THE EXTENDED ERLANG TRUNCATED EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATION TO STATISTICAL PREDICTION PROBLEM

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Abstract

The Erlang Truncated (ETE) distribution is modified and the new lifetime distribution is called the Extended Erlang Truncated Exponential (EETE) distribution. Some statistical and reliability properties of the new distribution functions have been characterized based on two non-adjacent generalized and dual generalized order statistics. Moreover, we show that these characterization properties provide a beneficial strategy to predict future events, which are based on past or current events and on an arbitrary distribution function. A characterization in statistics is a specific distributional property of a statistic that uniquely identify related parametric family of distributions. In statistical applications, the researchers usually want to verify whether the data that they are dealing with belong to a certain family of DFs. Therefore, the researchers have to rely on a characterization of the assumed distribution and check if the corresponding conditions are satisfied.

Keywords: Generalized order statistics, Dual generalized order statistics, Dilation Characterization of distributions, Point prediction

1. Introduction

Generalized order statistics introduced by [15]. A concept of generalized order statistics (GOSs) as a unified approach to a variety of models of ascendingly ordered random variables (RVs) introduced by [15]. The concept of dual GOSs, denoted by DGOSs, was introduced by [14] as a parallel concept of GOSs to enable a common approach to descendingly ordered RVs. [14] has shown that (cf. Theorem 3.3) there is a direct link between DGOSs and GOSs.

The subclasses m −GOSs and m −DGOSs of GOSs and DGOSs, respectively, contain many important models of ordered RVs such as ordinary order statistics (OOSs), lower and upper record values, k–records, sequential order statistics (SOSs) and type II censored OOSs. For any $1 \le r \le n$, the marginal probability density functions (PDFs) of the rth m-GOS X(r, n; m, k) and m-DGOS X^*(r, n; m, k), based on a continuous distribution function (DF) $F_X(x) = P(X \le x)$ with a PDF $f_X(x)$, are given, respectively, by (cf. [15] and [28]).

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!}\,[\bar{F}_X(x)]^{\gamma_r^{(n)}-1}\left[\frac{1-[\bar{F}_X(x)]^{m+1}}{m+1}\right]^{r-1} f_X(x)\quad,\quad m \neq -1 \tag{1}$$

and

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!}\,F_X{}^{\gamma_r^{(n)}-1}\left[\frac{1-[F_X(x)]^{m+1}}{m+1}\right]^{r-1} f_X(x)\quad,\quad m \neq -1 \tag{2}$$

where $\overline{F} = 1 - F$, $\gamma_r^{(n)} = k + (n-r)(m+1)$ and $C_{r-1}^{(n)} = \prod_{i=1}^{r} \gamma_r^{(n)}$, $1 \leq r \leq n$.

Classical results in characterizations can be found in [2], [3], [4] and [5]. Different results of characterization and its applications in terms of GOSs and DGOSs are derived by many authors. Among these authors are [6], [7], [8], [9], [10], [11] ,[12] and [13].

In this paper, we prove some new characteristic properties of the Erlang truncated exponential DF $\exp(\beta\alpha_\lambda)$, with mean $\frac{1}{(\beta\alpha_\lambda)}$, $\beta > 0, \alpha_\lambda > 0$. The Erlang truncated exponential distribution is prominent in life testing experiments and reliability problems.

The result of this paper enables us to predict the time at which some survived components will have failed or to predict the mean failure time of unobserved lifetimes in a lifetime experiment by using the result of another independent lifetime experiment. Throughout this paper, "$X \overset{d}{=} Y$" means that the RVs X and Y have the same DFs and "$X \sim F$" means that the RV X has the DF F.

The rest of this paper is organized as follows. In Section 2, we reveal some characterization properties for the Erlang truncated exponential distribution based on two nonadjacent m–GOSs (consequently m–DGOSs) from two independent Erlang truncated exponential distributions. In Section 3, we use the results of Section 2 in an application of the prediction problem concerning the lifetime experiments. If support of the distribution $F_X(x)$ be over (a, b), then by convention, we will write

$$X(0, n, m, k) = a \text{ and } X^*(0, n, m, k) = b$$

It may be seen that if Y is a measurable function of X with the relation

$$Y = h(X)$$

Then

$$Y(r, n, m, k) = h[X(r, n, m, k)]$$

if h is an increasing function and

$$Y^*(r, n, m, k) = h[X(r, n, m, k)]$$

if h is a decreasing function

where $X(r, n, m, k)$ is the r^{th} m-GOS and $X^*(r, n, m, k)$ is the r^{th} m- DGOS.

Erlang-Truncated Exponential (ETE) distribution was originally introduced by [1] as an extension of the standard one parameter exponential distribution. The Erlang-Truncated Exponential (ETE) distribution results from the mixture of Erlang distribution and the left truncated one-parameter exponential distribution. The cumulative distribution function CDF $F_X(x)$ and probability density function PDF $f_X(x)$ of the Erlang-Truncated Exponential (ETE) distribution are given by

$$F_X(x) = [1 - e^{-\beta(\alpha_\lambda)x}], \quad 0 \leq x < \infty, \ \beta, \ \lambda > 0 \tag{3}$$

where $\alpha_\lambda = \left(1 - e^{-\lambda}\right)$

and

$$f_X(x) = \beta\,(\alpha_\lambda)\, e^{-\beta(\alpha_\lambda)x}, 0 \leq x < \infty, \ \beta, \ \lambda > 0 \tag{4}$$

respectively, where β is the shape parameter and λ is the scale parameter. The Erlang-Truncated Exponential (ETE) distribution collapses to the classical one-parameter exponential distribution with parameter β and $\lambda \to \infty$.

$X \sim \text{Par}\,(\beta(\alpha_\lambda))$

if X has a Pareto distribution with the DF

$$F_X(x) = [1 - x^{-\beta(\alpha_\lambda)}]\,, 1 < x < \infty\,,\ \beta > 0, \propto > 0, \lambda > 0 \tag{5}$$

$X \sim pow\ (\beta(\alpha_\lambda))$

if X has a power function distribution with the DF

$$F_X(x) = x^{\beta(\alpha_\lambda)}\,, 0 < x < 1\,,\ \beta > 0, \propto > 0, \lambda > 0 \tag{6}$$

It may further be noted that

if $\log X \sim$ *Erlang-truncated exp* $(\beta(\alpha_\lambda))$ then $X \sim Par\ (\beta(\alpha_\lambda))$

if $-\log X \sim$ *Erlang-truncated exp* $(\beta(\alpha_\lambda))$ then $X \sim pow\ (\beta(\alpha_\lambda))$

It has been assumed here throughout that the DF is differentiable $w.r.t.$ its argument.

The new distribution is called the Extended Erlang-Truncated Exponential (EETE) distribution. The Extended Erlang-Truncated Exponential (EETE) distribution has a tractable PDF whose shape is either decreasing or unimodal. The failure rate function (FRF) is characterized by decreasing, constant and increasing shapes and the new three-parameter distribution demonstrates a superior fit when compared with some other well-known three parameter distributions, as we shall see later. Related works are: the Transmuted Erlang Truncated Exponential distribution, due to [25], Marshall–Olkin generalized Erlang-truncated exponential distribution, due to [26] and the generalized Erlang-Truncated Exponential distribution, due to [27].

2. MODEL

The cumulative distribution function CDF $F_X(x)$ and probability density function PDF$f_X(x)$ of the Extended Erlang-Truncated Exponential (EETE) distribution are given by

$$\mathrm{F_X}(x) = [1 - e^{-\beta(\alpha_\lambda)x}]^{\alpha}\,, 0 \le x < \infty,\ \alpha,\ \beta,\ \lambda > 0 \tag{7}$$

and

$$f_X(x) = \alpha\, \beta\, (\alpha_\lambda)\, e^{-\beta(\alpha_\lambda)x}[1 - e^{-\beta(\alpha_\lambda)x}]^{\alpha - 1}\,,\ 0 \le x < \infty,\ \alpha, \beta,\ \lambda > 0 \tag{8}$$

where α and β are the shape parameters and λ is the scale parameter.

The Extended Erlang-Truncated Exponential (EETE) distribution reduces to Erlang-Truncated Exponential (ETE) when $\alpha = 1$.

3. RELIABILITY CHARACTERISTICS

The reliability function R(x) is an important tool for characterizing life phenomenon. R(x) is analytically expressed as $R(x) = 1 - F(x)$. Under certain predefined conditions, the reliability function R(x) gives the probability that a system will operate without failure until a specified time x . The reliability function of the Extended Erlang-Truncated Exponential (EETE) distribution is given by

$$R(x) = 1 - \left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha}\,, 0 \le x < \infty,\ \alpha,\ \beta,\ \lambda > 0 \tag{9}$$

Another important reliability characteristics is the failure rate function. The failure rate function gives the probability of failure for a system that has survived up to time x. The failure rate function h(x) is mathematically expressed $h(x) = f(x)/R(x)$. The failure rate function the Extended Erlang-Truncated Exponential (EETE) distribution is given by:

$$h(x) = \frac{\alpha\, \beta\, (\alpha_\lambda)\, e^{-\beta(\alpha_\lambda)x}[1-e^{-\beta(\alpha_\lambda)x}]^{\alpha-1}}{1-[1-e^{-\beta(\alpha_\lambda)x}]^{\alpha}} \quad ,0 \le x < \infty,\ \alpha, \beta,\ \lambda > 0 \tag{10}$$

4. CHARACTERISTION RESULTS

We assume that all considered DFs are differentiable with respect to their arguments. Moreover, all the considered RVs are non-negative

THEOREM 4.1 :-

Let $X(r,n;m,k),\ m \neq -1$ be the r^{th} *m- GOS* from a sample of size n drawn from a continuous DF F_X (x) with PDF $f_X(x)$. Furthermore, let $Y(r,n,m,k),\ m \neq -1$ be the r^{th} *m- GOS* based on a sample of size n, which is drawn from a continuous DF $F_y(y) = P(Y \le y)$, where Y is independent of X. Finally, let the relation

$$X(s,n,m,k) \overset{d}{=} X(r,n,m,k) + \tilde{Y} \tag{11}$$

be satisfied for all $1 \le r < s \le n$, Then, $\tilde{Y} \overset{d}{=} X(s-r, n-r, m, k)$ and Y ~ exp($\beta\alpha_\lambda$) if and if X ~ exp($\beta\alpha_\lambda$), $\beta > 0, \alpha_\lambda > 0$.

Proof. We first prove the necessary part. Let the moment generating function (MGF) of $X(s,n,m,k)$ be $M_{X_{(s,n,,m,k)}}(t)$. Then, (11) implies that

$$M_{X_{(s,n,m,k)}}(t) = M_{X_{(r,n,m,k)}}(t) \cdot M_{\tilde{Y}}(t) \tag{12}$$

Let us now derive the MGF of the rth m-GOS $X(s,n,m,k)$ based on Erlang truncated exp($\beta\alpha_\lambda$). Clearly, in view of (1), we get

$$M_{X_{(s,n,m,k)}}(t) = \frac{\beta(\alpha_\lambda)\, c_{s-1}^{(n)}}{(r-1)!\,(m+1)^{s-1}} \int_0^\infty e^{-x((\beta\alpha_\lambda)\gamma_s^{(n)}-t)}\, [1-e^{-\beta(\alpha_\lambda)(m+1)x)}]^{s-1} dx$$

Which by using the transformation $y = e^{-\beta\alpha_\lambda(m+1)x}$ takes the form

$$M_{X_{(s,n,m,k)}}(t) = \frac{C_{s-1}^{(n)}}{(s-1)!\,(m+1)^s} \int_0^1 y^{\left(\frac{\gamma_s^{(n)}}{m+1} - \frac{t}{\beta(\alpha_\lambda)(m+1)}\right)^{-1}} (1-y)^{s-1} dy$$

$$= \frac{C_{s-1}^{(n)}}{(m+1)^s} \frac{\Gamma\left(\frac{\gamma_s^{(n)}}{(m+1)} - \frac{t}{\beta\alpha_\lambda(m+1)}\right)}{\Gamma\left(\frac{\gamma_s^{(n)}}{(m+1)} - \frac{t}{\beta\alpha_\lambda(m+1)} + s\right)} = \prod_{i=1}^{s} \left(\frac{\frac{\gamma_s^{(n)}}{(m+1)}}{\frac{\gamma_s^{(n)}}{(m+1)} - \frac{1}{\beta(\alpha_\lambda)(m+1)} + s - i} \right)$$

$$= \prod_{i=1}^{s} \left(\frac{\frac{\gamma_s^{(n)}}{(m+1)}}{\frac{\gamma_s^{(n)}}{(m+1)} - \frac{1}{\beta(\alpha_\lambda)(m+1)} + s - i} \right) = \prod_{i=1}^{s} \left(1 - \frac{t}{\beta(\alpha_\lambda)\gamma_i^{(n)}}\right)^{-1} \tag{13}$$

Where $\Gamma(.)$ is the usual gamma function. On the other hand, in view of (12)

$$M_{\tilde{Y}}(t) = \frac{M_{X_{(s,n,m,k)}}(t)}{M_{X_{(r,n,m,k)}}(t)} = \frac{(m+1)^r}{(m+1)^s} \frac{C_{s-1}^{(n)}}{C_{r-1}^{(n)}} \frac{\Gamma\left(\frac{\gamma_s^{(n)}}{(m+1)} - \frac{t}{\beta(\alpha_\lambda)(m+1)}\right)}{\Gamma\left(\frac{\gamma_s^{(n)}}{(m+1)} - \frac{t}{\beta(\alpha_\lambda)(m+1)} + s\right)} \frac{\Gamma\left(\frac{\gamma_r^{(n)}}{(m+1)} - \frac{t}{\beta(\alpha_\lambda)(m+1)} + r\right)}{\Gamma\left(\frac{\gamma_r^{(n)}}{(m+1)} - \frac{t}{\beta(\alpha_\lambda)(m+1)}\right)}$$

$$= \prod_{i=r+1}^{s}\left(1-\frac{t}{\beta(\alpha_\lambda)\gamma_i^{(n)}}\right)^{-1} = \prod_{j=1}^{s-r}\left(1-\frac{t}{\beta(\alpha_\lambda)\gamma_{r+j}^{(n)}}\right)^{-1} = \prod_{j=1}^{s-r}\left(1-\frac{t}{\beta(\alpha_\lambda)\gamma_j^{(n-r)}}\right)^{-1} \quad (14)$$

Since, $\gamma_{r+j}^{(n)} = k + (n-r-j)(m+1) = \gamma_j^{(n-r)}$. On comparing (14) with (13), we deduce that $M_{\tilde{Y}}(t)$ is the MGF of $Y\,(s-r, n-r, m, k)$, i.e., the $(s-r)^{th}$ m-GOS from a sample of size $(n-r)$ drawn from the DF Erlang truncated $\exp(\beta(\alpha_\lambda))$. This completes the proof of the necessity part.

We now turn to prove the sufficiency part. Let the representation (11) be satisfied with $\tilde{Y} \overset{d}{=} Y(s-r, n-r, m, k)$ and Y ~ $\exp(\beta(\alpha_\lambda))$. Furthermore, let $X(s,n,m,k)$ and $X(r,n,m,k)$ in (11) be m–GOSs, which are based on an unknown DF $F_X(x)$ and they are independent of $Y(s,n,m,k)$. Therefore, the convolution relation (11) implies that

$$f_{X(s,n,m,k)}(x) = \int_0^\infty f_{X(r,n,m,k)}(y) f_{Y((s,n,m,k))}(x-y)dy$$

$$= \frac{\beta(\alpha_\lambda)\, c_{(s-r-1)}^{(n-r)}}{(s-r-1)!(m+1)^{s-r-1}} \int_0^\infty [e^{-\beta(\alpha_\lambda)(x-y)}]^{\gamma_s^{(n)}}$$

$$\times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{s-r-1} f_{X(r,n,m,k)}(y)dy \quad (15)$$

as $\gamma_{s-r}^{(n-r)} = \gamma_s^{(n)}$

Differentiating both the sides of (15) $w.r.t.$ x, we get

$$C_{s-r-1}^{(n-r)} = \gamma_{s-r}^{(n-r)} C_{s-r-2}^{(n-r)} = \gamma_s^{(n)} C_{s-r-2}^{(n-r)}$$

and, $\gamma_s^{(n)} + (m+1) = \gamma_{s-1}^{(n)}$, we get

$$\frac{d}{dx} f_{X(s,n,m,k)}(x) = \frac{(\beta(\alpha_\lambda))^2\, \gamma_s^{(n)} C_{s-r-2}^{(n-r)}}{(s-r-2)!\,(m+1)^{s-r-2}} \int_0^x [e^{-\beta(\alpha_\lambda)(x-y)}]^{\gamma_{s-1}^{(n)}+(m+1)}$$

$$\times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{s-r-2} f_{X(r,n,m,k)}(y)dy$$

$$- \frac{(\beta(\alpha_\lambda))^2\, \gamma_s^{(n)}\, C_{s-r-1}^{(n-r)}}{(s-r-1)!\,(m+1)^{s-r-1}} \int_0^x [e^{-\beta(\alpha_\lambda)(x-y)}]^{\gamma_s^{(n)}}$$

$$\times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{s-r-1} f_{X(r,n,m,k)}(y)dy \quad (16)$$

On the other hand, by using the obvious relation

$$e^{-\beta(\alpha_\lambda)\,\gamma_s^{(n)} z}\left(1-(e^{-\beta(\alpha_\lambda)(z)})^{m+1}\right)^{s-r-1} = e^{-\beta(\alpha_\lambda))\,\gamma_s^{(n)} z}\left(1-e^{-\beta(\alpha_\lambda)\,(m+1)\,z}\right)^{s-r-2}$$

$$-e^{-\beta(\alpha_\lambda))\,\gamma_s^{(n)}+(m+1)z}\left(1-e^{-\beta(\alpha_\lambda)\,(m+1)\,z}\right)^{s-r-2}$$

and by using the representation (3.5), we get

$$\frac{\beta(\alpha_\lambda)\, C_{s-r-1}^{(n-r)}}{(s-r-2)!(m+1)^{s-r-2}} \int_0^x [e^{-\beta(\alpha_\lambda)(x-y)}]^{\gamma_{s-r}^{(n)}+(m+1)} \times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{s-r-2} f_{X(r,n,m,k)}(y)dy$$

$$= \frac{C_{s-r-1}^{(n-r)}}{C_{s-r-2}^{(n-1)}} f_{X(s-r-1,n-1,m,k)}(x) - (m+1)\,(s-r-1) f_{X(s-r,n,m,k)}(x) \quad (17)$$

Thus, by using the relation $\gamma_{s-r}^{(n)} + (m+1)\,(s-r-1) = \gamma_1^{(n)}$ and

$$\frac{C_{s-r-1}^{(n-r)}}{C_{s-r-2}^{(n-1)}} = \frac{\prod_{i=1}^{s-r}\gamma_i^{(n)}}{\prod_{i=2}^{s-r}\gamma_i^{(n)}} = \gamma_s^{(n)}$$

and by combing (23) and (16), weget

$$\frac{d}{dx} f_{X(s,n,m,k)}(x) = \beta(\alpha_\lambda)\gamma_s^{(n)}[f_{X(s-1,n-1,m,k)}(x) - f_{X(s,,m,k)}(x)]$$

or equivalently, by integrating from 0 to x

$$f_{X(s,n,m,k)}(x) = \beta(\alpha_\lambda)\gamma_s^{(n)}[F_{X(s-1,n-1,m,k)}(x) - F_{X(s,n,m,k)}(x)] \tag{18}$$

Now, by using the relation (II) of [15] on page 75, we get

$$F_{X(s-1,n-1,m,k)}(x) - F_{X(s,n,m,k)}(x) = \frac{C_{s-2}^{(n)}}{(s-1)!(m+1)^{s-1}}\,[\bar{F}_X(x)]^{\gamma_s^{(n)}-1}[1-(\bar{F}_X(x))^{m+1}]^{s-1} \tag{19}$$

Therefore, by combing (1), (18) and (19), we have

$$\frac{f_X(x)}{\bar{F}_X(x)} = \beta(\alpha_\lambda)$$

Which implies that $F_X(x) = [1 - e^{-\beta(\alpha_\lambda)x}]$, $x > 0, \beta > 0, \propto > 0, \lambda > 0$. This completes the proof of the sufficient part, as well as the proof of theorem 3.1.

Corollary 4.1. Assume that the RVs X and Y are independent, as we assumed in Theorem 4.1. By replacing the additive relation (11) by the multiplication relation

$$X(s,n,m,k) \stackrel{d}{=} X(r,n,m,k)\cdot\tilde{Y} \tag{20}$$

Then, $\tilde{Y} \stackrel{d}{=} Y(s-r,n-r,m,k)$ and $Y \sim Pareto(\beta(\alpha_\lambda))$ (i.e., $F(x) = [1 - y^{-\beta(\alpha_\lambda)}, y > 1]$ if and if $X \sim Pareto(\beta(\alpha_\lambda),$ $\beta > 0, \propto > 0, \lambda > 0$.

Proof. Here the proof immediately follows, by noting that if $X \sim Pareto(\beta(\alpha_\lambda))$, then $\log X \sim exp\,(\beta(\alpha_\lambda))$ and

$\log X(s,n,m,k) \stackrel{d}{=} \log X(r,n,m,k) + \log\tilde{Y}$

which implies

$X(s,n,m,k) \stackrel{d}{=} X(r,n,m,k)\cdot\tilde{Y}$

Remark 1. In (20), the product $X(r,n,m,k)\times\tilde{Y}$ is called random dilation of $X(r,n,m,k)$, cf. [7]. Moreover, at $s = r+1$, the representation (20) gives

$$X(r+1,n,m,k) \stackrel{d}{=} X(r,n,m,k)\cdot Y(1,n-s+1,m,k) \tag{21}$$

as was obtained by [7] for $X \sim Pareto(\beta(\alpha_\lambda))$. Also, at $r = s-1$, the relation (20) gives

$X(r+1,n,0,1) \stackrel{d}{=} X(r,n-1,0,1)\cdot Y(1,n,0,1)$

(i.e., for the OOSs model), which was obtained by [8], for $X(1,n,0,1) \sim Pareto(\beta(\alpha_\lambda)n)$. Finally, the representation (21) can be written as (for OOSs model) $X(s,n,0,1) \stackrel{d}{=} X(r,n,0,1)\cdot V$, $1 \le r < s$, which was an unsolved problem due to [6]).

Corollary 4.2. Assume that the RVs X and Y are independent, Let $X^*(r,n,m,k)$ and $Y^*(r,n,m,k)$be the r^{th} m- DGOS based on a sample of size n drawn from F_X and F_Y, respectively. By replacing the additive relation (11) by the multiplicative relation

$X^*(s,n,m,k) \stackrel{d}{=} X^*(r,n,m,k)\cdot Y^*$

Then, $Y^* \stackrel{d}{=} Y^*(s-r,n-r,m,k)$ and $Y^* \sim Power(\beta(\alpha_\lambda))$, $\beta > 0, \propto > 0$ (i.e., $F_y(y) = y^{\beta(\alpha_\lambda)}$ if and if $X^* \sim Power(\beta(\alpha_\lambda)$.

Proof. The proof immediately follows from the simple relation between the GOSs and DGOSs, by nothing that if , by noting that if $X \sim Power(\beta(\alpha_\lambda))$, then $-\log X \sim exp(\beta(\alpha_\lambda))$ and

$-logX^*(s,n,m,k) \stackrel{d}{=} -log\,X^*(r,n,m,k) - \log Y^*$

which implies

$X^*(s,n,m,k) \stackrel{d}{=} X^*(r,n,m,k)\cdot Y^*$

THEOREM 4.2 :-

Let $X(r,n,m,k)$, $m \neq -1$ be the r^{th} *m- GOS* from a sample of size n drawn from a continuous DF F_X (x) with PDF $f_X(x)$. Furthermore, let $Y(r,n,m,k)$, $m \neq -1$ be the r^{th} *m- GOS* based on a sample of size n , which is drawn from a continuous DF $F_y(y)$, where Y is independent of X. Finally, let the relation

$$X(s,n,m,k) \overset{d}{=} X(s-r,n-r,m,k) + \tilde{Y} \tag{22}$$

be satisfied for all $1 \leq r < s$, Then, $\tilde{Y} \overset{d}{=} X(r,n,m,k)$ and Y ~ exp($\beta\alpha_\lambda$) if and if X ~ exp($\beta\alpha_\lambda$), $\beta > 0, \propto > 0, \lambda > 0$.

Proof. Clearly, the proof of the necessity part follows from Theorem 4.1, while the proof of the sufficiency part follows closely as the sufficiency part of Theorem 4.1. Namely, let the representation (22) be satisfied with $\tilde{Y} \overset{d}{=} X(r,n,m,k)$ and Y ~ exp($\beta\alpha_\lambda$). Furthermore, let $X(s,n,m,k)$ and $X(s-r,n-r,m,k)$ in (22) be m–GOSs, which are based on an unknown DF $F_X(x)$ and they are independent of Y(r,n,m,k). Therefore, the convolution relation (22) implies that

$$f_{X(s,n,m,k)}(x) = \int_0^x f_{X(s-r,n-r,m,k)}(y) f_{Y(r,n,m,k)}(x-y)dy$$

$$= \frac{\beta(\alpha_\lambda)\, C_{r-1}^{(n)}}{(r-1)!(m+1)^{r-1}} \int_0^x e^{-\beta(\alpha_\lambda)\,\gamma_r^{(n)}(x-y)} \times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{r-1} f_{X(s-r,n-r,m,k)}(y)dy \tag{23}$$

By differentiating both the sides of (23) with respect to x, we get

$$\frac{df_{X(s,n,m,k)}(x)}{dx} = \frac{(\beta(\alpha_\lambda))^2\, C_{r-1}^{(n)}}{(r-2)!\,(m+1)^{r-2}} \int_0^x [e^{-\beta(\alpha_\lambda)(\gamma_r^{(n_2)}+(m+1)}]^{(x-y)}$$

$$\times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{r-2} f_{X(s-r,n-r,m,k)}(y)dy$$

$$- \frac{(\beta(\alpha_\lambda))^2\ \gamma_r^{(n)}\ C_{r-1}^{(n)}}{(r-1)!(m+1)^{r-1}} \int_0^x e^{-\beta(\alpha_\lambda)(\gamma_r^{(n)}(x-y)}$$
$$\times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{r-1} f_{X(s-r,n-r,m,k)}(y)dy$$

$$= \beta(\alpha_\lambda)\gamma_1^{(n)}[f_{X(s-1,n,m,k)}(x) - f_{X(s,n,m,k)}(x)]$$

Or equivalently, by integrating from 0 to x,

$$f_{X(s,n,m,k)}(x) = \beta(\alpha_\lambda)\gamma_1^{(n)}[F_{X(s-1,n-1,m,k)}(x) - F_{X(s,n,m,k)}(x)] \tag{24}$$

Now, by using the relation of [15] on page 75, we get

$$F_{X(s-1,n-1,m,k)}(x) - F_{X(s,n,m,k)}(x) = \frac{C_{s-2}^{(n-1)}}{(s-1)!(m+1)^{s-1}}\ [\bar{F}_X(x)]^{\gamma_s^{(n)}-1}[1-(\bar{F}_X(x))^{m+1}]^{s-1} \tag{25}$$

Therefore, by combing (1), (24) and (25), we get

$$\frac{f_X(x)}{\bar{F}_X(x)} = \beta(\alpha_\lambda)$$

which implies that

$$\mathrm{F_X}(x) = [1-e^{-\beta(\alpha_\lambda)y}\}, \beta > 0, \propto > 0, \lambda > 0, \mathrm{x} > 0$$

This complete the proof of the sufficiency part, as well as the proof of Theorem 4.2.

Corollary 4.3 Assume that the RVs X and Y are independent, as we assumed in Theorem 4.2. By replacing the additive relation (22) by the multiplicative relation

$$\mathrm{X}(s,n,m,k) \overset{d}{=} X(s-r,n-r,m,k) \cdot \tilde{Y}$$

(26)

Then, $\tilde{Y} \overset{d}{=} Y(r,n,m,k)$ and $Y \sim$ Pareto$(\beta(\alpha_\lambda))$ if and only if $X \sim$ Pareto$(\beta(\alpha_\lambda))$
Proof. The proof follows exactly as the proof of Corollary 4.1.

Remark 2. For OOSs model the relation (26) takes the form
$X(s,n;0,1) \overset{d}{=} X(s-r,n-r;0,1) \cdot Y(r,n;0,1)$
Which implies the relation $X(s,n;0,1) \overset{d}{=} X(r,n;0,1) \cdot Y(s-r,n-r;0,1)$ that is belonging to [8]).

Corollary 4.4. Assume that the RVs X and Y are independent, Let $X^*(r,n,m,k)$ and $Y^*(r,n,m,k)$be the r^{th} m- DGOs based on a sample of size n drawn from F_X and F_Y, respectively. By replacing the additive relation (22) by the multiplicative relation
$X^*(s,n,m,k) \overset{d}{=} X^*(s-r,n-r,m,k) \cdot Y^*$
Then, $Y^* \overset{d}{=} Y^*(r,n,m,k)$ and $Y^* \sim Power(\beta(\alpha_\lambda))$ if and if $X^* \sim Power(\beta(\alpha_\lambda)$, $\beta > 0, \propto > 0, \lambda > 0$.
Proof. The proof follows as the proof of Corollory 4.2.

5. APPLICATIONS TO THE PREDICTION PROBLEM

Prediction problem usually arises in life-testing experiments of medical and industrial applications. Often, in the life-testing experiments, the observations arrive in ascending order of magnitude. Consequently, in reliability theory, especially for OOSs and SOSs, $X(r,n,m,k)$ represents the life length of a $n-r+1$- out-of-n system made up of n independent life lengths (these components are identical for OOSs and non identical for SOSs). Motivation for the prediction problems arises when the experiment is terminated before its conclusion by stopping after a given time (Type I censoring) or after a given number of failures (Type II censoring). Several authors have considered prediction problems involving GOSs, see for example [30], [31], [32], [33] and [34].

Theorems 4.1 and 4.2 suggest a new method for treating two prediction problems of different types. Namely, Theorem 4.2 treats a classical prediction problem, that predicting $X(s,n,m,k), 1 \le r < s \le n$, based on the observed m–GOSs $X(1,n,m,k) \le X(2,n,m,k) \le \cdots \le X(r,n,m,k$. On the other hand, Theorem 3.1 considers the prediction problem of $X(r,n,m,k)$, when the sample size of the test is enlarged from n to N, by adding some extra items $X_{n+1},\cdots,X_N$ after observing $X(r,n,m,k)$. Clearly, the sequence {X(r,n,m,k)} is non-increasing in n. For example, if $F_X(x)$ is continuous and for any fixed value $r < n$, the observed value of $X(r,n,0,1)$, denoted by $x(r,n,0,1)$, did not change if min $(x_{n+1},\cdots,x_N) > x(r,n,0,1)$, otherwise we get $x(r,n,0,1) < x(r,N:0,1)$. In the preceding two prediction problems, the failure times of the un-observed lifetimes in a lifetime experiment are predicted by using the result of another independent lifetime experiment.

References

[1] Alosey-EL, R. A. (2007). Random sum of new type of mixture of distribution. Int. J. Stat. Syst. 2: 49-57.
[2] El-Adll, M. E. (2018), ‘Characterization of Distributions by Equalities of Two Generalized or Dual Generalized Order Statistics’, Communications in Statistics - Theory and Methods 26(3), 522–528.
[3] Galambos, J. & Kotz, S. (1978), Characterizations of Probability Distributions, Cambridge University Press, Berlin Heidelberg, New York.
[4] Nagaraja, H. N. (2006), Characterizations of Probability Distributions -Handbook of Engineering Statistics, 79-87(In Book Chapter), Springer, Editor: Hoang Pham, Uk.
[5] Rao, C. R. & Chanabhng, D. N. (1998), Recent Approaches to Characterizations Based on Order Statistics and Record Values (Handbook of Statistics, 16, 231- 256), Elsevier.

[6] Arnold, B. C., Castillo, E. & Sarabia, J. (2008), 'Some Characterizations Involving Uniform and Powers of Uniform Random Variables', Statistics 42(6), 527–534.

[7] Beutner, E. and Kamps, U. (2008). Random contraction and random dilation of generalized order statistics. Comm. Statist. Theory Methods 37: 2185-2201.

[8] Casta˜no-Martnez, A., L´opez-Bl´azquez, F. and Salamanca-Mi˜no, B. (2012). Random translations, contractions and dilations of order statistics and records. Statistics 46(1): 57-67.

[9] Khan, A. H., Shah Imtiyaz, A. and Ahsanullah, M. (2012). Characterization through distributional properties of order statistics. J. Egyptian Math. Soc. 20 :211-214.

[10] Öncel, S. Y., Ahsanullah, M., Aliev, F. A. & Aygun, F. (2005), 'Switching Record and Order Statistics via Random Contraction', Statistical Probability Letters 73(3), 207–217.

[11] Samuel, P. (2008), 'Characterization of Distributions by Conditional Expectation of Generalized Order Statistics', Statistical. Papers 49, 101–108.

[12] Tavangar, M. & Hashemi, M. (2013), 'On Characterizations of the Generalized Pareto Distributions Based on Progressively Censored Order Statistics', *Statistical Papers* 54, 381–390.

[13] Wesolowski, J. & Ahsanullah, M. (2004), 'Switching Order Statistics Through Random Power Contractions', *Australian & New Zealand Journal of Statistics* 46(2), 297–303

[14] Burkschat, M., Cramer, E. & Kamps, U. (2003), 'Dual Generalized Order Statistics', *Metron* LXI(I), 13–26.

[15] Kamps, U. (1995), *A Concept of Generalized Order Statistics*, B. G. Teubner, Stuttgart.

[16] Ahsanullah, M. (2000). Generalized order statistics from exponential distribution. J. Statist. Plan. Inf. 85: 85-91.

[17] Ahsanullah, M. (2006). The generalized order statistics from exponential distribution. J. Statist. Research 40(2): 21-27.

[18] Aitcheson, J. and Dunsmore, I. (1975). Statistical Prediction Analysis. Cambridge University Press, Cambridge.

[19] AL-Hussaini E. K. and Al-Awadi, F. (2010). Bayes two-sample prediction of generalized order statistics with fixed and random sample size. J. Statist. Comput. Sim. 80(1): 13-28.

[20] Arnold, B. C., Castillo, E. and Sarabia, J. M. (2008). Some characterizations involving uniform and powers of uniform random variables. Statistics 42(6): 527-534.

[21] Fagiuoli, E., Pellerey, F. and Shaked, M. (1999). A characterization of the dilation order and its applications. Statist. Papers 40: 393-406.

[22] Kaminsky, K. S. and Nelson, P. I. (1998). Prediction Intervals. In: Handbook of Statistics,

[23] Balakrishnan, N. and C. R. Rao (Eds.), Amesterdam, North Holland, 431-450.

[24] Lawless, J. F. (1971). A prediction problem concerning samples from the exponential distribution with applications in life testing. Technometrics 13: 725-730.

[25] Okorie, I. E., Akpanta, A.C and Ohakwe J. , Transmuted Erlang-truncated exponential distribution, Econ. Qual. Control 31 (2) (2016) 71–84.

[26] Okorie, I.E, Akpanta,A.C. and Ohakwe, J. Marshall–Olkin generalized Erlang truncated exponential distribution: properties and applications, Cogent Math. 4 (1) (2017) 1285093.

[27] Nasiru, S., Luguterah, A. and Iddrisu, M.M. Generalized Erlang-truncated exponential distribution, Adv. Appl. Stat. 48 (4) (2016) 273–301.

[28] Ahsanullah, M. (2004), 'A characterization of the Uniform Distribution by Dual Generalized Order Statistics', Communications in Statistics - Theory and Methods 33(12), 2921–2928.

[29] Cramer, E. (2003). Contributions to Generalized Order Statistics. Habililationsschrift, Reprint, University of Oldenburg.

[30] Aitcheson, J. & Dunsmore, I. (1975), Statistical Prediction Analysis, Cambridge University Press, Cambridge, U. K.

[31] Lawless, J. F. (1971), 'A Prediction Problem Concerning Samples from the Exponential Distribution with Applications in Life Testing', Technometrics 13(4), 725–730.

[32] Nagaraja, H. N. (1986), 'Comparison of Estimators and Predictors from Two Parameter Exponential Distribution', Sankhya, Series B 48(1), 10–18.

[33] Raqab, M. Z. & Barakat, H. M. (2018), 'Prediction Intervals for Future Observations Based on Samples of Random Sizes', Journal of Mathematics and Statistics 14(1), 16–28.

[34] Raqab, Z. M. (2001), 'Optimal Prediction-Intervals for the Exponential Distribution Based on Generalized Order Statistics', IEEE Transactions on Reliability 50(1), 112–115.