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On the TeSU−G family of distributions applied to life data analysis

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Abstract

This paper derives distributions from the U-quadratic and the T-X family of distributions labeled as the T-extended Standard U-quadratic−G family of distributions or simply, TeSU−G family. In particular, the TeSU−Weibull distribution (TeSU−W) is explored with respect to some statistical properties such as its limiting distribution, moment, mean and variance and moment generating function. Also, the statistical properties of the TeSU−Exponential distribution (TeSU−E) which is a special case of the TeSU−W are also derived. The Weibull and Exponential distributions are mostly used in life data analysis because of its ability to adapt to different situations. Moreover, the formula for the median is derived via a proposed algorithm. Simulation study is conducted to verify the performance of the ML estimates of the TeSU−W distribution for varied sample sizes. Further, real life data analysis reveals that derived extended distribution can provide a better fit than several well-known distributions.

Keywords: U-quadratic distribution, T-X family of distributions, Weibull distribution

1. Introduction

Classical statistical distribution plays a vital role in many areas of science for describing the behavior of any data as well as for modelling data. But nowadays, due to the complexity of the data, the classical distribution needs to be modified in order to cater the complexity of the data. Up to this time, researchers are working in methodologies on statistical distribution theory in order to solve these types of problems.

In 1985, Azzalini [4] introduced a skewed family of distribution for generating a distribution with additional skewed parameter. Other identified family of distributions are the Marshall-Olkin extended (MOE) family [12] and the exponentiated family of distributions [10].

Moreover Eugene [9] in 2002 introduced a composite method of combining two or more known competing distributions through transformations, like the Gamma generated family [16], the Kumaraswamy−G (Kw−G) family [7], the Beta extended−G family [8], the Exponentiated Generalized family [5], the Kumarsway Marshall-Olkin−G family [1], the Generalized odd log-logistic family [6], the generalized transmuted−G family [13] and the Exponentiated Kumarasway−G class family [15].

This paper derives an extended or modified distribution named as TeSU−G family of distribution and explored a derived model using the Weibull as the baseline distribution. This is named as the TeSU−Weibull distribution (TeSU−W). The statistical properties like its limiting distribution, moment, mean and variance, and moment generating function are derived. Similarly, the properties of the TeSU− Exponential distribution (TeSU−E), which is a special case of the TeSU−W are obtained.

The rest of the paper is organized as follows: The extended Standard U-quadratic (eSU) distribution is derived in section 2; in section 3, the TeSU−G family of distribution is introduced; in section 4, the cdf and pdf of both TeSU−W and TeSU−E distributions are derived using the results in sections 2 and 3. Some statistical properties of TeSU−W are presented in section 5. In section 6, estimates of the TeSU−W parameters via the maximum likelihood estimation is generated. Simulation study is presented in section 7 while the application to real life dataset are discussed in section 8. Finally, some concluding remarks are presented in section 9.

2. The eSU distribution

This section shows the derivation of the extended Standard U-quadratic distribution (eSU). Consider the special case of the T-X family which was introduced by Alzaatreh [2] in 2013. Accordingly, for any arbitrary baseline cumulative distribution function (cdf) $G(x)$, a new cdf $F(x)$ can be generated using the equation

$$F(x) = \int_0^{G(x)} f(t)dt \tag{1}$$

where $f(t)$ is a probability density function (pdf) of a random variable T with support on the interval $[0,1]$. Also, consider the Transmuted−G family of distributions introduced by Shaw [14], that is, for any baseline cdf $G(x)$, we can define a new cdf $K(x)$ given by

$$K(x) = (1+\lambda)G(x) - \lambda G^2(x), \tag{2}$$

where $\lambda \in [-1,1]$. Note that (2) can be written as

$$K(x) = \int_0^{G(x)} f(t)dt$$

where

$$f(t) = 1+\lambda-2\lambda t\ I_{[0,1]}(t) = (1-\lambda)f_1(t) + \lambda f_2(t) \tag{3}$$

with pdfs $f_1(t)$ and $f_2(t)$ are given as $f_1(t) = 1\ I_{[0,1]}(t)$ and $f_2(t) = 2(1-t)\ I_{[0,1]}(t)$, respectively. Hence, $f(t)$ can be written as a mixture of two pdfs with support set on the interval $[0,1]$.

Now consider the pdf of the U-quadratic distribution. For a random variable T that follows a U-quadratic distribution, the pdf of T is given by

$$l(t) = m(t-n)^2, \tag{4}$$

where $t \in [a,b], a < b, a, b \in \mathbb{R}, m = \dfrac{12}{(b-a)^3}$ and $n = \dfrac{a+b}{2}$.

To standardize equation (4), let $a = 0$ and $b = 1$. Then, equation (4) becomes

$$l(t) = 12(t-\frac{1}{2})^2, \tag{5}$$

where $t \in [0,1]$. Substituting $l(t)$ of (5) in equation (3) for $f_2(t)$ derives the pdf of the eSU-quadratic distribution denoted as $f_{eSU}(t)$ and is given by

$$f_{eSU}(t) = 1-\lambda+3\lambda(2t-1)^2, \tag{6}$$

where $t \in [0,1]$ and $\lambda \in [-\frac{1}{2},1]$.

3. The TeSU−G Family of Distribution

This section introduces a T-extended Standard U-quadratic (TeSU)− G Family of distribution. Using equation (1) and the pdf of eSU in (6) derives the cdf of TeSU−G family of distribution given by

$$F_{TeSU-G}(x) = (1+2\lambda)G(x) - 6\lambda G^2(x) + 4\lambda G^3(x), x \in \mathbb{R} \tag{7}$$

with corresponding pdf

$$f(x) = g(x)[1-\lambda+3\lambda(2G(x)-1)^2], x \in \mathbb{R} \tag{8}$$

where $\lambda \in [-\frac{1}{2}, 1]$ and $g(x)$ is the pdf associated with a baseline cdf $G(x)$. Note that, if $\lambda = 0$, the cdf of TeSU-G reduces to the cdf of the baseline distribution.

4. The TeSU−Weibull and the TeSU−Exponential distributions

This section discusses the derivation of the cdf and pdf of the TeSU using the Weibull and Exponential as baseline distributions. Suppose that a random variable X has Weibull distribution with cdf $G_w(x)$ and pdf $g_W(x)$ given, respectively, as follows:

$$G_W(x) = 1 - e^{-\tau x^\beta} \text{ and} \tag{9}$$

$$g_W(x) = \tau\beta x^{\beta-1} e^{-\tau x^\beta}, \tag{10}$$

where $x \geq 0$ and with scale τ and shape β parameters.

The cdf of the TeSU−Weibull distribution (TeSU−W) is derived by substituting (9) in equation (7), so that we have

$$F_{TeSU-W}(x) = 1 - e^{-\tau x^\beta}(1+2\lambda - 6\lambda e^{-\tau x^\beta} + 4\lambda e^{-2\tau x^\beta}), \tag{11}$$

where $\tau > 0$, $\beta > 0$, $\lambda \in [-\frac{1}{2}, 1]$ and $x \geq 0$ with corresponding pdf given as

$$f_{TeSU-W}(x) = \tau\beta x^{\beta-1} e^{-\tau x^\beta}(1+2\lambda - 12\lambda e^{-\tau x^\beta} + 12\lambda e^{-2\lambda e^{-2\tau x^\beta}}). \tag{12}$$

Note that the exponential distribution is a special case of the classical Weibull distribution when $\beta = 1$. Thus, when $\beta = 1$, the TeSU−W reduces to the TeSU− Exponential distribution (TeSU−E).

The cdf of TeSU−E is given as

$$F_{TeSU-E}(x) = 1 - e^{-\tau x}(1+2\lambda - 6\lambda e^{-\tau x} + 4\lambda e^{-2\tau x}),$$

where $\tau > 0$, $\lambda \in [-\frac{1}{2}, 1]$ and $x \geq 0$ with corresponding pdf

$$f_{TeSU-E}(x) = \tau e^{-\tau x}(1+2\lambda - 12\lambda e^{-\tau x} + 12\lambda e^{-2\lambda e^{-2\tau x}}).$$

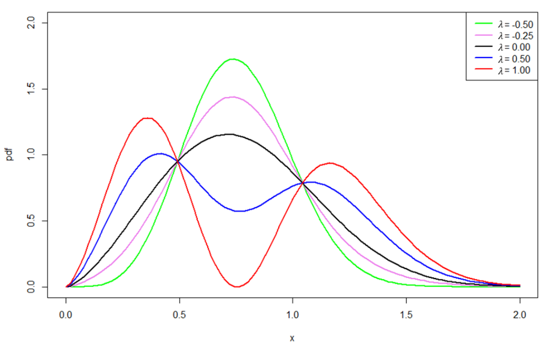

Figure 1: *Plots of the pdf of the TeSU−W for $\tau = 1.4$, $\beta = 2.5$ and for some values of λ*

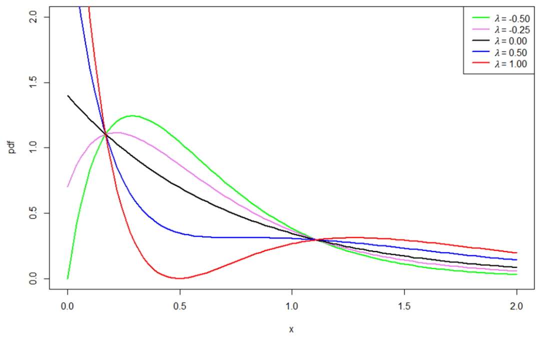

Figure 2: *Plots of the pdf of the TeSU−E for $\tau = 1.4$ and for some values of λ*

Figure 1 depicts the pdfs of TeSU−W at fixed values of $\tau = 1.4$ and $\beta = 2.5$ with varied values of $\lambda \in \{-0.5, -0.25, 0, 0.5, 1.0\}$. It can be observed that the TeSU−W displays a bimodal distribution when $0 < \lambda < 1$, while when $\lambda = 0$, it depicts the usual shape of the classical Weibull distribution. Moreover, for $-\frac{1}{2} \leq \lambda < 0$, a unimodal distribution which is leptokurtic in nature or a peaked top is observable.

A special case of the TeSU−W is the TeSU−Exponential distribution (TeSU−E), that is, when the parameter β is equal to one. Figure 2 shows the graph of TeSU−E with fixed value of $\tau = 1.4$ and varied values of λ stated previously. The following distribution shapes can be noticed: (1) when $\lambda = 0$, the graph of the TeSU-E is the same as the classical exponential distribution; (2) when $-\frac{1}{2} \leq \lambda < 0$, then it exhibits a unimodal distribution which is positively skewed, and (3) when $0 < \lambda \leq 1$, it follows an inverted skewed bathtub shape. These types of shape are important for describing the complex behavior of the data specially when data distribution reflects a bimodal shape. Hence, the next discussions are focused on the TeSU-W distribution which can cater the bimodal distribution.

5. Some Statistical Properties

5.1. Survival and Hazard Functions of TeSU-W

Let X be a random variable with cdf given in equation (11) and pdf given in equation (12). Then for $x > 0$, the survival function $S_{TeSU-W}(x)$ and hazard function $h_{TeSU-W}(x) = \frac{f_{TeSU-W}}{S_{TeSU-W}}$ of X are given, respectively, as follows:

$$\begin{aligned} S_{TeSU-W}(x) &= 1 - F_{TeSU-W}(x) \\ &= e^{-\tau x^{\beta}}(1 + 2\lambda - 6\lambda e^{-\tau x^{\beta}} + 4\lambda e^{-2\tau x^{\beta}}) \end{aligned}$$

and

$$h_{TeSU-W}(x) = \frac{\tau \beta x^{\beta-1}(1 + 2\lambda - 12\lambda e^{-\tau x^{\beta}} + 12\lambda e^{-2\tau x^{\beta}})}{1 + 2\lambda - 6\lambda e^{-\tau x^{\beta}} + 4\lambda e^{-2\tau x^{\beta}}}. \tag{13}$$

Note that if $\beta = 1$, then $S_{TeSU-W}(x) = S_{TeSU-E}(x)$ and $h_{TeSU-W}(x) = h_{TeSU-E}(x)$.

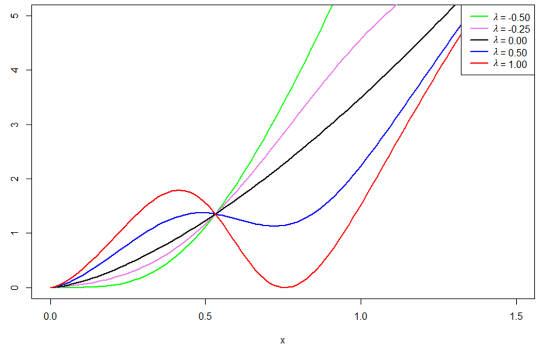

Figure 3: *Plots of the* $h(x)$ *of the TeSU−W when* $\tau = 1.4$, $\beta = 2.5$ *with varied* λ.

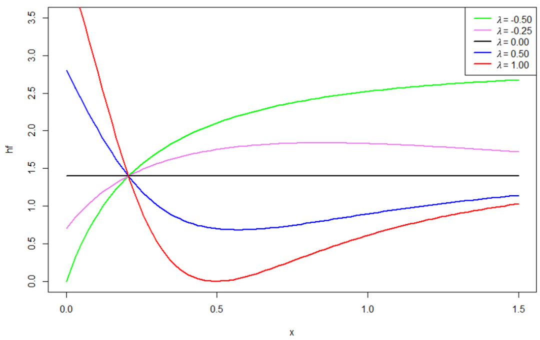

Figure 4: *Plots of the* $h(x)$ *of the TeSU−E for* $\tau = 1.4$ *with varied* λ.

Figure 3 shows that at fixed values of $\tau = 1.4$ and $\beta = 2.5$ and varied values $\lambda \in \{-0.5, -0.25, 0, 0.5, 1.0\}$, the hazard rate function $h(x)$ of the TeSU−W can model not only monotonic but also non-monotonic behavior of the failure rate of the observations, which are inherent in survival lifetime data. Moreover, Figure 4 reveals that the TeSU-E hazard rate function $h(x)$ can model complex data which are either non-monotonic decreasing, increasing or with constant rate.

5.2. The Limiting Distribution of TeSU−W

This section derives the limiting distribution of the probability distribution function (Theorem 1) and the hazard function (Theorem 2) of TeSU−W.

Theorem 1. (i) The limit of the probability density function $f(x)$ of the TeSU-Weibull distribution as $x \to \infty$ is equal to 0, that is,

$$\lim_{x\to\infty} f_{TeSU-W}(x) = 0.$$

(ii)

$$\lim_{x\to 0} f_{TeSU-W}(x) = \begin{cases} \infty & if \beta < 1 \\ \tau(1+2\lambda) & if \beta = 1 \\ 0 & if \beta > 1 \end{cases}.$$

Proof. Recall that the pdf of TeSU-W is given in equation (12). It is clear that

$$\lim_{x\to\infty} f_{TeSU-W}(x) = 0$$

since $\lim\limits_{x\to\infty} e^{-\tau x^{\beta}} = \lim\limits_{x\to\infty} \frac{1}{e^{\tau x^{\beta}}} = 0$. This proves (i).

To prove Theorem 1 (ii), we have

$$\lim_{x\to 0} f_{TeSU-W} = \tau\beta \lim_{x\to 0} x^{\beta-1} \lim_{x\to 0} e^{-\tau x^{\beta}} (1 + 2\lambda - 12\lambda \lim_{x\to 0} e^{-\tau x^{\beta}} + 12\lambda \lim_{x\to 0} e^{-2\tau x^{\beta}}).$$

Observe that for $\beta > 0$,

$$\lim_{x\to 0} e^{-\tau x^{\beta}} = \lim_{x\to} \frac{1}{e^{\tau x^{\beta}}} = 1.$$

It follows that

$$\lim_{x\to 0} f_{TeSU-W}(x) = \tau\beta(1 + 2\lambda) \lim_{x\to 0} x^{\beta-1}.$$

If $\beta = 1$, then we have

$$\lim_{x\to 0} f_{TeSU-W}(x) = \tau(1 + 2\lambda) \lim_{x\to 0} x^{0} = \tau(1 + 2\lambda).$$

Next, for $\beta > 1$ we have

$$\lim_{x\to 0} f_{TeSU-W}(x) = \tau\beta(1 + 2\lambda) \lim_{x\to 0} x^{\beta-1} = 0.$$

Lastly, for $\beta < 1$ we get

$$\lim_{x\to 0} f_{TeSU-W}(x) = \tau\beta(1 + 2\lambda) \lim_{x\to 0} x^{\beta-1}$$

but $\beta - 1 < 0$ since $\beta < 1$. Also, $\beta - 1$ can be express as $\beta - 1 = -(1 - \beta) = -c$, where $c = 1 - \beta > 0$.. It follows that

$$\lim_{x\to 0} f_{TeSU-W}(x) = \tau\beta(1 + 2\lambda) \lim_{x\to 0} x^{\beta-1} = \tau\beta(1 + 2\lambda)(\lim_{x\to 0} \frac{1}{x})^{c} = \infty.$$

■

Theorem 2. The limit of the hazard rate function of the TeSU-W distribution is given by the following:

(i)

$$\lim_{x\to\infty} h_{TeSU-W}(x) = \begin{cases} 0 & if \beta < 1 \\ \tau & if \beta = 1 \\ \infty & if \beta > 1 \end{cases}.$$

(ii)

$$\lim_{x\to 0} h_{TeSU-W}(x) = \begin{cases} \infty & if \beta < 1 \\ \tau(1 + 2\lambda) & if \beta = 1 \\ 0 & if \beta > 1 \end{cases}.$$

Proof. By taking the limit of equation (13) as $x \to \infty$, we have the following results. It can be verified that

$$\lim_{x\to\infty} h_{TeSU-W}(x) = \frac{\tau\beta(\lim_{x\to\infty} x^{\beta-1})(1 + 2\lambda - 12\lambda \lim_{x\to\infty} e^{-\tau x^{\beta}} + 12\lambda \lim_{x\to\infty} e^{-2\tau x^{\beta}})}{1 + 2\lambda - 6\lambda \lim_{x\to\infty} e^{-\tau x^{\beta}} + 4\lambda \lim_{x\to\infty} e^{-2\tau x^{\beta}}}.$$

Observe that

$$\lim_{x\to\infty} e^{-\tau x^{\beta}} = \lim_{x\to\infty} \frac{1}{e^{\tau x^{\beta}}} = 0,$$

then it follows that

$$\lim_{x\to\infty} h_{TeSU-W}(x) = \tau\beta \lim_{x\to\infty} x^{\beta-1}.$$

If $\beta = 1$ then
$$\lim_{x\to\infty} h_{TeSU-W}(x) = \tau.$$
If $\beta > 1$ then
$$\lim_{x\to\infty} h_{TeSU-W}(x) = \tau\beta(\lim_{x\to\infty} x)^{\beta-1} = \infty.$$
If $\beta < 1$ then
$$\lim_{x\to\infty} h_{TeSU-W}(x) = \tau\beta \lim_{x\to\infty} x^{\beta-1} = 0.$$
This proves (i). The proof of Theorem 2 (ii) is as follows: By definition of the hazard function, we have
$$\lim_{x\to 0} h_{TeSU-W}(x) = \frac{\lim_{x\to 0} f_{TeSU-W}(x)}{\lim_{x\to 0} S_{TeSU-W}(x)}.$$
Observed that,
$$\lim_{x\to 0} S_{TeSU-W}(x) = (\lim_{x\to 0} e^{-\tau x^{\beta}})(1 + 2\lambda - 6\lambda \lim_{x\to 0} e^{-\tau x^{\beta}} + 4\lambda \lim_{x\to 0} e^{-2\tau x^{\beta}}).$$
But $\lim_{x\to 0} e^{-\tau x^{\beta}} = 1$. Hence, it follows that $\lim_{x\to 0} S_{TeSU-W}(x) = 1$. Thus,
$$\lim_{x\to 0} h_{TeSU-W}(x) = \lim_{x\to 0} f_{TeSU-W}(x).$$
By Theorem 1, we have
$$\lim_{x\to 0} h_{TeSU-W}(x) = \lim_{x\to 0} f_{TeSU-W}(x) = \lim_{x\to 0} f_{TeSU-W}(x) = \begin{cases} \infty & if \beta < 1 \\ \tau(1+2\lambda) & if \beta = 1 \\ 0 & if \beta > 1 \end{cases}.$$
■

5.3. Moment and Moment Generating Function of TeSU-W

This section derives the *rth* moment (Theorem 3), the mean and variance (Corollary 1), and the moment generating function (Theorem 4) of TeSU−W.

Theorem 3. The *rth* moment of TeSU-W distribution, with pdf given in (12) is given by
$$\mu'_r = \tau^{-\frac{r}{\beta}}\Gamma(\frac{r}{\beta}+1)[1 + 2\lambda - 6\lambda 2^{-\frac{r}{\beta}} + 4\lambda 3^{-\frac{r}{\beta}}], \tag{14}$$
where $r = 1, 2, ..., n$ and $\Gamma(\cdot)$ is a gamma function.

Proof. The *rth* raw moment is defined by $\mu'_r = E(X^r) = \int_0^\infty x^r f(x)dx$. Thus, using the pdf $f(x)$ in equation (12) and simplifying, we have
$$\begin{aligned}
\mu'_r &= \int_0^\infty x^r \tau\beta x^{\beta-1} e^{-\tau x^{\beta}}(1 + 2\lambda - 12\lambda e^{-\tau x^{\beta}} + 12\lambda e^{-2\tau x^{\beta}})dx \\
&= (1+2\lambda)\tau^{-\frac{r}{\beta}}\Gamma(\frac{r}{\beta}+1) - 6\lambda 2^{-\frac{r}{\beta}}\tau^{-\frac{r}{\beta}}\Gamma(\frac{r}{\beta}+1) + 4\lambda 3^{-\frac{r}{\beta}}\tau^{-\frac{r}{\beta}}\Gamma(\frac{r}{\beta}+1) \\
&= \tau^{-\frac{r}{\beta}}\Gamma(\frac{r}{\beta}+1)[1 + 2\lambda - 6\lambda 2^{-\frac{r}{\beta}} + 4\lambda 3^{-\frac{r}{\beta}}].
\end{aligned}$$
■

Corollary 1. The mean and variance of the TeSU-W distribution are, respectively, given by

$$\mu = \mu_1' = \tau^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)[1+2\lambda-6\lambda 2^{-\frac{1}{\beta}}+4\lambda 3^{-\frac{1}{\beta}}] \text{ and}$$

$$\sigma^2 = \tau^{-\frac{2}{\beta}}[\Gamma(\frac{2}{\beta}+1)(1+2\lambda-6\lambda 2^{-\frac{2}{\beta}}+4\lambda 3^{-\frac{2}{\beta}}) - \Gamma^2(\frac{1}{\beta}+1)(1+2\lambda-6\lambda 2^{-\frac{1}{\beta}}+4\lambda 3^{-\frac{1}{\beta}})^2].$$

Proof. The mean of the TeSU-W distribution is obtained when $r = 1$ in (14). Thus,

$$\mu = \mu_1' = \tau^{-\frac{1}{\beta}}\Gamma(\frac{1}{\beta}+1)[1+2\lambda-6\lambda 2^{-\frac{1}{\beta}}+4\lambda 3^{-\frac{1}{\beta}}].$$

It is to note that $\sigma^2 = \mu_2' - (\mu_1')^2$. Now, the $2nd$ raw moment μ_2' of the proposed distribution is obtained using equation (14) when $r = 2$. It follows that

$$\mu_2' = \tau^{-\frac{2}{\beta}}\Gamma(\frac{2}{\beta}+1)[1+2\lambda-6\lambda 2^{-\frac{2}{\beta}}+4\lambda 3^{-\frac{2}{\beta}}].$$

Therefore, the variance σ^2 of TeSU-W distribution is derived as

$$\begin{aligned} \sigma^2 &= \mu_2' - (\mu_1')^2 \\ &= \tau^{-\frac{2}{\beta}}\left[\Gamma\left(\frac{2}{\beta}+1\right)\left(1+2\lambda-6\lambda 2^{-\frac{2}{\beta}}+4\lambda 3^{-\frac{2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta}+1\right)\left(1+2\lambda-6\lambda 2^{-\frac{1}{\beta}}+4\lambda 3^{-\frac{1}{\beta}}\right)^2\right]. \end{aligned}$$

■

Theorem 4. Let X follows the TeSU-W distribution, then its moment generating function, $M_{X_{TeSU-W}}(t)$ is given as

$$M_{X_{TeSU-W}}(t) = \sum_{r=0}^{\infty}\frac{t^r\tau^{-\frac{r}{\beta}}}{r!}\Gamma(\frac{r}{\beta}+1)(1+2\lambda-6\lambda 2^{-\frac{r}{\beta}}+4\lambda 3^{-\frac{r}{\beta}}),$$

where $t \in \mathbb{R}$.

Proof. By definition of moment generating function and using equation (14), we have

$$\begin{aligned} M_{X_{TeSU-W}}(t) &= \mathbb{E}(e^{tX}) \\ &= \int_0^{\infty} e^{tx} f_{TeSU-W}(x)dx. \end{aligned}$$

Recall that $e^{tX} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r$. Hence, we have

$$\begin{aligned} M_{X_{TeSU-W}}(t) &= \int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f_{TeSU-W}(x)dx \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f_{TeSU-W}(x)dx \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'. \end{aligned}$$

Thus,

$$M_{X_{TeSU-W}}(t) = \sum_{r=0}^{\infty}\frac{t^r\tau^{-\frac{r}{\beta}}}{r!}\Gamma(\frac{r}{\beta}+1)(1+2\lambda-6\lambda 2^{-\frac{r}{\beta}}+4\lambda 3^{-\frac{r}{\beta}}).$$

■

5.4. The Median of TeSU−G family and TeSU−W distribution

This section described the process of the derivation of the median of the TeSU−G family of distributions. Consider the Structured Set of Skew–Kurtotic Transmutations proposed by Shaw [14], that is, for parameters α_1, α_2 we shall consider the polynomial family given by

$$P(z, \alpha_1, \alpha_2) = z - z(1-z)\left[\alpha_1 + \alpha_2\left(z - \frac{1}{2}\right)\right],$$

where $z \in [0, 1]$ and the non-negativity of the pdf P' at the end points should satisfy

$$-1 - \frac{\alpha_2}{2} \leq \alpha_1 \leq 1 + \frac{\alpha_2}{2}.$$

Let u follows a uniform distribution $(0, 1)$. Then the solution for the equation $P(z, \alpha_1, \alpha_2) = u$ is as follows:

$$z = \begin{cases} u, & if \alpha_1 = \alpha_2 = 0 \\ \dfrac{\alpha_1 - 1 + \sqrt{1 + \alpha_1(\alpha_1 + 4u - 2)}}{2\alpha_1}, & if \alpha_2 = 0 \\ \sqrt[3]{u}, & if \alpha_1 = \frac{3}{2}, \alpha_2 = 1 \\ 1 - \sqrt[3]{1-u}, & if \alpha_1 = -\frac{3}{2}, \alpha_2 = 1 \\ C(u, \alpha_1, \alpha_2), & otherwise \end{cases}$$

where $C(\cdot)$ is a function that denotes the general cubic (GC) solver for other cases. This function is processed by the following algorithm.

Step 1. Compute

$$Q = \frac{4\alpha_1^2 + 3(\alpha_2 - 4)\alpha_2}{36\alpha_2^2},$$

$$R = \frac{4\alpha_1^3 - 9\alpha_2\alpha_1(\alpha_2 + 2) + 27(1 - 2u)\alpha_2^2}{108\alpha_2^3}.$$

Step 2. If $R^2 > Q^3$, the equation has one real and two complex roots. In this case we have,

$$A = -sign(R)\left(|R| + \sqrt{R^2 - Q^3}\right)^{\frac{1}{3}};$$

$$B = \begin{cases} A, & if A = 0 \\ \dfrac{Q}{A}, & otherwise \end{cases};$$

$$C(u, \alpha_1, \alpha_2) = A + B - \frac{1}{3}\left(\frac{\alpha_1}{\alpha_2} - \frac{3}{2}\right).$$

Otherwise, the cubic has three real roots and this is done by setting

$$\theta = arccos\left(\frac{R}{\sqrt[3]{Q}}\right);$$

$$C(u, \alpha_1, \alpha_2) = -2\sqrt{Q}\, cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{1}{3}\left(\frac{\alpha_1}{\alpha_2} - \frac{3}{2}\right).$$

Observed that the cdf (7) of the TeSU-G family can be rewritten as

$$F(x) = z - z(1-z)\left[\alpha_1 + \alpha_2\left(z - \frac{1}{2}\right)\right],$$

where $z = G(x)$, $\alpha_1 = 0$ and $\alpha_2 = 4\lambda$, $\lambda \in [-0.5, 1]$. The inverse of $F(x)$ is a solution to the following equation

$$z = C(u, \alpha_1, \alpha_2) = G(x) = C(u, 0, 4\lambda).$$

Hence, the given algorithm can be modified as follows. Let u follows a uniform distribution $(0, 1)$. Then,

Step 1*. Compute

$$Q = \frac{1}{2}\left(1 - \frac{1}{\lambda}\right), \lambda \neq 0;$$
$$R = \frac{1-2u}{16\lambda},$$

Step 2*. If $R^2 > Q^3$ then

$$A = -sign(R)\left(|R| + \sqrt{R^2 - Q^3}\right)^{\frac{1}{3}};$$
$$B = \begin{cases} A, & if A = 0 \\ \frac{Q}{A}, & otherwise \end{cases};$$
$$x = G^{-1}\left(A + B + \frac{1}{2}\right).$$

Otherwise,

$$\theta = arccos\left(\frac{R}{\sqrt{Q^3}}\right);$$
$$x = G^{-1}\left(\frac{1}{2} - 2\sqrt{Q}\,cos\left(\frac{\theta - 2\pi}{3}\right)\right),$$

where $G^{-1}(x)$ is the inverse function of any baseline distribution function $G(x)$. If $\lambda = 0$, then $x = G^{-1}(u)$. The updated algorithm can be used for generating random numbers that follows any TeSU distribution. Consequently, the median of TeSU−G family can be computed by taking $u = \frac{1}{2}$, that is,

$$x_{med} = \begin{cases} G^{-1}\left(\frac{1}{2}\right), & if \lambda = 0 \\ G^{-1}\left[\frac{1}{2} - 2\sqrt{\frac{1}{12}\left(1 - \frac{1}{\lambda}\right)}\,cos\left(\frac{90 - 2\pi}{3}\right)\right], & otherwise \end{cases}. \tag{15}$$

Setting $G(x)$ to be the cdf in equation (9) of the Weibull distribution, the algorithm is then modified to generate random numbers from the TeSU−Weibull distribution. The modified algorithm is as follows:

Step 1**. Compute

$$Q = \frac{1}{2}\left(1 - \frac{1}{\lambda}\right), \lambda \neq 0;$$
$$R = \frac{1-2u}{16\lambda}.$$

Step 2**. If $R^2 > Q^3$ then

$$A = -sign(R)\left(|R| + \sqrt{R^2 - Q^3}\right)^{\frac{1}{3}};$$
$$B = \begin{cases} A & if A = 0 \\ \frac{Q}{A} & otherwise \end{cases};$$

$$x = \left(-\frac{1}{\tau}log\left(\frac{1}{2} - A - B\right)\right)^{\frac{1}{\beta}}.$$

Otherwise,

$$\theta = arccos\left(\frac{R}{\sqrt{Q^3}}\right);$$

$$x = \left(-\frac{1}{\tau}log\left(\frac{1}{2} + 2\sqrt{Q}\,cos\left(\frac{\theta - 2\pi}{3}\right)\right)\right)^{\frac{1}{\beta}}.$$

If $\lambda = 0$, then $x = (-\frac{1}{\tau}log(u))^{\frac{1}{\beta}}$. Hence, the median of the TeSU−W is solved using equation (15) as

$$x_{med(TeSU-W)} = \begin{cases} \left(-\frac{1}{\tau}log(\frac{1}{2})\right)^{\frac{1}{\beta}}, & if \lambda = 0 \\ \left(-\frac{1}{\tau}log\left(\frac{1}{2} + 2\sqrt{Q}\,cos\left(\frac{90-2\pi}{3}\right)\right)\right)^{\frac{1}{\beta}}, & otherwise \end{cases}.$$

6. The TeSU−W Model Parameter Estimation

Let $X_1, X_2, \ldots, X_n$ be an independently and identically distributed random variables from a TeSU−Weibull distribution. Then the likelihood function of the TeSU−W is given by

$$\mathbb{L} = \prod_{i=1}^{n}\left[\tau\beta x_i^{\beta-1}e^{-\tau x_i^{\beta}}\left(1 + 2\lambda - 12\lambda e^{-\tau x_i^{\beta}} + 12\lambda e^{-2\tau x_i^{\beta}}\right)\right].$$

Then the log-likelihood function is given by

$$l = \sum_{i=1}^{n}\log\left[\tau\beta x_i^{\beta-1}e^{-\tau x_i^{\beta}}\left(1 + 2\lambda - 12\lambda e^{-\tau x_i^{\beta}} + 12\lambda e^{-2\tau x_i^{\beta}}\right)\right]. \tag{16}$$

The derivatives of (16) with respect to the parameters τ, β and λ are given as follow:

$$\frac{\partial l}{\partial \tau} = \frac{n}{\tau} - \sum_{i=1}^{n}x_i^{\beta} + 12\lambda\sum_{i=1}^{n}\frac{z_i x_i^{\beta}e^{-\tau x_i^{\beta}}}{y_i}; \tag{17}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\tau x_i^{\beta}log(x_i) + \sum_{i=1}^{n}log(x_i) + 12\tau\lambda\sum_{i=1}^{n}\frac{z_i x_i^{\beta}log(x_i)e^{-\tau x_i^{\beta}}}{y_i}; \tag{18}$$

$$\frac{\partial l}{\partial \lambda} = 2\sum_{i=1}^{n}\frac{1 - 6e^{-\tau x_i^{\beta}} + 6e^{-2\tau x_i^{\beta}}}{y_i}, \tag{19}$$

where $y_i = 1 + 2\lambda - 12\lambda e^{-\tau x_i^{\beta}} + 12\lambda e^{-2\tau x_i^{\beta}}$ and $z_i = 1 - 2e^{-\tau x_i^{\beta}}$.

Setting equations (17), (18) and (19) equal to zero, the numerical maximum likelihood estimates $\hat{\tau}$, $\hat{\beta}$ and $\hat{\lambda}$ of the parameters can be obtained by any numerical method like the Newton-Raphson iterative method.

7. The Asymptotic Properties of TeSU−W ML Estimates

This section presents the simulation study result conducted to verify the performance of the ML estimates of TeSU-W distribution when sample sizes are varied. The simulation process proceeded with 2 sets of data from TeSU-W distribution and considered the following sets of parameters values: $s_1 = \{\tau = 1.4, \beta = 2.5, \lambda = 0.5\}$ and $s_2 = \{\tau = 1.4, \beta = 1, \lambda = -0.5\}$. For each s_i, the study is processed for varied sample sizes $n \in \{50, 100, 200, 500, 1000\}$. Also, at each replication, the ML estimates $\hat{\tau}$, $\hat{\beta}$ and $\hat{\lambda}$ are computed. The process is repeated 1000 times for each s_i, and some diagnostic statistics like the average estimate (AE), biases and mean squared errors (MSE) are determined and are summarized in Tables 1 and 2. These indicate that the MSE of $\hat{\tau}$, $\hat{\beta}$ and $\hat{\lambda}$ for sets s_i, $i = 1, 2$ decay toward zero as the sample size increases, that is, $\lim_{n\to\infty} MSE = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\hat{P}_i - P\right)^2 = 0$ where $P = \tau, \beta, \lambda$. This implies that the AE of the parameters for each s_i tend to be closer to the true parameters as sample sizes n increases.

Table 1: *Some diagnostic statistics of TeSU-W for s_1 at varied n*

n	MLE	AE	$Bias$	MSE
	$\hat{\tau}$	1.446	0.046	0.047
50	$\hat{\beta}$	2.716	0.216	0.124
	$\hat{\lambda}$	0.569	0.069	0.041
	$\hat{\tau}$	1.440	0.040	0.022
100	$\hat{\beta}$	2.694	0.194	0.070
	$\hat{\lambda}$	0.552	0.052	0.020
	$\hat{\tau}$	1.433	0.033	0.010
200	$\hat{\beta}$	2.683	0.183	0.051
	$\hat{\lambda}$	0.555	0.055	0.010
	$\hat{\tau}$	1.429	0.029	0.005
500	$\hat{\beta}$	2.679	0.179	0.039
	$\hat{\lambda}$	0.547	0.047	0.005
	$\hat{\tau}$	1.428	0.028	0.003
1000	$\hat{\beta}$	2.671	0.171	0.032
	$\hat{\lambda}$	0.544	0.044	0.003

Table 2: *Some diagnostic statistics of TeSU-W for s_2 at varied n*

n	MLE	AE	$Bias$	MSE
	$\hat{\tau}$	1.509	0.109	0.044
50	$\hat{\beta}$	1.107	0.107	0.036
	$\hat{\lambda}$	−0.486	0.014	0.007
	$\hat{\tau}$	1.483	0.083	0.019
100	$\hat{\beta}$	1.084	0.084	0.014
	$\hat{\lambda}$	−0.498	0.002	0.001
	$\hat{\tau}$	1.473	0.073	0.012
200	$\hat{\beta}$	1.080	0.080	0.009
	$\hat{\lambda}$	−0.500	0.000	0.000
	$\hat{\tau}$	1.463	0.063	0.006
500	$\hat{\beta}$	1.073	0.073	0.006
	$\hat{\lambda}$	−0.500	0.000	0.000
	$\hat{\tau}$	1.463	0.063	0.005
1000	$\hat{\beta}$	1.072	0.072	0.006
	$\hat{\lambda}$	−0.500	0.000	0.000

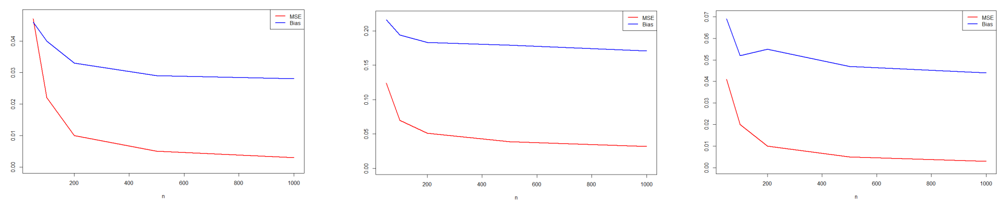

Figure 5: *Plots of the MSE and Bias for $\hat{\tau}$ (left), $\hat{\beta}$ (center) and $\hat{\lambda}$ (right) in s_1*

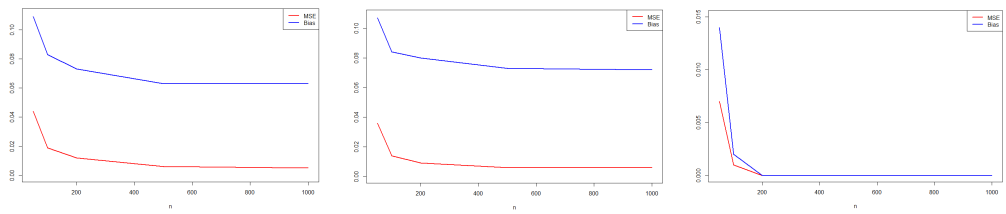

Figure 6: *Plots of the MSE and Bias for $\hat{\tau}$ (left), $\hat{\beta}$ (center) and $\hat{\lambda}$ (right) in s_2*

8. The TeSU−W in Life Data Analysis

This section illustrates the TESU−W distribution when applied to real life dataset using a package "fitdistrplus" of the R software. The result of the TeSU−W distribution will then be compared to the recent work of Arif [3] on the New Extended Exponentiated Weibull (NEEW) distribution and the work of Malik [11] on the New Transmuted Weibull (NTW) distribution. The New NEEW pdf is given by

$$f_{NEEW}(x) = \frac{\alpha\lambda x^{\lambda-1} e^{-\alpha x^{\lambda}} (1 - e^{-\alpha x^{\lambda}}) \left[e^{\theta(1-e^{\alpha x^{\lambda}})} (2 + \theta - \theta e^{-\alpha x^{\lambda}}) + 2 \right]}{e^{\theta} + 1}, x \geq 0, \alpha, \lambda, \theta > 0$$

while the pdf of the New Transmuted Weibull (NTW) distribution is given by

$$f_{NTW}(x) = \theta\lambda x^{\lambda-1} e^{-\theta x^{\lambda}} \left[1 - \beta + \frac{2\beta}{2 - e^{-\theta x^{\lambda}}} \right], x, \theta, \lambda > 0, -1 \leq \beta \leq 1.$$

Model diagnostics are done with the determination of the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K−S), Cramer-von Mises (W*) and the Anderson-Darling (A) statistics. As a rule of thumb, a smaller value of these statistics implies a better fit of the model using the proposed distribution to the given dataset.

The COVID-19 cases in India from May 1, 2020 to June 14, 2020 are used in this study. This data set can be accessed from the siteweb (Coronavirus Update (Live):7,114,524 Cases and 406,552 Deaths from COVID-19 Virus Pandemic - Worldometer). For calculation purpose, we consider data (10^{-2}). Table 3 lists the MLEs of the TeSU-W, NEEW, NTW and TeSU-E distributions fitted to the given dataset while Table 4 shows the different diagnostics statistics. Consistently in all diagnostic criterion, the TeSU−W gave the lowest values of the diagnostic statistics compared to NEEW, NTW and TeSU-E distributions. It may imply that the TeSU−W works well when fitted with the given dataset and that the ML estimates are asymptotically equal to the true values of the parameters. In addition, same result is observed from the plots of the fitted models and the histogram of the dataset given in Figure 7.

Table 3: *ML estimates of the fitted models using the different distributions*

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\tau}$
TeSU−W		3.05948300		0.47928580	0.00000183
NEEW	0.00111771		0.00000082	1.69201100	
NTW		−0.99999996	0.00047039	1.86047516	
TeSU−E				−0.49999997	0.01248669

Table 4: *Some diagnostic statistics of the fitted models using the different distributions*

Distribution	AIC	BIC	$K-S$	A	$W*$
TeSU−W	428.9631	434.3830	0.1005729	0.4046896	0.0566358
NEEW	433.5504	438.9703	0.1271787	0.6914853	0.1042066
NTW	433.0563	438.4763	0.1255249	0.7159459	0.1097673
TeSU−E	446.9146	450.528	0.1660065	2.2543706	0.3196720

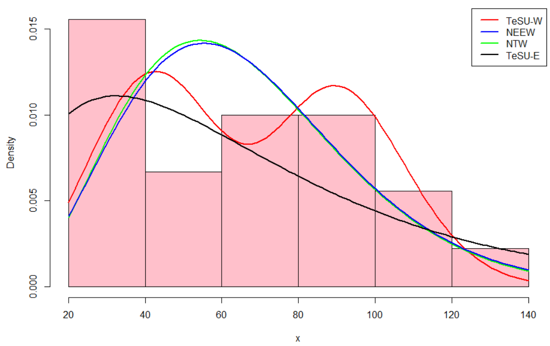

Figure 7: *Plots of the models fitted to the COVID-19 data*

9. Concluding Remarks

This paper derives a new family of distributions called the T-extended Standard U-quadratic−G family of distributions or simply, TeSU−G family. Derived models of the family called as TeSU−Weibull distribution (TeSU−W) and the TeSU-Exponential distribution (TeSU−E) are generated and its limiting behavior, moments, mean and variance, and moment generating function are computed. Also, formula of the median for the TeSU−G family as well as for TeSU−W distribution are derived. Furthermore, the Maximum Likelihood (ML) estimates of the TeSU−W distribution is derived. Simulation study shows that the ML estimates is asymptotically equal to the true value of the parameters as sample sizes increases. This can be observed by the values of MSE that goes to zero, on the average. Life data analysis using the TeSU−W distribution to a COVID-19 dataset provides better fit compared with the existing New Extended Exponentiated Weibull (NEEW) distribution and the New Transmuted Weibull (NTW) distribution as explored by Arif [3] in 2022 and Malik [11] in 2022, respectively.

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References

[1] Alizadeh, M., Tahir, M. H., Zubair, M. and Hamedani, G. G. (2015). The Kumaraswamy Marshall-Olkin family of distributions. *Journal of the Egyptian Mathematical Society*, 23:546–557.

[2] Alzaatreh, A. and Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71:63–79.

[3] Arif, M., Khan, D. M., Aamir, M., Khalil, U., Bantan, R. A. R. and Elgarhy, M. (2022). Modeling COVID-19 Data with a Novel Extended Exponentiated Class of Distributions. *Journal of Mathematics*, Article ID 1908161, https://doi.org/10.1155/2022/1908161.

[4] Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, 12:171–178.

[5] Cordeiro, G.M., Ortega, E. M. M. and Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11:1–27.

[6] Cordeiro, G. M., Alizadeh, M., Tahir, M. H., Mansoor, M., Bourguignon, M. and G., H. G. (2017). The generalized odd log logistic family of distributions: properties, regression models and applications. *Journal of Statistical Computation and Simulation*, 87:908–932.

[7] Cordeiro, G. M. and de Castro, M.. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81:883–893.

[8] Cordeiro, G. M., Silva, G. O. and Ortega, E. M. M. (2012). The beta extended Weibull distribution. *Journal of Probability and Statistical Science*, 10:15–40.

[9] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, 31:497–512.

[10] Gupta, R. C., Gupta, P. L. and Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. *Communication Statistics Theory and Methods*, 27:887–904.

[11] Malik, A. S. and Ahmad, S. P. (2022). A New Transmuted Weibull Distribution: Properties and Application. *Pakistan Journal of Statistics and Operation Research*, 18:369–381.

[12] Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84:641–652.

[13] Nofal, Z. M., Afify, A. Z., Yousof, H. M. and Cordeiro, G. M. (2017). The generalized transmuted - G family of distributions. *Communication Statistics Theory and Methods*, 49:4119–4136.

[14] Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *Research report*.

[15] Silva, R., Silva, F. G., Ramos, M., Cordeiro, G. M., Marinho, P. and De Andrade, T. A. N. (2019). The Exponentiated Kumaraswamy - G Class: General Properties and Application. *R.C de Estadistica*, 42:1–33.

[16] Zografos, K. J. and Balakrishnan, N. (2009). On families of beta and generalized gamma-generated distribution and associate inference. *Statistical Methodology*, 6:344–362.