EXPONENTIATED ADYA DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

In this article, we introduce a new generalization of Adya distribution known as Exponentiated Adya distribution. We have also derived and discussed its different statistical properties. The Exponentiated Adya distribution has two parameters (scale and shape). The different structural properties of the proposed distribution have been obtained. The maximum likelihood estimation technique is also used for estimating the parameters of the proposed distribution. Finally, a real lifetime data set is used for examining the superiority of the proposed distribution.

Keywords: Exponentiated distribution, Adya distribution, Order statistics, Entropies, Reliability analysis, Maximum likelihood Estimation.

1. Introduction

A new family of distributions namely the exponentiated exponential distribution was introduced by Gupta et al. [1]. The family has two parameters scale and shape, which are similar to the weibull or gamma family. Later Gupta and Kundu [2], studied some properties of the same distribution. They observed that many properties of the new family are similar to those of the weibull or gamma family. Hence the distribution can be used an alternative to a weibull or gamma distribution. The two-parameteric gamma and weibull are the most popular distributions for analyzing any lifetime data. The gamma distribution has a lot of applications in different fields other than lifetime distributions. The two parameters of gamma distribution represent the scale and the shape parameter and because of the scale and shape parameter, it has quite a bit of flexibility to analyze any positive real data. But one major disadvantage of the gamma distribution is that, if the shape parameter is not an integer, the distribution function or survival function cannot be expressed in a closed form. This makes gamma distribution little bit unpopular as compared to the Weibull distribution, whose survival function and hazard function are simple and easy to study. Nowadays exponentiated distributions and their mathematical properties are widely studied for applied science experimental data sets. Exponentiated weibull family as an extension of weibull distribution studied by Pal et al. [3]. Exponentiated generalized Lindley distribution studied by Rodrigues et al. [4]. Hassan et al. [5] discussed Exponentiated Lomax geometric distribution with its properties and applications. Nasiru et al. [6] obtained exponentiated generalized power series family of distributions. Rather and subramanian [7] discussed the exponentiated Mukherjee-Islam distribution which shows more flexibility than the classical distribution. Rather and subramanian [8] discussed the exponentiated ishita distribution with properties and Applications. Subramanian and Rather [9] obtained the exponentiated version of power distribution with its properties and estimation. Rather and subramanian [10] discussed the exponentiated Garima distribution which shows more flexibility than the classical distribution. Ganie and Rajagopalan [11] obtained exponentiated Aradhana distribution with its properties and applications. Recently, Rather et al. [12] discussed the exponentiated Ailamujia distribution with statistical inference and applications of medical science which shows better performance than the classical distributions.

Adya distribution is a newly proposed one parameteric distribution formulated by Shanker et al. [13] for several engineering applications and calculated its various characteristics including stochastic ordering, moments, order statistics, Renyi entropy, stress strength reliability and ML estimation. The two parameters of an exponentiated Adya distribution represent the shape and the scale parameter. It also has the increasing or decreasing failure rate depending of the shape parameter. The density function varies significantly depending of the shape parameter.

2. Exponentiated Adya Distribution (EAD)

The probability density function of Adya distribution is given by

$$g(x)=\frac{\theta^3}{\theta^4+2\theta^2+2}(\theta+x)^2 e^{-\theta x};\ x>0,\theta>0 \tag{1}$$

and the cumulative distribution function of Adya distribution is given by

$$G(x)=1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\ ;x>0,\theta>0 \tag{2}$$

A random variable X is said to have an exponentiated distribution, if its cumulative distribution function is given by

$$F_\alpha(x)\ =\ \left(G(x)\right)^\alpha\quad ;\ x\in R',\ \alpha>0 \tag{3}$$

Then X is said to have an exponentiated distribution.
The probability density function of X is given by

$$f_\alpha(x)=\alpha\left(G(x)\right)^{\alpha-1}g(x) \tag{4}$$

By Substituting (2) in (3), we will obtain the cumulative distribution function of Exponentiated Adya distribution

$$F_\alpha(x)=\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^\alpha\quad ;\ x>0,\theta>0,\alpha>0 \tag{5}$$

and the probability density function of Exponentiated Adya distribution can be obtained as

$$f_{\alpha}(x)=\frac{\alpha\theta^3(\theta+x)^2e^{-\theta x}}{\theta^4+2\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1} \tag{6}$$

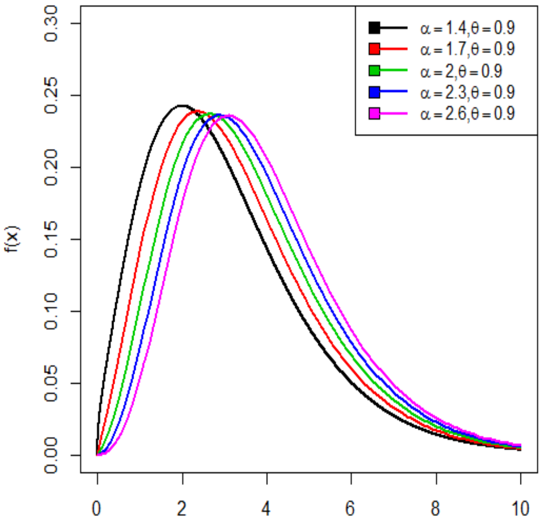

Fig 1: *pdf plot of exponentiated Adya distribution*

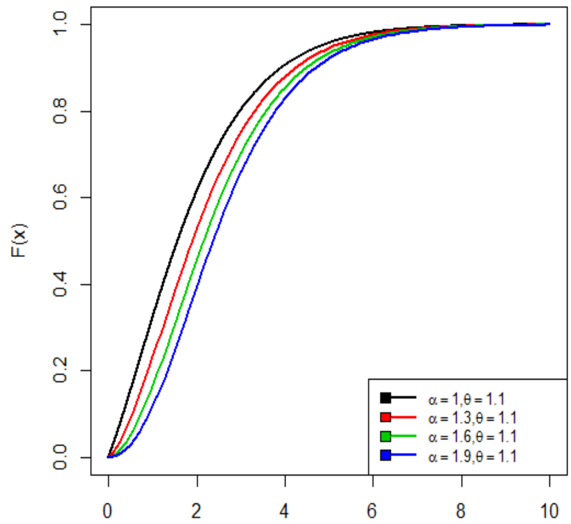

Fig 2: *cdf plot of exponentiated Adya distribution*

3. Reliability Analysis

In this section, we will obtain the survival function, hazard function and Reverse hazard rate function of the Exponentiated Adya distribution.

The survival function of Exponentiated Adya distribution is given by

$$S(x)=1-\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha} \tag{7}$$

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality and is given by

$$h(x)=\left(\frac{\frac{\alpha\theta^3(\theta+x)^2e^{-\theta x}}{\theta^4+2\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}}{1-\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha}}\right) \tag{8}$$

The reverse hazard rate of exponentiated Adya distribution is given by

$$h_r(x)=\frac{\alpha\theta^3\left(\theta+x\right)^2e^{-\theta x}}{\theta x(\theta x+2\theta^2+2)e^{-\theta x}} \tag{9}$$

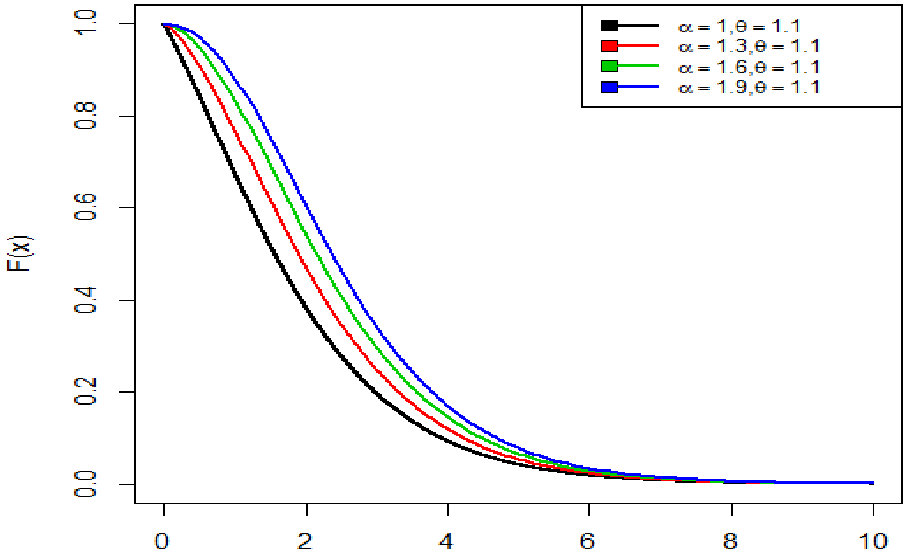

Fig 3: *Survival plot of exponentiated Adya distribution*

4. Statistical Properties

In this section, we will discuss the different statistical properties of the proposed Exponentiated Adya distribution.

4.1 Moments

Suppose X is a random variable following exponentiated Adya distribution with parameters α and θ, then the rth order moment $E(X^r)$ for a given probability distribution is given by

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_\alpha(x)dx$$

$$E(X^r) = \frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\int_0^\infty x^r(\theta+x)^2 e^{-\theta x}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1} dx \tag{10}$$

Using Binomial expansion of

$$\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}\left\{\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right\}^i (-1)^i$$

Equation (10) will become

$$E(X^r) = \frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\int_0^\infty x^r(\theta+x)^2 e^{-\theta x(1+i)}\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)^i dx \tag{11}$$

Again using Binomial expansion of

$$\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)^i = \sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)^k$$

Equation (11), will reduce to

$$E(X^r) = \frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\left(\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)^k\int_o^{\infty}x^r(\theta+x)^2 e^{-\theta x(1+i)}dx \tag{12}$$

After simplification, we obtain

$$E(X^r) = \alpha\theta^3\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{(\theta^2+2\theta^3+2\theta)^k}{(\theta^4+2\theta^2+2)^{k+1}}$$
$$\times\left(\frac{\theta^2(1+i)^2\Gamma(r+4k+1)+\Gamma(r+4k+3)+2\theta(\theta(1+i)\Gamma(r+4k+2))}{\theta(1+i)^{r+4k+3}}\right) \tag{13}$$

Since equation (13) is a convergent series for all r ≥ 0, therefore all the moments exist.
Therefore

$$E(X) = \alpha\theta^3\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{(\theta^2+2\theta^3+2\theta)^k}{(\theta^4+2\theta^2+2)^{k+1}}\left(\frac{\theta^2(1+i)^2\Gamma(4k+2)+\Gamma(4k+4)+2\theta(\theta(1+i)\Gamma(4k+3))}{\theta(1+i)^{4k+4}}\right) \tag{14}$$

$$E(X^2) = \alpha\theta^3\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{(\theta^2+2\theta^3+2\theta)^k}{(\theta^4+2\theta^2+2)^{k+1}}\left(\frac{\theta^2(1+i)^2\Gamma(4k+3)+\Gamma(4k+5)+2\theta(\theta(1+i)\Gamma(4k+4))}{\theta(1+i)^{4k+5}}\right) \tag{15}$$

Therefore, the Variance of X can be obtained as

$$V(X) = E(X^2)-(E(X))^2$$

4.2 Harmonic mean

The Harmonic mean for the proposed Exponentiated Adya distribution can be obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^{\infty}\frac{1}{x}f_{\alpha}(x)dx$$

$$H.M. = \frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\int_0^{\infty}\frac{1}{x}(\theta+x)^2 e^{-\theta x}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}dx \tag{16}$$

Using Binomial expansion in equation (16), we get

$$H.M = \frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\int_0^{\infty}\frac{1}{x}(\theta+x)^2 e^{-\theta x(1+i)}\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)^i dx \tag{17}$$

On using Binomial expansion in equation (17), we obtain

$$H.M = \frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\left(\frac{\theta^2x^2+2\theta^3x+2\theta x)}{\theta^4+2\theta^2+2}\right)^k\int_0^{\infty}\frac{1}{x}(\theta+x)^2 e^{-\theta x(1+i)}dx \tag{18}$$

After the simplification of equation (18), we obtain

$$H.M = \alpha\theta^3 \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i \binom{\alpha-1}{i}\binom{i}{k}\frac{\left(\theta^2+2\theta^3+2\theta\right)^k}{\left(\theta^4+2\theta^2+2\right)^{k+1}}\left(\frac{\theta^2(1+i)\Gamma(4k+1)+2\theta(1+i)\Gamma(4k+1)+\Gamma(4k+2)}{\theta(1+i)^{4k+2}}\right) \quad (19)$$

4.3 Moment Generating Function and Characteristics Function

Let X have an exponentiated Adya distribution, then the moment generating function of X is obtained as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f_\alpha(x)dx$$

Using Taylor's series, we get

$$M_X(t) = \int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+\ldots\right)f_\alpha(x)dx \quad (20)$$

$$M_X(t) = \alpha\theta^3 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{t^j}{j!}\frac{\left(\theta^2+2\theta^3+2\theta\right)^k}{(\theta^4+2\theta^2+2)^{k+1}}$$

$$\times\left(\frac{\theta^2(1+i)^2\Gamma(j+4k+1)+\Gamma(j+4k+3)+2\theta(\theta(1+i)\Gamma(j+4k+2))}{\theta(1+i)^{j+4k+3}}\right) \quad (21)$$

Similarly, the characteristic function of Exponentiated Adya distribution is given by

$$\varphi_X(t) = \alpha\theta^3 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{mt^j}{j!}\frac{\left(\theta^2+2\theta^3+2\theta\right)^k}{(\theta^4+2\theta^2+2)^{k+1}}$$

$$\times\left(\frac{\theta^2(1+i)^2\Gamma(j+4k+1)+\Gamma(j+4k+3)+2\theta(\theta(1+i)\Gamma(j+4k+2))}{\theta(1+i)^{j+4k+3}}\right) \quad (22)$$

5. Order Statistics

Order statistics represents the arranging of samples in an ascending order. Order statistics also has wide field in reliability and life testing. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots X_n$ drawn from the continuous population with probability density function $f_x(x)$ and cumulative distribution function $F_x(x)$, then the pdf of r^{th} order statistics $X_{(r)}$ can be written as

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \tag{23}$$

Substitute the values of equation (5) and (6) in equation (13), we will obtain the pdf of r[th] order statistics $X_{(r)}$ for exponentiated Adya distribution and is given by

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{\alpha\theta^3(\theta+x)^2e^{-\theta x}}{\theta^4+2\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}$$

$$\times\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha(r-1)}\times\left(1-\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha}\right)^{n-r} \tag{24}$$

Therefore, the probability density function of higher order statistics $X_{(n)}$ for exponentiated Adya distribution can be obtained as

$$f_{x(n)}(x)=n\frac{\alpha\theta^3(\theta+x)^2e^{-\theta x}}{\theta^4+2\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha(n-1)} \tag{25}$$

and the pdf of first order statistics $X_{(1)}$ for exponentiated Adya distribution can be obtained as

$$f_{x(1)}(x)=n\frac{\alpha\theta^3\left(\theta+x\right)^2e^{-\theta x}}{\theta^4+2\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}\left(1-\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha}\right)^{n-1} \tag{26}$$

6. Maximum Likelihood Estimation

In this section, we will discuss the maximum likelihood estimation for estimating the parameters of exponentiated Adya distribution. Let $X_1, X_2, \ldots, X_n$ be the random sample of size n from the Exponentiated Adya distribution, then the likelihood function can be written as

$$L(\alpha,\theta)=\frac{\left(\alpha\theta^3\right)^n}{(\theta^4+2\theta^2+2)^n}\prod_{i=1}^{n}\left((\theta+x)^2e^{-\theta x}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}\right) \tag{27}$$

The log likelihood function is given by

$$\log L(\alpha,\theta)=n\log\alpha+3n\log\theta-n\log(\theta^4+2\theta^2+2)+2\sum_{i=1}^{n}\log(\theta+x)-\theta\sum_{i=1}^{n}x_i$$

$$+(\alpha-1)\sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right) \tag{28}$$

The maximum likelihood estimates of α, θ which maximizes (28), must satisfy the normal equations given by

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)=0 \tag{29}$$

$$\Rightarrow \hat{\alpha} = \frac{n}{\sum_{i=1}^{n} log\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)} \quad (30)$$

$$\frac{\partial \log L}{\partial \theta} = \frac{3n}{\theta} - n\left(\frac{(4\theta^3+4\theta)}{\theta^4+2\theta^2+2}\right) + 2\sum_{i=1}^{n}\left(\frac{1}{(\theta+x)}\right) - \sum_{i=1}^{n} x_i + (\alpha-1)\psi\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right) = 0 \quad (31)$$

Where ψ (.) is the digamma function.

It is important to mention here that the analytical solution of the above system of non-linear equation is unknown. Algebraically it is very difficult to solve the complicated form of likelihood system of nonlinear equations. Therefore, we use R and wolfram mathematics for estimating the required parameters.

7. Information Measures of Exponentiated Adya Distribution

7.1 Renyi Entropy

The Renyi entropy is named after Alfred Renyi in the context of fractual dimension estimation, the Renyi entropy forms the basis of the concept of generalized dimensions. The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. Entropies quantify the diversity, uncertainty, or randomness of a system. For a given probability distribution, Renyi entropy is given by

$$e(\beta) = \frac{1}{1-\beta}\log\left(\int_0^\infty f^\beta(x)dx\right)$$

Where, $\beta > 0$ and $\beta \neq 1$

$$e(\beta) = \frac{1}{1-\beta}\log\left(\int_0^\infty \left\{\frac{\alpha\theta^3(\theta+x)^2 e^{-\theta x}}{\theta^4+2\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}\right\}^\beta dx\right) \quad (32)$$

$$e(\beta) = \frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\right)^\beta \int_0^\infty (\theta+x)^{2\beta} e^{-\theta\beta x}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\beta(\alpha-1)} dx\right) \quad (33)$$

Using binomial expansion in (33), we get

$$e(\beta) = \frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\right)^\beta \sum_{i=0}^\infty (-1)^i \binom{\beta(\alpha-1)}{i}\int_0^\infty (\theta+x)^{2\beta} e^{-\theta x(\beta+i)}\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)^i dx\right) \quad (34)$$

Again using binomial expansion in (34), we get

$$e(\beta) = \frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\right)^\beta \sum_{i=0}^\infty\sum_{k=0}^\infty (-1)^i \binom{\beta(\alpha-1)}{i}\binom{i}{k}\left(\frac{\theta^2x^2+2\theta^3x+2\theta x}{\theta^4+2\theta^2+2}\right)^k \int_0^\infty (\theta+x)^{2\beta} e^{-\theta x(\beta+i)} dx\right) \quad (35)$$

After the simplification of (35) we obtain

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\alpha\theta^3\right)^{\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\beta(\alpha-1)}{i}\binom{i}{k}\binom{2\beta}{j}\frac{\left(\theta^2+2\theta^3+2\theta\right)^k}{\left(\theta^4+2\theta^2+2\right)^{\beta+k}}\theta^{2\beta-j}\frac{\Gamma(4k+j+1)}{\theta(\beta+i)^{4k+j+1}}\right) \tag{36}$$

7.2 Tsallis Entropy

A generalization of Boltzmann-Gibbs (B-G) statistical mechanics initiated by Tsallis has gained a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy for a continuous random variable it is defined as

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_0^{\infty}f^{\lambda}(x)dx\right)$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_0^{\infty}\left\{\frac{\alpha\theta^3(\theta+x)^2e^{-\theta x}}{\theta^4+2\theta^2+2}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\alpha-1}\right\}^{\lambda}dx\right) \tag{37}$$

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\right)^{\lambda}\int_0^{\infty}(\theta+x)^{2\lambda}e^{-\lambda\theta x}\left(1-\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)e^{-\theta x}\right)^{\lambda(\alpha-1)}dx\right) \tag{38}$$

Using binomial expansion in (38), we get

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\right)^{\lambda}\sum_{i=0}^{\infty}(-1)^i\binom{\lambda(\alpha-1)}{i}\int_0^{\infty}(\theta+x)^{2\lambda}e^{-\theta x(\lambda+i)}\left(1+\frac{\theta x(\theta x+2\theta^2+2)}{\theta^4+2\theta^2+2}\right)^{i}dx\right) \tag{39}$$

Again using binomial expansion in (39), we obtain

$$=\frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^3}{\theta^4+2\theta^2+2}\right)^{\lambda}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\lambda(\alpha-1)}{i}\binom{i}{k}\left(\frac{\theta^2x^2+2\theta^3x+2\theta x}{\theta^4+2\theta^2+2}\right)^{k}\int_0^{\infty}(\theta+x)^{2\lambda}e^{-\theta x(\lambda+i)}dx\right) \tag{40}$$

After the simplification of (40), we get

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\left(\alpha\theta^3\right)^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\lambda(\alpha-1)}{i}\binom{i}{k}\binom{2\lambda}{j}\frac{\left(\theta^2+2\theta^3+2\theta\right)^k}{\left(\theta^4+2\theta^2+2\right)^{\lambda+k}}\theta^{2\lambda-j}\frac{\Gamma(4k+j+1)}{\theta(\lambda+i)^{4k+j+1}}\right) \tag{41}$$

7. Applications

In this section, we use a real-life time data set in exponentiated Adya distribution and the model has been compared over Adya distribution.
The following data set in table 1 represents the bladder cancer patients (n = 128) of the remission times (in months) reported by Lee and Wang [14].

Table 1: *Data represents the 123 blood cancer patients*

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
3.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	6.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64
17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40
5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02
2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69		

In order to compare the exponentiated Adya distribution with Adya distribution. We consider the Criteria like BIC (Bayesian information criterion), AIC (Akaike information criterion), AICC (Corrected Akaike information criterion) and -2logL. The better distribution is which corresponds to lesser values of AIC, BIC, AICC and -2logL. For calculating AIC, BIC, AICC and -2logL can be evaluated by using the formulas as follows.

$$AIC = 2k - 2\log L \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1} \quad \text{and} \quad BIC = k\log n - 2\log L$$

Where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model.

Table 2: *Fitted distributions of the data set and criteria for comparison*

Distribution	MLE	S.E	-2logL	AIC	BIC	AICC
Exponentiated	$\hat{\alpha} = 0.3967$ $\hat{\theta} = 0.1924$	$\hat{\alpha} = 0.0491$ $\hat{\theta} = 0.0201$	829.448	833.448	839.152	833.544
Adya	$\hat{\theta} = 0.3212$	$\hat{\theta} = 0.015$	891.2774	893.2774	896.1295	893.3091

From table 2, it can be observed that the exponentiated Adya distribution have the lesser AIC, BIC, AICC and -2log$L$ values as compared to Adya distribution. Hence we can conclude that the exponentiated Adya distribution leads to a better fit than the Adya distribution.

8. Conclusion

In this article, we have introduced a new generalization of Adya distribution called as exponentiated Adya distribution with two parameters (scale and shape). The subject distribution is generated by using the exponentiated technique and the parameters have been obtained by using the maximum likelihood estimator. Some statistical properties along with reliability measures are discussed. The new distribution with its applications in real life-time data has been demonstrated. Finally, the result of a real lifetime data set has been compared with Adya distribution and it has been found that the exponentiated Adya distribution provides a better fit than the Adya distribution.

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