A COMPOUND OF GAMMA AND SHANKER DISTRIBUTION

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Abstract

This paper considers a new lifetime distribution called Gamma-Shanker distribution which is a compound of Gamma and Shanker distribution. Many important properties of the suggested distribution including its shape, Inverse moments, hazard rate function, reversed hazard rate function, quantile function and stress-strength reliability have been discussed. The estimation of its parameters has been discussed using maximum likelihood estimation. Goodness of fit of the proposed distribution has been explained with two examples of real lifetime data from biomedical sciences and it shows that the proposed distribution gives much closure fit over the considered distributions

Keywords: Lifetime distribution, Statistical Properties, Stress-strength reliability, Maximum Likelihood estimation, Goodness of fit.

1. Introduction

The real lifetime data from different fields of knowledge are generally stochastic in nature and requires a distribution which can capture the variation to a great extent. There were two classical one parameter lifetime distributions namely exponential and Lindley by Lindley [1] in use for analyzing the lifetime data. Shanker et al [2] have detailed comparative study on the goodness of fit of exponential and Lindley distributions and observed that in some data sets exponential gives better fit than Lindley whereas in some datasets Lindley gives much closure fit than exponential and there were some datasets where neither exponential nor Lindley gives good fit. One of the important advantages of Lindley distribution over exponential distribution is that hazard rate for exponential distribution is constant while the hazard rate of Lindley distribution is not constant. The gamma and Weibull distributions which contain exponential distribution as particular case are the classical two-parameter lifetime distributions for the analysis and modeling of lifetime data. Shanker et al [3] have detailed comparative study on modeling of lifetime data using gamma and Weibull distributions and observed that both gamma and Weibull distributions are competing and each has some advantages over the others and there were some datasets where both gamma and Weibull did not give good fit.
Recently, Abdi et al [4] proposed gamma-Lindley distribution (G-LD) by compounding gamma distribution with Lindley distribution assuming that the scale parameter of gamma distribution follows Lindley distribution.

As we know that the nature of lifetime data is in general stochastic in nature and thus have different failure rates and to capture the analysis of such lifetime data different distributions are required. The decreasing and unimodal (upside down bathtub) failure rates have a lot of applications in survival analysis including the situation where the probability of an event in a fixed time interval in the future decreases over time and can be observed in case of infant mortality rate where earlier failures are eliminated or corrected, as observed by Finkelstein [5]. One of the practical examples of decreasing failure rate is the failure in the air conditioning systems explained by Proschan [6]. It has been observed by Lie and Xie [7] that if the main reasons of the failures of products are caused by fatigue and corrosion, the failure rates of those products exhibit unimodal shapes. For example, the data relating to breast cancer and infection in biomedical sciences with new viruses are generally of unimodal shape, as observed by Demicheli et al [8].

Recently, Abdi et al [4] derived gamma-Lindley distribution (G-LD) by compounding gamma (α,λ) with Lindley (β) distribution when the scale parameter λ of gamma distribution follows Lindley distribution. The probability density function (pdf) and cumulative distribution function (cdf) of G-LD are given by

$$f(x;\alpha,\beta)=\frac{\alpha\beta^2(1+\alpha+\beta+x)x^{\alpha-1}}{(\beta+1)(\beta+x)^{\alpha+2}};x>0,\alpha>0,\beta>0 \tag{1}$$

$$F(x;\alpha,\beta)=\frac{x^\alpha\left[(\beta+1)x+(1+\alpha+\beta)\beta\right]}{(\beta+1)(\beta+x)^{\alpha+1}};x>0,\alpha>0,\beta>0\ . \tag{2}$$

The pdf and cdf of Shanker distribution obtained by Shanker [9] are given by

$$f(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x};x>0,\theta>0 \tag{3}$$

$$F(x;\theta)=1-\left[\frac{\theta^2+1+\theta x}{\theta^2+1}\right]e^{-\theta x};x>0,\theta>0 \tag{4}$$

It should be noted that Shanker distribution is a two-component mixture of exponential distribution having scale parameter (θ) and a gamma distribution having shape parameter 2 and scale parameter (θ) with mixing proportion $\frac{\theta^2}{\theta^2+1}$. The statistical properties, estimation of parameter and application of Shanker distribution are available in Shanker [9].

The main motivations for proposing Gamma-Shanker distribution are as follows:
The first motivation G-SD lies in the fact that if X is the lifetime of component and λ is the scale parameter of the distribution of X and suppose that in the population from which the sample is being drawn has some variability in the scale parameter, then that variability can be described by the distribution of λ. The second motivation is that in real life situation components in a certain population differs sustainability from each other and this heterogeneity can easily be taken into consideration for the analysis of such population using compound distribution. In fact, the G-SD distribution can be shown as mixture representation like G-LD which is recommended for such variation in the population. The third motivation for proposing G-SD as a compound of Gamma and Lindley lies in context of Bayesian inference is that G-SD arises when Gamma $f(x\,|\,\alpha,\lambda)$ represents the distribution of future observations and the Lindley $f(\lambda\,|\,\beta)$ is the posterior distribution of the parameters of $f(x\,|\,\alpha,\lambda)$, given the information in a sample of observed data. The fourth motivation is that there are several lifetime data which have long right tail and the G-SD is most suitable for long right tail data. The fifth and the final motivation is that as the Shanker distribution provides much closure fit than exponential and Lindley distribution and G-LD distribution provides better fit than Gamma, Weibull and other two-

parameter distribution, it is expected and hoped that G-SD would provide much better fit than G-LD and other two-parameter distributions. In the present paper, statistical properties, estimation of parameters and applications of G-SD have been discussed.

2. Compound of Gamma and Shanker Distribution

Following the approach of obtaining G-LD distribution, the pdf and the cdf of gamma-Shanker distribution (G-SD) are obtained as

$$f(x;\alpha,\beta)=\frac{\alpha\beta^2\left(1+\alpha+\beta x+\beta^2\right)x^{\alpha-1}}{\left(1+\beta^2\right)\left(\beta+x\right)^{2+\alpha}};x>0,\alpha>0,\beta>0 \tag{5}$$

$$F(x;\alpha,\beta)=\frac{x^{\alpha}\left[x(1+\beta^2)+(1+\alpha+\beta^2)\beta\right]}{\left(1+\beta^2\right)\left(\beta+x\right)^{1+\alpha}};x>0,\alpha>0,\beta>0 \tag{6}$$

The shapes of the pdf and the cdf of G-SD for varying values of parameters are shown in the following figures 1 and 2 respectively. It is obvious that the pdf of G-SD is unimodal and positively skewed.

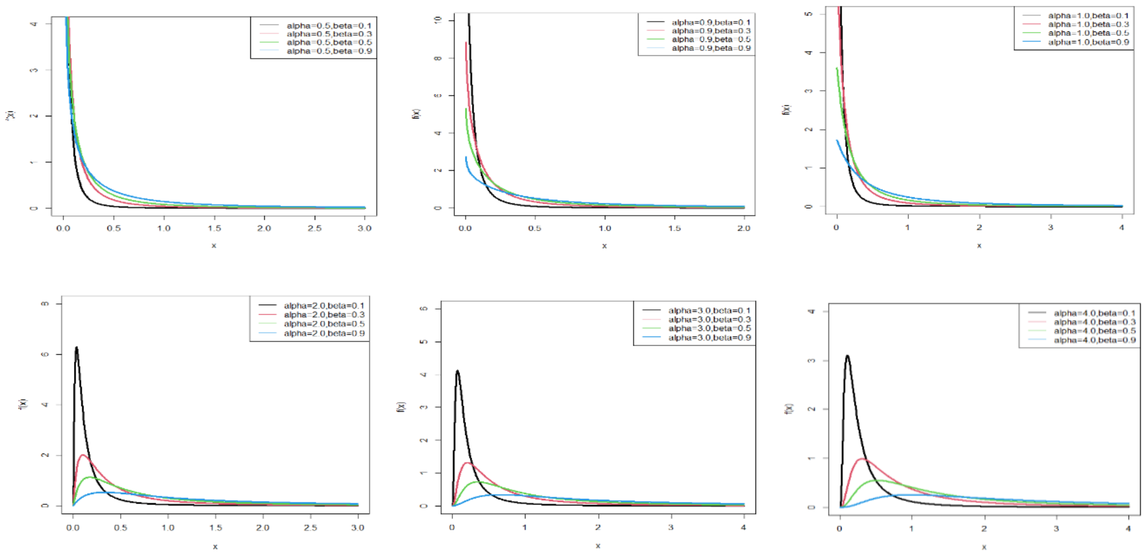

Figure 1: *pdf plots of G-SD for some selected values of parameters*

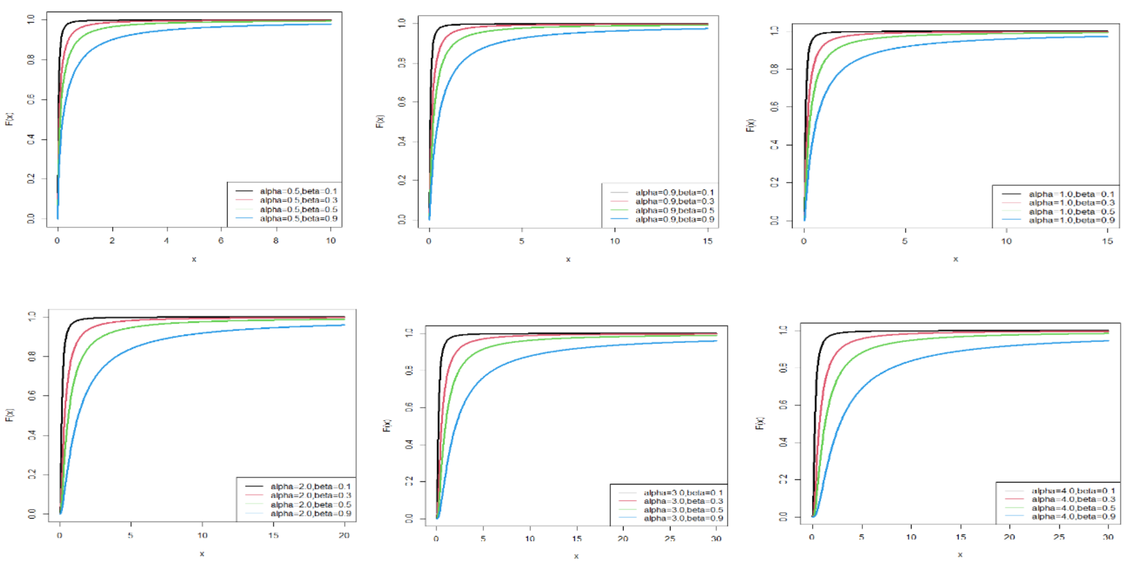

Figure 2: *cdf plots of G-SD for some selected values of parameters*

In the following theorem an attempt has been made to prove the decreasing nature of G-SD and the unimodality.

Theorem 1: The pdf of G-SD distribution is decreasing for $\alpha \le 1$ and unimodal for $\alpha > 1$
Proof: We have,

$$f(x) = \frac{\alpha\beta^2\left(1+\alpha+\beta x+\beta^2\right)x^{\alpha-1}}{\left(1+\beta^2\right)\left(\beta+x\right)^{2+\alpha}}$$

$$\log f(x) = \mathrm{C} + \log\left(1+\alpha+\beta^2+\beta x\right) + (\alpha-1)\log(x) - (\alpha+2)\log(\beta+x).$$

where C is a constant. We have

$$\frac{d}{dx}\log f(x) = \frac{\alpha-1}{x} - \frac{(\alpha+1)\left(\beta^2+\beta x+\alpha+2\right)}{(\beta+x)\left(1+\alpha+\beta^2+\beta x\right)}$$

For $\alpha \le 1$, $\frac{d}{dx}\log f(x) < 0$ and this means that $f(x)$ is decreasing for all x.
For $\alpha > 1$

$$\frac{d}{dx}\log f(x) = \frac{\alpha-1}{x} - \frac{(\alpha+1)\left(\beta^2+\beta x+\alpha+2\right)}{(\beta+x)\left(1+\alpha+\beta^2+\beta x\right)} = 0$$

This gives the following quadratic equation

$$2\beta x^2 - \left(\alpha\beta^2 - 3\alpha - 3\beta^2 - 3\right)x + \left(\beta + \alpha\beta + \beta^3 - \alpha\beta - \alpha^2\beta - \alpha\beta^3\right) = 0$$

A real root of the above equation is given by

$$x_0 = \frac{\left(\alpha\beta^2 - 3\alpha - 3\beta^2 - 3\right) + \sqrt{\alpha^2\beta^4 + 9\alpha^2 + \beta^4 + 2\alpha^2\beta^2 + 2\alpha\beta^4 + 12\alpha\beta^2 + 18\alpha + 10\beta^2 + 9}}{4\beta},$$

which is also the mode of G-SD.
Since, $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$, the pdf $f(x)$ is unimodal for $\alpha > 1$.

3. Hazard rate function and Reversed hazard rate function

The hazard rate function and the reverse hazard rate function are two important functions of a distribution. The reliability (survival) function of G-SD is given by

$$R(x;\alpha,\beta) = 1 - F(x;\alpha,\beta) = \frac{(\beta+x)^{\alpha+1}\left(1+\beta^2\right) - x^{\alpha}\left[x\left(\beta^2+1\right) + \left(1+\alpha+\beta^2\right)\beta\right]}{(\beta+x)^{\alpha+1}\left(1+\beta^2\right)} \tag{7}$$

The corresponding Hazard rate and Reversed Hazard rate function of G-SD are obtained as

$$h(x;\alpha,\beta) = \frac{f(x;\alpha,\beta)}{R(x;\alpha,\beta)} = \frac{\alpha\beta^2 x^{\alpha-1}\left(1+\alpha+\beta x+\beta^2\right)}{(\beta+x)\left[(\beta+x)^{\alpha+1}\left(1+\beta^2\right) - x^{\alpha}\left(1+\beta^2\right)(\beta+x) - \alpha\beta x^{\alpha}\right]} \tag{8}$$

$$r(x;\alpha,\beta) = \frac{f(x;\alpha,\beta)}{F(x;\alpha,\beta)} = \frac{\alpha\beta^2\left(1+\alpha+\beta x+\beta^2\right)}{x(\beta+x)\left[x\left(1+\beta^2\right) + \left(1+\alpha+\beta^2\right)\beta\right]} \tag{9}$$

The behavior of $h(x)$ when $x \to 0$ and $x \to \infty$, respectively are given by

$$\lim_{x\to 0} h(x) = \begin{cases} \infty, \alpha < 1 \\ \dfrac{\left(2+\beta^2\right)}{\beta\left(1+\beta^2\right)}, \alpha = 1 \\ 0, \alpha > 1 \end{cases} \text{ and } \lim_{x\to\infty} h(x) = 0$$

$$\lim_{x\to 0} r(x) = \infty \text{ and } \lim_{x\to\infty} r(x) = 0 .$$

The natures of hazard rate and the reversed hazard rate function of G-SD are shown in the figures 3 and 4 respectively.

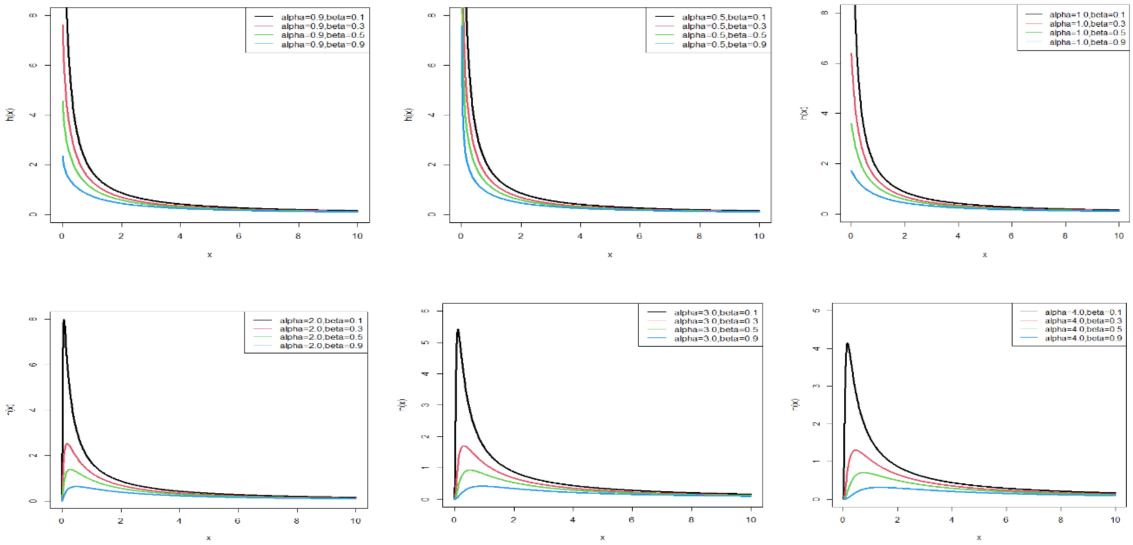

Figure 3: *Plots of hazard rate function of G-SD for some parameter values*

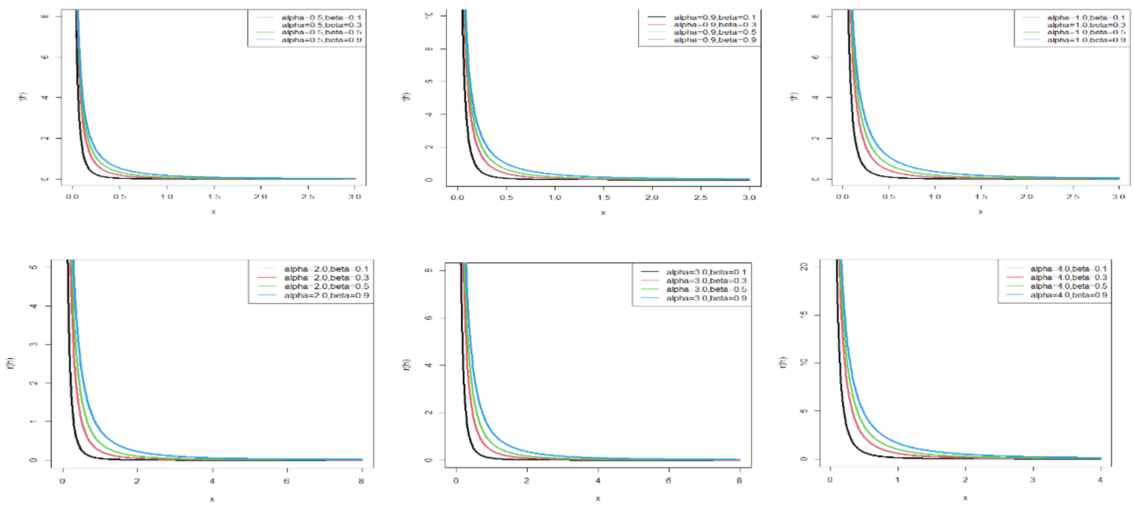

Figure 4: *Plots of Reversed hazard rate function of G-SD for some parameter values*

From figures 3 and 4 it is quite clear that the hazard rate function of G-SD is decreasing for $\alpha \le 1$ and unimodal for $\alpha > 1$ and the reversed hazard rate function of the G-SD is also decreasing, which is also shown in the following theorems 2 and 3, respectively.

Theorem 2: The hazard rate function of the G-SD is decreasing for $\alpha \le 1$ and unimodal for $\alpha > 1$
Proof: We have

$$f(x) = \frac{\alpha\beta^2\left(1+\alpha+\beta x+\beta^2\right)x^{\alpha-1}}{\left(1+\beta^2\right)\left(\beta+x\right)^{2+\alpha}}, \text{ and}$$

$$f'(x) = \frac{\alpha\beta^2 x^{\alpha-1}\left[(\beta+x)(\alpha-1)\left(1+\alpha+\beta^2+\beta x\right)+x\beta(\beta+x)-x(\alpha+2)\left(1+\alpha+\beta^2+\beta x\right)\right]}{x\left(1+\beta^2\right)(\beta+x)^{3+\alpha}}$$

Now, suppose that

$$\phi(x) = -\frac{f'(x)}{f(x)} = -\frac{(\alpha-1)}{x} + \frac{(\alpha+1)\left(2+\alpha+\beta x+\beta^2\right)}{(\beta+x)\left(1+\alpha+\beta x+\beta^2\right)}.$$

This gives

$$\phi'(x) = \frac{(\alpha-1)}{x^2} + \frac{(\alpha+1)\left(6\beta^2+6\alpha\beta^2+3\beta^4+4\beta^3x+6\beta x+6\alpha\beta x+3\beta^2x^2+3\alpha+\alpha^2+2\right)}{(\beta+x)^2\left(1+\alpha+\beta x+\beta^2\right)^2}$$

It is quite obvious that for $\alpha \le 1$, $\phi'(x) < 0$ and for $\alpha > 1$, $\phi(x) < 0$ has a global maximum at mode (say x_0).

Theorem 3: The reversed hazard rate function of the G-SD is decreasing
Proof: We have,

$$r(x) = \frac{\alpha\beta^2\left(1+\alpha+\beta x+\beta^2\right)}{x(\beta+x)\left[x\left(1+\beta^2\right)+\left(1+\alpha+\beta^2\right)\beta\right]}.$$

This gives

$$\frac{d}{dx}\log r(x) = \frac{-1-\alpha-\beta^2}{\left(1+\alpha+\beta x+\beta^2\right)\left[x\left(1+\beta^2\right)+\left(1+\alpha+\beta^2\right)\beta\right]} - \frac{1}{x} - \frac{1}{(\beta+x)} < 0 \text{ for all } \alpha,\beta$$

Therefore, reversed hazard function is decreasing for any value of the parameters α,β.

4. Quantiles and Moments

The p th quantiles x_p of G-SD. defined by $F\left(x_p\right) = p$,is the root of the equation

$$\frac{x_p^{\alpha}\left[x_p\left(\beta^2+1\right)+\left(1+\alpha+\beta^2\right)\beta\right]}{\left(\beta^2+1\right)\left(\beta+x_p\right)^{\alpha+1}} = p \tag{10}$$

This gives

$$x_p = \frac{\alpha\beta-\beta\left(1+\beta^2\right)\left[\left(1+\dfrac{\beta}{x_p}\right)^{\alpha}p-1\right]}{\left(1+\beta^2\right)\left[\left(1+\dfrac{\beta}{x_p}\right)^{\alpha}p-1\right]} \tag{11}$$

It should be noted that this x_p may be used to generate G-SD random variates. Further, the median of G-SD can be obtained from above equation by taking $p = \frac{1}{2}$.

The moments of G-SD can be obtained as follows:
If $X \sim$ G-SD(α,β) then,

$$E(X) = E\left(E(X \mid \lambda)\right) = E\left(\frac{\alpha}{\lambda}\right) = \alpha E\left(\frac{1}{\lambda}\right) = \infty$$

Thus, in general, $E\left(X^r\right) = \infty$ for $r \geq 1$. This means that all moments of G-SD are infinite and hence G-SD has no mean. As G-SD has no mean, if we take a sample $\left(X_1, X_2, ..., X_n\right)$ from G-SD, then mean $\bar{X}$ does not tend to a particular value. Since G-SD has no raw and central moments, we have to derive inverse moments. Negative moments are useful in several real life applications, such as life testing problems and estimation purpose. The negative moments for G-SD can be obtained as follows:

The r^{th} negative moment about origin, $\mu_{(-r)}'$, of the G-SD can be obtained as

$$\mu_{(-r)}' = E\left(X^{-r}\right) = E\left(E\left(X^{-r} \mid \lambda\right)\right)$$

$$= \int_0^\infty \left[\int_0^\infty x^{-r} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx\right] \frac{\beta^2}{\beta^2+1}(\beta+\lambda) e^{-\beta\lambda} d\lambda$$

$$= \frac{\Gamma(\alpha-r)}{\Gamma(\alpha)} \cdot \frac{r!\left(\beta^2+r+1\right)}{\beta^r\left(\beta^2+1\right)}; r = 1,2,3,... \tag{12}$$

Thus, for $r = 1,2,3,4$, we have

$$\mu_{(-1)}' = E\left(\frac{1}{X}\right) = \frac{\left(\beta^2+2\right)}{\beta\left(\beta^2+1\right)(\alpha-1)}, \alpha > 1 \tag{13}$$

$$\mu_{(-2)}' = E\left(\frac{1}{X^2}\right) = \frac{2\left(\beta^3+3\right)}{\beta^2(\alpha-1)(\alpha-2)\left(\beta^2+1\right)}, \alpha \geq 2 \tag{14}$$

$$\mu_{(-3)}' = E\left(\frac{1}{X^3}\right) = \frac{6\left(\beta^3+4\right)}{\beta^3(\alpha-1)(\alpha-2)(\alpha-3)\left(\beta^2+1\right)}, \alpha \geq 3 \tag{15}$$

$$\mu_{(-4)}' = E\left(\frac{1}{X^4}\right) = \frac{24\left(\beta^3+5\right)}{\beta^4(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\left(\beta^2+1\right)}, \alpha \geq 4 \tag{16}$$

It is obvious from the above expressions for negative moments that negative moments are not defined for $\alpha \leq 1$.

5. Extreme Order Statistics

Let, $X_{1:n},...,X_{n:n}$ be the order statistics of a random sample of size n from the G-SD(α,β) distribution with distribution function $F(x)$. The cdf of the minimum order statistic $X_{1:n}$ is given by

$$F_{X_{1:n}}(x) = 1-\left[1-F(x)\right]^n = 1-\left[\frac{\left(1+\beta^2\right)(\beta+x)^{1+\alpha} - x^\alpha(\beta+x)\left(1+\beta^2\right) - \alpha\beta x^\alpha}{\left(1+\beta^2\right)(\beta+x)^{1+\alpha}}\right] \tag{17}$$

The cdf of the maximum order statistic $X_{1:n}$ is given by

$$F_{X_{n:n}}(x) = \left[F(x)\right]^n = \left[\frac{x^\alpha}{(\beta+x)^\alpha} + \frac{\alpha\beta x^\alpha}{\left(1+\beta^2\right)(\beta+x)^{1+\alpha}}\right]^n \tag{18}$$

6. Stochastic Orderings

In probability theory and Statistics, a stochastic order quantifies the concept of one random variable being “bigger” than other. In many problems, it becomes necessary to compare two lifetime distributions with reference to some of their characteristics. Stochastic orders provide the necessary tools in such case.

A random variable X is said to be smaller than a random variable Y in the
i. Stochastic order $\left(X \prec_{st} Y\right)$ if $F_X(x) \ge F_Y(y)$ for all x
ii. Hazard rate order $\left(X \prec_{hr} Y\right)$ if $h_X(x) \ge h_Y(y)$ for all x
iii. Mean residual life order $\left(X \prec_{mrl} Y\right)$ if $m_X(x) \ge m_Y(y)$ for all x
iv. Likelihood ratio order $\left(X \prec_{lr} Y\right)$ if $\dfrac{f_X(x)}{f_Y(Y)}$ decrease in x

The following results due to Shaked and Shantikumar [10] are well known for establishing stochastic ordering of distributions:

$$X \prec_{lr} Y \Rightarrow X \prec_{hr} Y \Rightarrow X \prec_{mrl} Y$$
$$\Downarrow$$
$$X \prec_{st} Y$$

Theorem 4: Let $X_1 \sim$ G-SD$\left(\alpha_1, \beta_1\right)$ and $X_2 \sim$ G-SD$\left(\alpha_2, \beta_2\right)$. If $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 \le \beta_2$ and if $\beta_1 = \beta_2 = \beta \ge 1$ with $\alpha_1 \le \alpha_2$, then $X_1 \prec_{lr} X_2 \Rightarrow X_1 \prec_{hr} X_2 \Rightarrow X_1 \prec_{st} X_2$.

Proof: We have

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\alpha_1 \beta_1^2\left(\beta_2^2+1\right)\left(1+\alpha_1+\beta_1 x+\beta_1^2\right)\left(\beta_2+x\right)^{\alpha_2+2}}{\alpha_2 \beta_2^2\left(\beta_1^2+1\right)\left(1+\alpha_2+\beta_2 x+\beta_2^2\right)\left(\beta_1+x\right)^{\alpha_1+2}} x^{\alpha_1-\alpha_2}$$

If $\alpha_1 = \alpha_2 = \alpha$, we get

$$g_1(x) = \frac{\beta_1^2\left(\beta_2^2+1\right)\left(1+\alpha+\beta_1 x+\beta_1^2\right)}{\beta_2^2\left(\beta_1^2+1\right)\left(1+\alpha+\beta_2 x+\beta_2^2\right)}\left(\frac{\beta_2+x}{\beta_1+x}\right)^{\alpha+2}$$

$$\frac{d \log g_1(x)}{dx} = \left(\frac{\alpha+2}{\beta_2+x} - \frac{\beta_2}{1+\alpha+\beta_2 x+\beta_2^2}\right) - \left(\frac{\alpha+2}{\beta_1+x} - \frac{\beta_1}{1+\alpha+\beta_1 x+\beta_1^2}\right)$$
$$= q\left(\beta_2\right) - q\left(\beta_1\right),$$

where

$$q(\beta) = \left(\frac{\alpha+2}{\beta+x} - \frac{\beta}{1+\alpha+\beta x+\beta^2}\right)$$

$$\frac{d}{d\beta} q(\beta) = \frac{-(\alpha+2)}{(\beta+x)^2} - \frac{1+\alpha+\beta x}{\left(1+\alpha+\beta x+\beta^2\right)^2} < 0$$

If $\alpha_1 = \alpha_2 = \alpha$, then X_1 is stochastically smaller than X_2 with respect to the likelihood ratio if and only if $\beta_1 \le \beta_2$

Case II: If $\beta_1 = \beta_2 = \beta \ge 1$, we get

$$g_2(x) = \frac{\alpha_1\left(1+\alpha_1+\beta x+\beta^2\right)}{\alpha_2\left(1+\alpha_2+\beta x+\beta^2\right)}\left(\frac{x}{\beta+x}\right)^{\alpha_1-\alpha_2}$$

$$\frac{d \log g_2(x)}{dx} = \left(\frac{\beta}{1+\alpha_1+\beta x+\beta^2} + \frac{\alpha_1}{x} - \frac{\alpha_1}{\beta+x}\right) - \left(\frac{\beta}{1+\alpha_2+\beta x+\beta^2} + \frac{\alpha_2}{x} - \frac{\alpha_2}{\beta+x}\right)$$
$$= u\left(\alpha_1\right) - u\left(\alpha_2\right)$$

Where

$$u(\alpha) = \left(\frac{\beta}{1+\alpha+\beta x+\beta^2} + \frac{\alpha}{x} - \frac{\alpha}{\beta+x}\right)$$

$$\frac{d}{d\alpha} u(\alpha) = \frac{-1}{\left(1+\alpha+\beta x+\beta^2\right)^2} + \frac{1}{x} - \frac{1}{\beta+x} > 0 \text{ for } \beta \ge 1.$$

Thus, it is obvious that $\alpha_1 \le \alpha_2$, $\frac{d\log g_2(x)}{dx} < 0$. Hence, if $\beta_1 = \beta_2 = \beta \ge 1$ then X_1 is stochastically smaller than X_2 with respect to the likelihood ratio if and only if $\alpha_1 \le \alpha_2$.

7. Estimation of parameters

Let $(x_1, x_2, ..., x_n)$ be the observed values of a random sample $(X_1, X_2, ..., X_n)$ from the G-SD. Then the Likelihood function is given by

$$L(\alpha,\beta) = \left(\frac{\alpha\beta^2}{\beta^2+1}\right)^n \frac{\prod_{i=1}^{n}\left(1+\alpha+\beta x_i+\beta^2\right)\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{\prod_{i=1}^{n}\left(\beta+x_i\right)^{\alpha+2}} \tag{19}$$

The log-likelihood function of G-SD is thus obtained as

$$\begin{aligned}\ln L(\alpha,\beta) &= n\ln\alpha + 2n\ln\beta - n\ln\left(\beta^2+1\right) + \sum_{i=1}^{n}\ln\left(1+\alpha+\beta x_i+\beta^2\right) \\ &+ (\alpha-1)\sum_{i=1}^{n}\ln(x_i) - (\alpha+2)\sum_{i=1}^{n}\ln(\beta+x_i)\end{aligned} \tag{20}$$

The maximum likelihood estimators (MLEs) of α and β are the simultaneous solutions of the following log -likelihood equations

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{1}{\left(1+\alpha+\beta x_i+\beta^2\right)} + \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\ln(\beta+x_i) = 0$$

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\beta} = \frac{2n}{\beta} - \frac{2n\beta}{\left(\beta^2+1\right)} + \sum_{i=1}^{n}\frac{(x_i+2\beta)}{\left(1+\alpha+\beta x_i+\beta^2\right)} - (\alpha+2)\sum_{i=1}^{n}\frac{1}{(\beta+x_i)} = 0$$

It is very difficult to solve these two log-likelihood equations directly, so we will use Fisher's scoring method. We have

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha^2} = \frac{-n}{\alpha^2} - \sum_{i=1}^{n}\frac{1}{\left(1+\alpha+\beta x_i+\beta^2\right)^2}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\partial\beta} = -\sum_{i=1}^{n}\frac{2\beta+x_i}{\left(1+\alpha+\beta x_i+\beta^2\right)^2} - \sum_{i=1}^{n}\frac{1}{\beta+x_i} = \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta\partial\alpha}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta^2} = \frac{-2n}{\beta^2} + 2n\left[\frac{\left(\beta^2-1\right)}{\left(\beta^2+1\right)^2}\right] + \sum_{i=1}^{n}\left[\frac{2+2\alpha-2\beta^2-x_i^{\,2}-2\beta x_i}{\left(1+\alpha+\beta x_i+\beta^2\right)^2}\right] + \sum_{i=1}^{n}\frac{\alpha+2}{(\beta+x_i)^2}$$

The following equation can be solved for MLE's of α and β of G-SD

$$\begin{pmatrix} \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha^2} & \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta^2} \end{pmatrix}_{\substack{\hat\alpha=\alpha_0 \\ \hat\beta=\beta_0}} \begin{pmatrix} \hat\alpha-\alpha_0 \\ \hat\beta-\beta_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} \\ \frac{\partial \ln L(\alpha,\beta)}{\partial\beta} \end{pmatrix}_{\substack{\hat\alpha=\alpha_0 \\ \hat\beta=\beta_0}},$$

where α_0 and β_0 are initial value of α and β respectively. The initial values of the parameters taken in this paper for estimating parameters are $\alpha_0 = 0.5$ and $\beta_0 = 0.5$.

8. Estimation of the Stress-Strength parameter $R = P(X > Y)$

In Reliability, the Stress-Strength model describes the life of a component which has a random Strength X subjected to a random Stress Y.The component fails at the instant that the Stress applied to it exceeds the Strength, and the component will function satisfactory whenever $X > Y$. In this section our objective is to estimate $R = P(X > Y)$ when $X \sim$ G-SD(α_1, β_1) and $Y \sim$ G-SD(α_2, β_2) and X and Y are independently distributed. The, the Stress- Strength Parameter is given by

$$
\begin{aligned}
R = P(X > Y) &= \int_0^\infty P(X > Y \mid Y = y) f_Y(y)\,dy \\
&= \int_0^\infty \left[1 - F_X(y)\right] f_Y(y)\,dy \\
&= 1 - \int_0^\infty \frac{y^{\alpha_1}\left[y\left(\beta_1^2+1\right)+\left(1+\alpha_1+\beta_1^2\right)\beta_1\right]\alpha_2\beta_2^2\left(1+\alpha_2+\beta_2 y+\beta_2^2\right)y^{\alpha_2-1}}{\left(\beta_1^2+1\right)\left(\beta_1+y\right)^{\alpha_1+1}\left(\beta_2^2+1\right)\left(\beta_2+y\right)^{\alpha_2+2}}dy \\
&= 1 - \int_0^\infty \frac{\alpha_2\beta_2^2}{\left(\beta_1^2+1\right)\left(\beta_2^2+1\right)} \\
&\quad \times \frac{y^{\alpha_1+\alpha_2-1}\left[y\left(\beta_1^2+1\right)+\left(1+\alpha_1+\beta_1^2\right)\beta_1\right]\left(1+\alpha_2+\beta_2 y+\beta_2^2\right)}{\left(\beta_1+y\right)^{\alpha_1+1}\left(\beta_2+y\right)^{\alpha_2+2}}dy \\
&= \mathrm{H}\left(\alpha_1, \alpha_2, \beta_1, \beta_2\right)
\end{aligned} \tag{21}
$$

Let, $(x_1, x_2, ..., x_n)$ be the observed value of a random sample of size n from G-SD(α_1, β_1) and $(y_1, y_2, ..., y_m)$ be the observed value of a random sample of size m from G-SD(α_2, β_2).

The log-likelihood function of $\alpha_1, \alpha_2, \beta_1$ and β_2 is given by

$$
\begin{aligned}
\ln L(\alpha_1, \alpha_2, \beta_1, \beta_2) &= n\ln(\alpha_1) + 2n\ln(\beta_1) - n\ln\left(\beta_1^2+1\right) + \sum_{i-1}^{n}\ln\left(1+\alpha_1+\beta_1 x_i+\beta_1^2\right) \\
&\quad + (\alpha_1 - 1)\sum_{i=1}^{n}\ln(x_i) - (\alpha_1+2)\sum_{i=1}^{n}\ln(\beta_1+x_i) + m\ln(\alpha_2) + 2m\ln(\beta_2) - m\ln\left(\beta_2^2+1\right) \\
&\quad + \sum_{i-1}^{m}\ln\left(1+\alpha_2+\beta_2 y_i+\beta_2^2\right) + (\alpha_2-1)\sum_{i=1}^{m}\ln(y_i) - (\alpha_2+2)\sum_{i=1}^{m}\ln(\beta_2+y_i)
\end{aligned}
$$

The maximum likelihood estimates of $\alpha_1, \alpha_2, \beta_1$ and β_2 are the solutions of following log-likelihood equations

$$\frac{\partial}{\partial\alpha_1}\left(\ln L(\alpha_1, \alpha_2, \beta_1, \beta_2)\right) = \frac{n}{\alpha_1} + \sum_{i=1}^{n}\frac{1}{\left(1+\alpha_1+\beta_1 x_i+\beta_1^2\right)} + \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\ln(\beta_1+x_i) = 0$$

$$\frac{\partial}{\partial\alpha_2}\left(\ln L(\alpha_1, \alpha_2, \beta_1, \beta_2)\right) = \frac{m}{\alpha_2} + \sum_{i=1}^{m}\frac{1}{\left(1+\alpha_2+\beta_2 y_i+\beta_2^2\right)} + \sum_{i=1}^{m}\ln(y_i) - \sum_{i=1}^{m}\ln(\beta_2+y_i) = 0$$

$$\frac{\partial}{\partial\beta_1}\left(\ln L(\alpha_1, \alpha_2, \beta_1, \beta_2)\right) = \frac{2n}{\beta_1} - \frac{2n\beta_1}{\left(\beta_1^2+1\right)} + \sum_{i=1}^{n}\frac{x_i}{\left(1+\alpha_1+\beta_1 x_i+\beta_1^2\right)} - (\alpha_1+2)\sum_{i=1}^{n}\frac{1}{(\beta_1+x_i)} = 0$$

$$\frac{\partial}{\partial\beta_2}\left(\ln L(\alpha_1, \alpha_2, \beta_1, \beta_2)\right) = \frac{2m}{\beta_2} - \frac{2m\beta_2}{\left(\beta_2^2+1\right)} + \sum_{i=1}^{m}\frac{y_i}{\left(1+\alpha_2+\beta_2 y_i+\beta_2^2\right)} - (\alpha_2+2)\sum_{i=1}^{m}\frac{1}{(\beta_2+y_i)} = 0$$

Solving these non-linear equations using any iterative methods available in R packages we can obtain the MLEs of the parameters as $\left(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2\right)$ and hence the MLE of R can thus be obtained as

$$\hat{R} = \mathrm{H}\left(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2\right) \tag{22}$$

9. Applications

In this section, we present the goodness of fit of the G-SD to two real lifetime datasets to illustrate its applications. The goodness of fit of G-LD, Weibull, gamma, Shanker, Lindley and exponential have also been given for ready comparison. The datasets that we considered to demonstrate the applications of the proposed distribution are as follows:

Dataset 1: The dataset consists of plasma concentrations of indomethacin (mcg/ml) given by Team R.C. (2014) [11] are
1.50, 0.94, 0.78, 0.48, 0.37, 0.19, 0.12, 0.11, 0.08, 0.07, 0.05, 2.03, 1.63, 0.71, 0.70, 0.64, 0.36, 0.32, 0.20, 0.25, 0.12, 0.08, 2.72, 1.49, 1.16, 0.80, 0.80, 0.39, 0.22, 0.12, 0.11, 0.08, 0.08, 1.85, 1.39, 1.02, 0.89, 0.59, 0.40, 0.16, 0.11, 0.10, 0.07, 0.07, 2.05, 1.04, 0.81, 0.39, 0.30, 0.23, 0.13, 0.11, 0.08, 0.10, 0.06, 2.31, 1.44, 1.03, 0.84, 0.64, 0.42, 0.24, 0.17, 0.13, 0.10, 0.09

Dataset 2: The dataset is the survival times (in days) of 73 patients who diagnosed with acute bone cancer [12], available in https://doi.org/10.22436/jnsa.013.05.01
0.09, 0.76, 1.81, 1.10, 3.72, 0.72, 2.49, 1.00, 0.53,0.66, 31.61, 0.60, 0.20, 1.61, 1.88, 0.70, 1.36, 0.43, 3.16, 1.57, 4.93, 11.07, 1.63, 1.39, 4.54, 3.12, 86.01, 1.92, 0.92, 4.04, 1.16, 2.26, 0.20, 0.94, 1.82, 3.99, 1.46, 2.75, 1.38, 2.76, 1.86, 2.68, 1.76, 0.67, 1.29, 1.56, 2.83, 0.71, 1.48, 2.41, 0.66, 0.65, 2.36, 1.29, 13.75, 0.67, 3.70, 0.76, 3.63, 0.68,2.65, 0.95, 2.30, 2.57, 0.61, 3.93, 1.56, 1.29, 9.94, 1.67, 1.42, 4.18, 1.37

In order to compare lifetime distributions, values of $-2\log L$, AIC (Akaike information criterion), BIC (Bayesian information criterion), Kolmogorov – Smirnov (K-S) statistics with their P- values for the considered datasets has been computed. The formulae for computing AIC, AICC, BIC and K-S Statistics are as follows:

$$AIC = -2logL + 2k, \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}, \quad BIC = -2logL + k\log n,$$

$$D = \underset{x}{Sup}\,|F_n(x) - F_0(x)| \text{ where } k = \text{ number of parameter, } n = \text{ sample size}.$$

The distribution corresponding to the lower values of $-2\log L$, AIC and K-S is the best fit distribution. The standard errors of estimate of parameters are given in the parenthesis along with the ML estimates.

Table1: *ML estimates, $-2\log L$, AIC, BIC and K-S statistics with their P-values of the distributions for first data set 1*

Distributions	ML estimates $\hat{\alpha}$ (S.E) $\hat{\beta}$ (S.E)	$-2\log L$	AIC	BIC	K-S	P-value
G-SD	2.2087 (0.6607) 0.2677 (0.0986)	**60.72**	**64.72**	**65.51**	**0.15**	**0.09**
G-LD	2.5615 (0.8798) 0.2034(0.0837)	61.09	65.09	65.88	0.19	0.03
Weibull	1.6857 (0.2078) 0.9545 (0.0903)	62.51	66.51	67.31	0.42	0.00
Gamma	1.6513 (0.3257) 0.9772 (0.1495)	62.74	66.74	67.53	0.41	0.00
Shanker	2.0294 (0.1916)	63.27	67.27	68.06	0.17	0.08
Lindley	2.2152 (0.2208)	64.28	66.28	66.67	0.17	0.08
Exponential	1.6897 (0.2080)	62.76	64.76	65.15	0.17	0.15

Table 2: *ML estimates, $-2\log L$,AIC,BIC and K-S statistics with their P-values of the distributions for second data set*

Distributions	ML estimates $\hat{\alpha}$ (S.E) $\hat{\beta}$ (S.E)	$-2\log L$	AIC	BIC	K-S	P-value
G-SD	4.8969 (1.3904) 0.4968 (0.1360)	**282.82**	**286.81**	**296.62**	**0.09**	**0.63**
G-LD	5.1601 (1.8468) 0.4376 (0.1602)	284.31	288.31	298.13	0.13	0.23
Weibull	0.4395 (0.0687) 0.7656 (0.0568)	322.80	326.80	336.62	0.31	0.00
Gamma	0.1985 (0.0389) 0.7457 (0.1058)	334.53	338.53	348.35	0.78	0.00
Shanker	1.9473 (0.2707)	310.45	312.45	317.36	0.46	0.00
Lindley	0.4499 (0.0382)	374.77	376.77	381.67	0.31	0.00
Exponential	0.2663 (0.0312)	339.18	341.18	346.09	0.19	0.08

Based on the values of $-2\log L$,AIC , BIC and K-S statistics with their P- values it is obvious from tables 1 and 2 shows that among all considered distributions, gamma Shanker distribution (G-SD) give much closer fit.

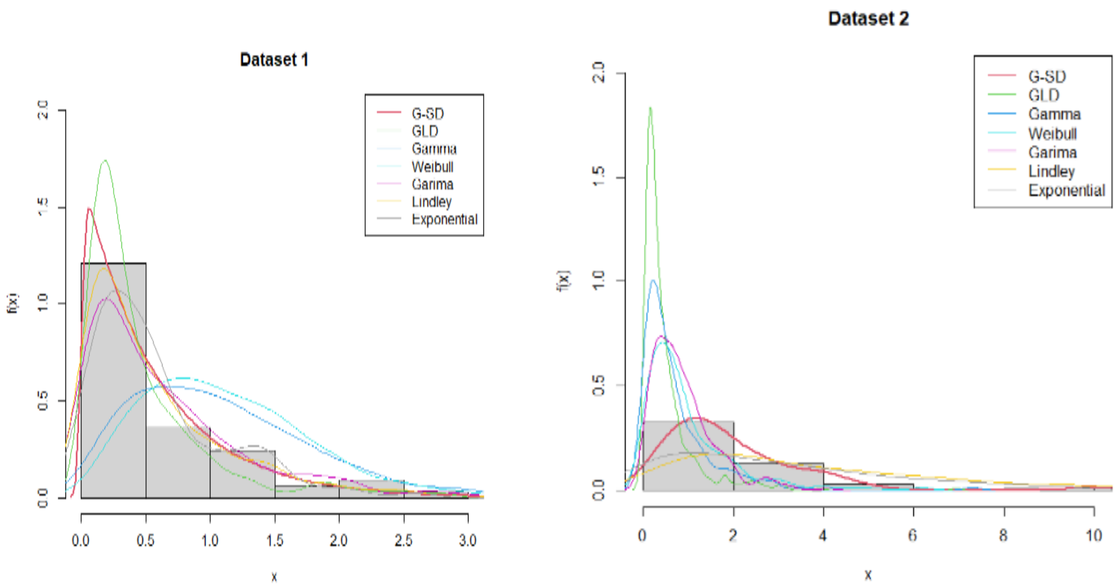

Figure 5: *Fitted Plots for Dataset 1 and 2*

In dataset 2 there are four values which act as an outlier, so for better graphical representation we have exclude the values from the dataset.

10. Concluding Remarks

In this paper, we propose a gamma-Shanker distribution by compounding gamma and Shanker distribution. Its statistical properties including shapes of hazard rate and reversed hazard rate function, Quantile, negative moments, stochastic ordering, and Stress-Strength reliability have been discussed. Maximum Likelihood estimation has been discussed for estimating its parameter. The goodness of fit of G-SD over G-LD distribution, Weibull distribution, gamma distribution, Shanker distribution, Lindley distribution and exponential distribution shows that G-SD gives much closure fit than these distributions for the considered datasets.

Conflict of Interest

The Authors declare that there is no conflict of Interest.

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