ENTROPHY APPROACH TO ASSESSING REDUCTION OF LIFE EXPECTANCY AT BIRTH DUE TO THE IMPACT OF NEGATIVE ENVIRONMENTAL FACTORS

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Abstract

The paper presents a method for estimating the population entropy, which assesses the impact of an increase in the intensity of mortality (caused by negative environmental factors or sudden death of people in a man-made accident or catastrophe) on the average life expectancy (ALE) at birth. The method is based on the use of the Gompertz-Makeham law.

Keywords: Keyfitz entropy, population entropy, life expectancy, mortality rate, mortality intensity, ecological risk.

I. Introduction

Life expectancy of the population is an indicator of the quality of health care system, *and* also *of* the conditions of human life. The authorities of the constituent entities were tasked according to the Decree of the President of the Russian Federation of May 7, 2018 No. 204, with achieving the following targets by 2024: increasing the life expectancy of the population to 78 years; reducing mortality rates due to health problems of the circulatory system (down to 450 cases per 100 thousand of the population), and neoplasms, including malignant ones (down to 185 cases per 100 thousand of the population).

An analysis of the dynamics of the population of the Sverdlovsk Oblast affected by negative environmental factors (soil, water, and atmospheric pollution) has revealed an alarming trend of its decline in recent years [1].

The assimilation potential of the environment - its ability to level the negative impacts of economic activity, especially in urban subjects with a high population density and expanding transportation system is decreasing over time [2]. When exposed to a complex of chemical factors of atmospheric air pollution, traffic noise, negative lifestyle factors, an unacceptable risk to human health is formed by the age of 47 years, and a high risk - by the age of 58 [3]. This is primarily due to the age-related decrease in body's ability to adapt to changing environmental conditions [4]. The report of the World Health Organization concluded that environmental risks cause the greatest damage to young children under five years of age and the elderly at age 50-75 years [5].

According to the analysis [1], the total influence of the environmental factors explains 87% of the variation in the average life expectancy ALE of the population.

II. Loading model of ecological risk

The aggravation of the ecological situation, due to the increasing scale of environmental pollution, has long put forward the priority task of developing reliable methods for predicting possible consequences of the impact of various physical and chemical factors on humans and wildlife. Its relevance came to light with force after major environmental disasters of recent times, and especially after the accident at the Chernobyl nuclear power plant.

In experiments on the dynamics of mortality in mammals, as a rule, three statistical biometric functions are measured [4]: the probability of lifespan (the survival function) $S(t)$, the probability density of lifespan $f(t)$, and the mortality rate (intensity) MR $\lambda(t)$. The latter function is also called the intensity or strength of mortality. The function $S(t)$ determines the ratio of the number of individuals $n(t)$, who survived to the age t , to their initial number N : $S(t) = n(t)/N$. The biometric function $f(t)$ characterizes the rate of decrease in the number of individuals because of death at age t, related to their initial number: $f(t) = -\frac{dn(t)}{dt} \Big/ N$. Function $\lambda(t)$ describes the rate of decrease in the number of individuals as a result of death at age t, divided by the number of individuals that survived to age t : $\lambda(t) = -\frac{dn(t)}{dt} \Big/ n(t)$. The biometric functions probability density $f(t)$ and probability $S(t)$ of lifespan are related to the mortality rate MR function $\lambda(t)$ by the following relations:

$$\lambda(t) = \frac{f(t)}{S(t)}, \quad S(t) = \exp\left(-\int_0^t \lambda(u)\,du\right). \tag{1}$$

By definition, the MR $\lambda(x)$ (synonym to the mortality intensity MI) is the probability of death occurring during an infinitely small interval at age x , divided by the duration of this interval

$$\lambda(t) = -\frac{1}{S(t)}\frac{dS}{dt}, \tag{2}$$

The main pattern revealed by published data on animal and human deaths is the exponential or close to it increase of $\lambda(x)$ with age in sexually mature individuals. This pattern, first noticed by B. Gomperz in the study of the human lifespan [5], was subsequently confirmed in many biological species that differ greatly in anatomic and physiological characteristics, conditions and lifespan. Gomperz's law of mortality

$$\lambda(t) = \lambda(t_0)e^{\alpha_0(t-t_0)}, \tag{3}$$

where $\lambda(t)$ is the mortality intensity MI at age t; $\lambda(t_0)$ is the initial mortality rate (its inverse value $\lambda^{-1}(t_0)$ characterizes the initial level of "viability"); α_0 is the rate of increase in mortality with age or the intensity of deterioration of viability (in other words, the rate of aging, which reflects the rate of increase in mortality with age).

B. Gompertz considered mortality as an inverse value to viability - the ability to withstand the totality of destructive processes.

Gompertz also suggested that the rate of decline in vitality over time decreases in proportion to itself. For mortality (the inverse of viability), this assumption corresponds to an exponential increase with age. He also assumed that along with mortality, which grows exponentially with age, there is a component of mortality that does not depend on age, that is, death is a consequence of two causes [6]: 1) random - without a previous predisposition to death or physical wear and tear; 2) wear or increased inability to resist fracture.

In other words, along with an exponentially growing component of mortality due to aging, there must be an age-independent component associated with extreme situations.

In 1860, W. Makeham-- another specialist in life insurance, added an age-independent term [7] to the Gompertz formula, which was called the Makeham parameter. Thus, the Gompertz-Makeham law (G-M) appeared:

$$\lambda(t) = M + \lambda(t_0) e^{\alpha_0 (t - t_0)}, \tag{4}$$

where M is the Makeham's coefficient characterizing the contribution to mortality of exposures, the effect of which does not depend on age (accidental mortality).

Makeham's law is most suitable for studying the process of human mortality, since it takes into account that for small ages accidents play a predominant role in mortality, and with increasing age their role weakens. The model best describes the dynamics of human mortality in the age range of 20–80 years. In the older age domain, mortality does not increase as rapidly as this law of mortality provides. Historically, human mortality prior to the 1950s was largely due to the time-independent component of the law of mortality (the Makeham's parameter), while the age-dependent component (Gompertz) remained almost unchanged. After the 1950s, the picture changed, leading to a decline in late-life mortality and a flattening of the survival curve.

In protected environments where there are no external causes of death (in laboratory conditions, in zoos, or for people in developed countries), the age-independent component often becomes small, and formula (4) simplifies to the Gompertz function (3).

Below is a mathematical load model of radiation risk, originally developed and used to calculate and normalize the radiation risk during space flights [8-12]. This model, in fact, is not specifically about radiation, *it is based on general assumptions that do not impose fundamental restrictions on either the type of influencing factor, the mode and intensity of exposure, or the biological species.* Therefore, it can be extended to other chemicals and physical factors present in the environment, in addition to the radiation load [4].

It seems very problematic to establish a direct quantitative dependence of the probability of death of an individual on indicators of the state of body systems and tissues under the influence of adverse environmental factors at a level admissible according to modern standards. Therefore, it is necessary to have a nonspecific (with respect to ionizing radiation) connection between the mathematical model of the age dependence of the MI and mathematical models of negative environmental effects (including radiobiological ones). This connection is carried out with the help of an auxiliary function - the load intensity [9, 11, 12].

Consider a homogeneous population of adults under certain given conditions. Factors of the external and internal environment form the load on a living organism, which manifests itself in the form of local or generalized deviations from the norm of the anatomic and physiological indicators of the organism state. As a result, some damage (defects) occur that reduce the reliability (viability) of the organism. Obviously, the aging of an organism with time (mortality rate) under the influence of adverse environmental factors (i.e., the accumulation of damage (defects) that reduce the organism viability) will depend only on the exponential component of the MR, since in this case the Makeham coefficient $M=0$.

The initial value through which the environmental risk and reduction of life expectancy are calculated, is the mortality intensity (MI). It is believed that at any age, the MI depends on both the load on the body and the state of the body itself. To describe the reliability of the body systems (the body state), a value $\rho(t)$ is introduced, as the probability of not overcoming a unitary load at age t under the influence of a harmful factor. It is assumed that the MI, i.e. the probability of dying per unit time at age *t*, having lived to this age, is equal to the product of the load $h(t)$ experienced by the body and the probability $\rho(t)$ that the body dies as a result of not overcoming the unit load [9, 11, 12]

$$\lambda(t) = h(t)\rho(t). \tag{5}$$

The generally recognized fundamental nature of the exponential (or almost exponential) growth of MI, applicable to all animals (of course, with their specific values of α_0), makes it appropriate to analyze the slope of the graph $\lambda(t)$ on a logarithmic scale, i.e., analyze the value:

$$\alpha(t) = \frac{d}{dt}\ln\lambda(t). \tag{6}$$

and dictates the need to model its dependence on the impact of harmful factors on human body.

In the proposed model, it is assumed that for weak impacts [12]

$$\alpha(t) = \alpha_0\frac{h(t)}{h(t_0)}, \tag{7}$$

If the load $h(t)$ is constant, i.e. $h(t) = h(t_0)$, we get the Gompertz model. When analyzing the role of a separate technogenic load (e.g.,, radiation), it is considered as additional and is designated as $\Delta h(x)$ (additional load component due to the impact of the considered harmful factor), i.e. the total load on the body $h(t) = h(t_0) + \Delta h(t)$. Then

$$\lambda(t,\Delta h) = \lambda(t_0)\exp\left(\alpha_0(t-t_0) + \alpha_0\int_{t_0}^{t}\frac{\Delta h(\tau)}{h(t_0)}d\tau\right), \tag{8}$$

i.e., MI increases by a factor of $\exp\left(\alpha_0\int_{t_0}^{t}\frac{\Delta h(\tau)}{h(t_0)}d\tau\right)$.

The value

$$\Delta t(t_0,t) = \int_{t_0}^{t}\frac{\Delta h(\tau)}{h(t_0)}d\tau\ , \tag{9}$$

is called *additional aging* due to the influence of the considered harmful factor. In this case, the result of exposure to some factor, such as ionizing radiation, is formally written as

$$\lambda(t,\Delta h) = \lambda(t_0)\exp\left(\alpha_0\left(t - t_0 + \Delta t(t_0,t)\right)\right). \tag{10}$$

The model proposed above gives an increase in MI, which is observed only with weak long-term impacts, while under acute impacts that cause partial death soon, MI sharply increases and then decreases. This issue is covered in detail in [9, 11, 12].

Next, consider how the change of MI parameters $\lambda(t_0)$, α_0 in the Gompertz model affects the ALE at birth. This problem is solved using the entropy approach.

III. Entropy approach in demography

Age at death is an undetermined random variable. In this context it is appropriate to quote Thomas Paine “*Nothing, they say is more certain than death, and nothing is more uncertain than the time of dying*”. However, it is known that the risk of death depends on many factors: environment, age, gender, genetics, lifestyle, etc. This knowledge is obtained from the analysis of specific groups of people and the information obtained has to be presented in the form of life tables that give a complete description of mortality in the population under consideration. The statistics of mortality tables (survival) gives an idea of life expectancy depending on age-specific mortality rates.

However, the average person usually has no idea of the risk of death relative to her/his age, nor of the rate at which this risk increases as she/he grows older or the environment deteriorates. Various measures have been proposed to quantify the uncertainty of death, including the concept of entropy. The most common tool for this is the Keyfitz' entropy [13].

This entropy was proposed to measure the change in life expectancy, which is a normalized version of the Shannon entropy and can be interpreted as the "elasticity" of life expectancy with respect to changes in the MI.

The Keyfitz' entropy is used to quantify the impact of MRs on life expectancy. Higher entropy means that life expectancy is more sensitive to changes in MRs, and vice versa. This entropy is a measure of the sensitivity of life expectancy to changes in mortality.

Let the MI $\lambda(t)$ change (increase) proportionally for all ages by a constant value $\Delta > 0$, so that the *new mortality rate* $\lambda_\Delta(t) = (1+\Delta)\lambda(t) = \lambda(t) + \Delta\lambda(t)$. Then probability of survival at age *t* due to an increase in MI:

$$S_\Delta(t) = e^{-\int_0^t (1+\Delta)\lambda(u)du} = \left(e^{-\int_0^t \lambda(u)du}\right)^{(1+\Delta)} = S^{(1+\Delta)}(t). \tag{11}$$

The residual ALE at birth is determined by the formula

$$e_0 = \int_0^{+\infty} S(t)dt. \tag{12}$$

Then the ALE at birth $e_\Delta(0)$, due to an increase in the MI

$$e_0(\Delta) = \int_0^{+\infty} S^{1+\Delta}(t)dt. \tag{13}$$

Suppose that a relative increase in mortality causes a relative decrease in ALE at birth. Then, in order to measure this decrease , it is necessary to calculate

$$\frac{de_0(\Delta)}{d\Delta} = \int_0^{+\infty} \frac{dS^{1+\Delta}(t)}{d\Delta}dt = \int_0^{+\infty} S^{1+\Delta}(t)\ln S^{1+\Delta}(t)dt. \tag{14}$$

For sufficiently small Δ

$$\frac{de_0(\Delta)}{d\Delta} = \lim_{\Delta\to 0}\frac{e_0(\Delta) - e_0}{\Delta}. \tag{15}$$

Then

$$\frac{\Delta e_0}{e_0} \approx \Delta \left.\frac{de_0(\Delta)}{e_0 d\Delta}\right|_{\Delta=0} = \Delta\left(\frac{\int_0^{+\infty} S(t)\ln S(t)dt}{\int_0^{+\infty} S(t)dt}\right). \tag{16}$$

Since $0 \le S(t) \le 1$, the ratio in brackets in (16) is negative, which confirms the assumption that a relative increase in mortality should lead to a relative decrease in life expectancy.

Expression

$$H_K = \frac{-\int_0^{+\infty} S(t)\ln S(t)dt}{\int_0^{+\infty} S(t)dt} \tag{17}$$

is known as the *Keyfitz entropy or population entropy*.

In this way,

$$\frac{\Delta e_0}{e_0} \approx -H_K\Delta. \tag{18}$$

Expression (18) allows following interpretation for H_K [14]: a slight increase Δ in mortality in all age groups leads to a proportional decrease in life expectancy by about a H_K factor of Δ. For example, at $H_K = 0.5$ and when the MR in all age groups increases by 1%, life expectancy decreases by 0.5% [13]. Thus, H_K measures how a proportional change in the MI affects the relative change in the population life expectancy.

Consider the boundary values of the quantity H_K. If $H_K = 0$, then all mortality is concentrated at one age. If, for example, everyone lives to be 70 years old and then dies, then $S(t) = 1$ for all ages up to 70 years, and its logarithm will be zero. On the other hand, if mortality λ is the same for all ages, then

$$S(t) = e^{-\lambda t},\ e_0 = \frac{1}{\lambda}.$$

Hence

$$\frac{\Delta e_0}{e_0} \approx -\Delta,$$

i.e., $H = 1$ and a proportional change in the MI leads to the same quantitative change in life expectancy (but in the opposite direction). For example, at $H = 1$, when mortality in all age groups increases by 1%, life expectancy decreases by 1%. With $H = 0.5$, $S(t)$ will be a linear function.

On the other hand, having a change in ALE at birth, one can estimate the corresponding change Δ in the MI. Consider the relation [13]

$$g(\Delta) = \frac{e_0(\Delta)}{e_0} = \frac{\int_0^{+\infty} S^{1+\Delta}(t)\,dt}{\int_0^{+\infty} S(t)\,dt}, \tag{19}$$

as a function of Δ.

Expanding this ratio into a Taylor series,

$$g(\Delta) = g(0) + g'(0)\Delta, \tag{20}$$

and taking into account that $g'(0) = \frac{de_0(\Delta)}{e_0 d\Delta} = -H_K$, we get

$$\frac{e_0(\Delta)}{e_0} \approx 1 - H_K\Delta. \tag{21}$$

For example, if ALE at birth increases by 10%, and $H_K = 0.2$, then MR will decrease by 50%:

$$\frac{e_0(\Delta)}{e_0} = 1.10 = 1 - 0.2\Delta \ \Rightarrow\ \Delta = -0.50.$$

Thus, a population that has 10% more ALE at birth will have half the MR.

It should be noted that with a decrease in ALE at birth by 10%, the MI will increase by the same percentage, i.e. by 50%. Thus, with an increase or decrease in ALE, the MI will, respectively, decrease or increase by the same amount, but with the opposite sign.

Function

$$\Lambda(t) = -\ln S(t) = \int_0^t \lambda(u)\,du, \tag{22}$$

is called *the cumulative risk function*.

Now the Keyfitz entropy can be rewritten in terms of the cumulative risk function:

$$H_K = \frac{\int_0^{+\infty} S(t)\Lambda(t)dt}{\int_0^{+\infty} S(t)dt}. \tag{23}$$

The numerator $\int_0^{+\infty} S(t)\Delta(t)dt$ is equal *to the average number of years of life lost due to death.*

Rewrite the Gompertz-Makeham law model in the form:

$$\lambda(t) = M + \beta e^{\alpha t}. \tag{24}$$

It is not difficult to show that the Keyfitz entropy in this case [15]:

$$H_K = \frac{1}{\alpha}\left(\frac{1}{e_0} - (M+\beta)\right) + M\bar{t} = \frac{\bar{\lambda} - \lambda_0}{\alpha} + M\bar{t}, \tag{24}$$

where $\bar{\lambda} = \frac{1}{e_0}$ is the average MR of the stationary population; $\lambda_0 = M + \beta$ is the initial MR; $\bar{t}$ is the average age of the stationary population:

$$\bar{t} = \frac{1}{e_0}\int_0^{+\infty} tS(t)dt. \tag{25}$$

According to [16, 17], the Keifitz entropy can be represented as:

$$H_K = \frac{-\int_0^{+\infty} S(t)\ln S(t)dt}{\int_0^{+\infty} S(t)dt} = \frac{-\int_0^{+\infty} \lambda(t)S(t)e(t)dt}{\int_0^{+\infty} S(t)dt} = \frac{-\int_0^{+\infty} f(t)e(t)dt}{\int_0^{+\infty} S(t)dt} = -\frac{e^{\dagger}}{e_0}, \tag{26}$$

where $e(t) = \frac{1}{S(t)}\int_t^{+\infty} S(\tau)d\tau$ is the ALE at age t.

Since $\int_0^{+\infty} f(t)dt = 1$, the value $e^{\dagger}$ can be considered as a weighted ALE at age t. In fact, it refers to the average number of years of future life lost due to observed deaths, or the average number of years that a person could live if he had a second chance at life. Considered as a measure of inequality in life expectancy.

IV. Impact of changes in mortality rates on life expectancy at birth

Consider a more general case. Assume that the MI $\lambda(t)$ at the age of t years increases due to the value Δ in such a way that following condition holds true:

$$\lim_{\Delta\to 0}\lambda(t,\Delta) = \lambda(t,0) = \lambda(t),\ \forall t. \tag{27}$$

where $\lambda(t,\Delta)$ is the new, perturbed MI.

As noted above, to calculate the relative decrease in ALE at birth, it is necessary to calculate

$$\frac{de_0(\Delta)}{d\Delta} = \lim_{\Delta\to 0}\frac{e_0(\Delta) - e_0}{\Delta}. \tag{28}$$

Expand the MR in a Taylor series using its first two terms:

$$\lambda(t,\Delta) \approx \lambda(t) + \Delta \frac{\partial \lambda}{\partial \Delta}(t,0). \tag{29}$$

Then

$$S(t,\Delta) \approx e^{-\int_0^t \lambda(u)du - \Delta \int_0^t \frac{\partial \lambda}{\partial \Delta}(u,0)du} = S(t) e^{-\Delta \int_0^t \frac{\partial \lambda}{\partial \Delta}(u,0)du}. \tag{30}$$

Applying the Taylor expansion to the exponent, we get

$$S(t,\Delta) \approx S(t)\left(1 - \Delta \int_0^t \frac{\partial \lambda}{\partial \Delta}(u,0)du\right). \tag{31}$$

Integrating this expression, we find a new ALE at birth

$$e_0(\Delta) \approx e_0 - \Delta \int_0^{+\infty} S(t) \int_0^t \frac{\partial \lambda}{\partial \Delta}(u,0)du dt. \tag{32}$$

Applying the method of integration by parts, we obtain

$$e_0(\Delta) \approx e_0 - \Delta \int_0^{+\infty} \frac{\partial \lambda}{\partial \Delta}(t,0) e(t) S(t) dt, \tag{33}$$

where following expressions were used: $\left(e(t)S(t)\right)' = -S(t)$; $\int_0^t \frac{\partial \lambda}{\partial \Delta}(u,0)du = \frac{\partial \lambda}{\partial \Delta}(t,0)$.

Then from (33) it follows that

$$\frac{de_0(\Delta)}{d\Delta} = \lim_{\Delta \to 0} \frac{e_0(\Delta) - e_0}{\Delta} = -\int_0^{+\infty} \frac{\partial \lambda}{\partial \Delta}(t,0) e(t) S(t) dt. \tag{3 4}$$

Consider how the value $\frac{de_0(\Delta)}{d\Delta}$ will change for various types of perturbations in the MI [18].

1. Additive growth: $\lambda(t,\Delta) = \lambda(t) + \Delta$. Then

$$\frac{de_0(\Delta)}{d\Delta} = -\int_0^{+\infty} e(t) S(t) dt = -\int_0^{+\infty} tS(t) dt = -\bar{t} e_0, \tag{35}$$

where $\bar{t}$ is the average age of the stationary population (see formula (25)); expression (35) is obtained by the method of integration by parts using equation $\left(e(t)S(t)\right)' = -S(t)$.

Thus, with an additive change in the intensity of mortality, the relative change in life expectancy at birth depends on two average values: the average age of the living (in a stationary population), and the unperturbed life expectancy, which is equivalent to the average age at death.

2. Proportional growth: $\lambda(t,\Delta) = (1+\Delta)\lambda(t)$. This case was considered above when describing the Keyfitz entropy, i.e.

$$\frac{de_0(\Delta)}{d\Delta} = -\int_0^{+\infty} \lambda(t) S(t) e(t) dt = -\int_0^{+\infty} f(t) e(t) dt = -e^{\dagger}. \tag{36}$$

Thus, if the perturbed MI at each age increases in proportion to the unperturbed intensity, then the relative change in life expectancy is associated with inequality in life expectancy, which is measured by the average number of years of future life $e^{\dagger}$ lost due to observed deaths.

3. Linear growth: $\lambda(t,\Delta) = \lambda(t) + \Delta t$. In this case:

$$\frac{de_0(\Delta)}{d\Delta} = -\int_0^{+\infty} tS(t) e(t) dt = -e_0\left(Cov_S(t,e) + \bar{t}^2\right), \tag{37}$$

where $\bar{t}$ is the mean age of the stationary population and the covariance

$$Cov_w(u,v)=\frac{\int_0^{+\infty}u(t)v(t)w(t)dt}{\int_0^{+\infty}w(t)dt}-\frac{\int_0^{+\infty}u(t)w(t)dt}{\int_0^{+\infty}w(t)dt}\cdot\frac{\int_0^{+\infty}v(t)w(t)dt}{\int_0^{+\infty}w(t)dt}.$$

Since linear growth, in contrast to additive growth, has a different effect on mortality at each age, the change in life expectancy depends not only on the value $\overline{t}$ (as with additive growth), but also on covariance, which reflects the age heterogeneity of the perturbed MR.

4. Exponential growth: $\lambda(t,\Delta)=\lambda(t)e^{\Delta t}$, which corresponds to the load model discussed above. In this case:

$$\frac{de_0(\Delta)}{d\Delta}=-\int_0^{+\infty}td(t)e(t)dt=-\left(Cov_d(t,e)+e_0e^{\dagger}\right). \quad (38)$$

Note that this case is equivalent to a perturbation of the form $\lambda(t,\Delta)=(1+\Delta t)\lambda(t)$, because $e^{\Delta t}\approx 1+\Delta t$. Thus, the exponential change in the MI is, in a sense, a variant of proportional growth, but affecting each age differently, so the resulting relative change in life expectancy consists of two terms covering different aspects of the change in mortality.

Consider the foregoing on the example of the Gompertz-Makeham law of mortality. Let us analyze the effect of an additive change in the parameters of the law M,α,β on ALE at birth.

1. Change in initial MR: $\beta(\Delta)=\beta+\Delta$. Then

$$\begin{aligned}&\lambda(t,\Delta)=M+\beta e^{\alpha x}+\Delta e^{\alpha x}=\lambda(t)+\Delta e^{\alpha t}=\lambda(t)+\Delta\left(\frac{\lambda(t)-M}{\beta}\right),\\&\frac{d\lambda}{d\Delta}(t,0)=\frac{\lambda(t)-M}{\beta}.\end{aligned} \quad (39)$$

Therefore, at $\beta\neq 0$ we have:

$$\frac{de_0(\Delta)}{d\Delta}=-\frac{1}{\beta}\left[\int_0^{+\infty}\lambda(t)e(t)S(t)dt-M\int_0^{+\infty}e(t)S(t)dt\right]=\frac{1}{\beta}\left[M\overline{t}e_0-e^{\dagger}\right]. \quad (40)$$

In the general case the change in life expectancy due to the additive increase in the initial mortality of the Gompertz-Makeham model is a linear combination of proportional and additive changes in the MI.

2. Change in MR: $\alpha(\Delta)=\alpha+\Delta$. Then

$$\begin{aligned}&\lambda(t,\Delta)=M+\beta e^{\alpha t}e^{\Delta t}=\lambda(x)+\Delta e^{\alpha t}=\left(\lambda(t)-M\right)e^{\Delta t}+M,\\&\frac{d\lambda}{d\Delta}(t,0)=t\left(\lambda(t)-M\right).\end{aligned} \quad (41)$$

On the other hand

$$\lambda'(t)=\lambda(t,\Delta)=\alpha\beta e^{\alpha t}=\alpha\left(\lambda(t)-M\right)\Rightarrow\lambda(t)-M=\frac{\lambda'(t)}{\alpha}.$$

Therefore, at $\alpha\neq 0$ we have:

$$\begin{aligned}&\frac{d\lambda}{d\Delta}(t,0)=t\frac{\lambda'(t)}{\alpha}\\&\frac{de_0(\Delta)}{d\Delta}=-\frac{1}{\alpha}\int_0^{+\infty}t\lambda'(t)e(t)S(t)dt=-\frac{1}{\alpha}\left[\int_0^{+\infty}td(t)dt-\int_0^{+\infty}e(t)d(t)dt\right]=\\&=\frac{1}{\alpha}\left(e^{\dagger}-e_0\right).\end{aligned} \quad (42)$$

This expression was obtained by integration by parts using the equation $\left(te(t)S(t)\right)'=\left(e(t)-t\right)S(t)$.

This ratio is a special case of the result obtained by changing the MR of the species $\lambda(t,\Delta)=\lambda((1+\Delta)t)$, which in the case of the Gompertz-Makeham law corresponds to an additive change in the parameter α.

3. Change in the age-independent parameter M characterizing death from accidental causes: $M(\Delta)=M+\Delta$. Since M is just an additive term independent of age, this is a case of an additive increase in MR:

$$\frac{de_0(\Delta)}{d\Delta}=-\bar{t}e_0. \tag{43}$$

V. Analysis of changes in life expectancy on the example of the Sverdlovsk region of Russia

To calculate the population entropy, it is necessary to have a mortality law that allows to determine the type of MI. Suppose that mortality in the Sverdlovsk region changes according to the Gompertz-Makeham law. Based on the analyzed table of mortality for 2020 of the urban population of the Sverdlovsk region, the estimates of parameters of this law:

$$M=4.78\cdot10^{-4},\ \alpha=2.65\cdot10^{-4},\ \beta=7.06\cdot10^{-2}.$$

According to (24), the Keifitz entropy for the Gompertz-Makeham mortality law, for our example will be equal to:

$$H_K=\frac{\bar{\lambda}-\lambda_0}{\alpha}+M\bar{t}\approx-0.21,$$

Where the average *mortality rate of the* stationary population $\bar{\lambda}=\frac{1}{e_0}=\frac{1}{69.95}=0.014$; initial mortality rate $\lambda_0=M+\beta=7.44\cdot10^{-4}$; average age of the stationary population: $\bar{t}=\frac{1}{e_0}\int_0^{+\infty}tS(t)\,dt\approx37.48$.

Thus, it was found that when the MI in all age groups increases by 1%, life expectancy decreases by 0.21%. Or, when the MI in all age groups increases by 10%, life expectancy decreases by 2.1%.

Let us consider how the intensity of mortality will change with a change (decrease and increase) in ALE at birth. To do this, plot the function $\Delta=\frac{1-\frac{e_0(\Delta)}{e_0}}{H_K}=\frac{1-\frac{e_0(\Delta)}{e_0}}{0.21}$. The Fig. 1 shows the percentage increase (decrease) of Δ depending on the percentage decrease (increase) of ALE at birth.

In Sverdlovsk Region ALE at birth is $69.95\approx70$ years. Thus, if ALE at birth increases by 10% (7 years), then the MI will decrease by almost half (47.65%). The same will happen with a decrease of ALE - the MI will increase by almost half (by 47.65%). With an increase (decrease) of ALE by 50% (that is, the population will live on average to 105 (35) years), the MR will decrease (increase) by 238.27%.

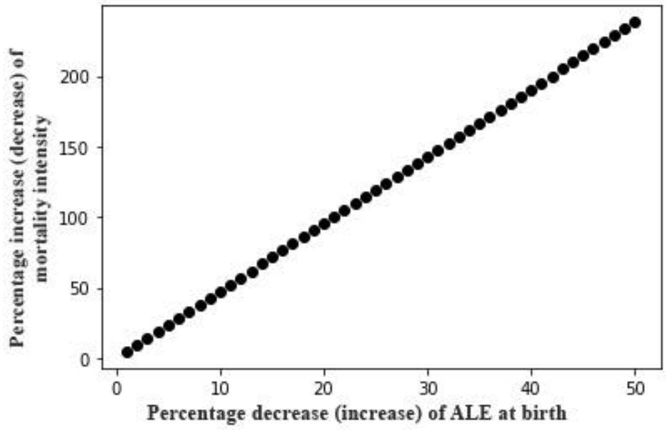

Figure 1: *Percentage increase (decrease) of* Δ *depending on the percentage decrease (increase) of ALE at birth*

Consider how the percentage increase of the Gompertz-Makeham law parameters will affect on ALE at birth.

1. Change in the level of initial MR: $\beta(\Delta)=\beta+\Delta$.

In this case

$$g'(0)=\frac{de_0(\Delta)}{e_0 d\Delta}=\frac{1}{\beta}\left[M\bar{t}-\frac{e^{\dagger}}{e_0}\right]=\frac{1}{\beta}\left[M\bar{t}-H_K\right].$$

Then

$$\frac{e_0(\Delta)}{e_0}=1+\frac{1}{\beta}\left[M\bar{t}-H_K\right]\Delta .$$

In our case $\Delta=k\beta$, where k changes from 0.01 (1%) to 1 (100%). Then

$$\frac{e_0(\Delta)}{e_0}=1+\left[M\bar{t}-H_K\right]k$$

A graph of the function of the percentage decrease of ALE at birth, depending on the percentage increase of initial mortality, is shown in Fig. 2.

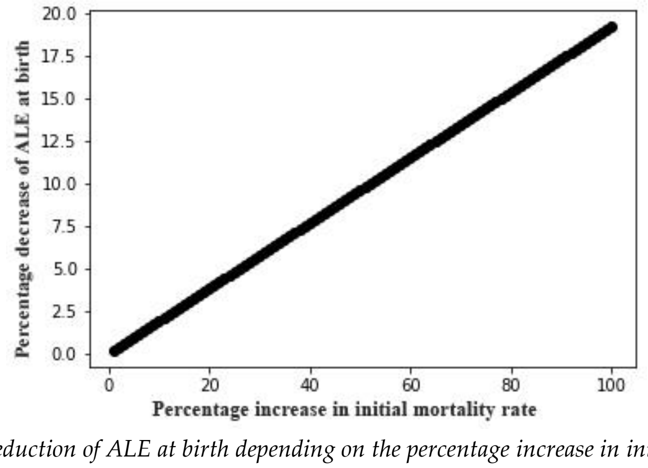

Figure 2: *Percentage reduction of ALE at birth depending on the percentage increase in initial mortality rate*

Thus, with an increase in the level of age-related mortality β by 1%, ALE at birth will decrease by 0.19% (0.13 years); with a twofold increase of β (by 100%), ALE will decrease by 19.19% (13.5 years).

2. Change in the rate of increase in mortality with age: $\alpha(\Delta)=\alpha+\Delta$.

In this case

$$g'(0)=\frac{de_0(\Delta)}{e_0 d\Delta}=\frac{1}{\alpha}\left(\frac{e^{\dagger}}{e_0}-1\right)=\frac{1}{\alpha}\left(H_K-1\right).$$

Then

$$\frac{e_0(\Delta)}{e_0}=1+\frac{1}{\alpha}\left(H_K-1\right)\Delta$$

In our case $\Delta=k\alpha$, where k changes from 0.01 (1%) to 1 (100%). Then

$$\frac{e_0(\Delta)}{e_0}=1+\left(H_K-1\right)k.$$

A graph of the function of the percentage decrease of ALE at birth, depending on the percentage increase in the rate of increase in mortality with age, is shown in Fig. 3.

Thus, with an increase in the MR α per 1% the ALE at birth will decrease by 0.79% (0.55 years); with a twofold increase of α (by 100%), the ALE will decrease by 79.01% (55.27 years).

3. Change in the age-independent parameter M characterizing death from accidental causes: $M(\Delta)=M+\Delta$.

In this case

$$g'(0)=\frac{de_0(\Delta)}{e_0 d\Delta}=-\bar{t}.$$

Then

$$\frac{e_0(\Delta)}{e_0}=1-\bar{t}\Delta$$

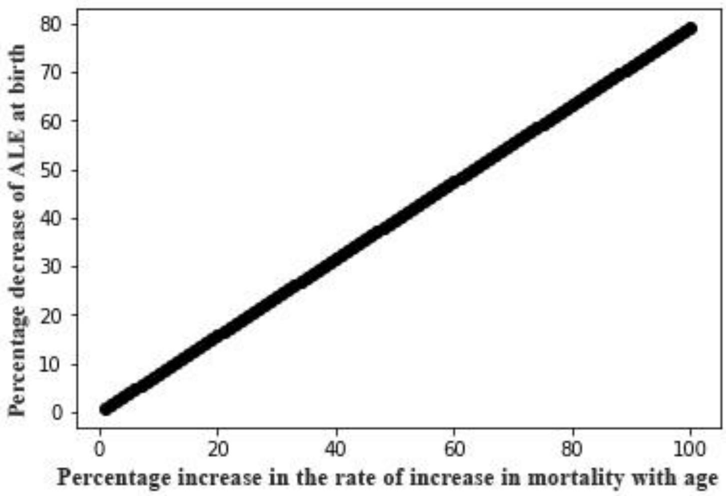

Figure 3: *Percentage reduction in ALE at birth depending on the percentage increase in the rate of increase in mortality with age*

In our case $\Delta=kM$, where k changes from 0.01 (1%) to 1 (100%). Then

$$\frac{e_0(\Delta)}{e_0}=1-\bar{t}kM.$$

A graph of the function of the percentage decrease in ALE at birth, depending on the percentage increase in the Makeham parameter M, is shown in Fig. 4.

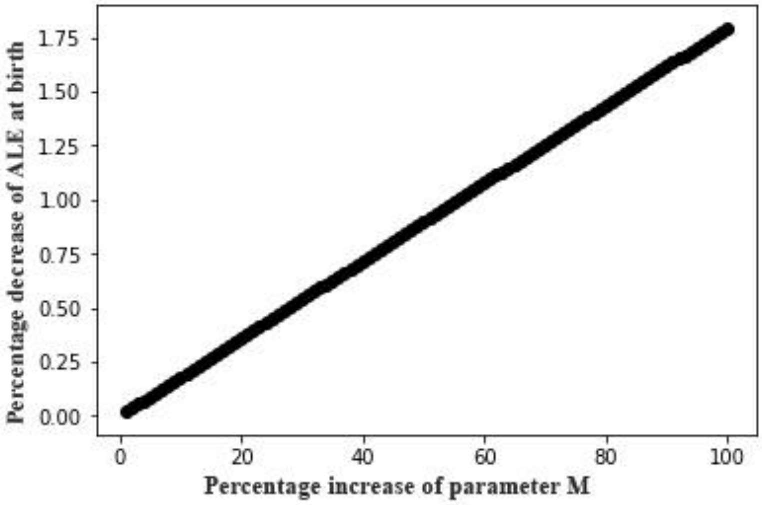

Figure 4: *Percentage reduction in ALE at birth depending on the percentage increase of parameter M*

Thus, with an increase in the parameter *M* by 1%, ALE at birth will decrease by 0.01% (0.01 years); with a twofold increase in *M* (by 100%), ALE will decrease by 1.79% (1.25 years).

According to the analysis, the greatest decrease in ALE at birth occurs with an increase in the MR (parameter α), followed by parameter β (initial MR), an increase in which causes a smaller decrease in ALE, and, finally, an increase in the parameter *M*, which characterizes death from random reasons.

VI. Conclusion

The paper presents an entropy approach to assessing the reduction in life expectancy from the impact of negative environmental factors, which is expressed as an increase in the MI, which is the initial value through which the environmental risk and reduction in life expectancy are determined.

Using the presented approach, an analysis was conducted of the increase in the MI, described by various mathematical forms of its change, as well as of the impact of such a change on the ALE at birth.

The results of the described research open the door to consistent assessing of the total damage inflicted on the society (in terms of biological, cognitive and social dimension) by different technogenic and environmental factors that evolve in time. This, in its term, permits optimizing expenditures needed for mitigating and minimizing collective risk.

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