One has $\iota_7(\mathring{C}_7(\alpha)) = \mathring{D}_7(\alpha)$ the interiors being with respect to affine spaces ζ_α and ℓ_α, respectively.

Proof. (i) The systems (5) and (9) imply the equalities. In case $\alpha \in (0,1)^3$, Lemma 3 yields that the bounded convex polyhedron $I_7(\alpha)$ in $\mathbb{R}^4$ has non-empty interior. In other words, it is a polytope.

(ii) It is enough to note that the (affine) restriction $\iota_7^{(\alpha)}\colon \zeta_\alpha \to \ell_\alpha$ is, in particular, a homeomorphism.

■

Corollary 1. Let $\alpha \in \mathbb{R}^3$.

(i) The system of constraint conditions $0 \leq \xi_k(\theta) \leq 1, k = 1, \ldots, 8$, on the solutions (6) of linear system (5) is equivalent to the property $\theta \in I_7(\alpha)$.

(ii) One has $0 < \xi_k(\theta) < 1$, $k = 1, \ldots, 8$, if and only if $\theta \in \mathring{I}_7(\alpha)$.

Proof. (i) The equalities $C_7(\alpha) = \zeta_\alpha \cap T_7$ and $D_7(\alpha) = \ell_\alpha \cap \Delta_7$ imply part (i). We have $\mathring{C}_7(\alpha) = \zeta_\alpha \cap \mathring{T}_7$ and $\mathring{D}_7(\alpha) = \ell_\alpha \cap \mathring{\Delta}_7$, where the interiors $\mathring{T}_7$ and $\mathring{\Delta}_7$ are with respect to affine spaces $\mathbb{R}^7$ and H_7, respectively. Now, Theorem 1, (ii), yields part (ii).

■

We have $R^{(\alpha)} \subset I_7(\alpha)$ and define $I_7^{(\cdot)}(\alpha) = R^{(\alpha)}$. The *dotted polytope* $C_7^{(\cdot)}(\alpha) = \{\alpha\} \times I_7^{(\cdot)}(\alpha)$, $(\alpha) \in R^3$, is the locus of all 7-tuples of probabilities $(\alpha, \theta^{(A)})$, where $A \in [(\alpha)]$.

By plugging $\theta^{(\alpha)}$ in the formulae (6), we obtain the point $\xi^{(\alpha)} \in H_7$ with coordinates

$$\xi_1^{(\alpha)} = \alpha_1(1-\alpha_2)(1-\alpha_3), \xi_2^{(\alpha)} = (1-\alpha_1)\alpha_2(1-\alpha_3),$$

$$\xi_3^{(\alpha)} = (1-\alpha_1)(1-\alpha_2)\alpha_3, \xi_4^{(\alpha)} = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3),$$

$$\xi_5^{(\alpha)} = \alpha_1\alpha_2\alpha_3, \xi_6^{(\alpha)} = (1-\alpha_1)\alpha_2\alpha_3, \xi_7^{(\alpha)} = \alpha_1(1-\alpha_2)\alpha_3, \xi_8^{(\alpha)} = \alpha_1\alpha_2(1-\alpha_3).$$

Let U_3 be the rational 3-dimensional algebraic manifold defined in $\mathbb{R}^7$ by the equations (10). In other words, U_3 is the locus of the points in $\mathbb{R}^7$ of the form $(\alpha, \theta^{(\alpha)})$, $\alpha \in \mathbb{R}^3$. Let us denote $W_3 = \iota_7(U_3)$, so W_3 is the locus of the points $\xi^{(\alpha)}$, $\alpha \in \mathbb{R}^3$, in H_7. Then $\chi_7(W_3) = U_3$, W_3 is an algebraic subvariety of H_7, and the restrictions of ι_7 and χ_7 on U_3 and W_3, respectively, form a pair of mutually inverse isomorphisms of 3-dimensional rational algebraic manifolds. Moreover, $W_3 \cap \ell_\alpha = \{\xi^{(\alpha)}\}$ for any $\alpha \in \mathbb{R}^3$. Let us denote $\kappa_3 = \iota_3 \circ \delta_3$, where δ_3 is the isomorphism of algebraic manifolds $\mathbb{R}^3 \to U_3$, $\alpha \mapsto (\alpha, \theta^{(\alpha)})$. Therefore, $\kappa_3\colon \mathbb{R}^3 \to W_3$ is also an isomorphism of algebraic manifolds.

We have the product vector bundle with total space $\mathbb{R}^7$, base $\mathbb{R}^3$, projection $(\alpha, \theta) \mapsto \alpha$, and fibre ζ_α. Now, we transport the structure of fibre bundle by means of the pair of isomorphisms (ι_7, κ_3) to H_7 and W_3, thus obtaining a structure of vector bundle with total space H_7, base W_3, projection $\pi\colon H_7 \to W_3$, with $\pi^{-1}(\xi^{(\alpha)}) = \ell_\alpha$. Via restriction we obtain a fibre bundle with total space T_7, base $[0,1]^3$, projection $(\alpha, \theta) \mapsto \alpha$, and fibre $C_7(\alpha)$, as well as a fibre bundle with total space Δ_7 and base $w_3 = \kappa_3([0,1]^3)$. Combining the equality $\iota_7(C_7(\alpha)) = D_7(\alpha)$, Lemma 3, and Theorem 1, (ii), we obtain that if $\alpha \in [0,1]^3$ (respectively, $\alpha \in (0,1)^3$), then $\xi^{(\alpha)} \in D_7(\alpha)$ (respectively, $\xi^{(\alpha)} \in \mathring{D}_7(\alpha)$). Thus, $w_3 \cap D_7(\alpha) = \{\xi^{(\alpha)}\}$ and the projection $\pi\colon \Delta_7 \to w_3$ has fibres $\pi^{-1}(\xi^{(\alpha)}) = D_7(\alpha)$. Moreover, the restriction of the pair (ι_7, κ_3) is an isomorphism of fibre bundles.

For the sake of transparency, we note that $T_7 = \cup_{(\alpha)\in[0,1]^3} C_7(\alpha)$, $\Delta_7 = \cup_{(\alpha)\in[0,1]^3} D_7(\alpha)$. The unions $T_7^{(\cdot)} = \cup_{(\alpha)\in R^3} C_7^{(\cdot)}(\alpha)$, $\Delta_7^{(\cdot)} = \cup_{(\alpha)\in R^3} D_7^{(\cdot)}(\alpha)$ are the corresponding *dotted polytopes*.

The above considerations yield the following classification theorem:

Theorem 2. (i) The affine isomorphism $\iota_7\colon \mathbb{R}^7 \to H_7$ transforms any polytope $C_7(\alpha)$ (resp., dotted polytope $C_7^{(\cdot)}(\alpha)$) onto the polytope $D_7(\alpha)$ (resp., onto the dotted polytope $D_7^{(\cdot)}(\alpha)$).

(ii) The dotted polytope $C_7^{(\cdot)}(\alpha)$ is the classification space of all Yule's triples of type $[(\alpha, \theta)]$. The dotted polytope $\Delta_7^{(\cdot)}(\alpha)$ is the classification space of all probability distributions (1) produced by Yule's triples of type $[(\alpha, \theta)]$.

(iii) ι_7 maps the polytope T_7 (resp., dotted polytope $T_7^{(\cdot)}$) onto the polytope Δ_7 (resp., onto the dotted polytope $\Delta_7^{(\cdot)}$).

(iv) The dotted polytope $T_7^{(\cdot)}$ is the classification space of all Yule's triples. The dotted polytope $\Delta_7^{(\cdot)}$ is the classification space of all probability distributions produced by Yule's triples.

5. Entropy and Dependence of Yule's Triples

In this section we suppose $\alpha \in (0,1)^3$, that is (Lemma 3), $\mathring{I}_7(\alpha) \neq \emptyset$.

5.1. The Entropy Function

The function $E\colon \mathring{\Delta}_7 \to \mathbb{R}$, $E(\xi) = -\sum_{k=1}^{8} \xi_k \ln \xi_k$, is strictly concave since the open simplex $\mathring{\Delta}_7$ is convex and all of its "entropy" summands $E^{(k)}(\xi) = -\xi_k \ln \xi_k$ are strictly concave. Let us fix $\alpha \in (0,1)^3$ and let

$$E_\alpha(\theta) = \sum_{k=1}^{8} E_\alpha^{(k)}(\theta), E_\alpha^{(k)}(\theta) = -\xi_k(\theta) \ln \xi_k(\theta), \tag{11}$$

be the composition of E with the affine isomorphism $\iota_7^{(\alpha)}$: $E_\alpha(\theta) = E(\iota_7^{(\alpha)}(\theta))$. In accord with Corollary 1, (ii), the entropy function (11) of the experiment $\mathfrak{I}_3$ has $\mathring{I}_7(\alpha)$ as a natural domain: $E_\alpha\colon \mathring{I}_7(\alpha) \to \mathbb{R}$.

Lemma 4. (i) The entropy function E_α is a strictly concave function.

(ii) The entropy function E_α can be extended as continuous at $I_7(\alpha)$ and this extension $\hat{E}_\alpha$ is unique.

(iii) The continuous extension $\hat{E}_\alpha$ of E_α at $I_7(\alpha)$ is also a strictly concave function.

Proof. Note that the polytope $I_7(\alpha)$ and its interior $\mathring{I}_7(\alpha)$ are bounded convex sets.

(i) The function E_α is composition of the affine map $\iota_7^{(\alpha)}$ followed by the strictly concave function $E(\xi)$.

(ii) We apply [3, § 8, n^o5, Theorem 1].

(iii) The point $\theta^{(0)}$ belongs to the frontier of the polytope $I_7(\alpha)$ if and only if $\xi_k(\theta^{(0)}) = 0$ for indices k from some set K and $\xi_k(\theta^{(0)}) > 0$ for the rest of the indices, where $k = 1, \ldots, 8$. Moreover, for any $k \in K$ we have $E^{(k)}(\theta) \to 0$ when $\theta \to \theta^{(0)}$, $\theta \in \mathring{I}_7(\alpha)$. In other words, $\hat{E}^{(k)}(\theta^{(0)}) = 0$.

A boundary transition yields that $\hat{E}_\alpha$ is a concave function. Moreover, since there are indices $k \notin K$, the function E_α is strictly concave. Indeed, let $\theta^{(1)} \in \mathring{I}_7(\alpha)$ and $\lambda \in (0,1)$. In accord with [2, Ch. 11, Lemma 11.2.4], we have $(1-\lambda)\theta^{(0)} + \lambda\theta^{(1)} \in \mathring{I}_7(\alpha)$, hence

$$\hat{E}^{(k)}((1-\lambda)\theta^{(0)} + \lambda\theta^{(1)}) = E^{(k)}((1-\lambda)\theta^{(0)} + \lambda\theta^{(1)}) < (1-\lambda)E^{(k)}(\theta^{(0)}) + \lambda E^{(k)}(\theta^{(1)})$$

for any $k = 1, \ldots, 8$.

In case $k \notin K$ we have $\hat{E}^{(k)}(\theta^{(0)}) = E^{(k)}(\theta^{(0)})$ and we are done. Now, let $k \in K$ and let $\theta \to \theta^{(0)}$, $\theta \in \mathring{I}_7(\alpha)$. We obtain

$$\hat{E}^{(k)}((1-\lambda)\theta^{(0)} + \lambda\theta^{(1)}) = \lim_{\theta \to \theta^{(0)}} E^{(k)}((1-\lambda)\theta + \lambda\theta^{(1)}) \leq$$

$$(1-\lambda) \lim_{\theta \to \theta^{(0)}} E^{(k)}(\theta) + \lambda E^{(k)}(\theta^{(1)}) = (1-\lambda)\hat{E}^{(k)}(\theta^{(0)}) + \lambda \hat{E}^{(k)}(\theta^{(1)}).$$

■

The symmetric group S_3 acts on the entropy functions $E_\alpha(\theta)$ by the rule $\sigma E_\alpha(\theta) = E_\alpha(\sigma^{-1}\theta)$, $\sigma \in S_3$.

Lemma 5. If $\sigma \in S_3$, then $E_{\sigma\alpha}(\theta) = \sigma E_\alpha(\theta)$ and $I_7(\sigma\alpha) = \sigma_\theta I_7(\alpha)$.

Proof. (i) According to Lemma 1, we have $\sigma^{-1}E_{\sigma\alpha}(\theta) = E_{\sigma\alpha}(\sigma\theta) = E(\iota_7^{(\sigma\alpha)}(\sigma\theta)) = E(\iota_7(\sigma\alpha, \sigma\theta)) = E(\sigma\iota_7(\alpha, \theta)) = E(\iota_7(\alpha, \theta)) = E_\alpha(\theta)$. Finally, the domain of $\sigma E_\alpha(\theta)$ is the polytope $\sigma_\theta I_7(\alpha)$ and we obtain $I_7(\sigma\alpha) = \sigma_\theta I_7(\alpha)$.

■

Corollary 2. Let $\sigma \in S_3$.

(i) One has $\hat{E}_{\sigma\alpha}(\sigma\theta) = \hat{E}_\alpha(\theta)$.

(ii) All permutations of the members of Yule's triple $A = (A_1, A_2, A_3)$ have the same entropy: If $A \in [(\alpha)]$, then $\sigma A \in [(\sigma\alpha)]$ and $\hat{E}_{\sigma\alpha}(\theta^{(\sigma A)}) = \hat{E}_\alpha(\theta^{(A)})$.

Proof. (i) Let $\theta^{(0)}$ be point from the frontier of the polytope $I_7(\alpha)$. Then $\sigma\theta^{(0)}$ is point from the frontier of the polytope $I_7(\sigma\alpha)$ with interior $\sigma_\theta \mathring{I}_7(\alpha)$. We have $\theta \to \theta^{(0)}$, $\theta \in \mathring{I}_7(\alpha)$, if and only if $\sigma\theta \to \sigma\theta^{(0)}$, $\sigma\theta \in \sigma\mathring{I}_7(\alpha)$. The equality from Lemma 5 can be written in the form $E_{\sigma\alpha}(\sigma\theta) = E_\alpha(\theta)$ and a boundary transition yields the result.

(ii) Implied by part (i).

■

5.2. The Entropy Function and its Critical Points

For any $\theta \in \mathring{I}_7(\alpha)$ we obtain

$$\frac{\partial E_\alpha(\theta)}{\partial\theta_0} = \ln\frac{\xi_1(\theta)\xi_2(\theta)\xi_3(\theta)}{\xi_4^2(\theta)\xi_5(\theta)}, \frac{\partial E_\alpha(\theta)}{\partial\theta_1} = \ln\frac{\xi_2(\theta)\xi_3(\theta)}{\xi_4(\theta)\xi_6(\theta)},$$

$$\frac{\partial E_\alpha(\theta)}{\partial\theta_2} = \ln\frac{\xi_1(\theta)\xi_3(\theta)}{\xi_4(\theta)\xi_7(\theta)}, \frac{\partial E_\alpha(\theta)}{\partial\theta_3} = \ln\frac{\xi_1(\theta)\xi_2(\theta)}{\xi_4(\theta)\xi_8(\theta)}.$$

Thus, the set of critical points of the function $E_\alpha(\theta)$ is the intersection of the interior $\mathring{I}_7(\alpha) \subset \mathbb{R}^4$ and the algebraic variety in $\mathbb{R}^4$ with equations

$$\xi_1(\theta)\xi_2(\theta)\xi_3(\theta) - \xi_4^2(\theta)\xi_5(\theta) = 0, \xi_2(\theta)\xi_3(\theta) - \xi_4(\theta)\xi_6(\theta) = 0,$$

$$\xi_1(\theta)\xi_3(\theta) - \xi_4(\theta)\xi_7(\theta) = 0, \xi_1(\theta)\xi_2(\theta) - \xi_4(\theta)\xi_8(\theta) = 0.$$

Lemma 6. (i) The point $\theta^{(\alpha)}$ is a critical point of the entropy function E_α.

(ii) One has

$$E_\alpha(\theta^{(\alpha)}) = -\ln\left(\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}(1-\alpha_1)^{1-\alpha_1}(1-\alpha_2)^{1-\alpha_2}(1-\alpha_3)^{1-\alpha_3}\right).$$

Proof. (i) We have

$$\xi_1^{(\alpha)}\xi_2^{(\alpha)}\xi_3^{(\alpha)} - \left(\xi_4^{(\alpha)}\right)^2\xi_5^{(\alpha)} =$$

$$\alpha_1(1-\alpha_2)(1-\alpha_3)(1-\alpha_1)\alpha_2(1-\alpha_3)(1-\alpha_1)(1-\alpha_2)\alpha_3 -$$

$$(1-\alpha_1)^2(1-\alpha_2)^2(1-\alpha_3)^2\alpha_1\alpha_2\alpha_3 = 0,$$

$$\xi_2^{(\alpha)}\xi_3^{(\alpha)} - \xi_4^{(\alpha)}\xi_6^{(\alpha)} =$$

$$(1-\alpha_1)\alpha_2(1-\alpha_3)(1-\alpha_1)(1-\alpha_2)\alpha_3 - (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)(1-\alpha_1)\alpha_2\alpha_3 = 0,$$

$$\xi_1^{(\alpha)}\xi_3^{(\alpha)} - \xi_4^{(\alpha)}\xi_7^{(\alpha)} =$$

$$\alpha_1(1-\alpha_2)(1-\alpha_3)(1-\alpha_1)(1-\alpha_2)\alpha_3 - (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\alpha_1(1-\alpha_2)\alpha_3 = 0,$$

$$\xi_1^{(\alpha)}\xi_2^{(\alpha)} - \xi_4^{(\alpha)}\xi_8^{(\alpha)} =$$

$$\alpha_1(1-\alpha_2)(1-\alpha_3)(1-\alpha_1)\alpha_2(1-\alpha_3) - (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\alpha_1\alpha_2(1-\alpha_3) = 0.$$

(ii) We have

$$-E_\alpha(\theta^{(\alpha)}) = -E(\xi^{(\alpha)}) = \sum_{k=1}^{8}\xi_k^{(\alpha)}\ln\xi_k^{(\alpha)} =$$

$$\xi_1^{(\alpha)}\ln(\alpha_1(1-\alpha_2)(1-\alpha_3)) + \xi_2^{(\alpha)}\ln((1-\alpha_1)\alpha_2(1-\alpha_3)) +$$

$$\xi_3^{(\alpha)}\ln((1-\alpha_1)(1-\alpha_2)\alpha_3) + \xi_4^{(\alpha)}\ln((1-\alpha_1)(1-\alpha_2)(1-\alpha_3)) +$$

$$\xi_5^{(\alpha)}\ln(\alpha_1\alpha_2\alpha_3) + \xi_6^{(\alpha)}\ln((1-\alpha_1)\alpha_2\alpha_3) +$$

$$\xi_7^{(\alpha)}\ln(\alpha_1(1-\alpha_2)\alpha_3) + \xi_8^{(\alpha)}\ln(\alpha_1\alpha_2(1-\alpha_3)) =$$

$$(\xi_1^{(\alpha)} + \xi_5^{(\alpha)} + \xi_7^{(\alpha)} + \xi_8^{(\alpha)})\ln\alpha_1 + (\xi_2^{(\alpha)} + \xi_5^{(\alpha)} + \xi_6^{(\alpha)} + \xi_8^{(\alpha)})\ln\alpha_2 +$$

$$(\xi_3^{(\alpha)} + \xi_5^{(\alpha)} + \xi_6^{(\alpha)} + \xi_7^{(\alpha)}) \ln \alpha_3 + (\xi_2^{(\alpha)} + \xi_5^{(\alpha)} + \xi_6^{(\alpha)} + \xi_8^{(\alpha)}) \ln(1-\alpha_1) +$$
$$(\xi_1^{(\alpha)} + \xi_3^{(\alpha)} + \xi_4^{(\alpha)} + \xi_7^{(\alpha)}) \ln(1-\alpha_2) + (\xi_1^{(\alpha)} + \xi_2^{(\alpha)} + \xi_4^{(\alpha)} + \xi_8^{(\alpha)}) \ln(1-\alpha_3) =$$
$$\ln\left(\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}(1-\alpha_1)^{1-\alpha_1}(1-\alpha_2)^{1-\alpha_2}(1-\alpha_3)^{1-\alpha_3}\right).$$

■

5.3. The Entropy Function and its Second Derivative

Given k, $k = 1, \ldots, 8$, the Hessian of the function $E_\alpha^{(k)}(\theta)$, $\theta \in \mathring{I}_7(\alpha)$, is the 4×4 symmetric matrix $\mathcal{H}^{(k)}(\theta) = (\frac{\partial^2 E_\alpha^{(k)}}{\partial\theta_i \partial\theta_j}(\theta))_{i,j=1}^4$, where $\frac{\partial^2 E_\alpha^{(k)}}{\partial\theta_i \partial\theta_j}(\theta) = -\frac{\partial\xi_k(\theta)}{\partial\theta_i}\frac{\partial\xi_k(\theta)}{\partial\theta_j}\frac{1}{\xi_k(\theta)}$. Then the Hessian $\mathcal{H}(\theta)$ of the entropy function $E_\alpha(\theta)$ is the 4×4 symmetric matrix $\mathcal{H}(\theta) = \sum_{k=1}^8 \mathcal{H}^{(k)}(\theta)$. In accord with [4, Ch. 3, 3.1.4], since the functions $E_\alpha^{(k)}(\theta)$ are strictly concave, the corresponding quadratic forms ${}^t\tau\mathcal{H}^{(k)}(\theta)\tau$ are negative semi-definite: ${}^t\tau\mathcal{H}^{(k)}(\theta)\tau \leq 0$ for all $\tau \in \mathbb{R}^4$. In particular, the quadratic form ${}^t\tau\mathcal{H}(\theta)\tau = \sum_{k=1}^8 {}^t\tau\mathcal{H}^{(k)}(\theta)\tau$ is negative semi-definite. Moreover, since ${}^t\tau\mathcal{H}^{(5)}(\theta)\tau = -\frac{1}{\theta_0}\tau_1^2$, ${}^t\tau\mathcal{H}^{(6)}(\theta)\tau = -\frac{1}{\theta_1}\tau_2^2$, ${}^t\tau\mathcal{H}^{(7)}(\theta)\tau = -\frac{1}{\theta_2}\tau_3^2$, and ${}^t\tau\mathcal{H}^{(8)}(\theta)\tau = -\frac{1}{\theta_3}\tau_4^2$, the quadratic form ${}^t\tau\mathcal{H}(\theta)\tau$ is negative definite for any $\theta \in \mathring{I}_7(\alpha)$ and we obtain

Lemma 7. The set of local maximums of the entropy function $E_\alpha(\theta)$ coincides with the set of its critical points.

The compactness of the polytope $I_7(\alpha)$ yields that the extended entropy function $\hat{E}_\alpha(\theta)$ attains its absolute maximum and absolute minimum.

Theorem 3. The extended entropy function $\hat{E}_\alpha(\theta)$ has a unique absolute maximum attained at the point $\theta^{(\alpha)}$ from (10).

Proof. Lemma 6 and Lemma 7 yield that the entropy function $E_\alpha(\theta)$ and, therefore, also the extended entropy function $\hat{E}_\alpha(\theta)$, has a local maximum at the point $\theta^{(\alpha)}$. In accord with Lemma 4 and Lemma 10, $\hat{E}_\alpha(\theta)$ has a unique absolute maximum at $\theta^{(\alpha)}$.

■

Theorem 4. If the extended entropy function $\hat{E}_\alpha(\theta)$ attains an absolute minimum at some point from the polytope $I_7(\alpha)$, then this point is a vertex of $I_7(\alpha)$.

Proof. Lemma 2 allows us to use [2, Theorem 12.1.5, 12.1.8, Proposition 12.1.9] and we conclude that since the restriction of $\hat{E}_\alpha(\theta)$ on an i-face, $i = 1, 2, 3$, of the polytope $I_7(\alpha)$ is also a strictly concave function, we can apply at most four times Lemma 11.

■

The continuous extension $\hat{E}_\alpha(\theta)$, $\theta \in I_7(\alpha)$, of the entropy function $E_\alpha(\theta)$, $\theta \in \mathring{I}_7(\alpha)$, is said to be the *extended entropy function of Yule's triples of type* $[(\alpha)]$.

6. Degree of Mutual Dependence of a Triple of Events

6.1. Two Motivation Statements

Lemma 8. The three components of the Yule's triple $A = (A_1, A_2, A_3)$ are mutually independent if and only if $\theta^{(A)} = \theta^{(\alpha)}$.

Proof. In accord with [8, I,§5, (4)], the events A_1, A_2, A_3 are mutually independent if and only if $\Pr(A_i \cap A_j) = \Pr(A_i)\Pr(A_j)$, $1 \leq i < j \leq 3$, $\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1)\Pr(A_2)\Pr(A_3)$. Using (2), we write these conditions in the form

$$\left|\begin{array}{lcccl} \theta_0 & + & \theta_1 & = & \alpha_2\alpha_3 \\ \theta_0 & + & \theta_2 & = & \alpha_1\alpha_3 \\ \theta_0 & + & \theta_3 & = & \alpha_1\alpha_2 \\ \theta_0 & & & = & \alpha_1\alpha_2\alpha_3. \end{array}\right.$$

The point $\theta^{(\alpha)}$ from (10) is the unique solution of this system.

■

Now, we suppose, in addition, that $(\Omega, \mathcal{A}, \Pr)$ is a discrete uniform probability space. The faces of the polytope $I_7(\alpha) \subset \mathbb{R}^4$ are parts of the hyperplanes with equations $\xi_k(\theta) = 0$, $k = 1, \ldots, 8$. According to (1), the following equivalences hold:

Lemma 9. Let $A = (A_1, A_2, A_3)$ be a Yule's triple of events. One has:

$$\xi_1(\theta^{(A)}) = 0 \text{ iff } A_1 \subset A_2 \cup A_3, \xi_2(\theta^{(A)}) = 0 \text{ iff } A_2 \subset A_1 \cup A_3,$$

$$\xi_3(\theta^{(A)}) = 0 \text{ iff } A_3 \subset A_1 \cup A_2, \xi_4(\theta^{(A)}) = 0 \text{ iff } A_1^c \subset A_2 \cup A_3,$$

$$\xi_5(\theta^{(A)}) = 0 \text{ iff } A_1 \cap A_2 \subset A_3^c, \xi_6(\theta^{(A)}) = 0 \text{ iff } A_2 \cap A_3 \subset A_1,$$

$$\xi_7(\theta^{(A)}) = 0 \text{ iff } A_1 \cap A_3 \subset A_2, \xi_8(\theta^{(A)}) = 0 \text{ iff } A_1 \cap A_2 \subset A_3.$$

6.2. Definition of Degree of Mutual Dependence

The value of extended entropy function $\hat{E}_\alpha(\theta)$ of Yule's triples of type $[(\alpha)]$ at $\theta = \theta^{(A)}$ is called *entropy of Yule's triple $A = (A_1, A_2, A_3)$ of type* $[(\alpha)]$. In accord with Corollary 2, the entropy does not depend on the order of the components of A. This fact together with the opposites described in Lemmas 8 and 9 motivate the use of the extended entropy function $\hat{E}_\alpha(\theta)$ as a measure of strength of mutual dependence of three events A_1, A_2, A_3.

Let us denote by M the absolute maximum $\hat{E}_\alpha(\theta^{(\alpha)})$ and let m be the absolute minimum of $\hat{E}_\alpha(\theta)$, attained at some vertex of the polytope $I_7(\alpha)$, see Theorems 3 and 4. The former also yields that $m < M$.

Following [6, 5.2], for any $\theta \in I_7(\alpha)$ we define $e_\alpha \colon I_7(\alpha) \to [0,1], e_\alpha(\theta) = \frac{\hat{E}_\alpha(\theta) - M}{m - M}$. The value of the function e_α at $\theta \in I_7(\alpha)$, $\theta = \theta^{(A)}$, $A = (A_1, A_2, A_3)$, is said to be *degree of mutual dependence of the events A_1, A_2, A_3, with $\alpha_1 = \Pr(A_1)$, $\alpha_2 = \Pr(A_2)$, $\alpha_3 = \Pr(A_3)$*. Intuitively, $e_\alpha(\theta^{(A)})$ measures the strength of the mutual relations among the events A_1, A_2, A_3.

The above definition of e_α yields

Corollary 3. The degree of mutual dependence of three events does not depend on the choice of base of logarithms in the extended entropy function.

Example 5. In case $\alpha = (\frac{1}{10}, \frac{1}{5}, \frac{3}{10})$ the polytope $I_7(\alpha)$ has 12 vertices

$$v_{1,2,3,8}, v_{1,2,5,8}, v_{1,3,5,8}, v_{2,3,5,8}, v_{1,2,3,5}, v_{1,2,5,7},$$

$$v_{1,2,7,8}, v_{1,5,6,7}, v_{1,5,6,8}, v_{1,6,7,8}, v_{2,5,7,8}, v_{5,6,7,8}.$$

Here by v_{k_1,k_2,k_3,k_4} we denote the vertex which is the intersection point of the hyperplanes with equations $\xi_{k_1} = 0$, $\xi_{k_2} = 0$, $\xi_{k_3} = 0$, and $\xi_{k_4} = 0$. At the first four vertices the extended entropy function attains its absolute minimum (approximately equal to 0.8018185525433372). Equivalently, we have

$$e_\alpha(v_{1,2,3,8}) = e_\alpha(v_{1,2,5,8}) = e_\alpha(v_{1,3,5,8}) = e_\alpha(v_{2,3,5,8}) = 1.$$

On the other hand, let, for example, the vertex $v_{1,3,5,8}$ belongs to the dotted polytope $I_7^{(\cdot)}(\alpha)$, that is, let $\theta^{(A)} = v_{1,3,5,8}$, where $A = (A_1, A_2, A_3)$ is a Yule's triple.

Moreover, let us assume that $(\Omega, \mathcal{A}, \Pr)$ is a sample space with equally likely outcomes. In accord with Lemma 9, we can conclude that the system of set-theoretic relations

$$A_1 \subset A_2 \cup A_3, A_3 \subset A_1 \cup A_2, A_1 \cap A_2 \subset A_3^c, A_1 \cap A_2 \subset A_3,$$

or equivalently, the system of relations $A_3 \subset A_1 \cup A_2$, $A_1 \subset A_3 \cap A_2^c$, is one of the most powerful under the condition $\alpha = (\frac{1}{10}, \frac{1}{5}, \frac{3}{10})$.

On the other hand, $v_{1,3,5,8}$ is again a vertex in case $\alpha = (\frac{1}{5}, \frac{3}{10}, \frac{2}{5})$ but now the above system of relations is not the most powerful one: $e_\alpha(v_{1,3,5,8}) < 1$.

Example 6. [9, Section 3, 3.2], (Bernstein 1928) Let us consider a sample space with four equally likely outcomes $112, 121, 211, 222$. The events $A_1 = \{112, 121\}, A_2 = \{112, 211\}, A_3 = \{121, 211\}$, are pairwise independent but not mutually independent because $A_1 \cap A_2 \cap A_3 = \emptyset$. Below we evaluate their degree of mutual dependence. We set $A = (A_1, A_2, A_3)$ and note that $\alpha = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Using (1), we obtain $\xi_1^{(A)} = \xi_2^{(A)} = \xi_3^{(A)} = \xi_5^{(A)} = 0$, $\xi_4^{(A)} = \xi_6^{(A)} = \xi_7^{(A)} = \xi_8^{(A)} = \frac{1}{4}$. Therefore $\hat{E}_\alpha(\theta^{(A)}) = -2\ln\frac{1}{2}$. On the other hand, the polytope $I_7(\alpha)$ has 50 vertices and the extended entropy function $\hat{E}_\alpha(\theta)$ attains its absolute minimum $m = -\ln\frac{1}{2}$ at 48 of them. Since $M = \hat{E}_\alpha(\xi^{(\alpha)}) = -3\ln\frac{1}{2}$, we have $e_\alpha(\theta^{(A)}) = \frac{1}{2}$.

Remark 1. One can find below the link to a Java program which calculates the degree of mutual dependence of three events in a sample space with equally likely outcomes:
`http://www.math.bas.bg/algebra/valentiniliev/`

7. Conclusions

This paper finishes the trilogy that begins with [6] and [7]. It presents an original approach to the problem of measuring the magnitude of dependence of several events in a probability space, which rests upon Boltzmann-Shannon entropy of a probability distributions produced by these events. The first two parts are devoted to the fundamental case of two events where, for a given level of entropy intensity, one can discern negative from positive dependence, thus defining a direction. Moreover, the function of dependence of two events is closely related to the information exchanged between the two binary trials generated by these events.

The case of three events is studied here and this examination shows, in particular, that the general case of a finite number of events differs only in technical difficulties.

A. Appendix

A.1. Folklore Results about Extrema of a Concave Function

Our source of definitions and results about convex sets is [1, Ch. 11].

Let $C \subset \mathbb{R}^n$. We remind that the function $f\colon C \to \mathbb{R}$ is said to be *concave* (respectively, *strictly concave*) if C is a convex set and for any two different points $c_1, c_2 \in C$ and any $\lambda \in (0,1)$ one has $f((1-\lambda)c_1 + \lambda c_2) \geq (1-\lambda)f(c_1) + \lambda f(c_2)$ (respectively, $f((1-\lambda)c_1 + \lambda c_2) > (1-\lambda)f(c_1) + \lambda f(c_2)$).

Lemma 10. (i) Any local maximum point of a concave function is an absolute one.

(ii) There exists at most one local maximum point of a strictly convex function.

(iii) There exists at most one absolute maximum point of a strictly concave function.

Proof. Let $f\colon C \to \mathbb{R}$ be a concave function.

(i) Let $c_0 \in C$ be a point at which f attains a local maximum and let $U \subset C$ be a neighbourhood of c_0 such that $f(c_0) \leq f(c)$ for all $c \in U$. Let us suppose that there exists a point $c_1 \in C$ such that $f(c_1) > f(c_0)$. Then $f((1-\lambda)c_0 + \lambda c_1) \leq (1-\lambda)f(c_0) + \lambda f(c_1) > f(c_0)$ for all $\lambda \in (0,1)$. If λ is sufficiently close to 0, then $f((1-\lambda)c_0 + \lambda c_1) \in U$ and hence $f((1-\lambda)c_0 + \lambda c_1) \geq f(c_0)$ which is a contradiction.

(ii) Let, in addition, f be strictly concave and $c_1, c_2 \in C$ be two different points at which f attains a local maximum. In accord with part (i), we have $f(c_1) = f(c_2)$ and then $f((1-\lambda)c_1 + \lambda c_2) > (1-\lambda)f(c_1) + \lambda f(c_2) = f(c_1)$ for all $\lambda \in (0,1)$. Since f attains an absolute maximum at c_1, this is a contradiction.

Part (ii) implies part (iii).

■

Lemma 11. Let $f : C \to \mathbb{R}$ be a strictly concave function and let for any point $c \in \mathring{C}$ there exists an open line segment W_c such that $c \in W_c \subset C$. If f attains an absolute minimum at $c_0 \in C$, then $c_0 \notin \mathring{C}$.

Proof. Let us suppose that $c_0 \in \mathring{C}$ and let the points $c_1, c_2 \in W_c$, $c_1 \neq c_2$, be such that $c_0 = (1-\lambda)c_1 + \lambda c_2$ for some $\lambda \in (0,1)$. Then $f(c_1) \geq f(c_0)$, $f(c_2) \geq f(c_0)$, and $f(c_0) = f((1-\lambda)c_1 + \lambda c_2) > (1-\lambda)f(c_1) + \lambda f(c_2) \geq (1-\lambda)f(c_0) + \lambda f(c_0) = f(c_0)$, which is a contradiction. ■

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Declaration of Conflicting Interests

The Author declares that there is no conflict of interest.

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Power Length biased weighted lomax distribution

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Abstract

In this research paper, we have proposed the Power Length Biased Weighted Lomax Distribution(PLBWLD) as a new probability model . Moments, moment generating function, characteristic function, cumulant generating function, and reliability analysis such as survival function, hazard rate, reverse hazard rate, cumulative hazard function, and mills ratio are among the statistical features of PLBWLD that have been obtained here. Order statistics and PLBWLD's generalized entropy are also calculated. Maximum likelihood estimation is used to estimate the parameters of the model. Finally for demonstration purposes an application to the real data sets is provided to understand the new probability model's performance and flexibility.

Keywords: Length biased weighted Lomax distribution, power length biased weighted Lomax distribution, hazard rate function, moments, maximum likelihood estimation, order statistics, generalized entropy.

1. Introduction

Pareto distribution of second type is another name for the Lomax distribution. Lomax distribution was first used to model the failure rate of businesses by Lomax [8]. In the literature, the Lomax distribution has been employed in a variety of ways. According to Balkema and de Haan[3], it has been extensively utilized for life testing and reliability modeling including insurance, actuarial, demographics, economics, medical sciences, finance and engineering. The number of novel models with a high degree of flexibility is growing year after year. As a result, the researchers have shifted their focus to create new families of distributions and propose a variety of new families of distributions in order to better examine and investigate real-world data in various applications. Statistical distributions have gained a lot of attention recently as researchers try to figure out how to create flexible models for modelling a variety of data sets. It is because the classical distributions aren't very good at modelling data sets with a lot of variation. As a result, generalised probability models continue to grow and expand. In recent years, designing a new probability model from previously established models using various methodologies has gained a lot of attention. The power transformation technique, in which an extra parameter is added to the parent distribution, is one such strategy employed by several researchers. The addition of an extra parameter to the parent model usually increases the goodness of fit and gives more flexibility. Krishnarani [6] , Zaka and Akhter [9] are few of the researchers who have worked on power generalization of probability models. The concept of weighted distributions was first developed by Fisher [5].

If X is a non-negative random variable with the probability density function $f(x)$, then the probability density function of the weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))} \; ; \; x \geq 0$$

When $w(x) = x$, the resultant distribution is clearly length biased, with a probability density function given as;

$$f_L(x) = \frac{xf(x)}{E(x)} \; ; \; x \geq 0$$

If the random variable X has the length biased weighted Lomax distribution with shape parameter η and scale parameter λ respectively, then it's probability density function(pdf) and cumulative distribution function (cdf) proposed by Ahmad *et al.* [1] , are respectively given as

$$f(x;\eta,\lambda) = \frac{\eta(\eta-1)}{\lambda^2} x \left(1+\frac{x}{\lambda}\right)^{-(\eta+1)} \; ; \; x > 0, \eta > 1, \lambda > 0$$

$$F(x;\eta,\lambda) = 1 - \left(1+\frac{x}{\lambda}\right)^{-\eta} \left(1+\frac{x\eta}{\lambda}\right) \; ; \; x > 0, \eta > 1, \lambda > 0$$

2. POWER LENGTH BIASED WEIGHTED LOMAX DISTRIBUTION(PLBWLD)

The primary goal of this research paper is to improve the flexibility of the length biased weighted Lomax distribution by developing an expanded version of the model using power transformation technique. Suppose the random variable X assumes the length biased weighted Lomax distribution with parameters η and λ, then the transformed variable $V = X^{\frac{1}{\beta}}$ will follow power length biased weighted Lomax distribution with parameters η, β and λ.
The probability density function of the power length biased weighted Lomax distribution is obtained as;

$$f(v;\eta,\beta,\lambda) = \frac{\eta(\eta-1)\beta}{\lambda^2} v^{2\beta-1} \left(1+\frac{v^\beta}{\lambda}\right)^{-(\eta+1)} \; ; \; x > 0, \eta > 1, \lambda, \beta > 0 \tag{1}$$

The cumulative distribution function of the power length biased weighted Lomax distribution is obtained as

$$F(v;\eta,\beta,\lambda) = 1 - \left(1+\frac{v^\beta}{\lambda}\right)^{-\eta} \left(1+\frac{v^\beta\eta}{\lambda}\right) \tag{2}$$

For the visual illustration of the possible shapes of pdf and cdf of PLBWLD, Figure 1 and Figure 2 have been plotted. Plots of the survival function and hazard rate function of the PLBWLD distribution for different parameter values are also displayed in Figure 3
Remark: For $\beta = 1$ in 1, we obtain the length biased weighted Lomax distrbution.

3. RELIABILITY ANALYSIS OF THE POWER LENGTH BIASED WEIGHTED LOMAX DISTRIBUTION(PLBWLD)

This section focuses on obtaining the reliability (survival function), hazard rate (failure rate), reverse hazard function, cumulative hazard function and mills ratio expressions respectively for PLBWLD.

3.1. Survival function

The survival function or reliability function is the complement of the cumulative distribution function and it is defined as the probability that a system will survive beyond a specified time

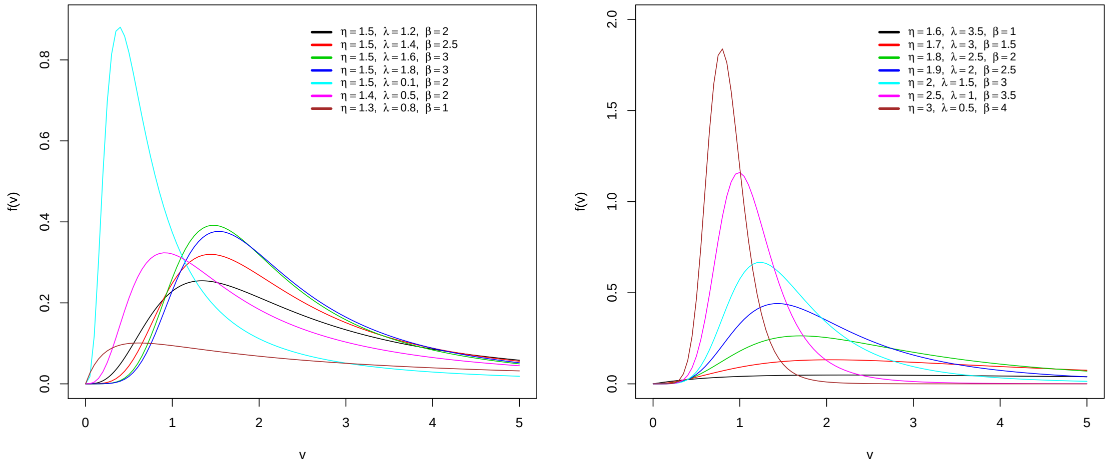

Figure 1: *Pdf Plots of the PLBWLD density for various values of η , β and λ.*

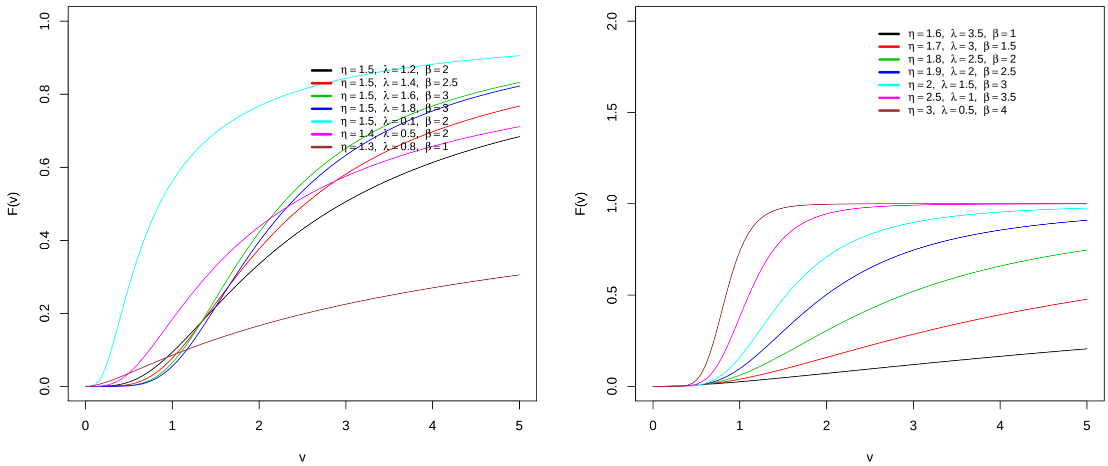

Figure 2: *Distribution function Plots of the PLBWLD for various values of η , β and λ.*

and is obtained for the PLBWLD as

$$R(v;\eta,\beta,\lambda) = 1 - F(v;\eta,\beta,\lambda) = \left(1+\frac{v^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{v^{\beta}\eta}{\lambda}\right) \tag{3}$$

3.2. Hazard Rate

Hazard rate also known as hazard function , force of mortality or failure rate. The Hazard rate assess the ability of a lifetime component to fail or to expire depending on the life completed and thus has wide variety of applications in lifetime distributions. Using (1) and (3), the expression for the hazard rate of PLBWLD is obtained as

$$h(v;\eta,\beta,\lambda) = \frac{f(v;\eta,\beta,\lambda)}{R(v;\eta,\beta,\lambda)} = \frac{\eta(\eta-1)\beta}{\lambda^2}v^{2\beta-1}\left(1+\frac{v^{\beta}}{\lambda}\right)^{-1}\left(1+\frac{v^{\beta}\eta}{\lambda}\right)^{-1} \tag{4}$$

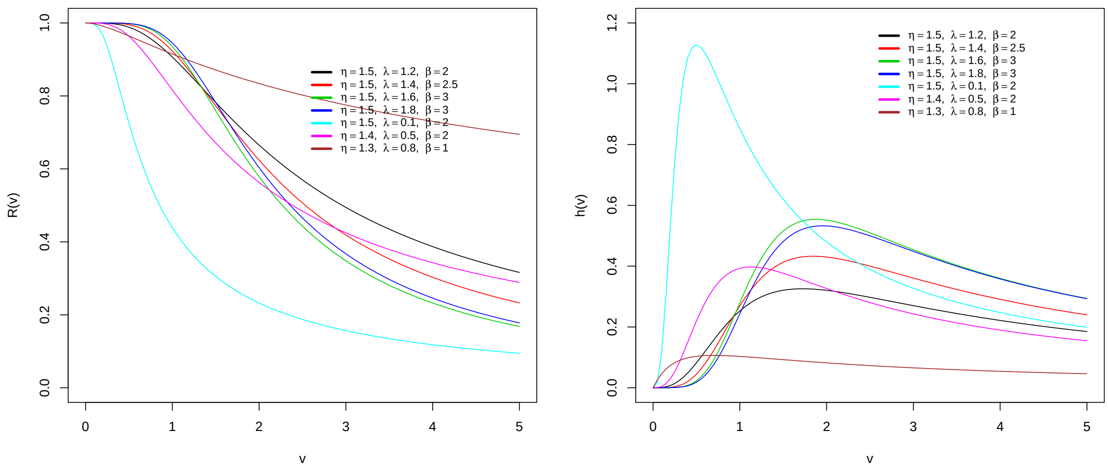

Figure 3: *Survival function and Hazard Rate Plots of the PLBWLD for various values of η , β and λ.*

3.3. Reverse Hazard function

The concept of reversed hazard rate of a random life is defined as the ratio between the life probability density to its distribution function . It is expressed as

$$h_r(v;\eta,\beta,\lambda) = \frac{f(v;\eta,\beta,\lambda)}{F(v;\eta,\beta,\lambda)}$$

Using equation (1) and (2) , the reverse hazard function for the Power length biased weighted Lomax distribution is obtained as

$$h_r(v;\eta,\beta,\lambda) = \frac{\frac{\eta(\eta-1)\beta}{\lambda^2} v^{2\beta-1}\left(1+\frac{v^\beta}{\lambda}\right)^{-(\eta+1)}}{1-\left(1+\frac{v^\beta}{\lambda}\right)^{-\eta}\left(1+\frac{v^\beta\eta}{\lambda}\right)} \tag{5}$$

3.4. Cumulative Hazard function

The cumulative hazard function can be thought of as providing the total accumulated risk of experiencing the event of interest that has been gained by progressing to time t. The cumulative hazard function for the PLBWLD is defined as

$$\Lambda_{PLBWLD}(v;\eta,\beta,\lambda) = -\log R(v;\eta,\beta,\lambda)$$

$$\Lambda_{PLBWLD}(v;\eta,\beta,\lambda) = \log\left\{\frac{\left(1+\frac{v^\beta}{\lambda}\right)^{\eta}}{\left(1+\frac{v^\beta\eta}{\lambda}\right)}\right\} \tag{6}$$

3.5. Mills Ratio

The mills ratio for the power length biased weighted Lomax distribution is defined as

$$M.R = \frac{F(v;\eta,\beta,\lambda)}{R(v;\eta,\beta,\lambda)} = \frac{1}{\left(1+\frac{v^\beta}{\lambda}\right)^{-\eta}\left(1+\frac{v^\beta\eta}{\lambda}\right)} - 1 \tag{7}$$

4. RESIDUAL AND REVERSED RESIDUAL LIFE FUNCTIONS OF THE POWER LENGTH BIASED WEIGHTED LOMAX DISTRIBUTION(PLBWLD)

4.1. Residual life function

In life testing situations, the additional lifetime given that a component has survived until time t is called residual life function. More specifically, if v is the life of a component, then the random variable $r(t) = (v - t|v > t); t \geq 0$ is used to explain the residual life of a lifetime component. For the PLBWLD, the survival function of the residual life time $r_{(t)}, t \geq 0$ is defined as

$$R_{r_{(t)}}(v;\eta,\beta,\lambda) = \frac{R(v+t)}{R(v)}$$

$$R_{r_{(t)}}(v;\eta,\beta,\lambda) = \left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{(v+t)^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)^{-1} \tag{8}$$

For the residual life time random variable $r_{(t)}, t \geq 0$, the cdf and pdf are respectively obtained as

$$F_{r_{(t)}}(v;\eta,\beta,\lambda) = 1 - R_{r_{(t)}}(v;\eta,\beta,\lambda)$$

$$F_{r_{(t)}}(v;\eta,\beta,\lambda) = 1-\left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{(v+t)^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)^{-1} \tag{9}$$

On differentiating the above equation w.r.t v, we obtain

$$f_{r_{(t)}}(v;\eta,\beta,\lambda) = \frac{l-m}{n^2} \tag{10}$$

where

$$l = \left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{(v+t)^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\frac{\theta}{\lambda}\beta t^{\beta-1}\left[1-\left(1+\frac{t^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{-1}\right]$$

$$m = \left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)\left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-\eta}\frac{\theta}{\lambda}\beta(v+t)^{\beta-1}\left[1-\left(1+\frac{(v+t)^{\beta}\eta}{\lambda}\right)\left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-1}\right]$$

$$n = \left[\left(1+\frac{t^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\right]^2$$

Also, the associated failure rate of $r_{(t)}, t \geq 0$ for the power length biased weighted Lomax distribution is given by

$$h_{r_{(t)}}(v;\eta,\beta,\lambda) = \frac{f_{r_{(t)}}(v;\eta,\beta,\lambda)}{R_{r_{(t)}}(v;\eta,\beta,\lambda)}$$

$$h_{r_{(t)}}(v;\eta,\beta,\lambda) = \left(\frac{l-m}{n^2}\right)\left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{\eta}\left(1+\frac{(v+t)^{\beta}\eta}{\lambda}\right)^{-1}\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)^{1} \tag{11}$$

where

$$l = \left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{(v+t)^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\frac{\theta}{\lambda}\beta t^{\beta-1}\left[1-\left(1+\frac{t^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{-1}\right]$$

$$m = \left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)\left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-\eta}\frac{\theta}{\lambda}\beta(v+t)^{\beta-1}\left[1-\left(1+\frac{(v+t)^{\beta}\eta}{\lambda}\right)\left(1+\frac{(v+t)^{\beta}}{\lambda}\right)^{-1}\right]$$

$$n = \left[\left(1+\frac{t^{\beta}\eta}{\lambda}\right)\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\right]^{2}$$

4.2. Reversed Residual life function

The random variable $r_{(t)}^{-} = (t-v|v \leq t); t \geq 0$ is used to explain the residual life of a lifetime component . For the power length biased weighted Lomax distribution , the survival function of the reversed residual life time $r_{(t)}^{-}, t \geq 0$ is defined as

$$R_{r_{(t)}^{-}}(v;\eta,\beta,\lambda) = \frac{F(t-v)}{F(t)}$$

$$R_{r_{(t)}^{-}}(v;\eta,\beta,\lambda) = \frac{1-\left(1+\frac{(t-v)^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{(t-v)^{\beta}\eta}{\lambda}\right)}{1-\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)} \tag{12}$$

For the reversed residual life time random variable $r_{(t)}^{-}, t \geq 0$, the cdf of power length biased weighted Lomax distribution is obtained as

$$F_{r_{(t)}^{-}}(v;\eta,\beta,\lambda) = 1 - R_{r_{(t)}^{-}}(v;\eta,\beta,\lambda)$$

$$F_{r_{(t)}^{-}}(v;\eta,\beta,\lambda) = \left[\frac{\left(1+\frac{(t-v)^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{(t-v)^{\beta}\eta}{\lambda}\right)}{1-\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)}\right] - \left[\frac{\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)}{1-\left(1+\frac{t^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{t^{\beta}\eta}{\lambda}\right)}\right] \tag{13}$$

5. STATISTICAL PROPERTIES OF PLBWLD

This section is devoted to discuss the related measures of the new formulated model like raw moments, central moments, measures of skewness, kurtosis, coefficient of variation, index of dispersion, mode and harmonic mean.

5.1. Raw Moments

The r^{th}moment of the PLBWLD about origin $\mu_r^{'}$ is given by

$$\mu_r^{'} = E(V^r) = \int_0^{\infty} v^r f(v;\eta,\beta,\lambda)dv$$

Using (1) and further simplification, r^{th}moment of the PLBWLD about origin $\mu_r^{'}$ is obtained as

$$\mu_r^{'} = \frac{\lambda^{\frac{r}{\beta}}\left(\frac{r}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)...(\eta-\frac{r}{\beta}-1)} \tag{14}$$

Using equation(14) and substituting $r = 1,2,3,4$, the first four moments about origin of the PLBWLD are obtained as

$$\mu_1^{'} = \frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)} \tag{15}$$

The equation (15) represents the mean of the PLBWLD.

$$\mu_2^{'} = \frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} \tag{16}$$

$$\mu_3^{'} = \frac{\lambda^{\frac{3}{\beta}}\left(\frac{3}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)} \tag{17}$$

$$\mu_4^{'} = \frac{\lambda^{\frac{4}{\beta}}\left(\frac{4}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)(\eta-\frac{4}{\beta}-1)} \tag{18}$$

5.2. Moments about Mean (Central Moments)

The moments about the mean, also known as central moments is defined as

$$\mu_2 = \mu_2^{'} - (\mu_1^{'})^2$$

using equations (15) and (16), we have

$$\mu_2 = \frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\theta-\frac{1}{\beta}-1)}\right)^2 \tag{19}$$

The equation(19) represents the variance of our new formulated model.

$$\mu_3 = \frac{\lambda^{\frac{3}{\beta}}\left(\frac{3}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)} - 3\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)}\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)} + 2\left[\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right]^3 \tag{20}$$

$$\mu_4 = \frac{\lambda^{\frac{4}{\beta}}\left(\frac{4}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)(\eta-\frac{4}{\beta}-1)} - 4\left(\frac{\lambda^{\frac{3}{\beta}}\left(\frac{3}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)}\right)\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)$$
$$+6\left(\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)}\right)\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right) - 3\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)^4$$

The following four coefficients are obtained for the PLBWLD based upon the first four moments about the mean and using the above expressions defined as:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$\gamma_1 = \sqrt{\beta_1}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$\gamma_2 = \beta_2 - 3$$

$$\beta_1 = \frac{\left\{ \frac{\lambda^{\frac{3}{\beta}}\left(\frac{3}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)} - 3\left(\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)}\right)\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right) + 2\left[\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right]^3 \right\}^2}{\left\{ \frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)^2 \right\}^3} \tag{21}$$

We need another measure that is dependent on the sign of the third central moment since the nature of skewness cannot be estimated using this relation.

$$\gamma_1 = \sqrt{\beta_1}$$

$$\gamma_1 = \frac{\left\{ \frac{\lambda^{\frac{3}{\beta}}\left(\frac{3}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)} - 3\left(\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)}\right)\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right) + 2\left[\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right]^3 \right\}}{\left\{ \frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)^2 \right\}^{\frac{3}{2}}} \tag{22}$$

Also,

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \tag{23}$$

where

$$\mu_4 = \frac{\lambda^{\frac{4}{\beta}}\left(\frac{4}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)(\eta-\frac{4}{\beta}-1)} - 4\left(\frac{\lambda^{\frac{3}{\beta}}\left(\frac{3}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)(\eta-\frac{3}{\beta}-1)}\right)\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)$$
$$+6\left(\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)}\right)\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right) - 3\left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)^4$$

And,

$$\mu_2 = \frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)^2$$

Again,

$$\gamma_2 = \beta_2 - 3$$

5.2.1 Coefficient of variation

$$CV = \frac{\sqrt{\mu_2}}{\mu_1'}$$

On using the equations (15) and(19), the coefficient of variation can be obtained for PLWLD.

$$C.V = \frac{\sqrt{\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)^2}}{\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}} \tag{24}$$

5.2.2 Index of Dispersion

The index of dispersion is defined as

$$D = \frac{\sigma^2}{\mu_1'}$$

Using the formula we obtain the index of dispersion for the PWLBWLD as

$$D = \frac{\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right)^2}{\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}} \tag{25}$$

5.2.3 Mode

To discuss PLBWLD's monotonicity, we use the logarithm of its probability density function as;

$$\log f(v;\eta,\beta,\lambda) = \log\left\{\frac{\eta(\eta-1)\beta}{\lambda^2}v^{2\beta-1}\left(1+\frac{v^\beta}{\lambda}\right)^{-(\eta+1)}\right\}$$

In order to find the value of mode, we differentiate the above equation w.r.t v and equate to zero, it yields

$$\hat{v} = \left\{\frac{(2\beta-1)\lambda}{(\eta-2)\beta+2}\right\}^{\frac{1}{\beta}} \tag{26}$$

Equation (26) represents the modal value for the PLBWLD.

5.2.4 Harmonic Mean

The harmonic mean for the PLBWLD is defined as

$$\begin{aligned} E(V^{-1}) = E(\frac{1}{V}) &= \int_0^\infty \frac{1}{v} f(v;\eta,\beta,\lambda)dv \\ &= \int_0^\infty \frac{1}{v}\frac{\eta(\eta-1)\beta}{\lambda^2}v^{2\beta-1}\left(1+\frac{v^\beta}{\lambda}\right)^{-(\eta+1)} dv \end{aligned}$$

on solving the integral and further simplification, we obtain the harmonic mean for PLBWLD as

$$H = \frac{1}{\eta(\eta-1)\lambda^{-\frac{1}{\beta}}\sum_{k=0}^{-\frac{1}{\beta}+1}(-1)^{k+1}\binom{-\frac{1}{\beta}+1}{k}\left(\frac{1}{-\frac{1}{\beta}-\eta-k+1}\right)} \tag{27}$$

6. Moment Generating function, Characteristic function and cumulant Generating function OF PLBWLD

6.1. Moment Generating Function

The moment generating function of PLBWLD distribution is defined as

$$M_v(t) = \int_0^\infty e^{tv} f(v)dv$$

using the following series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + ...$$

$$M_v(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} v^r f(v; \eta, \beta, \lambda) dv$$

Using equation (14) we obtain the moment generating function for PLBWLD as

$$M_v(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\lambda^{\frac{r}{\beta}} \left(\frac{r}{\beta} + 1\right)!}{(\eta - \frac{1}{\beta} - 1)(\eta - \frac{2}{\beta} - 1)...(\eta - \frac{r}{\beta} - 1)} \tag{28}$$

6.2. Characteristic Function

The characteristic function for the PLBWLD can be obtained using the relation $\phi_v(t) = M_v(it)$

$$\phi(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\lambda^{\frac{r}{\beta}} \left(\frac{r}{\beta} + 1\right)!}{(\eta - \frac{1}{\beta} - 1)(\eta - \frac{2}{\beta} - 1)...(\eta - \frac{r}{\beta} - 1)} \tag{29}$$

6.3. Cumulant Function

The cumulant function for the PLBWLD is obtained by using the relation $k_v(t) = \log M_v(it)$

$$k_v(t) = \log \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\lambda^{\frac{r}{\beta}} \left(\frac{r}{\beta} + 1\right)!}{(\eta - \frac{1}{\beta} - 1)(\eta - \frac{2}{\beta} - 1)...(\eta - \frac{r}{\beta} - 1)} \tag{30}$$

7. Order Statistics of PLBWLD

The order statistics connected to the power length biased weighted Lomax distribution is devoted in this section. Let $V_{(t;n)}$ be the t^{th} order statistics with the random sample $v_{(1)}, v_{(2)}, v_{(3)}, ...v_{(m)}$ derived from the PLBWLD having the probability density function (pdf) $f(v; \eta, \beta, \lambda)$ and cumulative distribution function (cdf) $F(v; \eta, \beta, \lambda)$. Therefore, the probability density function (pdf) and cumulative distribution function (cdf) of $v_{(t;n)}$ say $f_{(t;n)}(v)$ and $F_{(t;n)}(v)$ are respectively defined as

$$f_{(t;n)}(v) = \frac{n!}{(t-1)!(n-t)!} \left[F(v; \eta, \beta, \lambda)\right]^{t-1} \left[1 - F(v; \eta, \beta, \lambda)\right]^{n-t} f(v; \eta, \beta, \lambda) \tag{31}$$

$$F_{(t;n)}(v) = \sum_{j=t}^{n} \binom{n}{j} \left[F(v; \eta, \beta, \lambda)\right]^j \left[1 - F(v; \eta, \beta, \lambda)\right]^{n-j} \tag{32}$$

Using equation(1) and equation(2) in equation(31) and equation(32), the pdf and cdf of t^{th} ordered statistics for the PLBWLD are derived and are expressed as

$$f_{(t;n)}(v) = \frac{n!}{(t-1)!(n-t)!} \left[1 - \left(1 + \frac{v^\beta}{\lambda}\right)^{-\eta} \left(1 + \frac{v^\beta \eta}{\lambda}\right)\right]^{t-1} \left[1 + \frac{v^\beta}{\lambda}\right)^{-\eta} \left(1 + \frac{v^\beta \eta}{\lambda}\right]^{n-t} \frac{\eta(\eta-1)\beta}{\lambda^2} v^{2\beta-1} \left(1 + \frac{v^\beta}{\lambda}\right)^{-(\eta+1)}$$

$$F_{(t;n)}(v) = \sum_{j=t}^{n} \binom{n}{j} \left[1 - \left(1 + \frac{v^\beta}{\lambda}\right)^{-\eta} \left(1 + \frac{v^\beta \eta}{\lambda}\right)\right]^j \left[\left(1 + \frac{v^\beta}{\lambda}\right)^{-\eta} \left(1 + \frac{v^\beta \eta}{\lambda}\right)\right]^{n-j}$$

In order to obtain the expression for pdf of smallest(minimum) order statistics $v_{(1)}$ and the largest (maximum) order statistics $v_{(m)}$ of PLBWLD , we assume $t = 1$ and n respectively and are expressed in the form as

$$f_{(1;n)}(v) = n\left[1+\frac{v^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{v^{\beta}\eta}{\lambda}\right]^{n-1}\frac{\eta(\eta-1)\beta}{\lambda^2}v^{2\beta-1}\left(1+\frac{v^{\beta}}{\lambda}\right)^{-(\eta+1)} \tag{33}$$

$$f_{(n;n)}(v) = n\left[1-\left(1+\frac{v^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{v^{\beta}\eta}{\lambda}\right)\right]^{n-1}\frac{\eta(\eta-1)\beta}{\lambda^2}v^{2\beta-1}\left(1+\frac{v^{\beta}}{\lambda}\right)^{-(\eta+1)} \tag{34}$$

7.1. Median order statistics

The pdf of median order statistics, $v_{(n+1)}$ is defined as

$$f_{(n+1;n)}(v) = \frac{(2n+1)!}{n!n!}\left[F(v;\eta,\beta,\lambda)\right]^{n}\left[1-F(v;\eta,\beta,\lambda)\right]^{n} f(v;\eta,\beta,\lambda)$$

$$f_{(n+1;n)}(v) = \frac{(2n+1)!}{(n)!(n)!}\left[1-\left(1+\frac{v^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{v^{\beta}\eta}{\lambda}\right)\right]^{n}\left[1+\frac{v^{\beta}}{\lambda}\right)^{-\eta}\left(1+\frac{v^{\beta}\eta}{\lambda}\right]^{n}$$
$$\frac{\eta(\eta-1)\beta}{\lambda^2}v^{2\beta-1}\left(1+\frac{v^{\beta}}{\lambda}\right)^{-(\eta+1)}$$

8. Characterization of PLBWLD

Theorem 1. Let $v_{(1)}, v_{(2)}, ...v_{(n)}$ be n independently and identically distributed random samples selected from PLBWLD having a sample mean of $\bar{v_n}$ and sample variance of s_n^2 then,

$$\lim_{n\to\infty} E\left(\frac{s_n^2}{\bar{v_n}}\right) = \left(\frac{\sigma}{\mu}\right)^2$$

Proof: $E(\bar{v_n}) = \mu$ and $var(\bar{v_n}) = \frac{\sigma^2}{n}$

We know that

$$E(\bar{v_n})^2 = var(\bar{v_n}) + [E(\bar{v_n})]^2$$

$$E(\bar{v_n})^2 = \frac{1}{n}\left[\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\theta-\frac{1}{\beta}-1)}\right)^2\right] + \left[\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right]^2$$

since

$$E(s_n^2) = \sigma^2 = \left[\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\theta-\frac{1}{\beta}-1)}\right)^2\right]$$

Therefore

$$E\left(\frac{s_n^2}{\bar{v_n}^2}\right) = \frac{\left[\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\theta-\frac{1}{\beta}-1)}\right)^2\right]}{\frac{1}{n}\left[\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\theta-\frac{1}{\beta}-1)}\right)^2\right] + \left[\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right]^2}$$

On taking the limits to both sides of the above equation , we have

$$\lim_{n\to\infty} E(\frac{s_n^2}{\bar{v_n}}) = \frac{\left[\frac{\lambda^{\frac{2}{\beta}}\left(\frac{2}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)} - \left(\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\theta-\frac{1}{\beta}-1)}\right)^2\right]}{\left[\frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}\right]^2}$$

$$\lim_{n\to\infty} E(\frac{s_n^2}{\bar{v_n}}) = \left(\frac{\sigma}{\mu}\right)^2$$

Hence , the above theorem is proved

9. Information measure of PLBWLD

Entropy is a quantitative measures of the amount of uncertainty in a random variable. This section is dedicated to obtaining the PLBWLD generalized entropy expression.

Theorem 2. The generalized entropy for the PLBWLD is expressed as

$$I(\alpha) = \frac{1}{\alpha(\alpha-1)}\left\{\frac{\left(\frac{\alpha}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)...(\eta-\frac{\alpha}{\beta}-1)}\left\{\frac{\left(\eta-\frac{1}{\beta}-1\right)}{(\frac{1}{\beta}+1)!}\right\}^{\alpha} - 1\right\}$$

Proof: The generalized entropy is defined as

$$I(\alpha) = \frac{v_\alpha \mu^{-\alpha} - 1}{\alpha(\alpha-1)}$$

where

$$v_\alpha = \int_{-\infty}^{\infty} v^\alpha f(v) dv$$

and μ represents mean. For PLBWLD, we have

$$v_\alpha = \frac{\lambda^{\frac{\alpha}{\beta}}\left(\frac{\alpha}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)...(\eta-\frac{\alpha}{\beta}-1)}$$

$$\mu = \frac{\lambda^{\frac{1}{\beta}}\left(\frac{1}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)}$$

Therefore, the expression for the generalized entropy of PLBWLD is obtained as

$$I(\alpha) = \frac{1}{\alpha(\alpha-1)}\left\{\frac{\left(\frac{\alpha}{\beta}+1\right)!}{(\eta-\frac{1}{\beta}-1)(\eta-\frac{2}{\beta}-1)...(\eta-\frac{\alpha}{\beta}-1)}\left\{\frac{\left(\eta-\frac{1}{\beta}-1\right)}{(\frac{1}{\beta}+1)!}\right\}^{\alpha} - 1\right\} \tag{35}$$

10. Estimation of Parameters

This section is devoted to maximum likelihood estimation technique for estimating the unknown parameters η, β, λ of PLBWLD.

10.1. Maximum Likelihood Estimation(MLE)

Suppose $v_1, v_2, v_3, ...v_m$ be the random sample derived from the PLBWLD having the probability density function (pdf) $f(v;\eta,\beta,\lambda)$. Therefore, for m observations , the likelihood function of PLBLWD is obtained as

$$L(v;\eta,\beta,\lambda) = \left[\frac{\eta(\eta-1)\beta}{\lambda^2}\right]^m \prod_{i=1}^{m} v^{2\beta-1}\left(1+\frac{v^\beta}{\lambda}\right)^{-(\eta+1)}$$

Maximizing the log likelihood function yields estimates $\hat{\eta}, \hat{\beta}, \hat{\lambda}$ estimations of the unknown parameters η, β, λ. The log likelihood function is given by

$$\log L(v;\eta,\beta,\lambda) = m\log\eta + m\log(\eta-1) + m\log\beta - 2m\log\lambda + \sum_{i=1}^{m}\log(v_i)^{2\beta-1} - (\eta+1)\sum_{i=1}^{m}\log\left(1+\frac{v_i^\beta}{\lambda}\right) \quad (36)$$

The MLE's of η, β and λ are derived after partially differentiating (36) with respect to the corresponding parameters and equating to zero. We obtain the three normal equations as

$$\frac{m(2\eta-1)}{\eta(\eta-1)} = \sum_{i=1}^{m}\log\left(1+\frac{v_i^\beta}{\lambda}\right) \quad (37)$$

$$\frac{2m}{\lambda} = \frac{(\eta+1)}{\lambda^2}\sum_{i=1}^{m}\frac{v_i^\beta}{\left(1+\frac{v^\beta}{\lambda}\right)} \quad (38)$$

$$\frac{m}{\beta} + 2\sum_{i=1}^{m}\log v_i = (\eta+1)\sum_{i=1}^{m}\frac{\beta v_i^{\beta-1}}{(1+v_i^\beta)} \quad (39)$$

The above three non-linear equations (37),(38) and (39) are not in closed form. Therefore, we shall solve these equations numerically using Newton-Raphson technique of solving equations iteratively and numerically .

11. SIMULATION ILLUSTRATION

The performance of maximum likelihood estimates are examined in this section. To demonstrate the behavior of maximum likelihood estimates (MLEs) in terms of random generating sample sizes n= 100, 150 and a simulation research was conducted using R software. The procedure was repeated 100 times with various parameter combinations selected. The average MLE values and accompanying empirical mean squared errors (MSEs) were calculated in each scenario. Table 1 and table 2 shows the simulation findings. The estimates are stable and near to the genuine parameter values, as shown in table 1 and 2 . In all circumstances, the MSE drops as the sample size increases.

12. APPLICATION

For illustrating the flexibility, adaptability, and suitability of the PLBWLD, we use two actual data sets to show that the power length biased weighted lomax distribution (PLBWLD) can be better model than lomax distribution (LD) and length biased weighted lomax distribution (LBWLD).
To demonstrate how the proposed distribution can be effective in a real-world situation, two real life data sets have been examined . The following models have been investigated for comparison.

- Length biased weighted Lomax distribution (LBWLD) With pdf given in

$$f(v;\eta,\lambda) = \frac{\eta(\eta-1)}{\lambda^2}v\left(1+\frac{v}{\lambda}\right)^{-(\eta+1)} \ ; \ v>0, \eta>1, \lambda>0$$

- Lomax distribution (LD) with pdf given as

$$f(v;\eta,\lambda) = \frac{\eta}{\lambda}\left(1+\frac{v}{\lambda}\right)^{-(\eta+1)} \ ; \ v>0, \eta>1, \lambda>0$$

Here, several goodness-of-fit criterion such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Akaike Information Criterion Corrected (AICC) , Hannan Quinn Information Criterion (HQIC) and Kolmogorov -Smirnov (KS) statistics are used. The statistic with the lowest value is considered the best fit. The numerical results are produced using R programme for analysis purposes.

Table 1: *Average values of MLEs and the corresponding MSEs(n=100).*

Parameter			MLE			MSE		
η	λ	β	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$
1.5	1.2	2	1.60342	1.46263	1.88991	0.10386	0.29586	0.07998
		2.5	1.61165	1.52594	2.37138	0.04400	0.32747	0.11551
		3	1.60230	1.44918	2.80023	0.04089	0.23487	0.18081
	1.8	2	1.60342	2.13739	1.88990	0.03866	0.54490	0.07997
		2.5	1.56354	2.06030	2.41250	0.01490	0.28395	0.08610
		3	1.60906	2.06175	2.79079	0.039382	0.29147	0.19585
2	1.2	2	2.11303	1.37128	2.10674	0.33824	0.53606	0.17352
		2.5	2.29354	1.64115	2.61302	1.94607	3.27331	0.27052
		3	2.24369	1.54492	3.05438	0.88963	1.57343	0.42315
	1.8	2	2.15325	2.11382	2.08488	0.40547	1.02721	0.19684
		2.5	2.10288	1.95731	2.54206	0.20766	0.57590	0.21280
		3	2.17188	2.13039	3.11005	0.45798	1.41945	0.32907

Table 2: *Average values of MLEs and the corresponding MSEs(n=150).*

Parameter			MLE			MSE		
η	λ	β	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\beta}$
1.5	1.2	2	1.57382	1.40020	1.92791	0.02535	0.20046	0.06029
		2.5	1.57460	1.35389	2.36473	0.02172	0.10933	0.08347
		3	1.55781	1.36290	2.89490	0.01292	0.11236	0.08685
	1.8	2	1.56615	2.00830	1.90932	0.01953	0.24642	0.04964
		2.5	1.56273	1.92833	2.38188	0.01404	0.16196	0.08055
		3	1.58063	2.05766	2.87854	0.02763	0.23273	0.08986
2	1.2	2	2.12305	1.36716	2.05878	0.25457	0.38323	0.12918
		2.5	2.08487	1.37785	2.59013	0.15097	0.28118	0.16941
		3	2.05282	1.30232	3.10483	0.14772	0.29009	0.22523
	1.8	2	2.06271	1.90663	2.04340	0.14480	0.42436	0.085475
		2.5	2.01103	1.84029	2.62634	0.10312	0.31950	0.20608
		3	2.13476	2.06073	3.04121	0.23962	0.70488	0.26052

12.1. Data Set 1

Data set 1: The first data is on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been previously used by [2] . The data is as follows:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90

12.2. Data set 2

Data set 2: The following data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by [4]. The data are as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08,1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63,1.68, 1.71, 1.72,

Table 3: *$-2ln(l)$, AIC, AICC, BIC for the first data set.*

Model	$-2ln(l)$	AIC	AICC	BIC	HQIC	K-S
PLBWLD	175.004	181.004	181.391	187.573	183.6	0.078
LBWLD	224.008	228.008	228.199	232.388	229.731	0.250
LD	265.990	269.989	270.180	274.369	271.720	0.358

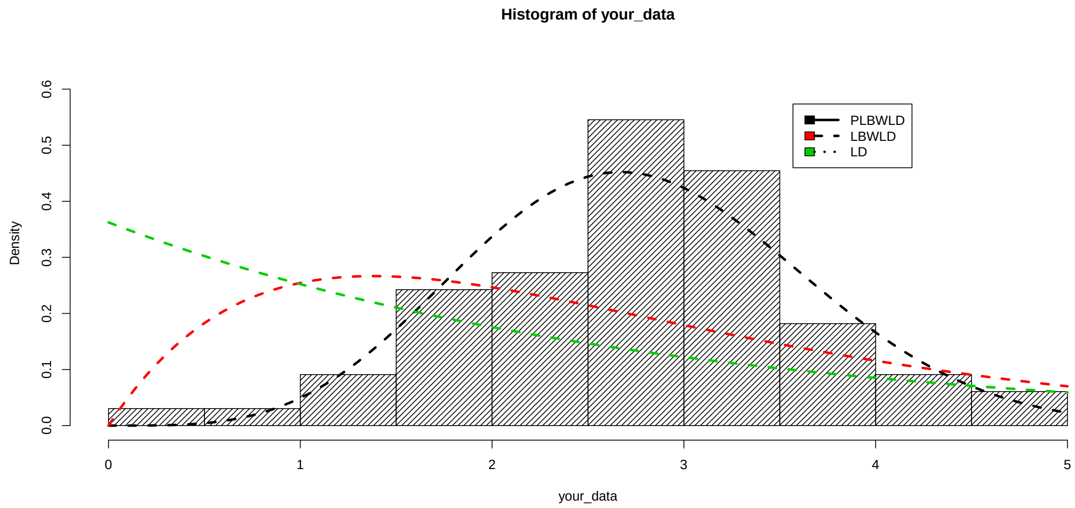

Figure 4: *Fitted density plots for dataset1*

1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54,2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55

Table 4: *$-2ln(l)$, AIC, AICC, BIC for the second data set.*

Model	$-2ln(l)$	AIC	AICC	BIC	HQIC	K-S
PLBWLD	187.753	193.753	194.106	200.583	196.472	0.084
LBWLD	195.049	199.049	199.223	203.602	200.861	0.168
LD	226.075	230.075	230.249	234.628	231.888	0.294

13. CONCLUSION

This research paper uses power transformation to develop a novel life time probability model called power length biased weighted Lomax distribution. Ordinary moments, moment generating function, hazard rate, order statistics, and generalized entropy are among the significant aspects of PLBWLD that are obtained here. In addition, two real data sets are used to highlight the practical value.
The three-parameter PLBWLD distribution has been introduced here to have more flexibility in terms of the hazard rate function and density function. Using goodness of fit criteria, the suggested model's effectiveness is compared with other competing distributions. The new distribution can exhibit a much more flexible model for life time data . The new model was fitted to two different real-life data sets and showed that it

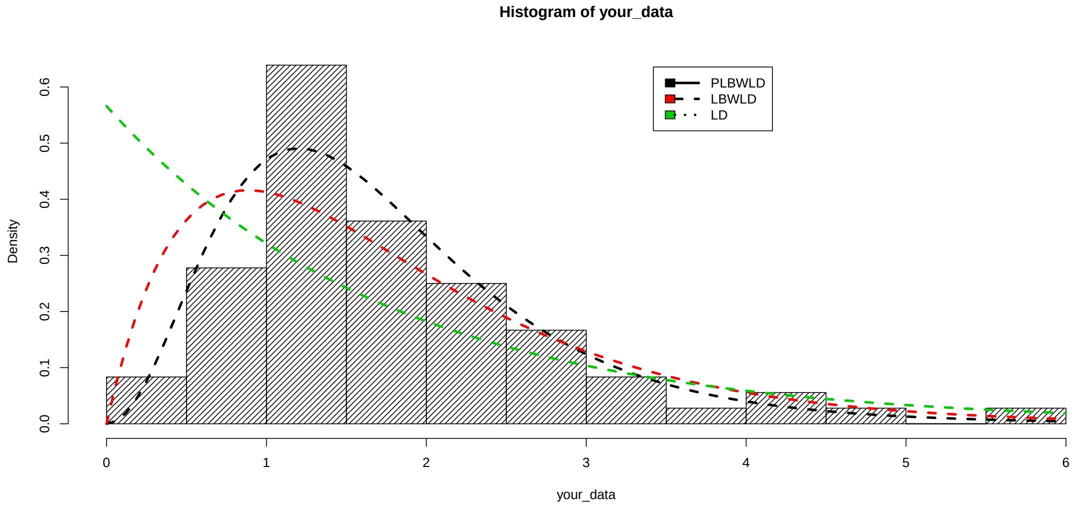

Figure 5: *Fitted density plots for dataset2*

could offer a better fit than a set of extensions of Lomax distribution. We believe that the suggested model will have broader statistical applications.

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Inferences for two parameter Teissier distribution in case of fuzzy progressively censored data

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Abstract

In process of observing data, it is sometimes not possible to obtain data precisely and fuzzy methods are useful for analyzing such data sets. In this article, we propose location-scale family of the Teissier distribution for fitting fuzzy censored data sets. The maximum likelihood, least squares and Bayes estimators of the parameters of the Teissier distribution are constructed in the presence of the progressively fuzzy censored samples. In addition to that statistical properties of the distribution are also derived. Fitting of the tensile strengths of the carbon fibers is done using the proposed distribution with comparison to the location-scale families of the exponential, Maxwell and Lindley distributions. We found that the Teissier distribution can be effectively used for fitting complete and fuzzy censored data as well.

Keywords: Location-scale Teissier distribution, Fuzzy lifetime data, Type-II progressive censoring scheme, Mean residual life, Moments, Maximum likelihood estimator, Least squares estimator, Bayes estimator

1. Introduction

Uncertainty is associated with our daily life activities. We have to take some actions/decisions to minimize the risk or maximize the gains in business and other activities. We make some probabilistic statements to assess the nature of the random phenomenon under consideration, so that our decision makes our life better. In this article, we are concerned about the life of man-made systems/products and propose a method of estimating the parameter associated with the random phenomenon. Lifetimes are not deterministic and they are random in nature. This type of uncertainty arises due the random nature of the phenomenon and are efficiently dealt by the statistical methods where the lifetimes are considered as random variables having certain probability distributions that are characterized by some constants (called parameters).

In conventional statistical inferences, random observations are drawn from the population and inferences are made about the parameters provided that the data are observed precisely. In many cases, it may not be possible to obtain the data in precise form because of many unavoidable reasons such as the lack of information, human errors, measurement errors and other practical difficulties. In fact, the real measurements of the continuous variables are never precisely obtained and have some errors. For instance, locations of the objects in space, live positions of ships on radar screen and space time data are a few examples among many. There is uncertainty always attached with the individual observation. Such type of uncertainty cannot be dealt by the conventional statistical procedures. In such cases, the fuzzy set theory is useful for explaining the random behavior of the observations. Until 1960, the probability theories and statistics are used to model the uncertainty. In 1968, [7] introduced the probability measure for fuzzy events, see also [6]. [22] discussed various methods for fuzzy data estimation and hypothesis testing. From the application point of view, [5] proposed the statistical analysis of fuzzy data. [8, 21] proposed to use Bayesian approach to estimate the parameters and reliability function for the distributions using the fuzzy lifetime data. Recently, [11] presented classical and Bayesian procedure for estimating the parameter of Rayleigh distribution based on Type-II progressively hybrid censored data under fuzzy setup.

In the field of surveys and life testing experiments, it may be possible to come across the situation of incomplete/lost data due to lack of time, cost and some other constraints. Some of the experimental units have some complete information and the rest reports non-occurrence. The units for which exact failure information is available are called complete samples and the remaining units are called censored observations. For analyzing such censored data, it is essential to use censoring schemes. Conventional Type-I and Type-II censoring schemes have been studied by many authors, see [13], [15] and [14] for modeling censored data sets.

Under Type-I censoring scheme, n units are put under observation and the experiment is carried-out up to a predetermined time. In the case of the Type-II censoring scheme, the experiment is terminated when pre-fixed numbers of units are observed to have failed. It is worth to mention here that these schemes do not enough flexible to incorporate removals of the experimental units at the stages other than the final terminal stage. In this context, [9] proposed a censoring scheme, called the progressive censoring scheme, which has an advanced feature that allows the removal of the units at the intermediate stages. Since then, an extensive list of literature is seen devoted to the progressive censoring. We follow [17] and [35] for the detailed theoretical aspects, estimation methods and applications of this scheme. [2] provided two simulation algorithms for generating the progressive Type-II censored samples based on exponential and uniform random variable transformations. [18] proposed classical and Bayes estimation for the flexible Weibull parameters under the progressive Type-II censoring scheme. [19] discussed the different estimation methods for estimating the parameters of Rayleigh distribution on the basis of progressive Type-II censoring scheme.

Suppose n experimental units are put under observation and experiment is terminated after a prefixed number (m) of observations are observed. At each stage, we progressively drop some prefixed number of units from the remaining units. Suppose that as the first failure occurs, some units, say R_1 are removed from the remaining $(n-1)$ units. At the second stage, R_2 units are removed from the remaining $(n-R_1-2)$ units. Similarly, at th mth stage x_m is observed, R_m units are removed from the remaining $(n-R_1-\cdots-R_{m-1}-m)$ units such that $(m+\sum_{i=1}^{m} R_i = n)$. We terminate the experiment at time point x_m. This is how we obtain the progressively censored data $(x_1, x_2, \ldots, x_m)$ with prefixed removals $(R_1, R_2, \ldots, R_m)$.

The objective of this paper is to develop the various estimation methods to estimate the parameters of two-parameter Teissier distribution with fuzzy censored data. The Teissier distribution was first introduced in [23] to model the frequency of the mortality due to ageing only i.e. deaths are protected from the accidents and disease. Later on, [24] used this distribution for the reliability analysis. After that it has been overlooked in the literature. [25] rediscovered the Teissier distribution and derived its statistical properties. For adding more flexibility to the model, [26] introduced a two parameter extension of Teissier distribution called Power Muth distribution. [34] proposed another two parameter extension of the Teissier distribution, called exponentiated Teissier distribution. Recently, [36] proposed the Teissier generalised family of the distributions and produced various flexible probability distributions.

The rest of the paper is organized as follows. In section 2, the progressive censoring and different methods of estimation are discussed. Section 3 deals with the brief introduction of fuzzy random variable and related concepts. We introduce location-scale family of the Teissier distributions in section 4 and discussed it's properties here. Section 5 discusses the estimation of parameters of the Teissier distribution for the Type-II progressive censoring under under fuzzy setup. Simulation study and real data application are discussed in section 6 and section 7, respectively. At last, overall conclusions are given in section 8.

2. Progressive Censoring and Estimation Methods

Statistical inference mainly focuses on the estimation of the unknown model parameters using the available observed data. As discussed in the above section, we may encounter the situation where the observed data is censored. The estimation procedures can be easily implemented for censored data, [35]. In this section, we describe the maximum likelihood estimation (MLE), least squares estimation (LSE) and Bayes estimation procedures for the Type-II progressively censored data.

Under the MLE, we maximize the joint density function (called likelihood function) of observed random sample over the parameter space. If we are given realizations of the n i.i.d. random variables, the likelihood function is nothing but the product of the respective densities. Suppose that X is a random variable governed by the probability distribution function (pdf) $f(x,\theta)$ and $x_1, x_2, ..., x_n$ are the n i.i.d. random samples drawn from it. The likelihood function is $L(\theta; x) = \prod_{i=1}^{n} f(x_i, \theta)$, where θ is the parameter of interest.

Since, in this paper, we consider progressively Type-II censored data, we define the likelihood function (following [35]) as given by

$$L(\theta) = C \prod_{i=1}^{m} f\left(x_i\right)\left(1 - F\left(x_i\right)\right)^{R_i}, \tag{1}$$

where $C = \prod_{i=1}^{m-1} \left(n - \sum_{j=1}^{i} (r_i + 1) \right)$ and $f(x)$ and $F(x)$ are the pdf and cdf of the assumed probability distribution. The MLE $\hat{\theta}$ of θ maximize the likelihood function given in the equation (1). For some distributions, the closed-form MLEs are not possible to derive and iterative methods are used for numerical computation of the estimates.

In the process of estimating the parameters, [1] proposed the method of least squares for estimating the beta parameters. That are obtained by minimizing the sum of squares of the discrepancies between the observed and expected distributions. For the progressively Type-II censored data, [12] have constructed the LSEs of the parameters of the generalised inverted exponential distribution. For the Type-II progressively censored sample, we have

$$E\left[F(x_i)\right] = 1 - \prod_{j=m-i+1}^{m} \alpha_j, \tag{2}$$

where $\alpha_j = \frac{a_i}{1+a_i}$ and $a_i = i + \sum_{j=m-i+1}^{m} R_i$. The LSEs are obtained by minimising the following function

$$S(\theta) = \sum_{i=1}^{m} \left[F(x_i) - E(F(x_i))\right]^2. \tag{3}$$

The equation (3) are usually optimized numerically as the closed form solution is not possible in most of the cases.

In recent decades, Bayesian perspective received a great attention by researchers for statistical inferences. In Bayesian framework, we use the Bayes theorem to update the probability for a hypothesis after observing the data as evidence. It facilitates to use the prior information for formulating a better method of estimation. Bayes procedure is discussed in section 5.

3. Fuzzy random variable and membership function

Before defining the fuzzy random variable, some basic definitions are needed to explore. Let $S = (\Omega, A, P_\theta)$ be the probability space. Here (Ω, A) is a measurable space and P_θ is the probability measure defined on measurable space (Ω, A).

Definition 1. Membership function for any set Ω is the function from Ω to real interval $[0, 1]$. Here the value of $\mu_{\tilde{A}}(x)$ at x define the "Grade of membership" or "degree of truthfulness ".

$$\mu_{\tilde{A}}(x) : \Omega \to [0, 1]$$

Here membership function is not limited to 0 and 1 only, but takes any value between $[0,1]$.

Definition 2. Here Ω is a universal set. Fuzzy set $\tilde{A}$ in Ω is denoted by an ordered set of pairs $(x, \mu_{\tilde{A}}(x))$. The first component of which denotes elements of the set $\tilde{A}$ and second denotes the degree of membership of that elements in set $\tilde{A}$.

If $sup\mu_{\tilde{A}}(x) = 1$, then $\tilde{A}$ is called normal fuzzy set.

Definition 3. Let X be the universal set. Then the support of a fuzzy set $\tilde{A}$, i.e. $S(\tilde{A})$ is the set of all points $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

Definition 4. Let X be the universal set. A fuzzy set $\tilde{A}$ is said to be convex if $\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)), \quad x_1, x_2 \in X, 0 < \lambda < 1.$

Definition 5. A fuzzy set $\tilde{A}$ is a fuzzy number if it is normal, convex and its support is bounded.

In general, there are two fuzzy numbers that are mostly used Triangular fuzzy number and Trapezoidal fuzzy number. Triangular fuzzy number is denoted by $\tilde{x} = (u, v, w)$ with corresponding membership function as

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{x-u}{v-u}, & u \leq x \leq v \\ \\ \frac{w-x}{w-v} & v \leq x \leq w \\ \\ 0, & otherwise \end{cases} \tag{4}$$

Fuzzy random variable can be defined as random variable, the value of which is not real but fuzzy number which is nothing but a particular kind of fuzzy set, see [7], [6] and [22].

4. Location-Scale Teissier distribution

In any probability distribution, location parameter determines the shift of the distribution or origin on the horizontal axis. Sometimes, in reliability and life testing experiments, it is possible that failure will absolutely not happen before a given time. In this regard, location parameter have some profound effect in reliability and life-testing experiments. Here, we introduce the Teissier distribution with location parameter denoted by μ.

Definition 6. Consider random variable Y that follows the Teissier distribution defined in [23] with scale parameter θ. Then the random variable $Y = X + \mu$ is said to follow the 2-parameter Teissier (2-T) distribution indexed by location parameter μ and scale parameter θ. The pdf and cdf of the 2-T distribution are given as follows

$$f(x; \mu, \theta) = \theta e(e^{\theta(x-\mu)}) \left(e^{\theta(x-\mu)} - 1\right) \exp\left(-e^{\theta(x-\mu)}\right), x > \mu > 0, \theta > 0, \tag{5}$$

$$F(x; \mu, \theta) = 1 - \exp\left[\theta(x - \mu) - e^{\theta(x-\mu)} + 1\right], x > \mu > 0, \theta > 0. \tag{6}$$

The corresponding hazard function of the 2-T distribution is given by

$$h(x; \mu, \theta) = \theta\left(e^{\theta(x-\mu)} - 1\right), x > \mu > 0, \theta > 0.$$

Note that $h^{'}(x; \mu, \theta) = \theta^2 e^{\theta(x-\mu)} + \left(e^{\theta(x-\mu)} - 1\right) > 0, \forall \ x > \mu > 0, \theta > 0$. It can be concluded that the hazard function of the 2-T distribution is monotonic increasing.

Fig. 1 displays the various shapes of the pdf of the 2-T distribution. We can see that the distribution is unimodal and right skewed. So, it is flexible enough to fit a wide varieties of right skewed data sets. Fig. 2 displays the hazard rate function which has increasing shapes for different parameter values. Since, the 2-T has increasing hazard rate, so it may be a good choice for modelling data set in reliability and survival analysis where the things are more likely to fail with the increasing time.

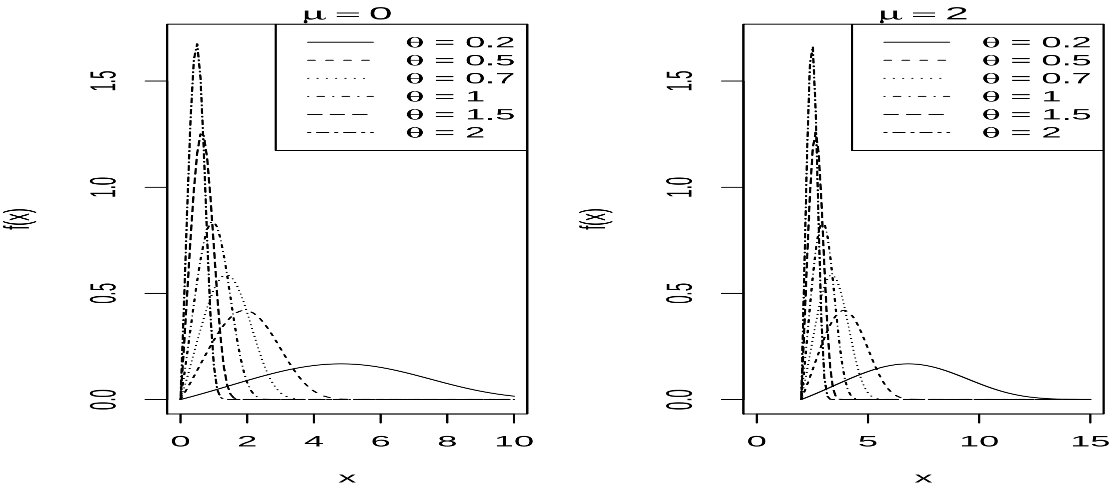

Figure 1: *Density function for different values of θ for given $\mu = 0$ and $\mu = 2$*

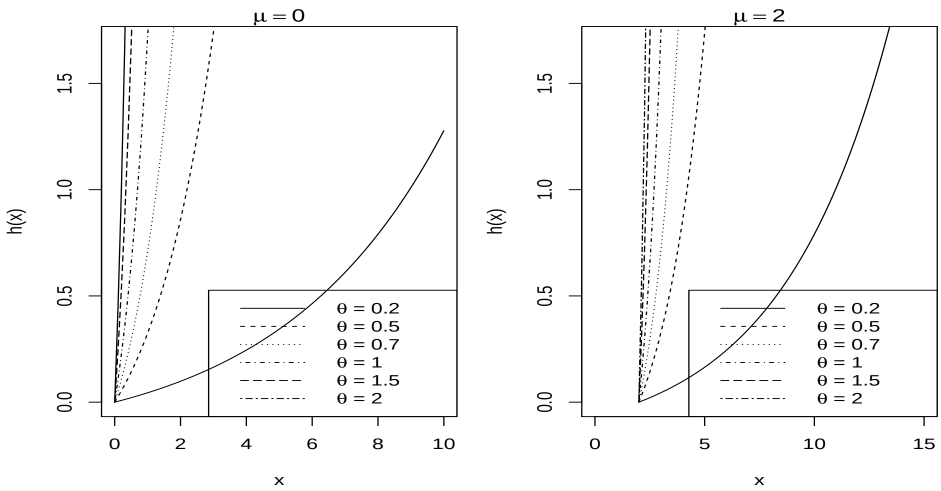

Figure 2: *Hazard function for different values of θ for given $\mu = 0$ and $\mu = 2$*

4.1. Properties

In this section, the basic properties of the 2-T distribution are derived. We derive moments, quantile function, moments generating function and mean residual life function.

4.1.1 Quantile function

The quantile function or inverse cumulative distribution function is more precisely used to generate the computer pseudo random data from associated probability distribution and explore its basic statistical properties such as measures of central value, dispersion, skewness and kurtosis. The qth quantile of any random variable X is defined as the solution of the equation $F_X(\eta_q) = q, 0 < q < 1$ i.e. $\eta_q = F_X^{-1}(q)$, where $F(.)$ is the cdf. The closed-form expression of the 2-T distribution is obtained in terms of the Lambert-W function. The Lambert-W function is used to obtain the solution of the equation $W(z)e^{W(z)} = z, z \in C$. For more details about the Lambert-W function see [31]. For the 2-T distribution η_q is the solution of the equation

$$1 - \exp\left[\theta(\eta_q - \mu) - e^{\theta(\eta_q - \mu)} + 1\right] = q, \quad 0 < q < 1. \tag{7}$$

On simplifying the equation (7), it turns out to be

$$e^{\theta(\eta_q-\mu)} = \theta(\eta_q - \mu) + (1 - \log(1-q)).$$

Here we note that $e^{\theta} > 0, \forall\, \theta > 0$ and $e^{\theta} \neq 1, \forall\, \theta > 0$ also $(\eta_q - \mu) \neq 0 \;\forall\; \eta_q > \mu$. So, this equation can be solved by using the Lambert-W function. η_q is given by

$$\eta_q = \mu - \frac{1}{\theta} + \frac{1}{\theta}\log(1-q) - \frac{1}{\theta} W_{-1}\left(\frac{u-1}{e}\right) \tag{8}$$

where W_{-1} represents the negative branch fo Lambert-W function.

4.1.2 Mean Residual Life

The mean residual life (MRL) is an important criterion of reliability measure of non-negative random variables. Sometimes it is more relevant than the hazard rate function, mainly in repair and replacement problems because it relates only to the risk of immediate failure than entire residual life function. The MRL or mean remaining life of the random variable X beyond the value x, denoted by $r(x)$ is given as

$$r(x) = E\left((X-x)/X > x\right) = \int_x^{\infty} \frac{\bar{F}(y)dy}{\bar{F}(x)}. \tag{9}$$

The analytical expression of the MRL function for the 2-T distribution is given as follows

$$r(x) = \frac{1}{\bar{F}(x)} \int_x^{\infty} \exp\left(\theta(y-\mu) - e^{\theta(y-\mu)} + 1\right) dx. \tag{10}$$

In equation (10), we have made change of variable $(x - \mu) = z$ and again make substitution as $e^{\theta z} = t$ in the equation. Then we obtain the expression of MRL of the 2-T distribution as

$$r(x) = \frac{\exp\left(1 - e^{\theta(x-\mu)}\right)}{\theta \bar{F}(x)}, \tag{11}$$

where $\bar{F}(x)$ is the survival function of the random variable X.

4.1.3 Moment Generating Function

The analytical expression for moment generating function of random variable X following the 2-T distribution, having the density function as $f_X(x)$ is defined as

$$M_X(t) = E(e^{tx}) = \int_{\mu}^{\infty} e^{tx} f_X(x)dx. \tag{12}$$

Proposition 1. The moment generating function of the 2-T Teissier distribution is given by

$$M_X(t) = e^{\mu t+1}\left[\Gamma\left(2 + \frac{t}{\theta}, 1\right) - \Gamma\left(1 + \frac{t}{\theta}, 1\right)\right], -\infty < t < \infty,$$

where Γ denotes the upper incomplete gamma function.

Proof. The proof is straight forward. ∎

4.1.4 Moments

For calculating the moments of the 2-T distribution, we make use of Generalized integro-exponential function, the integral representation of which is given as

$$E_a^b(t) = \frac{1}{\Gamma(b+1)} \int_1^{\infty} (\log v)^b v^{-a} e^{-tv} dv, m = 0, 1, 2, ... \tag{13}$$

where $t, a \in C$ and $\Gamma(p) = \int_0^{\infty} e^{-x} x^{p-1}$ is the ordinary gamma function.

Proposition 2. For random variable X following the distribution (5), the kth moment is given by

$$E[X^k] = e\mu^k \sum_{i=0}^{k} \frac{\binom{k}{i}}{(\mu\theta)^i} \left[E_{-1}^i(1) - E_0^i(1)\right]. \tag{14}$$

Proof. Let X be any random variable and following any particular distribution $f_X(x)$. Then k^{th} moment of the random variable X is given as

$$E[X^k] = \int x^k f_X(x)dx. \tag{15}$$

Now for the moments of the 2-T distribution, $f_X(x)$ here is as defined in (5) also make substitution as $(x-\mu) = y$, the above equation turns out to be as

$$E[X^k] = \theta e \int_0^\infty (\mu+y)^k e^{\theta y}\left(e^{\theta y}-1\right)\exp\left(-e^{\theta y}\right)dy.$$

Now using Binomial expansion in the expression $(\mu+y)^k$ and using the substitution $e^{\theta y} = t$ in the above equation. Then, it turns out to be as

$$E[X^k] = \mu^k e \sum_{i=0}^{k} \frac{\binom{k}{i}}{(\mu\theta)^i}\left[\int_1^\infty (\log t)^i t e^{-t}dt - \int_1^\infty (\log t)^i e^{-t}dt\right].$$

Here, by using the generalized integro-exponential function in the above equation it will be same as (14). ■

In this section, the analytical expression of the particular case i.e. for $k = 1$ (Mean) is also derived.

Corollary 1. Let X be the random variable following the 2-T distribution given in (5), then for given μ and θ, the mean of the distribution will be given by

$$E[X] = \left(\mu + \frac{1}{\theta}\right)$$

where E stands for the expectation.
Proof. For given μ and θ, mean value of the 2-T distribution is defined as $E[X] = \int_\mu^\infty x f_X(x)dx$. Here, $f_X(x)$ is same as defined in (5). Now, making substitution as $(x-\mu) = y$ we get

$$E[X] = \theta e \int_0^\infty (\mu+y)\, e^{\theta y}\left(e^{\theta y}-1\right)\exp\left(-e^{\theta y}\right)dy$$

By making substitution $e^{\theta y} = t$ in the above equation and using analytical expression $\int_1^\infty \log z(z-1)e^{-z}dz = \frac{1}{e}$, we get required result. ■

5. Estimation for fuzzy progressively censored data

In this section, for the given fuzzy progressively censored data, we obtain the MLEs, LSEs and Bayes estimators of the parameters μ and θ.

5.1. Maximum likelihood estimation

Following the [7], the likelihood for fuzzy data can expressed as

$$\ell(\theta,\mu;x) = A\prod_{i=1}^{m}\int f(x_i)\left[1-F(x_i)\right]^{R_i}\mu_{\tilde{x}_i}(x)dx. \tag{16}$$

Let (x_i, r_i) be the progressive sample. For 2-T distribution, the log $-likelihood$ function can be obtained by putting the value (6) and (5) in (16). Differentiating the above equation with respect to μ and θ and equating to zero. We obtain

$$\frac{d}{d\mu}\log l = \sum_{i=1}^{m} \frac{\int \left(f_{\mu}(x_i)\left[1 - F(x_i)\right]^{R_i} - R_i\left[1 - F(x_i)\right]^{R_i - 1} F_{\mu}(x_i) \right) \mu_{\tilde{x}_i}(x)dx}{\int f(x_i)\left[1 - F(x_i)\right]^{R_i} \mu_{\tilde{x}_i}(x)dx} = 0,$$

$$\frac{d}{d\theta}\log l = \sum_{i=1}^{m} \frac{\int \left(f_{\theta}(x_i)\left[1 - F(x_i)\right]^{R_i} - R_i\left[1 - F(x_i)\right]^{R_i - 1} F_{\theta}(x_i) \right) \mu_{\tilde{x}_i}(x)dx}{\int f(x_i)\left[1 - F(x_i)\right]^{R_i} \mu_{\tilde{x}_i}(x)dx} = 0,$$

where $f_{\mu}(x) = \frac{d}{d\mu}f(x)$ and $F_{\mu}(x) = \frac{d}{d\mu}F(x)$. Above equations can not be solved analytically. That's why some numerical techniques such as Newton-Raphson are required to solve these equations.

5.2. Least squares estimation

The equations (2) and (6) provide the method of obtaining the LSEs for fuzzy data. The LSEs can be obtained by minimising S. So differentiating S with respect to μ and θ and equating to 0, we obtain

$$\frac{dS}{d\mu} = \sum_{i=1}^{m} \int \left[F(x_i) - E(F(x_i))\right] F_{\mu}(x_i)\mu_{\tilde{x}_i}(x)dx = 0,$$

$$\frac{dS}{d\theta} = \sum_{i=1}^{m} \int \left[F(x_i) - E(F(x_i))\right] F_{\theta}(x_i)\mu_{\tilde{x}_i}(x)dx = 0,$$

where $E\left(F(x_i)\right) = 1 - \prod_{j=m-i+1}^{m} \alpha_j$, $\alpha_j = \frac{a_i}{1+a_i}$, $a_i = i + \sum_{j=m-i+1}^{m} R_i$, and $\frac{dS}{d\mu}$, $\frac{dS}{d\theta}$ are the first order derivatives of the $F(x)$ with respect to μ and θ. Equations $\frac{dS}{d\mu} = 0$ and $\frac{dS}{d\theta} = 0$ provide the LSEs for μ and θ. But there is no closed form solution of the above equations. A suitable iterative search method such as Newton-Raphson is is needed to obtain the LSEs of the parameters.

5.3. Bayes Estimation

In statistical inferences, Bayesian estimation turns out as a valid and powerful alternative of classical or traditional perspectives of the parameter estimation. In this section, Bayes estimates of the parameters of the 2-T distribution are derived using the Type-II progressive censoring scheme where data is given in form of the fuzzy numbers. Here, we assume that the parameters (μ, θ) follow the independent gamma priors denoted by $\pi_1(\mu)$ and $\pi_2(\theta)$ respectively. Then, the probability density function of μ and θ are given as follows

$$\pi_1(\mu) = \frac{p^n}{\Gamma(n)} \exp(-p\mu)\mu^{n-1}, \quad \mu, p, n > 0,$$

$$\pi_2(\theta) = \frac{q^a}{\Gamma(n)} \exp(-q\theta)\theta^{b-1}, \quad \theta, b, q > 0.$$

The joint posterior density of μ and θ for given observed data is defined by

$$\pi(\mu, \theta|\tilde{x}) = \frac{\pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)}{\int_0^{\infty}\int_0^{\infty} \pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)d\mu d\theta},$$

where $l(\theta, \mu; x)$ is the log-likelihood defined in the equation (16). Bayes estimates of any function of μ and θ i.e. $g(\mu, \theta)$ under the squared error loss function is defined as

$$E(g(\mu, \theta)) = \frac{\int_0^{\infty}\int_0^{\infty} g(\mu, \theta)\pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)d\mu d\theta}{\int_0^{\infty}\int_0^{\infty} \pi_1(\mu)\pi_2(\theta)l(\theta, \mu; x)d\mu d\theta}. \tag{17}$$

The above integral (17) can not be solved analytically. So we have to use some approximations to calculate the Bayes estimates. Here, we proposed Markov Chain Monte Carlo technique to obtain the Bayes estimates of the parameters of the 2-T distribution. In this paper, we use Metropolis-Hastings (MH) algorithm to simulate the posterior samples.

In order to obtain the reasonable results by simulation in a limited amount of time, the choice of an effective proposal distribution is crucial. Since the target density is unknown, the choice of the proposal distribution is very difficult. To overcome this difficulty [33] provided a possible adaptive algorithm as a remedy which adapts continuously to the target distribution. The basic idea is to update the proposal distribution by using the knowledge we acquired so far about the target population. For the simulation algorithm researchers may see [37].

6. Simulation Study

To estimate the unknown constants of the proposed distribution, various estimation methods are proposed such as MLE, LSE, and Bayes estimation in the previous sections. To access the long-run performance and to choose the best possible estimators for the 2-T parameters, simulation experiments are carried-out. During simulations, we generate the Type-II progressively fuzzy censored pseudo-random data from the 2-T distribution with various choices of the 2-T parameters. Without loss of generality, we here present the simulation results for a parameter combination $(\mu = 1, \theta = 1.5)$. To compare the performance of these estimators, we calculate the bias and mean squared error (MSE) for each estimators. For generating the progressive Type-II censored data, we use an algorithm given by ([2]). After getting the pseudo data, each realization of x is then fuzzified by using the triangular fuzzy number defined in (4). In generating the random sample form the 2-T distribution, we have to use Lambert-W function which is easily available in R software.

Further, for the fixed value of μ and θ, we take the different combinations of (n, m) along with four removal schemes for variation purposes. The removal schemes are as follows:
Scheme 1: $R_1 = (n - m)$ and $R_2 = R_2 = ...R_m = 0$,
Scheme 2: $R_1 = R_2 = ...R_{m-1} = 0$ and $R_m = n - m$,
Scheme 3: $R_1 = R_2 = ...R_{n-m} = 1$ and $R_{n-m+1} = ...R_{2m-n} = 0$,
Scheme 4: $R_1 = R_2 = ...R_{(2m-n)} = 0$ and $R_{2m-n+1} = ...R_m = 0$.

On the basis of simulation results summarized in the Tables (1 and 2), we observe that even for the small sample size, the performance of all the estimators is quite satisfactory. The average estimates are nearer to the true values of the parameters and MSEs of all the estimators decrease as the sample size increases (i.e. m increases for the given n) under the different removal schemes.

Table 2: *Average estimate, Bias and MSE of the estimators of μ for given $\mu = 1, \theta = 1.5$*

(n,m,s)	MLE	BIAS	MSE	LSE	BIAS	MSE	BAYES	BIAS	MSE
(20,12,1)	1.05117	0.08489	0.01194	0.96279	0.147329	0.03393	0.95877	0.10030	0.01661
(20,12,2)	1.07121	0.09659	0.01546	0.96337	0.128139	0.02589	0.93566	0.12577	0.02813
(20,12,3)	1.05428	0.08559	0.01219	0.96157	0.13481	0.02856	0.95331	0.10465	0.01845
(20,12,4)	1.05842	0.08893	0.01322	0.95713	0.13284	0.02776	0.94327	0.11840	0.02488
(20,16,1)	1.04958	0.08302	0.01146	0.9722	0.12815	0.02577	0.96858	0.09272	0.01336
(20,16,2)	1.05754	0.08701	0.01252	0.96564	0.12048	0.02289	0.96034	0.10296	0.01680
(20,16,3)	1.04896	0.08189	0.01119	0.97146	0.12415	0.0242	0.96601	0.09257	0.01345
(20,16,4)	1.05377	0.08582	0.01221	0.96663	0.12211	0.02320	0.96038	0.10094	0.01654
(30,18,1)	1.03790	0.06643	0.00736	0.97335	0.11831	0.02198	0.97923	0.07149	0.00784
(30,18,2)	1.05625	0.07593	0.00966	0.97553	0.10164	0.01621	0.98319	0.06926	0.00736
(30,18,3)	1.03848	0.06589	0.00734	0.97729	0.10477	0.01742	0.98215	0.08784	0.01240
(30,18,4)	1.04099	0.06827	0.00786	0.96974	0.10467	0.01727	0.97611	0.07657	0.00913
(30,24,1)	1.03706	0.06560	0.00719	0.97992	0.10319	0.01662	0.97835	0.07379	0.00745
(30,24,2)	1.04694	0.07004	0.00821	0.97807	0.09683	0.01470	0.09792	0.06645	0.00721
(30,24,3)	1.03493	0.06352	0.00674	0.97905	0.09887	0.01541	0.98565	0.06580	0.00653
(30,24,4)	1.03961	0.06692	0.00756	0.97831	0.09554	0.01432	0.98573	0.06517	0.00643
(40,24,1)	1.02919	0.05493	0.00509	0.98192	0.10178	0.01636	0.98354	0.06807	0.00702
(40,24,2)	1.05041	0.06566	0.00721	0.98207	0.08668	0.01182	0.98933	0.05802	0.00510
(40,24,3)	1.02968	0.05493	0.00503	0.98149	0.08933	0.01259	0.98579	0.06193	0.00582
(40,24,4)	1.03283	0.05694	0.00540	0.97724	0.08872	0.01242	0.98870	0.05761	0.00505
(40,32,1)	1.02840	0.05495	0.00503	0.98604	0.08961	0.01249	0.99025	0.05438	0.00451
(40,32,2)	1.04047	0.05944	0.00593	0.98374	0.08232	0.01073	0.98886	0.05601	0.00478
(40,32,3)	1.02827	0.05382	0.00483	0.98503	0.08418	0.01117	0.99035	0.05371	0.00441
(40,32,4)	1.03192	0.05598	0.00526	0.98272	0.083117	0.01073	0.98901	0.05568	0.00472
(50,32,1)	1.02491	0.04841	0.00389	0.98714	0.08689	0.01173	0.99131	0.04960	0.00373
(50,32,2)	1.04350	0.05681	0.00542	0.98590	0.07658	0.00919	0.99125	0.04857	0.00361
(50,32,3)	1.02472	0.04763	0.00381	0.98443	0.07822	0.00967	0.98994	0.05062	0.00391
(50,32,4)	1.02843	0.04942	0.00416	0.98371	0.07742	0.00948	0.98932	0.05171	0.00407

Table 1: *Average estimate, Bias and MSE of the estimators of θ for given $\mu = 1, \theta = 1.5$*

(n,m,s)	MLE	BIAS	MSE	LSE	BIAS	MSE	BAYES	BIAS	MSE
(20,12,1)	1.68417	0.26014	0.13438	1.51204	0.32128	0.19246	1.51421	0.46034	0.09226
(20,12,2)	1.87181	0.43052	0.39091	1.50953	0.32969	0.19795	1.51804	0.43837	0.15056
(20,12,3)	1.70755	0.28655	0.15911	1.51094	0.33063	0.21401	1.51649	0.13176	0.10492
(20,12,4)	1.74806	0.32708	0.21514	1.49362	0.32598	0.19692	1.52243	0.00117	0.13382
(20,16,1)	1.65856	0.23182	0.10179	1.50694	0.27225	0.13749	1.51162	0.42873	0.06524
(20,16,2)	1.72482	0.28867	0.16376	1.48766	0.26682	0.12266	1.51452	0.16994	0.08835
(20,16,3)	1.65894	0.23057	0.10174	1.50268	0.26759	0.12866	1.50970	0.07517	0.06535
(20,16,4)	1.69341	0.26412	0.13576	1.49197	0.26952	0.12975	1.51735	0.03427	0.08529
(30,18,1)	1.62509	0.19206	0.06812	1.49969	0.24981	0.10879	1.51535	0.56535	0.04742
(30,18,2)	1.79499	0.33826	0.11789	1.50243	0.26144	0.11812	1.52215	0.78643	0.07443
(30,18,3)	1.63589	0.20816	0.08106	1.50654	0.25692	0.11976	1.51554	0.38108	0.05667
(30,18,4)	1.65963	0.23143	0.10024	1.48741	0.25252	0.11179	1.52062	0.64902	0.06721
(30,24,1)	1.61181	0.17401	0.05427	1.49836	0.21281	0.07755	1.51166	0.13018	0.03632
(30,24,2)	1.67945	0.22550	0.09407	1.49155	0.21491	0.07673	1.51314	0.12987	0.03987
(30,24,3)	1.60598	0.17006	0.05202	1.49456	0.21026	0.07416	1.51199	0.13882	0.03696
(30,24,4)	1.63389	0.19729	0.07220	1.49172	0.2089	0.07286	1.51423	0.12571	0.04457
(40,24,1)	1.59452	0.15782	0.04424	1.50070	0.21230	0.07591	1.51460	0.03188	0.03336
(40,24,2)	1.76290	0.29066	0.15152	1.50191	0.22435	0.07381	1.52099	0.06421	0.05131
(40,24,3)	1.60391	0.17089	0.05283	1.49815	0.21323	0.07699	1.51675	0.03746	0.03862
(40,24,4)	1.62594	0.19079	0.06559	1.48809	0.21711	0.07946	1.51199	0.21665	0.03781
(40,32,1)	1.58081	0.14022	0.03471	1.49854	0.18253	0.05580	1.50969	0.19968	0.02466
(40,32,2)	1.66169	0.19758	0.06913	1.49313	0.18304	0.05512	1.51179	0.21566	0.03059
(40,32,3)	1.58256	0.140137	0.03532	1.4959	0.17741	0.05193	1.50996	0.19452	0.02508
(40,32,4)	1.60563	0.16096	0.04668	1.49024	0.17888	0.05241	1.51156	0.21259	0.03001
(50,32,1)	1.57342	0.13123	0.02990	1.50051	0.17974	0.05394	1.50665	0.09304	0.02300
(50,32,2)	1.72174	0.24448	0.10128	1.49803	0.19029	0.05928	1.51015	0.11409	0.03324
(50,32,3)	1.58079	0.13939	0.03447	1.49515	0.18043	0.05417	1.50771	0.11663	0.02528
(50,32,4)	1.60111	0.15821	0.04490	1.49576	0.18461	0.05669	1.50946	0.12165	0.03084

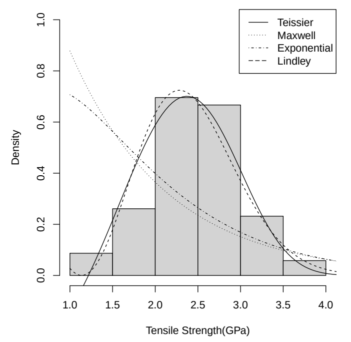

Figure 3: *Fitted density of various distributions*

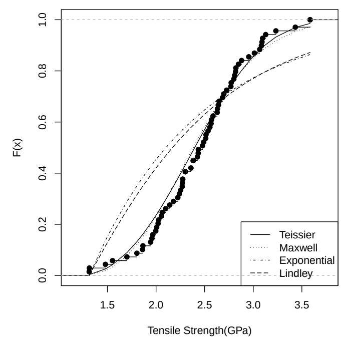

Figure 4: *Fitted CDF of various distributions*

7. Real Data Application

To demonstrate the application of the proposed distribution and estimation methods, a real data set has been considered, that represent the tensile strength, measured in Giga-Pascal (GPa), of 69 carbon fibers tested under tension at gauge lengths of 20*mm*, see [27]. They conducted single-filament tensile tests on carbon fibers of differing gauge lengths. These experimental results were further reported and analyzed by [28] and [29]. [30] also used this data to show the usefulness of the three-parameter Birnbaum-Saunders distribution and the inverse Gaussian distribution. Set of data points is given as follows :

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585

As we have stated in the first section that this paper aims to propose the use of the 2-T distribution for fuzzy data sets, we assume that the tensile strength of carbon fibers are observed with some degrees of imprecision and fit the 2-T distribution over the other competing distributions. Here, the triangular fuzzy number is used to model the unknown value of $\tilde{x} = (u, v, w)$. The corresponding membership function of each value of observed data point, say x, is given by

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{x-(x_i-h)}{h}, & x_i - h \le x \le x_i, \\ \frac{(x_i+h)-x}{h}, & x_i \le x \le x_i + h, \\ 0, & otherwise, \end{cases} \tag{18}$$

where $h = 0.05x_i$.

Table 3: *MLEs, AIC, BIC and KS-statistics along with p-value for complete data set without fuzzy*

Model	MLEs $\hat{\alpha}$	MLEs $\hat{\lambda}$	NLL	AIC	BIC	KS	p-value
Teissier	1.217	0.834	49.982	103.965	108.433	0.089	0.641
Maxwell	1.136	0.761	51.291	106.582	111.051	0.095	0.557
Exponential	1.312	0.878	78.001	158.001	164.469	0.310	0.000
Lindley	1.312	1.265	72.943	147.887	154.355	0.276	0.000

Table 4: *MLEs, AIC, BIC and KS-statistics along with p-value for complete fuzzy data*

Model	MLEs $\hat{\alpha}$	MLEs $\hat{\lambda}$	NLL	AIC	BIC
Teissier	1.216	0.834	196.3064	396.613	401.081
Maxwell	1.136	0.761	197.6153	399.230	403.699
Exponential	1.312	0.877	224.3245	450.649	457.117
Lindley	1.312	1.264	219.2673	440.535	447.002

In order to proceed with the fuzzy set-up, we would like to first assess the goodness-of-fit of the proposed distribution to model the given data set. For this purpose, we use some goodness-of-fit criteria such as Kolmogorov-Smirnov (KS) statistic, Akaike information criterion (AIC) and Bayesian information criterion (BIC) to compare the fitting of the competing distributions. For the comparison purposes, we take two parameter families of the exponential, Maxwell and Lindley distributions, which are very popular distributions in statistical literature. Table 3 consists of the MLEs and negative log-Likelihood (NLL) values for all four distributions for the carbon fibers data. Table (4) also shows the different model selection criteria such as AIC, BIC, and KS-statistic along with the p-value. From the table, it is observed that the 2-T distribution has the lowest AIC, BIC and KS values for given data-set. As we can note that the 2-T distribution has the smallest statistic values among others, it may be used to model the tensile strength data set over the considered distributions. Since, we aimed to propose the estimation under fuzzy set-up, the fitting of all four distributions under fuzzy environment is also presented. The fitting results are presented in table (4). We can also note here that the 2-T distribution also has the smallest AIC and BIC values under fuzzy set up. It can be concluded that the 2-T is the best fitting model among others for the fuzzy and without fuzzy data problems as well.

From the Tables (3) and (4) it is found that 2-T distribution is quite enough flexible to model the uncertainty arises due to the randomness and fuzziness. Table (5) includes the estimates of the parameters under the MLE, LSE and Bayesian estimation methods for the 2-T distribution under different progressive Type-II censoring schemes. These parameters are calculated for different schemes as well as for the different sample sizes. The 95 percent confidence intervals are also provided for the 2-T parameters. For the Bayesian estimation, the non-informative priors are considered to estimate μ and θ. Fig. (5) represents the corresponding density and trace plots for the simulated posterior samples based on the real data-set. Here, the trace plots display the random scatter of the sample around the mean values and do not have trend which emphasis the fact that the model has converged. Also, the density plots show the unique modality of the parameters.

Table 5: *MLEs, LSEs and Bayes estimates under progressive fuzzy sample*

m	Schemes	MLEs		LSEs		Bayes		CI		HPD	
		$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\theta}$
	1	1.242	1.208	1.513	1.510	1.213	1.170	(1.15,1.33)	(1.03,1.39)	(1.11,1.30)	(0.99,1.33)
40	2	1.211	0.828	1.343	0.876	1.140	0.752	(1.08,1.35)	(0.68,0.97)	(0.95,1.29)	(0.60,0.88)
	3	1.250	1.038	1.641	1.589	1.216	0.999	(1.16,1.34)	(0.88,1.19)	(1.10,1.30)	(0.83,1.16)
	4	1.233	0.915	1.587	1.302	1.190	0.875	(1.13,1.34)	(0.77,1.06)	(1.05 ,1.30)	(0.72,1.02)
	1	1.243	1.142	1.512	1.401	1.215	1.111	(1.15,1.33)	(0.98,1.30)	(1.10,1.30)	(0.94,1.26)
45	2	1.217	0.841	1.366	0.900	1.152	0.771	(1.09,1.34)	(0.70,0.98)	(0.98,1.29)	(0.64,0.90)
	3	1.255	1.057	1.623	1.510	1.225	1.026	(1.17,1.33)	(0.91,1.20)	(1.13,1.30)	(0.88,1.18)
	4	1.230	0.888	1.552	1.176	1.193	0.858	(1.12,1.34)	(0.75,1.02)	(1.06,1.30)	(0.72,0.99)
	1	1.244	1.085	1.513	1.311	1.215	1.058	(1.15,1.34)	(0.94,1.23)	(1.10,1.30)	(0.91,1.20)
50	2	1.220	0.847	1.389	0.926	1.156	0.786	(1.10,1.34)	(0.72,0.97)	(1,1.29)	(0.66,0.91)
	3	1.255	1.049	1.590	1.391	1.226	1.020	(1.17,1.33)	(0.91,1.19)	(1.13,1.30)	(0.88,1.16)
	4	1.228	0.874	1.508	1.082	1.189	0.846	(1.11,1.34)	(0.75,1.00)	(1.04,1.30)	(0.71,0.97)
	1	1.241	0.974	1.490	1.134	1.204	0.945	(1.15,1.33)	(0.85,1.09)	(1.01,1.30)	(0.83,1.07)
60	2	1.220	0.843	1.410	0.949	1.169	0.800	(1.10,1.34)	(0.73,0.95)	(1.01,1.29)	(0.69,0.91)
	3	1.246	0.971	1.507	1.148	1.215	0.945	(1.15,1.33)	(0.85,1.09)	(1.11,1.30)	(0.83,1.07)
	4	1.226	0.859	1.435	0.975	1.187	0.831	(1.11,1.34)	(0.75,0.97)	(1.05,1.30)	(0.73,0.95)

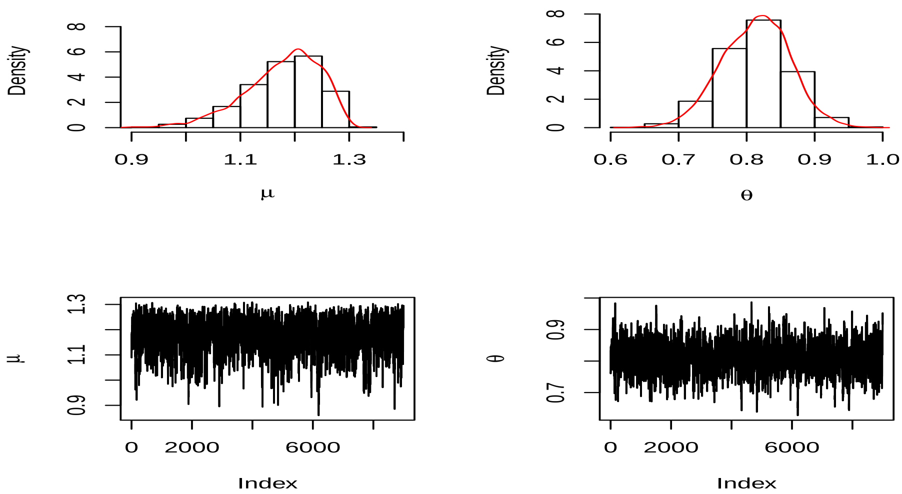

Figure 5: *The density and trace plots of simulated μ and θ for complete data*

8. Conclusion

In real world situations, it is always possible that the observed lifetime data might be observed imprecisely and may be represented in the form of fuzzy numbers. Therefore, a suitable statistical methodology is required to handle these type of data sets. In this article, we introduced a two parameter Teissier distribution to model the fuzzy censored lifetime data sets. We derived the some useful properties of the distribution such as mean residual life, moments, moment generating function and quantile function. We also discussed different methods to estimate the parameters of Teissier distribution by using the maximum likelihood, least square and Bayesian techniques. In order to assess the validity and applications of the estimation procedures, we presented an extensive simulation study. Lastly, a real data set is considered to discuss the applicability of the distribution. From the data, it is found that Two parameter Teissier distribution may be a better choice over other competing distributions such as Maxwell, Exponential and Lindley. The objective of the article was to introduce a new two parameter distribution to model the fuzzy censored data set having increasing failure rate which is generally found in the reliability and survival analysis. Future objectives may be threefold. The first one may be In future, with the objective of modelling the fuzzy data set, the more flexibility can be added by introducing more parameters in the distribution. Also, we can work in future with different censoring schemes available in the literature.

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Record-based Transmuted Power Lomax Distribution: Properties and its Applications in Reliability

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Abstract

In this paper, we consider a record-based transmuted version of Power Lomax distribution and it is named as Record-based Transmuted Power Lomax (RTPL) distribution. Further, we present several statistical properties of the proposed distribution such as moments, quantiles, stochastic ordering, order statistics, and its explicit expressions. Some of its reliability measures such as survival function, hazard function, cumulative hazard function, mean residual time, and mean inactivity time is also discussed. The maximum likelihood method is used to estimate the parameters of the RTPL distribution and this new extended model is applied to a real datasets to access the suitability and applicability of the model based on well-known information criteria and test for goodness of fit. The simulation study is performed to verify the efficiency and asymptotic behavior of the maximum likelihood estimators.

Keywords: Record-based Transmuted map, Power Lomax distribution, Lambert *W* function, Maximum Likelihood Estimation.

1. INTRODUCTION

Record values and record statistics are routine and central points for monitoring many aspects of human life in date to date activities and it has a lot of real-life applications. In particular, the industry has many products which fail at times due to stress. For example, an electronic component ceases to function in an environment of high temperature, and a battery dies under the stress due to over use. But the precise breaking stress or failure point varies even among identical items. Hence in such experiments, measurements may be made sequentially and only the record values are observed. Thus, the number of measurements made is considerably smaller than the complete sample size. This "measurement saving" method can be important when the measurement of these experiments is costly if the entire sample was destroyed. There are situations in which an observation is sorted only if it is a record value. This includes studies in meteorology, hydrology, economics, athletic events, and life testing studies.

In 1952, Chandler introduced the study of record values and discussed lots of the most important and basic properties of records. Let $X_1, X_2,...$ be the sequence of the random variables, there are two types of the record values such as upper and lower records. We say that X_n be the upper record value if $X_n > \max\{X_1, X_2,..., X_{n-1}\}, n = 2,3,....$ this means that X_n which is more than all previous $X's$, and X_n be the lower record value if $X_n < \min\{X_1, X_2,..., X_{n-1}\}, n = 2,3,....$. In two situations X_1 is considered the first upper or lower record value. The upper records can be used in many real-life phenomena when compares to the lower records. Now if together with some sequence $X_1, X_2,...$ one considers $Y_1 = -X_1,\ Y_1 = -X_2,...,\ Y_n = -X_n,...$, then it becomes evident that the lower record times for $Y's$ is coinciding with the corresponding upper record times of $X's$.

Balakrishnan et.al [9] proposed a record-based transmuted map to generate new probability models. Let $X_1, X_2, \dots, X_n$ be a sequence of an independent and identically distributed random variable with a distribution function $G(x)$. Let $X_{U(1)}$ and $X_{U(2)}$ be the two upper records from the above sequence of independent and identically distributed random variables. Define a random variable Y as follows:

$$Y = \begin{cases} X_{U(1)}, \text{with probability } 1-p \\ X_{U(2)}, \text{with probability } p \end{cases}$$

Where, $p \in [0,1]$, then

$$F_Y(x) = (1-p)P(X_{U(1)} \le x) + p\,P(X_{U(2)} \le x)$$

The record-based transmuted cumulative distribution function is obtained as

$$F_Y(x) = G(x) + p.\overline{G}(x)\log\ \overline{G}(x);\ \text{for } x \in R\ ,\ 0 \le p \le 1 \tag{1}$$

The corresponding probability density function is given by

$$f_Y(x) = g(x)\left[1 - p - p.\log \overline{G}(x)\right];\ \text{for } x \in R\ ,\ 0 \le p \le 1 \tag{2}$$

Balakrishnan et al. [9] also introduced a few new record-based transmuted (RT) probability distributions like RT-exponential (RTE) distribution, RT-Linear exponential (RTLE) distribution, RT-Weibull (RTW) distribution, etc. Vijay Kumar et al. [8] studied the Record-Based Transmuted Generalized Linear Exponential Distribution with increasing, decreasing, and bathtub-shaped failure rates.

The Lomax distribution, is also known as Pareto Type II distribution and it is proposed by K.S. Lomax (1954). It is also classified as heavy-tailed distribution and referred as a shifted Pareto distribution, which is widely used in survival analysis. It is popularly used as an alternative to power-law, exponential, gamma, and Weibull distribution for modeling heavy-tailed data in the domain of business, Economics, and Actuarial science. A random variable X follows the Lomax distribution with the shape parameters $\beta > 0$ and the scale parameter $\lambda > 0$ and its cumulative distribution function is given by

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\beta}\ ;\ x > 0 \tag{3}$$

The corresponding probability density function is given below

$$f(x) = \frac{\beta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-\beta-1};\ x > 0 \tag{4}$$

El-Houssainy et al. [14] mentioned that the Power Lomax (PL) distribution is obtained by using the power transformation that is $Y = X^{1/\beta}$. The random variable X is said to follow the three-parameter PL distribution with the shape parameters $\alpha, \beta > 0$ and scale parameter $\lambda > 0$ if the cumulative distribution function of $x > 0$ is given by

$$F(x) = 1 - \lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha} \tag{5}$$

The probability density function of the power Lomax distribution is given by

$$f(x) = \alpha\beta\lambda^{\alpha} x^{\beta-1}\left(\lambda + x^{\beta}\right)^{-\alpha-1} \tag{6}$$

In the literature, some extensions of the Lomax distribution were developed and further showed that these resultant distributions are better than the baseline distribution, and the following will be the list of a few such extensions of the Lomax distribution. Abdul-Moniem et al. [1] introduced Exponentiated Lomax (EL) distribution, Muhammad Rajab et al.[19] proposed Beta-Lomax (BL) distribution, Cordeiro G. M et al. [11] developed gamma-Lomax (GL) distribution, El-Bassiouny et al. [13] studied Exponential Lomax distribution, Singh Yadav et al.[20] investigated on Inverse Lomax (IL) distribution, Masood Anwar et al. [6] presented the Half-logistic Lomax (HLL) distribution, and Sanaa Al-Marzouki et al. [4] developed the Exponentiated power Lomax distribution.

The remaining part of this paper is organized as follows: In Section 2, we introduce Record based Transmuted power Lomax (RTPL) distribution and present some of its special cases. In Section 3, we derive some structural properties including quantile function, moments, Lorenz curve, Bonferroni curve, entropy, and order statistics. In Section 4, we present the simulation study to measure the precision and asymptotic nature of parameter estimates of the proposed distribution. In Section 5, we discuss the maximum likelihood estimates (MLEs) of the model parameters. In Section 6, we considered two data set for illustrating the suitability and goodness of fit of the RTPL distribution. In Section 7, we conclude the study with a summary of results.

2. RECORD-BASED TRANSMUTED POWER LOMAX DISTRIBUTION

A non-negative integer-valued random variable X is said to follow Record based transmuted Power Lomax distribution with scale parameter $\lambda > 0$, shape parameters $\alpha, \beta > 0$ and $p \in [0,1]$ if its cumulative distribution function is of the following form

$$F(x) = 1 - \frac{\lambda^{\alpha}}{\left(x^{\beta} + \lambda\right)^{\alpha}}\left[1 - p\log\left(\frac{\lambda}{x^{\beta} + \lambda}\right)^{\alpha}\right] \tag{7}$$

And the corresponding probability density function of the RTPL distribution is given by

$$f(x) = \frac{\alpha\beta\lambda^{\alpha} x^{\beta-1}}{\left(x^{\beta} + \lambda\right)^{\alpha+1}}\left[1 - p.\left(1 + \log\left(\frac{\lambda}{x^{\beta} + \lambda}\right)^{\alpha}\right)\right] \tag{8}$$

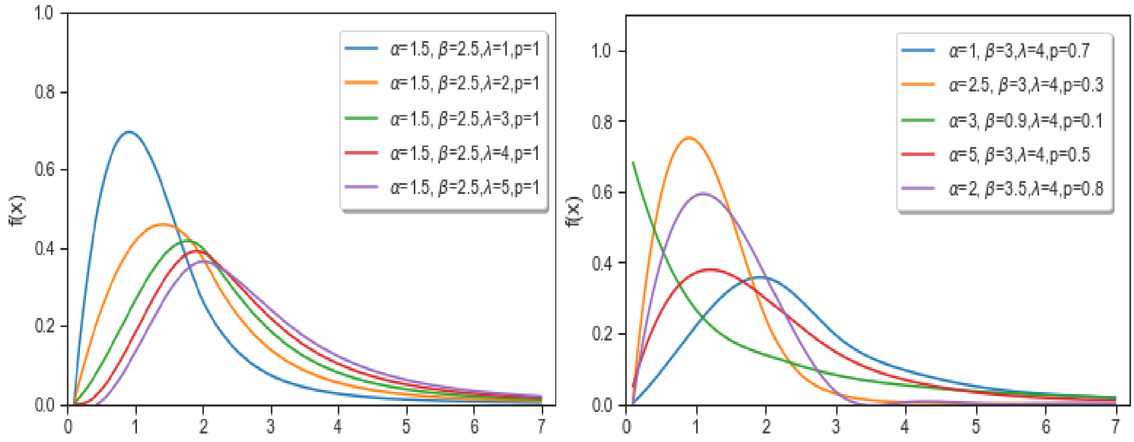

Figure: 1a Figure: 1b

Figure 1. The probability density plot of the RTPL distribution

The shapes of the probability density function of RTPL distributions for different values of the parameters can be described in Figure 1. From Figure 1a, it is observed that the curves are unimodal and positively skewed which represents the density plot with fixed $\alpha = 0.5$, $\beta = 2.5$, $p = 1$ and λ those with different values. From Figure 1b, it can be observed that the curve is left-skewed and reversed J shaped and the curves represent fixed values such that $\lambda = 4$and assign different values to the other three parameters of RTPL distribution.

2.1.Reliability Analysis

In this section, we define the survival function, hazard rate function, reversed hazard rate function, and cumulative hazard rate function of the RTPL distribution.

The survival function of RTPL distribution is obtained as follows

$$S(x) = \frac{\lambda^{\alpha}}{\left(x^{\beta}+\lambda\right)^{\alpha}}\left[1 - p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right] \tag{9}$$

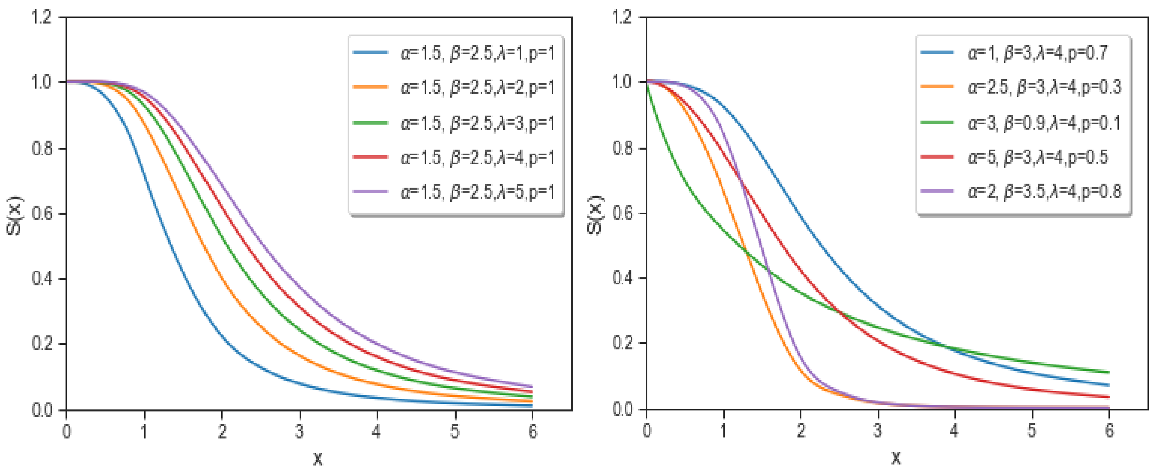

Figure: 2a Figure: 2b

Figure 2. The plots of the Survival function of RTPL distribution

The hazard rate function of RTPL distribution defined as

$$h(x)=\frac{\alpha\beta x^{\beta-1}\left[1-p\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right]}{\left(x^{\beta}+\lambda\right)\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]} \tag{10}$$

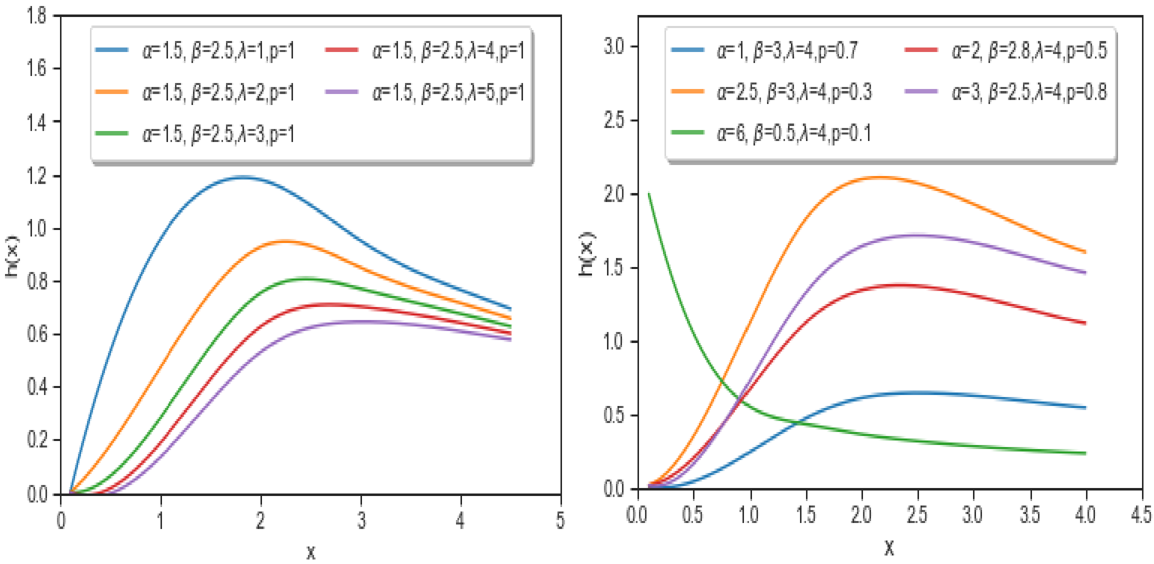

Figure: 3a

Figure: 3b

Figure 3. The hazard rate plot for RTPL distribution

The cumulative hazard function of RTPL distribution is as follows

$$H(x)=-\log\left[\frac{\lambda^{\alpha}}{\left(x^{\beta}+\lambda\right)^{\alpha}}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right]$$

(11)

The reversed hazard function of the RTPL distribution is given as

$$\tau(x)=\frac{\alpha\beta\lambda^{\alpha}x^{\beta-1}\left(x^{\beta}+\lambda\right)^{-(\alpha+1)}\left(1-p.\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right)}{1-\lambda^{\alpha}\left(x^{\beta}+\lambda\right)^{-\alpha}\left(1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)} \tag{12}$$

The survival function plot with parameters $\alpha=1.5, \beta=2.5$ and $p=1$ is represented in Figure 2a and different values of the parameter λ and Figure 2b shows the survival function plot of RTPL for $\lambda=4$ and assigning different values for the parameters α, β, and p. The survival curves are decreasing as time increases. Figure 3a displays the hazard rate plot for $\alpha=1.5, \beta=2.5, p=1$ and λ with changing value, which describes the curves are increasing, decreasing, and Figure 3b shows that hazard rate plot for $\lambda=4$, and varying parameter values for α, β and p the curves are increasing and reversed *J* shaped.

Special cases: For different values of the parameters, the following distributions are obtained as the special case of the RTPL distribution.

Case 1: If the value of $p = 0$, then the distribution function given in (7) reduced to the power Lomax distribution.
Case 2: If the value of $p = 0$ and $\beta = 1$, then the distribution function given in (7) reduced to the Lomax distribution.

3. STATISTICAL AND MATHEMATICAL PROPERTIES

This section deals with some important properties of the proposed RTPL distribution such as quantile function, moments, inverted moments, entropy, stochastic ordering, and order statistics.

3.1. Quantile Function

The quantile function plays an important role when simulating random variables from a probability distribution. The quantile function of the RTPL distribution function is defined as follows

$$1-\frac{\lambda^{\alpha}}{\left(x^{\beta}+\lambda\right)^{\alpha}}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]=u \tag{13}$$

The closed-form expression of the quantile function has been obtained by using the Lambert W-function such as:

$$W(\nu)e^{W(\nu)}=\nu$$

Where v is the complex number. For real numbers, $\nu \geq -\frac{1}{e}$, the Lambert W function has only two branches W_0 which takes the value in $\left[-1,\infty\right)$ and W_{-1} which takes the value in $\left[-\infty,-1\right)$ and for $\nu \in \left[-\frac{1}{e},0\right)$. It can be verified that $\frac{u-1}{pe^{1/p}} \in \left[-\frac{1}{e},0\right)$, and $-\frac{1}{p}+\log\left(\frac{\lambda^{\alpha}}{x^{\beta}+\lambda^{\alpha}}\right) \in \left[-\infty,-1\right)$

Now using the negative branch of the Lambert W function in the above equation we get,

$$W_{-1}\left(\frac{u-1}{pe^{1/p}}\right)=-\frac{1}{p}+\log\left(\frac{\lambda^{\alpha}}{x^{\beta}+\lambda^{\alpha}}\right) \tag{14}$$

Thus, by solving the equation (13), we get the quantile function as given below

$$x_p=\left(\lambda e^{-\frac{1}{\alpha}\left(W_{-1}\left(\frac{u-1}{pe^{1/p}}\right)+\frac{1}{p}\right)}-\lambda\right)^{1/\beta} \tag{15}$$

The median of the probability distribution can be obtained by taking the u as 0.5 in the above quantile function.

3.2. Method of Moments

The r^{th} raw moment of the random variable X having RTPL distribution is obtained by substituting the equation (8) as follows:

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r \frac{\alpha\beta\lambda^\alpha x^{\beta-1}}{\left(x^\beta+\lambda\right)^{\alpha+1}}\left[1-p.\left(1+\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right)\right]dx$$

$$= \frac{\alpha\beta}{\lambda}\int_0^\infty x^{r-\beta-1}\left(1+\frac{x^\beta}{\lambda}\right)^{\alpha+1}\left[1-p.\left(1+\log\left(1+\frac{x^\beta}{\lambda}\right)^\alpha\right)\right]dx$$

By taking $y=\dfrac{x^\beta}{\lambda}$ and applying the transformation method and $y=\dfrac{w}{1-w}$ in the above equation, we get

$$\mu_r' = E\left(X^r\right) = \alpha\lambda^{r/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}\frac{\Gamma\left(\frac{r}{\beta}+k+1\right)\Gamma\left(\alpha-\frac{r}{\beta}-k\right)}{\Gamma(\alpha+1)}+(1-p)\frac{\Gamma\left(\frac{r}{\beta}+k\right)\Gamma\left(\alpha-\frac{r}{\beta}\right)}{\Gamma(\alpha+1)}\right] \tag{16}$$

The first two moments of the distribution can be derived from equation (16) and it is given as

$$\mu_1' = \alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}\frac{\Gamma\left(\frac{1}{\beta}+k+1\right)\Gamma\left(\alpha-\frac{1}{\beta}-k\right)}{\Gamma(\alpha+1)}+(1-p)\frac{\Gamma\left(\frac{1}{\beta}+k\right)\Gamma\left(\alpha-\frac{1}{\beta}\right)}{\Gamma(\alpha+1)}\right] \tag{17}$$

$$\mu_2' = \alpha\lambda^{2/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}\frac{\Gamma\left(\frac{2}{\beta}+k+1\right)\Gamma\left(\alpha-\frac{2}{\beta}-k\right)}{\Gamma(\alpha+1)}+(1-p)\frac{\Gamma\left(\frac{2}{\beta}+k\right)\Gamma\left(\alpha-\frac{2}{\beta}\right)}{\Gamma(\alpha+1)}\right] \tag{18}$$

The r^{th} incomplete moment of the RTPL distribution can be obtained by using the equation (8) as follows:

$$\phi_r(t) = \int_x^\infty x^r \frac{\alpha\beta\lambda^\alpha x^{\beta-1}}{\left(x^\beta+\lambda\right)^{\alpha+1}}\left[1-p.\left(1+\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right)\right]dx$$

$$= \alpha\lambda^{r/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{r}{\beta}+k+1\right),\left(\alpha-\frac{r}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{r}{\beta}+k\right),\left(\alpha-\frac{r}{\beta}\right)\right)\right] \tag{19}$$

By taking $r=1$ in the equation (19) to get the1^{st} incomplete moment of RTPL distribution and it is given as

$$\phi_1(t) = \alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right] \tag{20}$$

The r^{th} central moment of RTPL distribution is defined as follows

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu_1')^n \frac{\alpha\beta\lambda^{\alpha} x^{\beta-1}}{(x^{\beta}+\lambda)^{\alpha+1}} \left[1-p.\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right]dx$$

$$=\sum_{k=0}^{r}\alpha\lambda^{r-k/\beta}(-\mu_1')^k\binom{r}{k}\left\{(1-p)B\left(\left(\frac{r-k}{\beta}+1\right),\left(\alpha-\frac{r-k}{\beta}\right)\right)+p\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!}B\left(\left(\frac{r-k}{\beta}+1\right),\left(\alpha-\frac{r-k}{\beta}-m\right)\right)\right\} \quad (21)$$

The r^{th} inverted moment of RTPL distribution is defined and obtained as follows

$$\mu_r^* = \int_{-\infty}^{\infty} x^{-r} \frac{\alpha\beta\lambda^{\alpha} x^{\beta-1}}{(x^{\beta}+\lambda)^{\alpha+1}} \left[1-p.\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right]dx$$

$$=\alpha\lambda^{-r/\beta}\left\{(1-p)B\left(\left(1-\frac{r}{\beta}\right),\left(\alpha+\frac{r}{\beta}\right)\right)+p\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(k-\frac{r}{\beta}+1\right),\left(\alpha+\frac{r}{\beta}-k\right)\right)\right\} \quad (22)$$

Mean residual life (MRL) or life expectancy at time *t* is the expected additional life length for a unit, which is still alive at time *t*. The mean residual lifetime of the RTPL distribution is defined as follows

$$m_x(t) = E(X-t/X>t), \quad t>0$$

$$= \frac{\mu-\phi_1(t)}{S(t)} - t, \quad \text{where} \quad \mu=\mu_1'$$

$$=\frac{\mu-\left[p\alpha^2\lambda^{1/\beta}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+\alpha(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\lambda^{\alpha}(x^{\beta}+\lambda)^{-\alpha}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]}-t \quad (23)$$

Mean inactivity time (MIT) is the waiting time to elapsed since the failure of an item is on the condition that the failure can be occurred in (*0,t*). The mean inactivity time of the proposed RTPL distribution is obtained as

$$\psi_x(t) = E(t-X/X<t)$$

$$= t - \frac{\phi_1(t)}{F(t)}$$

$$=t-\frac{p\alpha^2\lambda^{1/\beta}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+\alpha(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)}{1-\left(\lambda^{\alpha}(x^{\beta}+\lambda)^{-\alpha}\left[1-p\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)} \quad (24)$$

3.3. Measures of Inequality and Uncertainty

In this section, the measures of uncertainty and three inequality measures of the RTPL distribution have been derived. The Lorenz curve of the RTPL distribution can be derived by using the first incomplete moment in (20) and the moment in (17) is obtained as follows

$$LO(x)=\frac{\phi_1(x)}{E(x)}$$

$$=\frac{\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]} \quad (25)$$

The Bonferroni curve of the RTPL distribution is obtained by using (7) and the Lorenz curve in (25) is given below

$$BO(x)=\frac{LO(x)}{F(x)}$$

$$=\frac{\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left(1-\frac{\lambda^\alpha}{(x^\beta+\lambda)^\alpha}\left[1-p\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right]\right)\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]} \quad (26)$$

The Zenga Index of the RTPL distribution is obtained as

$$Z=1-\frac{\mu_{(x)}^{-}}{\mu_{(x)}^{+}} \quad (27)$$

Where, $\mu_{(x)}^{-}=\frac{1}{F(x)}\int_0^x xf(x)dx$ and $\mu_{(x)}^{+}=\frac{1}{1-F(x)}\int_0^\infty xf(x)dx$

$$\mu_{(x)}^{-}=\frac{\alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B_t\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B_t\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left(1-\frac{\lambda^\alpha}{(x^\beta+\lambda)^\alpha}\left[1-p\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right]\right)} \quad (28)$$

$$\mu_{(x)}^{+}=\frac{\alpha\lambda^{1/\beta}\left[p\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}B\left(\left(\frac{1}{\beta}+k+1\right),\left(\alpha-\frac{1}{\beta}-k\right)\right)+(1-p)B\left(\left(\frac{1}{\beta}+k\right),\left(\alpha-\frac{1}{\beta}\right)\right)\right]}{\left(\frac{\lambda^\alpha}{(x^\beta+\lambda)^\alpha}\left[1-p\log\left(\frac{\lambda}{x^\beta+\lambda}\right)^\alpha\right]\right)} \quad (29)$$

By substituting the equations (28) and (29) in (27), we get the Zenga Index of the RTPL distribution.

Entropy is one of the important tools for measuring the uncertainty of the random variables and the information provided by such variables. In some cases, the random variables in the probability distribution are associated with some sort of uncertainty, and entropy can be used to quantify them.

The Rényi entropy can be derived by using the equation (8) is defined as follows

$$RE_x(\delta)=\frac{\delta}{1-\delta}\log\left(\int_{-\infty}^{\infty} f(x)^{\delta}dx\right)$$

$$=\frac{\delta}{1-\delta}\left[\log\left(\frac{\alpha\beta}{\lambda}\right)+\log\left(\sum_{k=0}^{\infty}\sum_{i,j=0}^{\infty}\binom{\delta}{k}\binom{k}{j}\frac{p^k\alpha^j j(-1)^{i+j+k+1}}{n\lambda^n\lambda^{-\frac{\delta\beta+i\beta-\beta+1}{\beta}}}\left(\frac{\Gamma\left(\delta+i-2+\frac{1}{\beta}\right)\Gamma\left(\frac{1}{\beta}-\delta-i-\alpha+2\right)}{\Gamma\left(\frac{1}{\beta}+\alpha\right)}\right)\right)\right] \tag{30}$$

3.4. Order Statistics

The order statistics play a vital role in predicting the failure time of certain items by using previously observed failures. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n, and let $X_{r:n}$ denotes that i^{th} order statistic, then the pdf of $X_{r:n}$ is given by

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}F(x)^r f(x)(1-F(x))^{n-r}$$

We can rewrite the above equation as follows:

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}F(x)^{r-1} f(x)(1-F(x))^{n-r}$$

Substituting the equations (7) and (8) in the above equation, we can write

$$f_{k:n}(x)=\left(\frac{\alpha\beta}{\lambda B(r,n-r+1)}\right)\sum_{i=0}^{r-1}\sum_{k=0}^{\infty}(-1)^i\binom{r-1}{i}\left(\frac{\Gamma(n+i-r+1)}{k!\Gamma(n+i-r)}\left(p.\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\right)^k\right)\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right) \tag{31}$$

3.5. Record Statistics

Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the upper record values from a sequence of identically and independently distributed random variables from the RTPL distribution. The pdf of n^{th}upper record value $X_{U(n)}$ of the RTPL distribution is defined by

$$f_{U(n)}(x)=\frac{1}{\Gamma n}\left[-\log(1-F(x))\right]^{n-1} f(x)$$

$$=\left(\frac{\alpha\lambda^{\alpha}\beta x^{\beta-1}}{\Gamma n\left(\lambda+x^{\beta}\right)^{\alpha+1}}\right)\left[-\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\left(1-p.\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\right)^{n-1}\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)\right] \tag{32}$$

The pdf of n^{th}lower record value $X_{L(n)}$ of the RTPL distribution is defined by

$$f_{L(n)}(x) = \frac{1}{\Gamma n}\left[-\log(F(x))\right]^{n-1} f(x)$$

$$f_{L(n)}(x) = \left(\frac{\alpha\lambda^{\alpha}\beta x^{\beta-1}}{\Gamma n\left(\lambda + x^{\beta}\right)^{\alpha+1}}\right)\left[-\log\left(1-\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\left(1-p.\log\left[\frac{\lambda}{x^{\beta}+\lambda}\right]^{\alpha}\right)\right)^{n-1}\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)\right] \tag{33}$$

3.6.Stochastic Ordering

The ordering of probability distributions particularly among lifetime distributions plays an important role in the statistical literature. We consider stochastic orders, namely, the hazard rate, the mean residual life, and the likelihood ratio order for two independent RTPL random variables under a restricted parameter space. It can be recalled that if a family has a likelihood ratio ordering, it has the monotone likelihood ratio property. If X and Y are independent random variables with a cumulative distribution function F_X and F_Y respectively, then X is said to be smaller than Y in the

- stochastic order $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x
- hazard rate order $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)$ for all x
- mean residual life order $X \leq_{mrl} Y$ if $m_X(x) \geq m_Y(x)$ for all x
- likelihood ratio order $X \leq_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results are well known for establishing stochastic ordering of probability distributions. The likelihood ratio is given as follows

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\beta_1\lambda_1^{\alpha_1}x^{\beta_1-1}\left(x^{\beta_2}+\lambda_2\right)^{\alpha_2+1}\left[1-p_1.\left(1+\log\left(\frac{\lambda_1}{x^{\beta_1}+\lambda_1}\right)^{\alpha_1}\right)\right]}{\alpha_2\beta_2\lambda_2^{\alpha_2}x^{\beta_2-1}\left(x^{\beta_1}+\lambda_1\right)^{\alpha_1+1}\left[1-p_2.\left(1+\log\left(\frac{\lambda_2}{x^{\beta_2}+\lambda_2}\right)^{\alpha_2}\right)\right]} \tag{34}$$

By taking the logarithm on both sides of the likelihood ratio which is given in the equation (34) then we get,

$$\log\left[\frac{f_X(x)}{f_Y(x)}\right] = \log\left[\frac{\alpha_1\beta_1\lambda_1^{\alpha_1}x^{\beta_1-1}\left(x^{\beta_2}+\lambda_2\right)^{\alpha_2+1}\left[1-p_1.\left(1+\log\left(\frac{\lambda_1}{x^{\beta_1}+\lambda_1}\right)^{\alpha_1}\right)\right]}{\alpha_2\beta_2\lambda_2^{\alpha_2}x^{\beta_2-1}\left(x^{\beta_1}+\lambda_1\right)^{\alpha_1+1}\left[1-p_2.\left(1+\log\left(\frac{\lambda_2}{x^{\beta_2}+\lambda_2}\right)^{\alpha_2}\right)\right]}\right] \tag{35}$$

$$\frac{d}{dx}\log\left[\frac{f_X(x)}{f_Y(x)}\right]=\frac{\beta_1-1}{x}-\frac{\beta_2-1}{x}-\left(\frac{(\alpha_1+1)\beta_1 x^{\beta_1-1}}{x^{\beta_1}+\lambda_1}\right)+\left(\frac{(\alpha_2+1)\beta_2 x^{\beta_2-1}}{x^{\beta_2}+\lambda_2}\right)+ \frac{\alpha_1\beta_1 p_1 x^{\beta_1-1}\left(x^{\beta_1}+\lambda_1\right)^{-1}}{\left[1-p_1.\left(1+\log\left(\frac{\lambda_1}{x^{\beta_1}+\lambda_1}\right)^{\alpha_1}\right)\right]}-\frac{\alpha_2\beta_2 p_2 x^{\beta_2-1}\left(x^{\beta_2}+\lambda_2\right)^{-1}}{\left[1-p_2.\left(1+\log\left(\frac{\lambda_2}{x^{\beta_2}+\lambda_2}\right)^{\alpha_2}\right)\right]} \tag{36}$$

Now if $\alpha_1=\alpha_2=\alpha$, $\beta_1=\beta_2=\beta$, $p_1=p_2=p$ and $\lambda_1>\lambda_2$ then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}\leq 0$ implies that $X\leq_{lr} Y$ and hence $X\leq_{lr} Y$, $X\leq_{hr} Y$ $X\leq_{mlr} Y$ and $X\leq_{st} Y$.

4. MAXIMUM LIKELIHOOD ESTIMATION METHOD

In this section, the maximum likelihood method is used to estimate the unknown parameters of the RTPL distribution and the information matrix is obtained to observe the asymptotic behavior of the parameters of RTPL distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the RTPL distribution with unknown parameters $\alpha,\beta,\lambda,$ and p then the likelihood function is given by

$$L=\prod_{i=1}^{n}\frac{\alpha\beta\lambda^{\alpha}x^{\beta-1}}{\left(x^{\beta}+\lambda\right)^{\alpha+1}}\left[1-p\left(1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right)\right]$$

The log-likelihood function of the RTPL distribution is given below

$$l=n\log\alpha+n\log\beta+n\alpha\log\lambda-(\alpha+1)\sum_{i=1}^{n}\log\left(\lambda+x_i^{\beta}\right)+(\beta-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(1-p\left[1+\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right) \tag{37}$$

Taking first-order partial derivatives of the equation (37) to find the unknown parameters,

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha}+n\log\lambda-\sum_{i=1}^{n}\log\left(x_i^{\beta}+\lambda\right)+\sum_{i=1}^{n}\frac{p\log\lambda-p\log\left(x_i^{\beta}+\lambda\right)}{\lambda\left(1-p\left[1+\alpha\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)}=0 \tag{38}$$

$$\frac{\partial\log L}{\partial\beta}=\frac{n}{\beta}-(\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\beta}\log x_i}{\left(x_i^{\beta}+\lambda\right)}+\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\frac{\alpha p x_i^{\beta}\log x_i\left(x_i^{\beta}+\lambda\right)^{-1}}{\left(1-p\left[1+\alpha\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)}=0 \tag{39}$$

$$\frac{\partial\log L}{\partial\lambda}=\frac{n\alpha}{\lambda}-\sum_{i=1}^{n}\frac{(\alpha+1)}{\left(x_i^{\beta}+\lambda\right)}-\sum_{i=1}^{n}\frac{\alpha p x_i^{\beta}\left(x_i^{\beta}+\lambda\right)^{-1}}{\lambda\left(1-p\left[1+\alpha\log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)}=0 \tag{40}$$

$$\frac{\partial \log L}{\partial p} = -\sum_{i=1}^{n} \frac{\left[1+\alpha \log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]}{\lambda\left(1-p\left[1+\alpha \log\left(\frac{\lambda}{x^{\beta}+\lambda}\right)^{\alpha}\right]\right)} = 0$$
(41)

Then the maximum likelihood estimates of the parameters $\alpha, \beta, \lambda,$ and p can be obtained by solving the partial differential equations in (38) to (41). The Fisher information I_{ij} matrix for RTPL distribution is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}$$

$$I_{11} = E\left[-\frac{\partial^2 \log L}{\partial \alpha^2}\right],\ I_{22} = E\left[-\frac{\partial^2 \log L}{\partial \beta^2}\right],\ I_{33} = E\left[-\frac{\partial^2 \log L}{\partial \lambda^2}\right],\ I_{44} = E\left[-\frac{\partial^2 \log L}{\partial p^2}\right]$$

$$I_{12} = I_{21} = E\left[-\frac{\partial^2 \log L}{\partial \alpha\, \partial \beta}\right],\ I_{13} = I_{31} = E\left[-\frac{\partial^2 \log L}{\partial \alpha\, \partial \lambda}\right],\ I_{14} = I_{41} = E\left[-\frac{\partial^2 \log L}{\partial \alpha\, \partial p}\right],$$
$$I_{23} = I_{32} = E\left[-\frac{\partial^2 \log L}{\partial \beta\, \partial \lambda}\right],\ I_{24} = I_{42} = E\left[-\frac{\partial^2 \log L}{\partial \beta\, \partial p}\right],\ I_{34} = I_{43} = E\left[-\frac{\partial^2 \log L}{\partial \lambda\, \partial p}\right]$$

The Fisher information matrix can be obtained by deriving the second-order partial of the log-likelihood function in equation (37) for unknown parameters. So we obtain the asymptotic $100(1-\alpha)\%$ confidence intervals for the unknown parameters of RTPL α, β, λ, and p can be easily obtained by using the equation given below

$$\alpha \in \left[\hat{\alpha} - z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}}, \hat{\alpha} + z_{\frac{\alpha}{2}}\sqrt{I_{11}^{-1}}\right], \beta \in \left[\hat{\beta} - z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}, \hat{\beta} + z_{\frac{\alpha}{2}}\sqrt{I_{22}^{-1}}\right] \text{ and}$$

$$\lambda \in \left[\hat{\lambda} - z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}, \hat{\lambda} + z_{\frac{\alpha}{2}}\sqrt{I_{33}^{-1}}\right], p \in \left[\hat{p} - z_{\frac{\alpha}{2}}\sqrt{I_{44}^{-1}}, \hat{p} + z_{\frac{\alpha}{2}}\sqrt{I_{44}^{-1}}\right]$$

Where $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ quantile of the standard normal distribution.

5. MONTE CARLO SIMULATION

This section deals with the simulation study by generating the samples from the proposed distribution. The idea behind the Monte Carlo simulation is to generate a series of experimental samples using the random number sequence and it creates a fluctuating convergence process. The inverse transformation method is the most commonly used technique to generate random variates of the distribution. If a random variates R follows a uniform distribution with [0,1], the random variates $X = F^{-1}(R)$ have a continuous cumulative probability distribution $F(X)$. In this case, the inverse function is defined as

$$X = F^{-1}(R) = \min\{x : F(x) \geq R\}; \quad \text{for } 0 \leq R \leq 1$$

The procedure for generating random variates using the inverse transformation method is

Step 1: Generate a uniformly distributed random number sequence *R* between the interval [0,1].

Step 2: Calculate the random variates *X* of the RTPL distribution by using the equation given below,

$$x_p = \left(\lambda e^{-\frac{1}{\alpha}\left(W_{-1}\left(\frac{u-1}{pe^{1/p}} \right) + \frac{1}{p} \right)} - \lambda \right)^{1/\beta}$$

We study the performance of MLE of the RTPL distribution by conducting various simulations for different sample sizes and different parameter values. After generating random samples, it can be used to obtain the mean estimate, average bias, and root mean square error of the maximum likelihood estimators of the distribution.

a) Mean estimate of the MLE $\hat{\upsilon}$ of the parameter $\upsilon = \alpha, \beta, \lambda, \text{ and } p$:

$$\frac{1}{N}\sum_{i=1}^{N} \hat{\upsilon}$$

b) The average bias of the MLE $\hat{\upsilon}$ of the parameter $\upsilon = \alpha, \beta, \lambda, \text{ and } p$:

$$\frac{1}{N}\sum_{i=1}^{N} (\hat{\upsilon} - \upsilon)$$

c) Root mean squared error of the MLE $\hat{\upsilon}$ of the parameter $\upsilon = \alpha, \beta, \lambda, \text{ and } p$:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N} (\hat{\upsilon} - \upsilon)^2}$$

Table 1. Average Bias, Root mean square error of the estimates based on MLE by Monte Carlo Simulation of RTPL distribution for different sample sizes.

N	**Parameter**	$Case\ I: \alpha = 3, \beta = 2.5, \lambda = 4, p = 0.5$			$Case\ II: \alpha = 3, \beta = 2.5, \lambda = 4, p = 1$		
		Mean	AB	RMSE	Mean	AB	RMSE
25	α	4.63043	1.63043	6.54560	4.13564	1.13564	4.42860
	β	2.87287	0.37287	0.91905	3.27358	0.77358	1.02376
	λ	7.87230	3.87230	12.0427	12.8557	8.85574	18.3768
	p	0.36628	-0.13371	0.32787	0.56351	-0.43648	0.63158
50	α	4.79498	1.79498	6.37452	3.21649	0.21649	2.91639
	β	2.65200	0.15200	0.56537	3.35949	0.85949	1.07604
	λ	8.58040	4.58040	13.3039	9.45730	5.45730	13.0895
	p	0.35866	-0.14134	0.31642	0.63158	-0.36841	0.44482
75	α	4.70200	1.70200	5.25124	2.85943	-0.14056	2.78340
	β	2.57257	0.07257	0.47150	3.39907	0.89907	1.08594
	λ	8.06813	4.08045	11.2990	8.56134	4.56134	12.5419
	p	0.39320	-0.10679	0.31475	0.63080	-0.36919	0.45047
100	α	4.88606	1.88606	5.17449	2.58303	-0.41696	2.06095
	β	2.52120	0.02120	0.42337	3.39977	0.89977	1.03959
	λ	8.08045	4.06813	10.4641	7.72486	3.72485	10.9990
	p	0.39184	-0.10816	0.31473	0.63409	-0.36590	0.44544

Table 2. Average Bias, Root mean square error can be obtained by Monte Carlo Simulation of RTPL distribution for different sample sizes.

n	**Parameter**	$Case\ I: \alpha=0.5, \beta=0.6, \lambda=0.4, p=0.1$			$Case\ II: \alpha=0.5, \beta=0.6, \lambda=0.4, p=1$		
		Mean	AB	RMSE	Mean	AB	RMSE
50	α	0.63005	0.13004	0.46860	0.46542	-0.03457	0.27323
	β	0.66297	0.06297	0.24380	0.39145	0.29992	0.29433
	λ	0.50398	0.10398	0.78359	0.34823	1.39085	2.27508
	p	0.29443	0.19443	0.35607	0.34648	-0.51504	0.69344
100	α	0.55927	0.05927	0.22920	0.69993	-0.10854	0.16292
	β	0.63287	0.03287	0.15135	0.75728	0.25728	0.25213
	λ	0.37730	-0.02269	0.30049	0.81821	0.86313	1.28071
	p	0.29320	0.19320	0.36410	0.83007	-0.59811	0.62799
500	α	0.52201	0.02201	0.10083	1.79085	-0.15176	0.16203
	β	0.60772	0.00772	0.05899	1.26313	0.21821	0.24087
	λ	0.34190	-0.05809	0.15553	0.79885	0.39885	0.66036
	p	0.24686	0.14686	0.32035	0.65827	-0.39769	0.48515
1000	α	0.51573	0.01573	0.07803	0.40188	-0.15352	0.15928
	β	0.60490	0.00490	0.04176	0.48495	0.20107	0.24376
	λ	0.34769	-0.05230	0.12756	0.60230	0.25827	0.46895
	p	0.21904	0.11904	0.28422	0.66413	-0.33586	0.40010

Table 1 shows the simulation study is repeated for $N = 1000$ times each which has its sample size is given by $n = 25, 50, 75, 100$ and for two different cases such parameter values are shown as $Case\ I: \alpha = 3, \beta = 2.5, \lambda = 4, p = 0.5$, and $Case\ II: \alpha = 3, \beta = 2.5, \lambda = 4, p = 1$. Table 2 describes the simulation study is repeated for $N = 10000$ times each with sample size n = 50, 100, 500, 1000 and by taking parameter values $Case\ I: \alpha = 0.5, \beta = 0.6, \lambda = 0.4, p = 0.1$ and $Case\ II: \alpha = 0.5, \beta = 0.6, \lambda = 0.4, p = 1$. In the simulation study, we present the mean, average bias, and RMSE values of the parameters $\alpha, \beta, \lambda,$ and p for different sample sizes. From the results, we can verify that as the sample size n increases, the RMSEs decay toward zero. The average bias for the parameters is slightly larger for small to moderate sample sizes but tends to get smaller as the sample size n increases. We also observe that for all the parametric values, the bias decrease as the sample size n increases. Hence the ML estimates of RTPL distribution are consistent and efficient.

6. APPLICATIONS

In this section, we consider two real data sets for illustrating the suitability of the RTPL distribution in real-time applications, the first data set consists of the breaking stress of carbon fibers with the length of 50mm, and the second data involves the exact failure time of Kevlar 373/epoxy that is subject to constant pressure can be discussed by using the maximum likelihood method of estimation and goodness of fit test. The model selection is carried out by using the AIC (Akaike information criterion), the BIC (Bayesian information criterion), and the CAIC (consistent Akaike information criteria).

$$AIC = -2L(\hat{\theta}) + 2q$$

$$BIC = -2L(\hat{\theta}) + q\log(n)$$

$$CAIC = -2L(\hat{\theta}) + \frac{2qn}{n-q-1}$$

Where $L(\hat{\theta})$ denotes the log-likelihood function evaluated at the MLEs, p is the number of parameters, and n is the sample size. Here, θ denotes the parameters $\theta = \alpha, \beta, \lambda, p$. An iterative procedure is applied to solve the equations (38), (39), (40), and (41). The model with minimum AIC (or BIC, CAIC) values is chosen as the best model to fit the given data sets.

Data Set 1: The data set contains exact times of failure. More precisely, it consists of the life of fatigue fracture of Kevlar 373/epoxy that is subject to constant pressure (at the 90% stress level) until all had failed. Analysis of this data set can also be found in [16]. These data are listed as: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773,1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 3. Summary of statistics for Data set 1.

n	Minimum	Q_1	Median	Mean	Q_3	Maximum
69	0.0251	0.8645	1.5728	1.5675	2.0903	3.7455

Table 4. The Parameter estimates of the RTPL distributions for Data set 1.

Probability Models	Parameter Estimates			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{p}$
RTPL	142.646	1.25041	134.721	0.84430
Expo-Lomax	14.3067	2.973127e+02	3.299822e-03	-
H-L	4.669135e+02	1.965831e-03	-	-
Lomax	4569.5	6914.7	-	-

Table 5. The log-likelihood, information criteria, and Goodness of fit test for Data set 1.

Accuracy Measures	Probability Models			
	RTPL	**Expo-Lomax**	**H-L**	**Lomax**
-log L	87.698	91.753	93.440	100.062
AIC	183.39	189.50	190.91	204.12
BIC	192.33	196.20	195.37	208.59
CAIC	184.02	189.87	191.09	204.31
$\boldsymbol{W_n^2}$	0.1678	0.2094	0.5941	1.1240
$\boldsymbol{A_n^2}$	1.1275	1.2655	3.0836	5.5955
$\boldsymbol{D_n}$	0.1175	0.1183	0.1639	0.2221

Table 5, provides the estimated values of the parameters and likelihood values for all the fitted distributions. From this, minimum values of the information criterion represent the fitness of the new model and we conclude that the RTPL distribution is best when compared to Lomax distribution, Half-logistic Lomax (HL) [6], and Exponentiated Lomax (Expo-Lomax) [1] distributions. The test statistics D_n, W_n^2, A_n^2 have the smallest values for the Kevlar 373/epoxy data set under the RTPL distribution when compare to other suitable models. The RTPL distribution is approximately a better model for this real dataset.

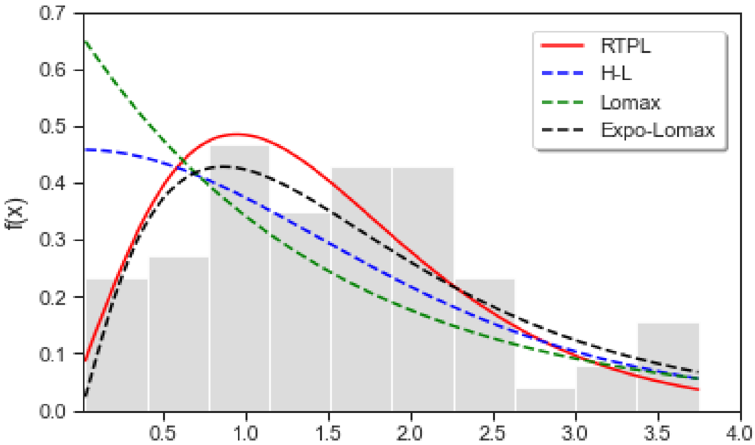

Figure 4: Estimated pdf plot for the data set 1

Figure 4 shows the fitted pdf plot, in which the histogram represents data points, and the curves show the fitness of the four comparable distributions. This plot shows that the RTPL model provides an adequate fit to the lifetime of fatigue fracture of the Kevlar 373/epoxy datasetwhen compared to the other advisory models.

Data Set 2: This dataset describes the breaking stress of carbon fibers with a length (GPA) of 50mm. The data has been taken from [17]. The data is given as follows: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.5, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88 , 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 6. Summary of statistics for Data set 2.

n	**Minimum**	Q_1	**Median**	**Mean**	Q_3	**Maximum**
66	0.390	2.178	2.835	2.759	3.277	4.900

Table 7. The Parameter estimates of the RTPL distributions for Data set 2.

Probability Models	**Parameter Estimates**				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{p}$	$\hat{\gamma}$
RTPL	98.6077	2.77527	1161.283	0.74846	-
Expo-PL	5690.105	1.11941	7.95276	-	6600.825
PL	105.423	2.08411	784.590	-	-
H-L	7.176263e+02	7.599327e-04	-	-	-
Lomax	2686.160	7663.882	-	-	-

Table 8: The log-likelihood, information criteria, and Goodness of fit test for Data set 2.

Accuracy Measures	**Probability Models**				
	RTPL	**Expo-PL**	**PLomax**	**H-L**	**Lomax**
-log L	85.5269	94.2648	98.4301	122.4363	133.0312
AIC	179.053	196.529	202.860	248.872	270.0624
BIC	187.812	205.288	209.429	253.252	274.4417
CAIC	179.709	197.185	203.247	249.063	270.2529

W_n^2	0.06810	0.38229	1.8567	12.206	13.422
A_n^2	0.40351	1.9864	8.6186	2.4891	2.7094
D_n	0.07319	0.16162	0.29427	0.3269	0.3474

Table 8 presents the estimated values of the parameters for all the fitted distributions. From this we conclude that the RTPL distribution provides the best fit to the given data set when compared to Lomax distribution, Half-logistic Lomax (HL) [6], Exponentiated Power Lomax (Expo-PL) [4], and Power Lomax (PLomax) [14] distributions. The values of tests statistics such as the Kolmogorov-Smirnov D_n, Cramér-von Mises W_n^2, Anderson and Darling A_n^2can be used to measure the goodness of fit of the RTPL distribution while concerning the other models through the breaking stress of carbon fibers data. Hence, the RTPL distribution approximately provides an adequate fit for the dataset.

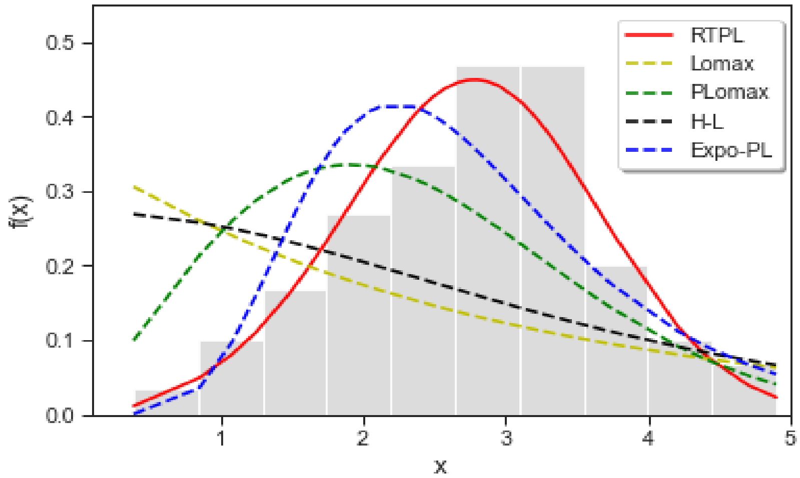

Figure 5: Estimated pdf plot for the data set 2

The fitted pdf plot is displayed in Figure 5, this plot shows that the histogram represents the data points and the curve shows the fitness of the five different distribution which is chosen for this comparative study. From this, we conclude that the RTPL model provides an adequate fit to the breaking stress of the carbon fibers data set, when compared to the other suitable models.

7. CONCLUSION

In this paper, a new extension of the four-parameter Lomax distribution is proposed and it is named as Record-based Transmuted Power Lomax distribution based on Record based transmuted map. The usefulness of this newly proposed model is illustrated by using two real data sets. This results illustrate that the proposed model provides a consistently better fit than the other existing suitable models. The graphical representation of the hazard rate of RTPL model has been explored and the obtained shapes are increasing, decreasing, and reversed J shaped. The maximum likelihood estimation method is used to estimate the unknown parameters of the RTPL distribution. The performance of the maximum likelihood estimates is investigated through the Monte Carlo simulation study to generate a random sample by using the quantile function and we observed that the proposed distribution shows a better fit when the sample size increases. The results of the Kolmogorov Smirnov test, Cramer Von Mises test, Anderson Darlings test, and important information criterions conclude that the RTPL model is provided goodness fit and emerge as better model compared to the other models. It is evident that, it has a lot of scope and real time applications in many field of science.

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Censoring and reliability inferences for power Lindley distribution with application on hematologic malignancies data

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Abstract

In this paper, by using progressively type II censored samples, we discuss on estimation of the parameters of a power Lindley model. Maximum likelihood estimates (MLE) and approximate confidence intervals of the unknown parameters are obtained. Then, considering squared error loss function, the Bayes estimates of the parameters are derived. Because there are not closed forms for the Bayes estimates, we use Tierney and Kadane's technique, to calculate the approximate Bayes estimates. Further, the results are extended to the stress-strength reliability parameter involving two power Lindley distributions. The ML estimate of the stress-strength parameter and its approximate confidence interval are obtained. Then, the Bayes estimates and highest posterior density credible interval of the involved parameter are obtained by using a Markov Chain Monte Carlo method. To evaluate the performances of maximum likelihood and Bayes estimators simulation studies are conducted and two examples of real data sets are provided to illustrate the procedures.

Keywords: Power Lindley model, progressive type II censoring, Bayesian approach, Maximum likelihood method, Stress-strength reliability

1. Introduction

A random variable (r.v.) X follows the power Lindley model with parameters γ and δ, denoted by $PL(\gamma,\delta)$, if its probability density function (p.d.f.) and survival function are defined as

$$f(x;\gamma,\delta) = \frac{\gamma\delta^2}{\delta+1}(1+x^\gamma)\, x^{\gamma-1}e^{-\delta x^\gamma}, \quad x>0, \quad \gamma,\delta>0. \tag{1}$$

and

$$S(x;\gamma,\delta) = \left(1+\frac{\delta}{\delta+1}x^\gamma\right)e^{-\delta x^\gamma}, \quad x>0, \quad \gamma,\delta>0, \tag{2}$$

respectively. This model is introduced by Ghitany et al. [10] as a new distribution useful to analyze lifetime data. They studied the statistical properties and maximum likelihood estimation (MLE) of the power Lindley model on the basis of complete random sample. However, in many life testing and reliability analysis, the experiment may be terminated before the failure of all items. Hence, the available observations are called censored samples. By the censoring, the test time can be reduced and further some experimental components are kept for future use. In the conventional type I and type II censoring schemes, removing items at stages other than the terminal stage of the test is not allowed. Therefore, in the literature, a more important scheme called progressively type II censoring (PTII) is provided as follows. Suppose that a sample of size n items are in a life test. When the first item is failed (time $x_{(1)}$), U_1 items are discarded from the surviving $n-1$ items. With the second failure ($x_{(2)}$), U_2 items of the $n-2-U_1$ surviving items are deleted. This procedure is continued until the time of dth failure ($x_{(d)}$) in which $U_d = n-d-(U_1+U_2+\ldots+U_{d-1})$ surviving items are removed. Note that the censoring numbers U_i, $i=1,...,d$, are determined before beginning of the study. When $d=n$ and $U_1=U_2=\ldots=U_d=0$, the complete sample of size n is observed. Also, if $U_1=U_2=\ldots=U_{d-1}=0$ and $U_d=n-d$, the ordinary TII censored sample of size d is observed.

There is a large amount of literature about the estimation of lifetime model parameters using PTII censoring scheme. Krishna and Kumar [16] studied estimation of reliability characteristics in Lindley model. Bayesian analysis for Rayleigh distribution under PTII scheme is discussed by Lee et al. [19]. Pradhan and Kundu [21] addressed statistical inference of generalized exponential model in presence of PTII censored data. Balakrishnan [3] presented inferential approaches for different lifetime models based on the above PTII censoring scheme. Ghitany et al. [11] applied ML procedure to derive the estimates of the Gompertz model parameters by using complete and PTII censored data. Kim and Han [13] provided different inference procedures for Rayleigh distribution parameter by using a progressively censored sample.

The interest of this paper is to provide classical and Bayesian inferences for the parameters of power Lindley distribution by using a PTII censored sample. We first describe the construction of likelihood function using a PTII censored sample from power Lindley distribution. Then, the ML estimates of the parameters and their approximate confidence intervals (CI) are obtained. Considering squared error loss function and using gamma priors of the parameters, an expression is provided as the Bayesian estimate of any function of the parameters. Since this expression can not simplified to a nice closed form, we employ Tierney and Kadane's procedure to obtain the approximate Bayes estimates.

Moreover, the above estimation techniques based on PTII censoring scheme can be naturally extended for inferences about the stress-strength model. This model has attracted the attention of statisticians for many years due to their applicability in diverse areas such as medicine, engineering, and quality control, among others. In reliability studies with strength X and stress Y , the parameter $R = P(X > Y)$ measures the reliability of a system ([15]). It is used in biometrical researches for comparison of the two quantities obtained from practical experiments. There is a large amount of literature about the estimation of R using different approaches and distributional assumptions on (X, Y). Estimation of R in the models with correlated stress and strength is conducted by [4]. Hanagal [12] derived maximum likelihood estimate of stress-strength parameter R in a bivariate Pareto model. Inference for the stress-strength models in a generalized exponential model is studied by Kundu and Gupta [18]. Pak et al. [20] have used fuzzy set theory to derive inferences on the parameter R when the observations of the strength and stress are imprecise quantities. Statistical estimation of R for the exponential model is discussed by Krishnamoorthy et al. [17]. Inference on the reliability in multicomponent models when the stress and strength have Weibull distribution is considered by Kizilaslan and Nadar[14]. Eryilmaz [6] computed the reliability of coherent structures in multivariate stress-strength models.

Recently, Ghitany et al. [9] developed inference procedures for the stress-strength power Lindley models when the complete information about all experimental units are available. However, in practice, we may deal with censored data sets in which the failures of some items are not observed. For example, assume that the random variables X and Y describe the treatment effects of two new drugs and the quantity of interest is $R = P(X > Y)$. In such situations, censored samples from both treatment groups are observed, rather than complete samples. Other examples include comparison of carbon fiber strengths at different gauge lengths and comparison of the concentration of sulphur dioxide from a Beach in two different years. In this study, we obtain Bayesian and classical estimates of the reliability R by using PTII censored samples from the stress and strength populations. We first determine the ML estimate of the reliability parameter and its asymptotic confidence interval. Then, we use a Markov Chain Monte Carlo (MCMC) procedure to obtain the Bayes estimate and highest posterior density (HPD) credible interval of the parameter R.

The layout of this paper is as follows. Section 2 concerns inference procedures for the power Lindley based on PTII censored sample. In Section 3, statistical inferences for the reliability parameter R are discussed. To evaluate the performances of the proposed estimators, simulation studies are conducted in Section 4. In Section 5, a real data set from Ebrahimi [7] is analysed to demonstrate the application of PTII censoring scheme. Then, to illustrate the estimation procedures of the stress-strength model, we present an example of two real data sets. Finally, some comments and conclusions are made in Section 6.

2. Inference for progressively censored data

2.1. Maximum likelihood estimation

Assume that n independent components are put on a life testing experiment with the lifetimes following the power Lindley model. Before the commencement of the experiment, the quantity $d \leq n$ is specified and the censoring scheme $(U_1, ..., U_d)$ with $U_i \geq 0$ is determined. Then, by using a PTII censored sample denoted as $\mathbf{x} = (x_{(1)}, ..., x_{(d)})$, the likelihood function of γ and δ can be expressed as

$$\begin{aligned} L_O(\gamma,\delta) &= K\prod_{i=1}^{d} f(x_{(i)};\gamma,\delta)\left[S(x_{(i)};\gamma,\delta)\right]^{U_i} \\ &= K\frac{\gamma^d\delta^{2d}}{(\delta+1)^d}e^{-\delta\sum\limits_{i=1}^{d}x_{(i)}^{\gamma}(1+U_i)}\prod_{i=1}^{d}(1+x_{(i)}^{\gamma})x_{(i)}^{\gamma-1}\left(1+\frac{\delta}{\delta+1}x_{(i)}^{\gamma}\right)^{U_i}, \end{aligned} \tag{3}$$

where $K = n(n - U_1 - 1)\ldots(n - U_1 - \ldots - U_{d-1} - d + 1)$. Therefore, the corresponding log-likelihood function of the parameters become

$$\begin{aligned} \ell(\gamma,\delta) &= \log(K) + d\log\gamma + 2d\log\delta - d\log(\delta+1) - \delta\sum_{i=1}^{d}x_{(i)}^{\gamma}(1+U_i) \\ &\quad + \sum_{i=1}^{d}\left[\log(1+x_{(i)}^{\gamma}) + (\gamma-1)\log x_{(i)}\right] + \sum_{i=1}^{d}U_i\log\left(1+\frac{\delta}{\delta+1}x_{(i)}^{\gamma}\right). \end{aligned} \tag{4}$$

The MLE of the parameters γ and δ, say $\hat{\gamma}$ and $\hat{\delta}$, are the solutions of nonlinear equations

$$\begin{aligned} \frac{\partial\ell}{\partial\gamma} &= \frac{d}{\gamma} + \sum_{i=1}^{d}\log x_{(i)} - \delta\sum_{i=1}^{d}x_{(i)}^{\gamma}\log x_{(i)}(1+U_i) \\ &\quad + \sum_{i=1}^{d}\frac{x_{(i)}^{\gamma}\log x_{(i)}}{1+x_{(i)}^{\gamma}} + \sum_{i=1}^{d}U_i\frac{\delta x_{(i)}^{\gamma}\log x_{(i)}}{\delta+1+\delta x_{(i)}^{\gamma}} = 0, \end{aligned} \tag{5}$$

$$\frac{\partial\ell}{\partial\delta} = \frac{2d}{\delta} - \frac{d}{\delta+1} - \sum_{i=1}^{d}x_{(i)}^{\gamma}(1+U_i) + \sum_{i=1}^{d}U_i\frac{x_{(i)}^{\gamma}}{(\delta+1)^2+\delta(\delta+1)x_{(i)}^{\gamma}} = 0. \tag{6}$$

Note that there are not explicit solutions for the above system of equations and it is required to employ nonlinear numerical computational techniques to calculate the MLEs. In a similar problem, Valiollahi et al. [25], have use EM algorithm to obtain the ML estimates of the parameters. Here, in real data application and simulation studies described later on, we employ *nlm* function in the R statistical software ([22]) to compute the MLEs.

Once the ML estimates of γ and δ are obtained, we can apply the asymptotic normality of the MLEs to compute the approximate CIs for the parameters. The observed variance-covariance matrix for the MLEs of the parameters is

$$\hat{\Sigma} = \begin{bmatrix} -\frac{\partial^2\ell(\gamma,\delta)}{\partial\gamma^2} & -\frac{\partial^2\ell(\gamma,\delta)}{\partial\gamma\partial\delta} \\ -\frac{\partial^2\ell(\gamma,\delta)}{\partial\gamma\partial\delta} & -\frac{\partial^2\ell(\gamma,\delta)}{\partial\delta^2} \end{bmatrix}^{-1}_{(\gamma=\hat{\gamma},\delta=\hat{\delta})} = \begin{bmatrix} \sigma_{11}(\hat{\gamma},\hat{\delta}) & \sigma_{12}(\hat{\gamma},\hat{\delta}) \\ \sigma_{12}(\hat{\gamma},\hat{\delta}) & \sigma_{22}(\hat{\gamma},\hat{\delta}) \end{bmatrix}, \tag{7}$$

where

$$\frac{\partial^2 \ell(\gamma,\delta)}{\partial \gamma^2} = -\frac{d}{\gamma^2} - \delta \sum_{i=1}^{d} x_{(i)}^{\gamma} (\log x_{(i)})^2 (1+U_i) + \sum_{i=1}^{d} [\frac{x_{(i)}^{\gamma} (\log x_{(i)})^2}{(1+x_{(i)}^{\gamma})^2} + U_i \frac{\delta x_{(i)}^{\gamma} (\log x_{(i)})^2}{(\delta+1)(1+\frac{\delta}{\delta+1} x_{(i)}^{\gamma})^2}], \tag{8}$$

$$\frac{\partial^2 \ell(\gamma,\delta)}{\partial \gamma \partial \delta} = -\sum_{i=1}^{d} x_{(i)}^{\gamma} \log x_{(i)} (1+U_i) + \sum_{i=1}^{d} U_i x_{(i)}^{\gamma} \log x_{(i)} \frac{1}{(\delta+1+\delta x_{(i)}^{\gamma})^2}, \tag{9}$$

$$\frac{\partial^2 \ell(\gamma,\delta)}{\partial \delta^2} = -\frac{2d}{\delta^2} + \frac{d}{(\delta+1)^2} - \sum_{i=1}^{d} U_i x_{(i)}^{\gamma} \frac{2(\delta+1) + (2\delta+1) x_{(i)}^{\gamma}}{((\delta+1)^2 + \delta(\delta+1) x_{(i)}^{\gamma})^2}. \tag{10}$$

Thus, by using the delta method and inverse logarithmic transformation (see [10]), the $100(1-\alpha)\%$ CIs for the parameters γ and δ are derived, respectively, as

$$(e_1^L, e_1^U) \qquad \text{and} \qquad (e_2^L, e_2^U), \tag{11}$$

where

$$(L_1, U_1) \equiv \log \hat{\gamma} \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\sigma_{11}(\hat{\gamma},\hat{\delta})}}{\hat{\gamma}}, \tag{12}$$

$$(L_2, U_2) \equiv \log \hat{\delta} \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\sigma_{22}(\hat{\gamma},\hat{\delta})}}{\hat{\delta}} \tag{13}$$

in which $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ upper quantile of the standard normal distribution.

2.2. Bayesian analyzes

In the Bayesian setting, the observer combine subjective opinion based on insight or experience with the available observations to get balanced values and to update the estimates as more information and data become accessible. In this section we obtain the Bayes estimates of the unknown parameters assuming that γ and δ are independent r.v.s from the gamma models with respective densities

$$\begin{cases} \pi_1(\gamma; a_1, b_1) \propto \gamma^{a_1-1}\, e^{-\gamma b_1}, & \gamma > 0, \\ \pi_2(\delta; a_2, b_2) \propto \delta^{a_2-1}\, e^{-\delta b_2}, & \delta > 0, \end{cases} \tag{14}$$

where the hyperparameters a_i, b_i, $i = 1, 2$, are positive. By combining (3) with (14), the joint density function of (γ, δ) and the data $\mathbf{x} = (x_{(1)}, ..., x_{(m)})$ becomes

$$\pi_3(\gamma,\delta,\mathbf{x}) \propto \frac{\gamma^{d+a_1-1}\, e^{-\gamma b_1} \delta^{2d+a_2-1}}{(\delta+1)^d} e^{-\delta(b_2 + \sum_{i=1}^{d} x_{(i)}^{\gamma}(1+S_i))} \prod_{i=1}^{d} (1+x_{(i)}^{\gamma}) x_{(i)}^{\gamma-1} \left(1 + \frac{\delta}{\delta+1} x_{(i)}^{\gamma}\right)^{U_i}. \tag{15}$$

Thus, we can write the posterior density function of γ and δ as

$$\pi^*(\gamma,\delta \mid \mathbf{x}) = \frac{\pi_3(\gamma,\delta,\mathbf{x})}{\int_0^{\infty}\int_0^{\infty} \pi_3(\gamma,\delta,\mathbf{x}) d\gamma d\delta}. \tag{16}$$

Now, assuming squared error loss function, the Bayes estimate of a function $h(\gamma,\delta)$ from the parameters is obtained as

$$E(h(\gamma,\delta) \mid \mathbf{x}) = \int_0^{\infty}\int_0^{\infty} \pi^*(\gamma,\delta \mid \mathbf{x})\, h(\gamma,\delta) d\gamma d\delta. \tag{17}$$

Since the posterior density function $\pi^*(\gamma,\delta \mid \mathbf{x})$ has a complex form, deriving a nice closed form for the Bayes estimate of $h(\gamma,\delta)$ is difficult. Therefore, in the following, the approximate Bayes estimates are calculated using Tierney and Kadane's procedure.
Setting

$$F(\gamma,\delta) = \frac{1}{n}\ln \pi_3(\gamma,\delta,\mathbf{x}) \quad \text{and} \quad F^*(\gamma,\delta) = F(\gamma,\delta) + \frac{1}{n}\ln h(\gamma,\delta),$$

the expression in (17) can be rewritten as

$$E(h(\gamma,\delta) \mid \mathbf{x}) = \frac{\int_0^\infty \int_0^\infty e^{nF^*(\gamma,\delta)} d\gamma d\delta}{\int_0^\infty \int_0^\infty e^{nF(\gamma,\delta)} d\gamma d\delta}. \tag{18}$$

Following Tierney and Kadane [24], equation (18) can be approximated as the following form:

$$\hat{h}_{BT}(\gamma,\delta) = \left[\frac{\det \Psi^*}{\det \Psi}\right]^{1/2} \exp\left\{n\left[F^*(\bar{\gamma}^*,\bar{\delta}^*) - F(\bar{\gamma},\bar{\delta})\right]\right\}, \tag{19}$$

where $(\bar{\gamma}^*,\bar{\delta}^*)$ and $(\bar{\gamma},\bar{\delta})$ maximize $F^*(\gamma,\delta)$ and $F(\gamma,\delta)$, respectively, and Ψ^* and Ψ are minus the inverse Hessians of $F^*(\gamma,\delta)$ and $F(\gamma,\delta)$ at $(\bar{\gamma}^*,\bar{\delta}^*)$ and $(\bar{\gamma},\bar{\delta})$, respectively.
In our case

$$\begin{aligned} F(\gamma,\delta) &= \frac{1}{n}\{c + (d+a_1-1)\log\gamma - \gamma b_1 + (2d+a_2-1)\log\delta \\ &\quad -d\log(\delta+1) - \delta\sum_{i=1}^{d} x_{(i)}^{\gamma}(1+U_i) \\ &\quad + \sum_{i=1}^{d}\left[\log(1+x_{(i)}^{\gamma}) + (\gamma-1)\log x_{(i)}\right] + \sum_{i=1}^{d} U_i \log\left(1+\frac{\delta}{\delta+1}x_{(i)}^{\gamma}\right)\} \end{aligned} \tag{20}$$

where c does not depend on γ and δ. Therefore, $(\bar{\gamma},\bar{\delta})$ can be derived from the equations

$$\begin{aligned} \frac{\partial}{\partial\gamma}F(\gamma,\delta) &= \frac{1}{n}\{\frac{d+a_1-1}{\gamma} - b_1 + \sum_{i=1}^{d}\log x_{(i)} - \delta\sum_{i=1}^{d} x_{(i)}^{\gamma}\log x_{(i)}(1+U_i) \\ &\quad + \sum_{i=1}^{d}\frac{x_{(i)}^{\gamma}\log x_{(i)}}{1+x_{(i)}^{\gamma}} + \sum_{i=1}^{n} U_i\frac{\delta x_{(i)}^{\gamma}\log x_{(i)}}{\delta+1+\delta x_{(i)}^{\gamma}}\} = 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\delta}F(\gamma,\delta) &= \frac{1}{n}\left\{\frac{2d+a_2-1}{\delta} - \frac{d}{\delta+1} - b_2 - \sum_{i=1}^{d} x_{(i)}^{\gamma}(1+U_i) + \sum_{i=1}^{d} U_i\frac{x_{(i)}^{\gamma}}{(\delta+1)^2+\delta(\delta+1)x_{(i)}^{\gamma}}\right\} \\ &= 0. \end{aligned}$$

Then, by using the second order derivatives of $F(\gamma,\delta)$, the determinant of the negative of the inverse Hessian of $H(\gamma,\delta)$ at $(\bar{\gamma},\bar{\delta})$ is given by $\det\Psi = (F_{11}F_{22} - F_{12}^2)^{-1}$ where

$$\begin{aligned} F_{11} &= \frac{1}{n}\{-\frac{d+a_1-1}{\bar{\gamma}^2} - \delta\sum_{i=1}^{d} x_{(i)}^{\bar{\gamma}}(\log x_{(i)})^2(1+U_i) \\ &\quad + \sum_{i=1}^{d}[\frac{x_{(i)}^{\bar{\gamma}}(\log x_{(i)})^2}{(1+x_{(i)}^{\bar{\gamma}})^2} + U_i\frac{\bar{\delta}x_{(i)}^{\bar{\gamma}}(\log x_{(i)})^2}{(\bar{\delta}+1)(1+\frac{\bar{\delta}}{\bar{\delta}+1}x_{(i)}^{\gamma})^2}]\}, \end{aligned}$$

$$F_{12} = \frac{1}{n}\{-\sum_{i=1}^{d} x_{(i)}^{\bar{\gamma}}\log x_{(i)}(1+U_i) + \sum_{i=1}^{d} U_i x_{(i)}^{\bar{\gamma}}\log x_{(i)}\frac{1}{(\bar{\delta}+1+\bar{\delta}x_{(i)}^{\bar{\gamma}})^2}\},$$

$$F_{22}=\frac{1}{n}\{-\frac{2d+a_2-1}{\bar{\delta}^2}+\frac{d}{(\bar{\delta}+1)^2}-\sum_{i=1}^{d}U_i x_{(i)}^{\bar{\gamma}}\frac{2(\bar{\delta}+1)+(2\bar{\delta}+1)x_{(i)}^{\bar{\gamma}}}{((\bar{\delta}+1)^2+\bar{\delta}(\bar{\delta}+1)x_{(i)}^{\bar{\gamma}})^2}\}.$$

Now, for computing the estimate of γ under squared error loss function, let $h(\gamma,\delta)=\gamma$. Thus, we have

$$\begin{aligned}F^{1*}(\gamma,\delta) &= \frac{1}{n}\{c+(d+a_1)\log\gamma-\gamma b_1+(2d+a_2-1)\log\delta \\ &\quad -d\log(\delta+1)-\delta\sum_{i=1}^{d}x_{(i)}^{\gamma}(1+U_i) \\ &\quad +\sum_{i=1}^{d}\left[\log(1+x_{(i)}^{\gamma})+(\gamma-1)\log x_{(i)}\right]+\sum_{i=1}^{d}U_i\log\left(1+\frac{\delta}{\delta+1}x_{(i)}^{\gamma}\right)\}\end{aligned} \tag{21}$$

and $(\bar{\gamma}^*,\bar{\delta}^*)$ are computed from the following system of equations:

$$\begin{aligned}\frac{\partial}{\partial\gamma}F^{1*}(\gamma,\delta) &= \frac{1}{n}\{\frac{d+a_1}{\gamma}-b_1+\sum_{i=1}^{d}\log x_{(i)}-\delta\sum_{i=1}^{d}x_{(i)}^{\gamma}\log x_{(i)}(1+U_i) \\ &\quad +\sum_{i=1}^{d}\frac{x_{(i)}^{\gamma}\log x_{(i)}}{1+x_{(i)}^{\gamma}}+\sum_{i=1}^{n}U_i\frac{\delta x_{(i)}^{\gamma}\log x_{(i)}}{\delta+1+\delta x_{(i)}^{\gamma}}\}=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\delta}F^{1*}(\gamma,\delta) &= \frac{1}{n}\left\{\frac{2d+a_2-1}{\delta}-\frac{d}{\delta+1}-b_2-\sum_{i=1}^{d}x_{(i)}^{\gamma}(1+U_i)\right. \\ &\quad \left.+\sum_{i=1}^{d}U_i\frac{x_{(i)}^{\gamma}}{(\delta+1)^2+\delta(\delta+1)x_{(i)}^{\gamma}}\right\}=0.\end{aligned}$$

Moreover, calculating the second order derivative of $F^{1*}(\gamma,\delta)$ at $(\bar{\gamma}^*,\bar{\delta}^*)$, we obtain

$$\begin{aligned}F_{11}^{1*} &= \frac{1}{n}\{-\frac{d+a_1}{(\bar{\gamma}^*)^2}-\delta\sum_{i=1}^{d}x_{(i)}^{\bar{\gamma}^*}(\log x_{(i)})^2(1+U_i) \\ &\quad +\sum_{i=1}^{d}[\frac{x_{(i)}^{\bar{\gamma}^*}(\log x_{(i)})^2}{(1+x_{(i)}^{\bar{\gamma}^*})^2}+U_i\frac{\bar{\delta}^* x_{(i)}^{\bar{\gamma}^*}(\log x_{(i)})^2}{(\bar{\delta}^*+1)(1+\frac{\bar{\delta}^*}{\bar{\delta}^*+1}x_{(i)}^{\bar{\gamma}^*})^2}]\},\end{aligned}$$

$$F_{12}^{1*}=\frac{1}{n}\{-\sum_{i=1}^{d}x_{(i)}^{\bar{\gamma}^*}\log x_{(i)}(1+U_i)+\sum_{i=1}^{d}U_i x_{(i)}^{\bar{\gamma}^*}\log x_{(i)}\frac{1}{(\bar{\delta}^*+1+\bar{\delta}^* x_{(i)}^{\bar{\gamma}^*})^2}\},$$

$$F_{22}^{1*}=\frac{1}{n}\{-\frac{2d+a_2-1}{(\bar{\delta}^*)^2}+\frac{d}{(\bar{\delta}^*+1)^2}-\sum_{i=1}^{d}U_i x_{(i)}^{\bar{\gamma}^*}\frac{2(\bar{\delta}^*+1)+(2\bar{\delta}^*+1)x_{(i)}^{\bar{\gamma}^*}}{((\bar{\delta}^*+1)^2+\bar{\delta}^*(\bar{\delta}^*+1)x_{(i)}^{\bar{\gamma}^*})^2}\}.$$

and hence $\det\Psi^{1*}=(F_{11}^{1*}F_{22}^{1*}-(F_{12}^{1*})^2)^{-1}$. Therefore, the Bayes estimate of γ becomes

$$\hat{\gamma}_{BT}=\left[\frac{\det\Psi^{1*}}{\det\Psi}\right]^{1/2}\exp\left\{n\left[F^{1*}(\bar{\gamma}^*,\bar{\delta}^*)-F(\bar{\gamma},\bar{\delta})\right]\right\}. \tag{22}$$

Following the same arguments with $h(\gamma,\delta)=\delta$ in $F^*(\gamma,\delta)$, $\hat{\delta}_{BT}$ can then be obtained straightforwardly.

3. Inference for the stress-strength reliability

3.1. MLE of R

Suppose that X and Y are random variables in the stress-strength model that are independently distributed as $PL(\gamma,\delta)$ and $PL(\gamma,\eta)$, respectively. Our quantity of interest is the parameter $R=P(X>Y)$ that is derived as (see [10]):

$$R = \frac{\eta^2}{\eta+1}\left(\frac{2\delta+1}{(\delta+1)(\delta+\eta)^2}+\frac{1}{\delta+\eta}+\frac{2\delta}{(\delta+1)(\delta+\eta)^3}\right). \tag{23}$$

In order to compute the maximum likelihood of the parameter R, we need to compute the MLEs of γ, δ and η. Let $\mathbf{x}=(x_{(1)},...,x_{(d_1)})$ be a PTII censored sample from $PL(\gamma,\delta)$ based on censoring scheme $(U_1,...,U_{d_1})$ and $\mathbf{y}=(y_{(1)},...,y_{(d_2)})$ be a PTII censored sample from $PL(\gamma,\eta)$ based on censoring scheme $(V_1,...,V_{d_2})$. Then, the log-likelihood function of the parameters γ, δ and η (ignoring the constant terms) becomes

$$\begin{aligned}\mathcal{L}(\gamma,\delta,\eta;\mathbf{x},\mathbf{y}) &= (d_1+d_2)\log\gamma + d_1\log\left(\frac{\delta^2}{\delta+1}\right) - \delta\sum_{i=1}^{d_1} x_{(i)}^{\gamma}(1+U_i)\\ &+\sum_{i=1}^{d_1}\left[\log(1+x_{(i)}^{\gamma})+(\gamma-1)\log x_{(i)}\right]+\sum_{i=1}^{d_1}U_i\log\left(1+\frac{\delta}{\delta+1}x_{(i)}^{\gamma}\right)\\ &+d_2\log\left(\frac{\eta^2}{\eta+1}\right)-\eta\sum_{j=1}^{d_2}y_{(j)}^{\gamma}(1+V_j)\\ &+\sum_{j=1}^{d_2}\left[\log(1+y_{(j)}^{\gamma})+(\gamma-1)\log y_{(j)}\right]+\sum_{j=1}^{d_2}V_j\log\left(1+\frac{\eta}{\eta+1}v_{(j)}^{\gamma}\right).\end{aligned} \tag{24}$$

The ML estimates of the parameters γ, δ and η, say $\hat{\gamma}$, $\hat{\delta}$ and $\hat{\eta}$, are computed from the system of equations

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\gamma} &= \frac{d_1+d_2}{\gamma}+\sum_{i=1}^{d_1}\log x_{(i)}-\delta\sum_{i=1}^{d_1}x_{(i)}^{\gamma}\log x_{(i)}(1+U_i)\\ &+\sum_{i=1}^{d_1}\frac{x_{(i)}^{\gamma}\log x_{(i)}}{1+x_{(i)}^{\gamma}}+\sum_{i=1}^{d_1}U_i\frac{\delta x_{(i)}^{\gamma}\log x_{(i)}}{\delta+1+\delta x_{(i)}^{\gamma}}\\ &+\sum_{j=1}^{d_2}\log y_{(j)}-\eta\sum_{j=1}^{d_2}y_{(j)}^{\gamma}\log y_{(j)}(1+V_j)\\ &+\sum_{j=1}^{d_2}\frac{y_{(j)}^{\gamma}\log y_{(j)}}{1+y_{(j)}^{\gamma}}+\sum_{j=1}^{d_2}V_j\frac{\eta y_{(j)}^{\gamma}\log y_{(j)}}{\eta+1+\eta y_{(j)}^{\gamma}}=0,\end{aligned} \tag{25}$$

$$\frac{\partial\mathcal{L}}{\partial\delta}=\frac{d_1(\delta+2)}{\delta(\delta+1)}-\sum_{i=1}^{d_1}x_{(i)}^{\gamma}(1+U_i)+\sum_{i=1}^{d_1}U_i\frac{x_{(i)}^{\gamma}}{(\delta+1)^2+\delta(\delta+1)x_{(i)}^{\gamma}}=0 \tag{26}$$

and

$$\frac{\partial\mathcal{L}}{\partial\eta}=\frac{d_2(\eta+2)}{\eta(\eta+1)}-\sum_{j=1}^{d_2}y_{(j)}^{\gamma}(1+V_j)+\sum_{j=1}^{d_2}V_j\frac{y_{(j)}^{\gamma}}{(\eta+1)^2+\eta(\eta+1)x_{(i)}^{\gamma}}=0. \tag{27}$$

Then, by using the invariance property of the MLEs, the maximum likelihood estimate of $R\equiv R(\delta,\eta)$ is obtained as $R(\hat{\delta},\hat{\eta})$. Moreover, from the asymptotic normality of the MLEs (see

[23]), $\hat{R}$ is asymptotically normal with mean R and asymptotic variance

$$\sigma_R^2 = \left\{ \tau_{11} \left(\frac{\partial R}{\partial \delta} \right)^2 + \tau_{22} \left(\frac{\partial R}{\partial \eta} \right)^2 + 2\tau_{12} \left(\frac{\partial R}{\partial \delta} \right) \left(\frac{\partial R}{\partial \eta} \right) \right\}$$

where

$$\frac{\partial R}{\partial \delta} = \frac{-\delta\eta^2[\delta^3 + 2\delta^2(\eta+3) + \delta(\eta+2)(\eta+6) + 2(\eta^2+3\eta+3)]}{(\eta+1)(\delta+1)^2(\delta+\eta)^4},$$

$$\frac{\partial R}{\partial \eta} = \frac{\delta^2\eta[6 + \delta^2(\eta+2) + 2\delta(\eta+1)(\eta+3) + \eta(\eta^2+6\eta+12)]}{(\delta+1)(\eta+1)^2(\delta+\eta)^4},$$

and τ_{ij}, $i = 1,2,3$, are the elements of the negative of the matrix

$$\begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial \delta^2} & \frac{\partial^2 \mathcal{L}}{\partial \delta \partial \eta} & \frac{\partial^2 \mathcal{L}}{\partial \delta \partial \gamma} \\ \frac{\partial^2 \mathcal{L}}{\partial \eta \partial \delta} & \frac{\partial^2 \mathcal{L}}{\partial \eta^2} & \frac{\partial^2 \mathcal{L}}{\partial \eta \partial \gamma} \\ \frac{\partial^2 \mathcal{L}}{\partial \gamma \partial \delta} & \frac{\partial^2 \mathcal{L}}{\partial \gamma \partial \eta} & \frac{\partial^2 \mathcal{L}}{\partial \gamma^2} \end{bmatrix}^{-1}. \tag{28}$$

Now, by using (24), we obtain

$$\begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial \gamma^2} = & -\frac{d_1+d_2}{\gamma^2} - \delta \sum_{i=1}^{d_1} x_{(i)}^{\gamma} (\log x_{(i)})^2 (1+U_i) \\ & + \sum_{i=1}^{d_1} [\frac{x_{(i)}^{\gamma} (\log x_{(i)})^2}{(1+x_{(i)}^{\gamma})^2} + U_i \frac{\delta x_{(i)}^{\gamma} (\log x_{(i)})^2}{(\delta+1)(1+\frac{\delta}{\delta+1} x_{(i)}^{\gamma})^2}] \\ & -\eta \sum_{j=1}^{d_2} y_{(j)}^{\gamma} (\log y_{(j)})^2 (1+V_j) \\ & + \sum_{j=1}^{d_2} [\frac{y_{(j)}^{\gamma} (\log y_{(j)})^2}{(1+y_{(j)}^{\gamma})^2} + V_j \frac{\eta y_{(j)}^{\gamma} (\log y_{(j)})^2}{(\eta+1)(1+\frac{\eta}{\eta+1} y_{(j)}^{\gamma})^2}], \end{aligned}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \delta^2} = -\frac{2d_1}{\delta^2} + \frac{d_1}{(\delta+1)^2} - \sum_{i=1}^{d_1} U_i x_{(i)}^{\gamma} \frac{2(\delta+1) + (2\delta+1)x_{(i)}^{\gamma}}{((\delta+1)^2 + \delta(\delta+1)x_{(i)}^{\gamma})^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \eta^2} = -\frac{2d_2}{\eta^2} + \frac{d_2}{(\eta+1)^2} - \sum_{j=1}^{d_2} V_j y_{(j)}^{\gamma} \frac{2(\eta+1) + (2\eta+1)y_{(j)}^{\gamma}}{((\eta+1)^2 + \eta(\eta+1)y_{(j)}^{\gamma})^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \gamma \partial \delta} = -\sum_{i=1}^{d_1} x_{(i)}^{\gamma} \log x_{(i)} (1+U_i) + \sum_{i=1}^{d_1} U_i x_{(i)}^{\gamma} \log x_{(i)} \frac{1}{(\delta+1+\delta x_{(i)}^{\gamma})^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \gamma \partial \eta} = -\sum_{j=1}^{d_2} y_{(j)}^{\gamma} \log y_{(j)} (1+V_j) + \sum_{j=1}^{d_2} V_j y_{(j)}^{\gamma} \log y_{(j)} \frac{1}{(\delta+1+\delta y_{(j)}^{\gamma})^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \delta \partial \eta} = \frac{\partial^2 L}{\partial \eta \partial \delta} = 0.$$

Thus, the $100(1-\alpha)\%$ asymptotic CI of the reliability R can be derived as

$$\left(\frac{e^L}{1+e^L}, \frac{e^U}{1+e^U} \right) \tag{29}$$

where

$$(L,U) \equiv \log\left(\frac{\hat{R}}{1-\hat{R}}\right) \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\hat{\sigma}_R^2}}{\hat{R}(1-\hat{R})}. \tag{30}$$

Table 1: *Different estimates of the parameter γ for various sample sizes when $(\gamma, \delta) = (2, 1)$.*

n	d	Scheme	MLE		Bayes		Confidence interval	
			AV	MSE	AV	MSE	AL	CP
20	12	(0,...,0,8)	2.2134	0.3406	2.2186	0.3619	2.2976	0.9261
		(8,0...,0)	2.2463	0.5812	2.2509	0.5875	2.8365	0.9232
		(0,8,0,...,0)	2.2377	0.5685	2.2311	0.5713	2.8121	0.9238
20	15	(0,...,0,5)	2.1023	0.2818	2.1058	0.2831	1.8658	0.9317
		(5,0...,0)	2.2339	0.3416	2.2354	0.3427	2.5813	0.9306
		(0,5,0,...,0)	2.1961	0.3225	2.1975	0.3240	2.5762	0.9311
20	18	(0,...,0,2)	2.0761	0.1938	2.0782	0.1947	1.5696	0.9359
		(2,0...,0)	2.1874	0.2773	2.1876	0.2785	2.3375	0.9346
		(0,2,0,...,0)	2.1325	0.2619	2.1338	0.2623	2.3129	0.9352
30	15	(0,...,0,15)	2.0830	0.2310	2.0861	0.2341	1.7589	0.9317
		(15,0...,0)	2.2116	0.3341	2.2174	0.3352	2.3436	0.9302
		(0,15,0,...,0)	2.1078	0.3196	2.1083	0.3197	2.3379	0.9305
30	20	(0,...,0,10)	2.0322	0.1875	2.0328	0.1878	1.6136	0.9321
		(10,0...,0)	2.1371	0.2918	2.1395	0.2925	2.3355	0.9308
		(0,10,0,...,0)	2.0916	0.2641	2.0937	0.2644	2.3278	0.9311
30	25	(0,...,0,5)	2.0208	0.1234	2.0214	0.1238	1.3373	0.9432
		(5,0...,0)	2.0864	0.2175	2.0873	0.2189	2.1897	0.9409
		(0,5,0,...,0)	2.0738	0.1983	2.0749	0.1984	2.1736	0.9414
50	30	(0,...,0,20)	2.0192	0.1185	2.0205	0.1187	1.3118	0.9373
		(20,0...,0)	2.0775	0.1931	2.0782	0.1946	2.1671	0.9358
		(0,20,0,...,0)	2.0368	0.1857	2.0391	0.1874	2.1503	0.9360
50	35	(0,...,0,15)	2.0143	0.0902	2.0151	0.0908	1.1579	0.9461
		(15,0...,0)	2.0560	0.1428	2.0568	0.1434	2.1486	0.9432
		(0,15,0,...,0)	2.0229	0.1297	2.0247	0.1302	2.1338	0.9438
50	45	(0,...,0,5)	2.0113	0.0606	2.0128	0.0618	0.9576	0.9467
		(5,0...,0)	2.0416	0.1089	2.0431	0.1097	1.1945	0.9440
		(0,5,0,...,0)	2.0177	0.0926	2.0190	0.0934	1.1871	0.9443

3.2. Bayes estimate of R

This section focuses on Bayesian estimation of the reliability parameter R as well as the corresponding HPD credible interval when the prior assigns to γ and δ the gamma model with the pdfs given by (14) and takes η to be independent of γ and δ with the prior

$$\pi_3(\eta; a_3, b_3) \propto \eta^{a_3-1}\, e^{-\eta b_3}, \qquad \eta > 0,\ a_3 > 0,\ b_3 > 0. \tag{31}$$

First, by using (14), (24) and (31), the joint density function of γ, δ, η and the *data* can be written as

$$\begin{aligned}\pi_4(\gamma,\delta,\eta,;\mathbf{x},\mathbf{y}) \quad \propto \quad & \frac{\gamma^{d_1+d_2+a_1-1}\, e^{-\gamma b_1} \delta^{2d_1+a_2-1}}{(\delta+1)^{d_1}}\, e^{-\delta(b_2+\sum\limits_{i=1}^{d_1} x_{(i)}^{\gamma}(1+U_i))} \\ & \frac{\eta^{2d_2+a_3-1}}{(\eta+1)^{d_2}}\, e^{-\eta(b_3+\sum\limits_{j=1}^{d_2} y_{(j)}^{\gamma}(1+V_j))} \prod_{i=1}^{d_1}(1+x_{(i)}^{\gamma})x_{(i)}^{\gamma-1}\left(1+\frac{\delta}{\delta+1}x_{(i)}^{\gamma}\right)^{U_i} \\ & \prod_{j=1}^{d_2}(1+y_{(j)}^{\gamma})y_{(j)}^{\gamma-1}\left(1+\frac{\eta}{\eta+1}y_{(j)}^{\gamma}\right)^{V_j}.\end{aligned} \tag{32}$$

Table 2: *Different estimates of the parameter γ for various sample sizes when $(\gamma, \delta) = (2, 0.5)$.*

n	d	Scheme	MLE		Bayes		Confidence interval	
			AV	MSE	AV	MSE	AL	CP
20	12	(0,...,0,8)	2.0431	0.2782	2.0438	0.2795	1.9530	0.9212
		(8,0...,0)	2.1251	0.4137	2.1279	0.4163	2.2571	0.9207
		(0,8,0,...,0)	2.0983	0.4062	2.1016	0.4078	2.2429	0.9210
20	15	(0,...,0,5)	1.9698	0.2119	1.6759	0.2127	1.7483	0.9326
		(5,0...,0)	2.1073	0.3376	2.1079	0.3378	2.23116	0.9311
		(0,5,0,...,0)	2.0891	0.3198	2.0893	0.3214	2.2164	0.9315
20	18	(0,...,0,2)	2.0116	0.1490	2.0129	0.1493	1.4368	0.9369
		(2,0...,0)	2.0852	0.2618	2.0873	0.2637	2.2103	0.9347
		(0,2,0,...,0)	2.0717	0.2560	2.0729	0.2584	2.1852	0.9353
30	15	(0,...,0,15)	1.9774	0.2057	1.9762	0.2069	1.6771	0.9312
		(15,0...,0)	2.0965	0.3284	2.0988	0.3287	2.2196	0.9303
		(0,15,0,...,0)	2.0827	0.3095	2.0844	0.3116	2.1975	0.9309
30	20	(0,...,0,10)	1.9813	0.1371	1.9803	0.1378	1.4097	0.9322
		(10,0...,0)	2.0817	0.2841	2.0835	0.2867	2.1678	0.9310
		(0,10,0,...,0)	2.0736	0.2537	2.0740	0.2558	2.1513	0.9314
30	25	(0,...,0,5)	1.9837	0.0970	1.9821	0.0973	1.2195	0.9438
		(5,0...,0)	2.0705	0.2118	2.0723	0.2140	2.1431	0.9413
		(0,5,0,...,0)	2.0591	0.1956	2.0595	0.1973	2.1108	0.9420
50	30	(0,...,0,20)	1.9792	0.1088	1.9766	0.1096	1.1414	0.9315
		(20,0...,0)	2.0633	0.1837	2.0635	0.1845	2.1570	0.9306
		(0,20,0,...,0)	2.0485	0.1791	2.0492	0.1793	2.1206	0.9311
50	35	(0,...,0,15)	1.9839	0.0837	1.9826	0.0846	1.0388	0.9349
		(15,0...,0)	2.0518	0.1398	2.0540	0.1403	2.1148	0.9328
		(0,15,0,...,0)	2.0409	0.1134	2.0418	0.1149	2.1953	0.9340
50	45	(0,...,0,5)	1.9915	0.0519	1.9913	0.0528	0.8762	0.9418
		(5,0...,0)	2.0478	0.1034	2.0496	0.1047	1.1826	0.9411
		(0,5,0,...,0)	2.0362	0.0892	2.0368	0.0907	1.1644	0.9417

Thus, the Bayes estimate of the reliability parameter against squared error loss function becomes

$$\hat{R}_{SE} = E(R \mid \mathbf{x}, \mathbf{y}) = \frac{\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty R\pi_4(\gamma, \delta, \eta, ; \mathbf{x}, \mathbf{y}) d\delta d\eta d\gamma}{\int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \pi_4(\gamma, \delta, \eta, ; \mathbf{x}, \mathbf{y}) d\delta d\eta d\gamma}. \tag{33}$$

It is observed that the Bayes estimate of R are involved the ratio of two integrals for which simplified closed forms can not be obtained. Therefore, in the following, we adopt Gibbs sampling method to extract random samples from the conditional densities of the parameters and use them to compute the Bayes estimate and HPD credible interval of R.

From (32), the conditional posterior densities of γ, δ and η can be extracted, respectively, as

$$\begin{aligned} \pi_1^*(\gamma \mid \delta, \eta, \mathbf{x}, \mathbf{y}) \quad \propto \quad & \pi_1(\gamma; d_1 + d_2 + a_1, b_1)\, e^{-\delta \sum\limits_{i=1}^{d_1} x_{(i)}^{\gamma}(1+U_i)} \prod_{i=1}^{d_1} (1 + x_{(i)}^{\gamma}) x_{(i)}^{\gamma-1} \left(1 + \frac{\delta}{\delta+1} x_{(i)}^{\gamma}\right)^{U_i} \\ & e^{-\eta \sum\limits_{j=1}^{d_2} y_{(j)}^{\gamma}(1+V_j)} \prod_{j=1}^{d_2} (1 + y_{(j)}^{\gamma}) y_{(j)}^{\gamma-1} \left(1 + \frac{\eta}{\eta+1} y_{(j)}^{\gamma}\right)^{V_j}, \end{aligned} \tag{34}$$

Table 3: *Different estimates of the parameter δ for various sample sizes when $(\gamma, \delta) = (2, 1)$.*

n	d	Scheme	MLE		Bayes		Confidence interval	
			AV	MSE	AV	MSE	AL	CP
20	12	(0,...,0,8)	0.9925	0.0793	0.9914	0.0824	0.9167	0.9241
		(8,0...,0)	0.9813	0.1137	0.9802	0.1141	0.9814	0.9225
		(0,8,0,...,0)	0.9841	0.1064	0.9819	0.1097	0.9732	0.9228
20	15	(0,...,0,5)	0.9947	0.0533	0.9923	0.0554	0.8280	0.9316
		(5,0...,0)	0.6832	0.1085	0.9815	0.1087	0.9328	0.9305
		(0,5,0,...,0)	0.9866	0.0936	0.9860	0.0952	0.9215	0.9311
20	18	(0,...,0,2)	0.9965	0.0455	0.9957	0.0463	0.8045	0.9374
		(2,0...,0)	0.9873	0.0872	0.9864	0.0879	0.8906	0.9357
		(0,2,0,...,0)	0.9911	0.0810	0.9897	0.0831	0.8755	0.9362
30	15	(0,...,0,15)	0.9951	0.0475	0.9930	0.0478	0.8113	0.9385
		(15,0...,0)	0.9856	0.0914	0.9852	0.0922	0.8842	0.9339
		(0,15,0,...,0)	0.9892	0.0851	0.9903	0.0858	0.8731	0.9350
30	20	(0,...,0,10)	0.9960	0.0307	0.9938	0.0319	0.6693	0.9498
		(10,0...,0)	0.9893	0.0836	0.9861	0.0874	0.8371	0.9347
		(0,10,0,...,0)	0.9907	0.0768	0.9905	0.0791	0.8219	0.9358
30	25	(0,...,0,5)	0.9978	0.0276	0.9956	0.0280	0.6523	0.9415
		(5,0...,0)	0.9915	0.0711	0.9807	0.0725	0.8112	0.9383
		(0,5,0,...,0)	0.9921	0.0547	0.9913	0.0569	0.7863	0.9392
50	30	(0,...,0,20)	0.9960	0.0211	0.9952	0.0216	0.5222	0.9403
		(20,0...,0)	0.9904	0.0766	0.9883	0.0790	0.7460	0.9376
		(0,20,0,...,0)	0.9914	0.0631	0.9809	0.0657	0.7291	0.9380
50	35	(0,...,0,15)	0.9973	0.0157	0.9934	0.0168	0.5063	0.9417
		(15,0...,0)	0.9920	0.0519	0.9913	0.0523	0.7186	0.9391
		(0,15,0,...,0)	0.9928	0.0469	0.9917	0.0475	0.7033	0.9394
50	45	(0,...,0,5)	0.9982	0.0150	0.9975	0.0152	0.5003	0.9438
		(5,0...,0)	0.9937	0.0471	0.9924	0.0485	0.6719	0.9407
		(0,5,0,...,0)	0.9946	0.0338	0.9937	0.0346	0.6548	0.9411

$$\pi_2^*(\delta \mid \gamma, \mathbf{x}, \mathbf{y}) \propto \pi_2(\delta; 2d_1 + a_2, b_2 + \sum_{i=1}^{d_1} x_{(i)}^{\gamma}(1 + U_i)) \, \frac{1}{(\delta+1)^{d_1}} \prod_{i=1}^{d_1} \left(1 + \frac{\delta}{\delta+1} x_{(i)}^{\gamma}\right)^{U_i} \tag{35}$$

and

$$\pi_3^*(\eta \mid \gamma, \mathbf{x}, \mathbf{y}) \propto \pi_2(\delta; 2d_2 + a_3, b_3 + \sum_{j=1}^{d_2} y_{(j)}^{\gamma}(1 + V_j)) \, \frac{1}{(\eta+1)^{d_2}} \prod_{j=1}^{d_1} \left(1 + \frac{\eta}{\eta+1} y_{(j)}^{\gamma}\right)^{V_j}. \tag{36}$$

Since the well known distributions are not available for conditional densities in (34)-(36), direct sampling from these distributions is not possible. We can approximate a posterior density function by normal distribution if the density be unimodal and roughly symmetric (see Gelman et al. [8]). In our case, we observed that the plot of posterior densities of γ, δ and η are similar to normal distribution (not reported here). Therefore, in the following algorithm, we employ Metropolis-Hastings (M-H) technique with the proposed normal distribution to generate samples from conditional densities.

1) Let initial values of the parameters to be $(\gamma^0, \delta^0, \eta^0)$ and set $l = 1$.
2) Considering the proposed distribution $q(\gamma) \equiv N(\gamma^{l-1}, \tau_{33})$ for the M-H method, generate γ^l , from $\pi_1^*(\gamma \mid \delta^{l-1}, \eta^{l-1}, \mathbf{x}, \mathbf{y})$.
3) Generate δ^l, from $\pi_2^*(\delta \mid \gamma^l, \mathbf{x}, \mathbf{y})$ using M-H method with the proposed distribution $q(\delta) \equiv N(\delta^{l-1}, \tau_{11})$.

Table 4: *Different estimates of the parameter δ for various sample sizes when $(\gamma, \delta) = (2, 0.5)$.*

n	d	Scheme	MLE		Bayes		Confidence interval	
			AV	MSE	AV	MSE	AL	CP
20	12	(0,...,0,8)	0.4810	0.0217	0.4782	0.0209	0.5393	0.9221
		(8,0...,0)	0.4729	0.0346	0.4711	0.0369	0.6748	0.9216
		(0,8,0,...,0)	0.4765	0.0317	0.4726	0.0323	0.6513	0.9220
20	15	(0,...,0,5)	0.4862	0.0188	0.4855	0.0195	0.5364	0.9287
		(5,0...,0)	0.4793	0.0309	0.4764	0.0341	0.6472	0.9254
		(0,5,0,...,0)	0.4850	0.0274	0.4819	0.0280	0.6391	0.9263
20	18	(0,...,0,2)	0.5037	0.0163	0.5052	0.0169	0.5327	0.9328
		(2,0...,0)	0.4866	0.0250	0.4861	0.0278	0.6118	0.9308
		(0,2,0,...,0)	0.4907	0.0239	0.4892	0.0254	0.5975	0.9315
30	15	(0,...,0,15)	0.4895	0.0126	0.4873	0.0149	0.5281	0.9321
		(15,0...,0)	0.4811	0.0287	0.4806	0.0293	0.6255	0.9296
		(0,15,0,...,0)	0.4829	0.0241	0.4814	0.0248	0.6194	0.9307
30	20	(0,...,0,10)	0.4936	0.0117	0.4917	0.0146	0.4737	0.9346
		(10,0...,0)	0.4874	0.0216	0.4860	0.0235	0.5914	0.9312
		(0,10,0,...,0)	0.4891	0.0194	0.4879	0.0206	0.5726	0.9317
30	25	(0,...,0,5)	0.5044	0.0105	0.5091	0.0109	0.4419	0.9385
		(5,0...,0)	0.4917	0.0183	0.4913	0.0187	0.5137	0.9357
		(0,5,0,...,0)	0.4926	0.0168	0.4922	0.0175	0.4975	0.9363
50	30	(0,...,0,20)	0.5080	0.0107	0.5103	0.0119	0.3529	0.9377
		(20,0...,0)	0.4855	0.0175	0.4854	0.0196	0.4816	0.9328
		(0,20,0,...,0)	0.4902	0.0159	0.4896	0.0171	0.4589	0.9336
50	35	(0,...,0,15)	0.4958	0.0090	0.4947	0.0093	0.3455	0.9389
		(15,0...,0)	0.5123	0.0144	0.5128	0.0177	0.4258	0.9352
		(0,15,0,...,0)	0.4920	0.0123	0.4917	0.0140	0.4177	0.9364
50	45	(0,...,0,5)	0.5033	0.0067	0.5046	0.0069	0.3398	0.9422
		(5,0...,0)	0.4923	0.0130	0.4912	0.0138	0.3941	0.9395
		(0,5,0,...,0)	0.4937	0.0108	0.4936	0.0114	0.3892	0.9413

4) Generate η^l, from $\pi_3^*(\eta \mid \gamma^l, \mathbf{x}, \mathbf{y})$ using M-H method with the proposed distribution $q(\eta) \equiv N(\eta^{l-1}, \tau_{22})$.

5) Compute R from (4) and set $l = l + 1$.

6) Repeat Steps 2-5, M times to get R^l for $l = 1, ..., M$.

By using the generated random samples from the above Gibbs technique, the approximate Bayes estimate of the reliability parameter R against squared error loss function becomes

$$\tilde{R} = \frac{1}{M} \sum_{l=1}^{M} R^l. \tag{37}$$

Also, let $R^{(1)} < ... < R^{(M)}$ be the ordered values of R^l for $l = 1, ..., M$. The HPD credible interval of R will be derived by selecting the interval with the shortest length through the following $100(1-\alpha)\%$ credible intervals of R:

$$(R^{(1)}, R^{((1-\alpha)M)}), ..., (R^{(\alpha M)}, R^{(M)}).$$

4. Simulation study

To evaluate the behaviour of the proposed estimators for various sample sizes, we performed extensive Monte Carlo simulations. The performance of the competitive estimates has been

Table 5: *Different estimates of the stress-strength parameter R for various sample sizes when* $(\gamma, \delta, \eta) = (2, 1, 1)$.

n_1, n_2	d_1, d_2	Scheme	MLE		Bayes		CI		CRI	
			AV	MSE	AV	MSE	AL	CP	AL	CP
20	12	(0,...,0,8)	0.4976	0.0127	0.4978	0.0136	0.3704	0.9318	0.3648	0.9312
		(8,0...,0)	0.4943	0.0156	0.4918	0.0178	0.3775	0.9302	0.3754	0.9267
		(0,8,0,...,0)	0.4961	0.0139	0.4937	0.0141	0.3716	0.9305	0.3690	0.9274
20	15	(0,...,0,5)	0.4986	0.0100	0.4952	0.0119	0.3352	0.9337	0.3325	0.9320
		(5,0...,0)	0.4967	0.0137	0.4938	0.0155	0.3419	0.9316	0.3408	0.9308
		(0,5,0,...,0)	0.4981	0.0120	0.4945	0.0127	0.3369	0.9317	0.3347	0.9314
20	18	(0,...,0,2)	0.4988	0.0084	0.4973	0.0089	0.3221	0.9352	0.3146	0.9342
		(2,0...,0)	0.4970	0.0116	0.4977	0.0128	0.3297	0.9328	0.3275	0.9326
		(0,2,0,...,0)	0.4985	0.0092	0.4961	0.0090	0.3228	0.9336	0.3218	0.9331
30	15	(0,...,0,15)	0.4978	0.0099	0.4986	0.0095	0.3478	0.9386	0.3421	0.9359
		(15,0...,0)	0.4955	0.0125	0.4942	0.0144	0.3507	0.9352	0.3472	0.9338
		(0,15,0,...,0)	0.4971	0.0108	0.4938	0.0107	0.3483	0.9358	0.3440	0.9340
30	20	(0,...,0,10)	0.4983	0.0073	0.4967	0.0076	0.3362	0.9407	0.3292	0.9380
		(10,0...,0)	0.4970	0.0107	0.4953	0.0093	0.3419	0.9365	0.3378	0.9347
		(0,10,0,...,0)	0.4978	0.0085	0.4972	0.0086	0.3393	0.9374	0.3314	0.9356
30	25	(0,...,0,5)	0.4992	0.0052	0.4993	0.0058	0.3047	0.9421	0.2982	0.9417
		(5,0...,0)	0.4982	0.0083	0.4971	0.0083	0.3120	0.9397	0.3102	0.9403
		(0,5,0,...,0)	0.4991	0.0054	0.4980	0.0051	0.3059	0.9409	0.3041	0.9407
50	30	(0,...,0,20)	0.4983	0.0039	0.4972	0.0032	0.2841	0.9441	0.2776	0.9417
		(20,0...,0)	0.4962	0.0071	0.4966	0.0083	0.2875	0.9423	0.2856	0.9403
		(0,20,0,...,0)	0.4981	0.0044	0.4982	0.0049	0.2849	0.9428	0.2814	0.9407
50	35	(0,...,0,15)	0.4991	0.0035	0.4963	0.0031	0.2621	0.9447	0.2605	0.9426
		(15,0...,0)	0.4965	0.0067	0.4974	0.0069	0.2689	0.9430	0.2657	0.9411
		(0,15,0,...,0)	0.4987	0.0036	0.4983	0.0036	0.2643	0.9432	0.2641	0.9414
50	45	(0,...,0,5)	0.4994	0.0026	0.4992	0.0025	0.2385	0.9468	0.2332	0.9461
		(5,0...,0)	0.4982	0.0029	0.4970	0.0033	0.2384	0.9439	0.2370	0.9425
		(0,5,0,...,0)	0.4993	0.0026	0.4985	0.0028	0.2366	0.9446	0.2351	0.9426

compared in terms of their average values (AV) and mean squared errors (MSE). In addition, the confidence intervals (CI) and HPD credible intervals (CRI) are compared on the basis of their average lengths and coverage percentages. The calculations are conducted using R 2.14.0 which is a common software package for statistical computing.

First, in order to compare the maximum likelihood and Bayesian procedures developed in Section 2, We have considered two sets of parameter values as $(\gamma, \delta) = (2, 1), (2, 0.5)$ and three sampling schemes

I: $(U_1, ..., U_d) = (0, ..., 0, n - d)$,

II: $(U_1, ..., U_d) = (n - d, 0, ..., 0)$

III: $(U_1, ..., U_d) = (0, n - d, 0, ..., 0)$

In each case, by employing the method of Balakrishnan and Sandhu [2], different random samples are generated from PL model and the ML estimates of the unknown parameters are obtained from the system of equations in (5) and (6). To obtain the Bayes estimates of γ and δ using Tierney and Kadane's approach, we assume that the hyper-parameters take values as 0.001 as suggested by Congdon [5]. Tables 1-4 present the AVs and MSEs of the estimates obtained from 10000 replications.

Further, for the generated samples, we have derived 95% confidence intervals and counted the ones that cover the correct value of a specific parameter. The number of such intervals divided by 10000 is reported as estimated coverage probabilities. For different sample sizes, the average

Table 6: *Different estimates of the stress-strength parameter R for various sample sizes when* $(\gamma, \delta, \eta) = (2, 0.2, 1)$.

n_1, n_2	d_1, d_2	Scheme	MLE		Bayes		CI		CRI	
			AV	MSE	AV	MSE	AL	CP	AL	CP
20	12	(0,...,0,8)	0.9260	0.0022	0.9244	0.0028	0.1728	0.9340	0.1676	0.9319
		(8,0...,0)	0.9289	0.0038	0.9273	0.0046	0.1756	0.9316	0.1732	0.9288
		(0,8,0,...,0)	0.9276	0.0025	0.9265	0.0027	0.1737	0.9332	0.1719	0.9294
20	15	(0,...,0,5)	0.9243	0.0015	0.9221	0.0019	0.1641	0.9381	0.1528	0.9347
		(5,0...,0)	0.9277	0.0032	0.9289	0.0031	0.1692	0.9350	0.1655	0.9326
		(0,5,0,...,0)	0.9253	0.0016	0.9255	0.0018	0.1650	0.9357	0.1637	0.9331
20	18	(0,...,0,2)	0.9222	0.0013	0.9230	0.0013	0.1519	0.9408	0.1492	0.9390
		(2,0...,0)	0.9227	0.0023	0.9241	0.0027	0.1563	0.9389	0.1535	0.9358
		(0,2,0,...,0)	0.9225	0.0014	0.9247	0.0014	0.1527	0.9394	0.1508	0.9362
30	15	(0,...,0,15)	0.9239	0.0017	0.9245	0.0016	0.1567	0.9412	0.1432	0.9386
		(15,0...,0)	0.9275	0.0026	0.9291	0.0032	0.1590	0.9390	0.1565	0.9379
		(0,15,0,...,0)	0.9264	0.0017	0.9258	0.0018	0.1574	0.9408	0.1546	0.9381
30	20	(0,...,0,10)	0.9227	0.0013	0.9174	0.0014	0.1431	0.9433	0.1327	0.9412
		(10,0...,0)	0.9261	0.0019	0.9266	0.0023	0.1466	0.9419	0.1449	0.9389
		(0,10,0,...,0)	0.9239	0.0014	0.9231	0.0014	0.1439	0.9423	0.1435	0.9403
30	25	(0,...,0,5)	0.918	0.0010	0.9207	0.0011	0.1256	0.9472	0.1240	0.9435
		(5,0...,0)	0.9203	0.0015	0.9225	0.0017	0.1278	0.9439	0.1269	0.9422
		(0,5,0,...,0)	0.9196	0.0011	0.9216	0.0012	0.1263	0.9446	0.1247	0.9427
50	30	(0,...,0,20)	0.9216	0.0009	0.9227	0.0010	0.1065	0.9419	0.1027	0.940
		(20,0...,0)	0.9241	0.0018	0.9233	0.0016	0.1093	0.9403	0.1064	0.9356
		(0,20,0,...,0)	0.9223	0.0013	0.9229	0.0014	0.1076	0.9407	0.1056	0.9378
50	35	(0,...,0,15)	0.9204	0.0008	0.9219	0.0009	0.1008	0.9430	0.0958	0.9416
		(15,0...,0)	0.9232	0.0011	0.9258	0.0013	0.1034	0.9412	0.1017	0.9405
		(0,15,0,...,0)	0.9217	0.0009	0.9213	0.0011	0.1016	0.9414	0.1005	0.9411
50	45	(0,...,0,5)	0.9179	0.0005	0.9275	0.0006	0.0978	0.9487	0.0923	0.9473
		(5,0...,0)	0.9192	0.0006	0.9210	0.0008	0.0991	0.9461	0.0975	0.9448
		(0,5,0,...,0)	0.9206	0.0006	0.9208	0.0006	0.0980	0.9464	0.0962	0.9457

lengths (AL) and coverage probabilities (CP) of the CIs are also provided in Tables 1-4.

It is observed from Tables 1-4 that, for each censoring scheme, the estimates computed from larger sample sizes have smaller MSEs as we expected. The estimates of the parameters computed using the Bayesian procedures and the MLEs yield similar results. Therefore, in this case, the maximum likelihood method is preferred since it has concise computations compared to the Tierney and Kadane's technique. It can be further observed that the asymptotic results of the MLEs have satisfactory performances and in most of the cases the CPs are close to the predetermined nominal level. Comparing the three different censoring schemes, we observe that the estimates computed over the first sampling scheme, corresponding to the well-known type II censored sampling, have better performances followed by schemes III and II, respectively.

Next, to assess the accuracy of the inferential procedures of the reliability parameter R, we generate PTII censored samples from PL distribution by considering two sets of values for the parameters γ,δ and η as $(\gamma, \delta, \eta) = (2, 1, 1), (2, 0.2, 1)$. With these choices of the parameter values, the true value of reliability R become 0.5 and 0.9182, respectively. We first obtain the ML estimates of the unknown parameters by using the log-likelihood function (24) and use them to compute the MLE of the reliability R from expression (23). Also, by using relation (29), we construct 95% confidence intervals of R and reported ALs and CPs computed over 10000 replications in Tables 5 and 6.

Moreover, we derive the approximate Bayes estimate and HPD credible interval of the

Table 7: *Point and interval estimations of the parameters γ and δ under different progressive type II censoring schemes for example 1.*

m	Scheme		MLE	Bayes	CI
51	(0^{*51})	γ	0.9467	0.9319	(0.7618,1.1317)
		δ	0.0093	0.0128	(0.0039,0.0196)
40	$(0^{*39},11)$	γ	1.0275	1.0007	(0.8027,1.3152)
		δ	0.0062	0.0079	(0.0014,0.0260)
40	$(0^{*34},1^{*5},6)$	γ	0.9996	0.9671	(0.7785,1.2835)
		δ	0.0066	0.0084	(0.0016,0.0272)
40	$(0^{*34},2^{*5},1)$	γ	0.9773	0.9519	(0.7592,1.2581)
		δ	0.0069	0.0087	(0.0017,0.0283)
30	$(0^{*29},21)$	γ	1.0348	0.9927	(0.7684,1.3935)
		δ	0.0059	0.0085	(0.0011,0.0316)
30	$(0^{*22},2^{*7},7)$	γ	0.9982	0.9571	(0.7767,1.3524)
		δ	0.0060	0.0083	(0.0012,0.0310)
30	$(0^{*19},1^{*10},11)$	γ	1.0197	0.9773	(0.7539,1.3794)
		δ	0.0056	0.0081	(0.0010,0.0307)

parameter R by applying Gibbs sampling technique. To this end, a Markov chain of size 75000 is generated and the first 25000 of the observations is removed to eliminate the effect of the starting distribution. In order to reduce the dependence among the generated samples, we take every 10th sampled value which result in a final chain of size 5000. To investigate the convergence of MCMC samples, we have used the idea of Gelman[8] and compute scale reduction factor estimate $\sqrt{Var(\Delta)/W}$ in which Δ is the estimand of interest and $Var(\Delta) = (n-1)W/n + Z/n$, where n is the iteration number of each chain, and W and Z are the within and between sequence variances, respectively. It is observed that the value of scale factor is less than 1.1 which is an acceptable value for convergence of MCMC chain. Finally, the means of the simulated samples are recorded as the Bayes estimates of the parameter R. The AVs and MSEs of the Bayes estimates obtained from 10000 replications as well as the 95% credible intervals are tabulated in Tables 5 and 6.

It is found that classical and Bayesian point estimates of R behave in a similar manner. The MSEs of all the estimates decrease as d_1 and d_2 increase. Also, the MSEs for the extreme value 0.9182 of R are smaller than the case where $R = 0.5$. It is seen that credible intervals of the parameter R attained smaller CPs compared to the approximate CIs and the length of all confidence and credible intervals decrease as the observed sample sizes increase.

5. Data Analysis

To illustrate the estimation procedures presented in this paper, two examples based on real-life data sets are provided.

Example 1: The following data set reports the times (in days) from remission to relapse for 51 patients with acute nonlymphoblastic leukaemia ([7]).

304, 273, 955, 642, 239, 269, 230, 534, 197, 1160, 24, 697, 57, 395, 284, 64, 209, 90, 82, 89, 111, 117, 128, 143, 148, 152, 166, 171, 186, 191, 223, 247, 254, 258, 264, 270, 332, 393, 487, 510, 516, 518, 518, 608, 46, 57, 304, 341, 294, 65, 90.

[**?**] provided various methods of estimation for this data considering that it is drown from a PL distribution. Here, assuming different PTII samples of size d = 30; 40; 51 from these data, we compute the parameter estimates using the ML and Bayesian procedures. First, we use the *nlm* function in R statistical package to determine the MLEs of γ and δ. Then, assuming that the hyper-parameters take values as $a_1 = b_1 = a_2 = b_2 = 2$, the Bayes estimates of the parameters

Table 8: *Point and interval estimations of the parameter R under different progressive type II censoring schemes for example 2.*

d_1, d_2	Scheme	MLE	Bayes	CI	CRI
69 65	(0^{*69}) (0^{*65})	0.6388	0.6355	(0.5536,0.7240)	(0.5393,0.6387)
50 50	$(0^{*49}, 19)$ $(0^{*49}, 15)$	0.6213	0.6188	(0.5377,0.7642)	(0.5114,0.7313)
69 50	(0^{*69}) $(0^{*49}, 15)$	0.6293	0.6350	(0.5228,0.6943)	(0.5099,0.6265)
50 65	$(0^{*49}, 19)$ (0^{*65})	0.6264	0.6260	(0.5371,0.7165)	(0.5268,0.6543)
69 50	(0^{*69}) $(0^{*39}, 1^{*10}, 5)$	0.5781	0.5743	(0.4952,0.7329)	(0.4628,0.6755)
50 65	$(0^{*39}, 1^{*10}, 9)$ (0^{*65})	0.6684	0.6672	(0.5618,0.7807)	(0.5724,0.7639)
50 50	$(0^{*39}, 1^{*10}, 9)$ $(0^{*39}, 1^{*10}, 5)$	0.6140	0.6092	(0.4931,0.7556)	(0.5044,0.7103)
50 50	$(0^{*44}, 2^{*5}, 9)$ $(0^{*44}, 2^{*5}, 5)$	0.6117	0.6104	(0.5137,0.7613)	(0.4988,0.7151)
50 65	$(0^{*44}, 2^{*5}, 9)$ (0^{*65})	0.6717	0.6695	(0.5280,0.7259)	(0.5734,0.7621)
40 40	$(0^{*39}, 29)$ $(0^{*39}, 25)$	0.6248	0.6196	(0.4763,0.7314)	(0.4992,0.7441)
40 40	$(0^{*29}, 1^{*10}, 19)$ $(0^{*29}, 1^{*10}, 15)$	0.6204	0.6173	(0.4933,0.7295)	(0.5033,0.7354)
40 40	$(0^{*29}, 2^{*10}, 9)$ $(0^{*29}, 2^{*10}, 5)$	0.6171	0.6147	(0.4719,0.7136)	(0.4958,0.7280)
40 40	$(0^{*39}, 29)$ $(0^{*29}, 1^{*10}, 15)$	0.5834	0.5781	(0.4406,0.6929)	(0.4581,0.6994)
40 40	$(0^{*39}, 29)$ $(0^{*29}, 2^{*10}, 5)$	0.5478	0.5438	(0.4572,0.7079)	(0.4244,0.6716)

are obtained by applying Tierney and Kadane's method described in section 2. The respective estimates of the parameters along with 95% CIs are tabulated in Table 7.

Example 2: In this example we consider two data sets reported in [1] on the failure stresses of single carbon fibers of lengths 20mm and 50mm, as follows:

Data set 1: (20mm, $(n = 69)$) 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Data set 2: (50mm, $(k = 65)$) 1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.840, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051,2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.270, 2.272, 2.280, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.390, 2.410, 2.430, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.620, 2.633, 2.670, 2.682, 2.699, 2.705, 2.735, 2.785, 3.020, 3.042, 3.116, 3.174.

Ghitany et al. [9] showed that the $PL(\gamma, \delta)$ fits data sets 1 and 2 very well and compute the MLE of the reliability parameter R by using the complete samples. Now, we obtain the Bayes and ML estimates of R by using different censoring schemes. To analyze the data under Bayesian perspective, all the hyper-parameters are considered to be 0.001. At first, samples of

70,000 realizations are generated from the posterior densities in (34)-(36) and to diminish the trace of initial samples, the first 20000 realizations are deleted. Then, one observation in every 5 iterations is saved to break the autocorrelation between generated samples. For the first sampling scheme, the plot of the simulated values of R and its Histogram are given in Fig. 2 which shows the convergence of Gibbs algorithm. Table 8 reports different estimates of R as well as the 95% confidence and credible intervals. It is observed that the Bayesian and ML estimates of the parameters are about the same, however, the width of CRIs are somewhat shorter than that of CIs.

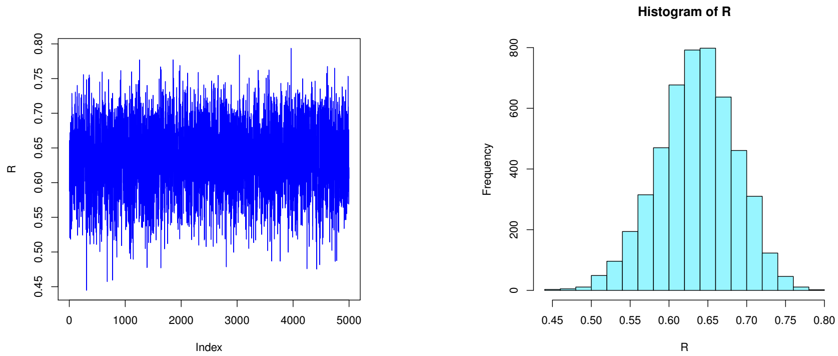

Figure 1: *Simulated values of R and Histogram of R.*

6. Conclusions

In this paper, we have used maximum likelihood and Bayesian procedures for estimating the unknown parameters of the two-parameter PL model based on PTII censoring scheme. The MLEs and asymptotic CIs for the interested parameters are computed. Since the Bayes estimates of the involved parameters could not be obtained analytically, we have employed an approximate technique to derive Bayes estimates. Further, we have developed inferential procedures for the stress-strength reliability parameter R based on PTII censored samples. ML and Bayes point estimates of the parameter R along with its classical and Bayesian interval estimates are derived. In order to assess the accuracy of the various approaches, Monte Carlo simulations are conducted. It is found that, on the basis of non-informative priors, the Bayes and ML estimates have similar performances. Also, by increasing the sample sizes, expected improvements are observed in the performances of all estimators. It must be pointed out that Bayesian methods based on Tierney and Kadane and MCMC procedures need expensive computations compared to the maximum likelihood method. However, by employing informative priors (not reported here), Bayesian approach produces estimates with better performances.

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