ESTIMATION OF A PARAMETER OF FARLIE-GUMBEL-MORGENSTERN BIVARIATE BILAL DISTRIBUTION BY RANKED SET SAMPLING

M. R. Irshad[1], R. Maya[2], A.I. Al-Omari[3], Ahmad A. Hanandeh[4], S. P. Arun[5]

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[1]Cochin University of Science and Technology, Kochi-22, Kerala.
irshadm24@gmail.com
[2]University College, Trivandrum-695 034.
publicationsofmaya@gmail.com
[3]Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq, Jordan.
alomari_amer@yahoo.com
[4]Yarmouk University, Irbid 21163, Jordan.
ahmad.hanandeh@yu.edu.jo
[5]University Library, Research Centre, University of Kerala, Trivandrum-695 034.
arunsptvm@gmail.com

Abstract

A bivariate version of the Bilal distribution has been proposed in the literature, called the Farlie-Gumbel-Morgenstern bivariate Bilal (FGMBB) distribution. In this article, we have dealt with the problem of estimation of the scale parameter associated with the study variable Z of primary interest, based on the ranked set sample defined by ordering the marginal observations on an auxiliary variable W, when (W, Z) follows a FGMBB distribution. When the dependence parameter ϕ is known, we have proposed the following estimators, viz., an unbiased estimator based on the Stoke's ranked set sample and the best linear unbiased estimator based on the Stoke's ranked set sample for the scale parameter of the variable of primary interest. The efficiency comparison of the proposed estimators with respect to the maximum likelihood estimator have been carried out.

Keywords: Farlie-Gumbel-Morgenstern bivariate Bilal distribution, Concomitants of order statistics, Ranked set sampling, Best linear unbiased estimator

1. Introduction

The Bilal distribution was introduced by [1], as a member of the families of distributions for the median of a random sample arising from an arbitrary lifetime distribution. Also, he shows that, this distribution belongs to the class of new better than average renewal failure rates and its probability density function (pdf) is always unimodal and has less of skewness and kurtosis than the pdf of the exponential distribution by about 25% and 28% respectively. The cumulative distribution function (cdf) of the Bilal distribution with the scale parameter σ is given by

$$F(x;\sigma) = 1 - e^{-\frac{2x}{\sigma}} \left(3 - 2e^{-\frac{x}{\sigma}}\right);\ \sigma > 0,\ x > 0. \tag{1}$$

The corresponding pdf is given by

$$f(x;\sigma) = \frac{6}{\sigma} e^{-\frac{2x}{\sigma}} \left(1 - e^{-\frac{x}{\sigma}}\right);\ \sigma > 0,\ x > 0. \tag{2}$$

Furthermore, the author obtained the closed form expressions for the quantile function, the hazard rate function and simple expression for moments in terms of the exponential function. Even though the Bilal distribution has only one parameter, this distribution possess high fitting ability compared to other competing models for two different real datasets, namely, the dataset consisting of thirty successive values of precipitation (in inches) given by [14] and the data for waiting times before service of 100 bank customers reported by [13]. Based on type-2 censored sample, [2] provide certain estimators of the parameter of the Bilal distribution. According to [3], the one parameter Bilal model can be generalized into the two parameter Bilal model, whose applications are elaborately discussed. Now the Proficiency of univariate Bilal distribution compared to other competing models well established in the literature in the theoretical as well as applied perspective. But even a single work is not been seen so far in the available literature on bivariate Bilal model except the work of [17]. A bivariate extension of one parameter Bilal distribution using Morgenstern approach was proposed by [17], so-called the Farlie-Gumbel-Morgenstern Bivariate Bilal (FGMBB) distribution and elucidated its inferential aspects using concomitants of order statistics (COS).
A bivariate random variable (W, Z) is said to follow a FGMBB distribution, if its pdf is given by

$$f(w,z) = \begin{cases} \dfrac{36}{\sigma_1\sigma_2} e^{-\frac{2w}{\sigma_1}} \left(1 - e^{-\frac{w}{\sigma_1}}\right) e^{-\frac{2z}{\sigma_2}} \left(1 - e^{-\frac{z}{\sigma_2}}\right) \\ \times \left[1 + \phi \left(2e^{-\frac{2w}{\sigma_1}} \left\{3 - 2e^{-\frac{w}{\sigma_1}}\right\} - 1\right) \left(2e^{-\frac{2z}{\sigma_2}} \left\{3 - 2e^{-\frac{z}{\sigma_2}}\right\} - 1\right)\right], \\ \qquad w > 0, z > 0; \sigma_1 > 0, \sigma_2 > 0; -1 \leq \phi \leq 1. \\ 0, \quad \text{otherwise}. \end{cases} \tag{3}$$

Clearly the marginal distributions of W and Z variables are univariate Bilal distributions with pdf's are respectively given by

$$f_W(w) = \begin{cases} \frac{6}{\sigma_1} e^{-\frac{2w}{\sigma_1}} \left(1 - e^{-\frac{w}{\sigma_1}}\right); \ if \ \ \sigma_1 > 0, \ w > 0, \\ 0, \qquad otherwise. \end{cases}$$

and

$$f_Z(z) = \begin{cases} \frac{6}{\sigma_2} e^{-\frac{2z}{\sigma_2}} \left(1 - e^{-\frac{z}{\sigma_2}}\right); \ if \ \ \sigma_2 > 0, \ z > 0, \\ 0, \qquad otherwise. \end{cases} \tag{4}$$

Clearly,

$$E(W) = \frac{5}{6}\sigma_1, Var(W) = \frac{13}{36}\sigma_1^2,$$

$$E(Z) = \frac{5}{6}\sigma_2, \tag{5}$$

$$Var(Z) = \frac{13}{36}\sigma_2^2. \tag{6}$$

The ranked set sampling (RSS) scheme was first developed by [19] as a process of increasing the precision of the sample mean as an estimator of the population mean. McIntyre's idea of ranking is possible whenever it can be done easily by a judgement method. For a detailed discussion on the theory and applications of RSS [11]. Basically the procedure involves choosing n sets of units, each of size n, and ordering the units of each of the set by judgement method or by applying some inexpensive method, without making actual measurement on the units. Then the unit ranked as one from the 1$^{\text{st}}$ set is actually measured, the unit ranked as two from the 2$^{\text{nd}}$ set is measured. The process continuous in this way until the unit ranked as n from the n^{th} set is measured. Then the observations obtained under the afore mentioned criterion is known as ranked set sample (*rss*) and the procedure is known as RSS. For recent developments in RSS, one can refer [6], [4] and [5].

In some practical problems, the variable of primary concern say Z, is more intricate to measure, but an auxiliary variable W related with Z is easily measurable and can be ordered exactly. In this case, [22] developed another scheme of RSS, which is as follows: Choose n independent bivariate sets, each of size n. In the first set of size n, the Z variate associated with smallest ordered W is measured, in the second set of size n, the Z variate associated with the second smallest, W is measured. This process is continued until the Z associated with the largest W from the n^{th} set is measured. The measurements on the Z variate of the resulting new set of n units chosen by the above method gives a *rss* as suggested by [22]. If $W_{(r:n)r}$ is the observation measured on the auxiliary variable W from the unit chosen from the r^{th} set, then we write $Z_{[r:n]r}$ to denote the corresponding measurement made on the study variable Z on this unit so that $Z_{[r:n]r}, r = 1, 2, \cdots, n$ form the *rss*. $Z_{[r:n]r}$ was referred by [12] as the concomitant of the r^{th} order statistic arising from the r^{th} sample.
The *rss* mean as an estimator for the mean of the study variate Z, when an auxiliary variable W is used for ranking the sample units has suggested by [22], under the assumption that (W, Z) follows a bivariate normal distribution. Based on *rss* obtained on the study variate Z, [10] have improved the estimator of [22] by deriving the best linear unbiased estimator (BLUE) of the mean of the study variate Z. COS and its applications in RSS from Farlie-Gumbel-Morgenstern bivariate Lomax distribution is elaborately elucidated by [20]. The estimation of a parameter of Morgenstern type bivarite Lindley distribution by RSS has been discussed in [15]. Parameter estimation of Cambanis-type bivariate uniform distribution with RSS is studied by [16]. For review of various variants of RSS and their application in parameter estimation [11].
The remaining part of this paper is assembled as follows. In section 2, we have proposed an unbiased estimator σ_2^* of σ_2 using Stoke's *rss*. As mentioned earlier if (W, Z) has a FGMBB distribution as defined in (3), then the marginal distributions of both W and Z have Bilal distributions and the pdf of Z is given in (4). We have evaluated the Cramer-Rao Lower Bound (CRLB) for the variance of an unbiased estimator of σ_2 involved in (4) based on a random sample of size n and is given by $\frac{13}{25}\frac{\sigma_2^2}{n}$. In this section, we have also shown that the variance of proposed unbiased estimator σ_2^* is strictly less than $\frac{13}{25}\frac{\sigma_2^2}{n}$, the CRLB for the variance of an unbiased estimator of σ_2 involved in (4), for all $\phi \in B$, where $B = [-1, 1] - \{0\}$. In this section, we have further discussed an efficiency comparison between σ_2^* and the maximum likelihood estimator (MLE) $\hat{\sigma_2}$ of σ_2 based on a random sample of size n arising from (3). In section 3, we have derived the BLUE $\tilde{\sigma_2}$ of σ_2 involved in FGMBB distribution based on Stoke's *rss* and made an efficiency comparison of $\tilde{\sigma_2}$ relative to $\hat{\sigma_2}$.

2. An unbiased estimator of σ_2 using Stoke's RSS.

Suppose the bivariate random vector (W, Z) follows a FGMBB distribution with pdf given in (3). Select a *rss* as per Stoke's RSS scheme. Let $W_{(r:n)r}$ be the observation obtained on the auxiliary variate W in the r^{th} unit of the *rss* and let $Z_{[r:n]r}$ be the measurement made on the variate related with $W_{(r:n)r}, r = 1, 2, \cdots, n$. Clearly $Z_{[r:n]r}$ is the r^{th} COS of a random sample of size n arising from the FGMBB distribution. Using the results of [21], we obtain the pdf of $Z_{[r:n]r}, r = 1, 2, \cdots, n$, and is given by

$$f_{[r:n]}(z) = \frac{6}{\sigma_2} e^{-\frac{2z}{\sigma_2}} \left(1 - e^{-\frac{z}{\sigma_2}}\right) \left[1 + \phi \frac{(n-2r+1)}{(n+1)} \left(2e^{-\frac{2z}{\sigma_2}} \left\{3 - 2e^{-\frac{z}{\sigma_2}}\right\} - 1\right)\right]. \tag{7}$$

The mean and variance of $Z_{[r:n]r}$ for $r = 1, 2, \cdots, n$, is obtained as

$$E[Z_{[r:n]r}] = \sigma_2 \left[\frac{5}{6} - \frac{19}{60}\phi \frac{(n-2r+1)}{(n+1)}\right] \tag{8}$$

and

$$Var[Z_{[r:n]r}] = \sigma_2^2 \left[\frac{13}{36} - \frac{253}{1800}\phi \frac{(n-2r+1)}{(n+1)} - \frac{361}{3600}\phi^2 \frac{(n-2r+1)^2}{(n+1)^2}\right]. \tag{9}$$

Since $Z_{[r:n]r}$ and $Z_{[s:n]s}$ for $r \neq s$ are arising from two independent samples, we obtain

$$Cov[Z_{[r:n]r}, Z_{[s:n]s}] = 0, r \neq s.$$

Next, we derive an unbiased estimator of σ_2 and its variance using the *rss* observations $Z_{[r:n]r}$ for $r = 1, 2, \cdots, n$, on the variable Z of primary interest and are given by the following theorem.

Theorem 1. Let (W, Z) follows a FGMBB distribution with pdf given by (3). Let $Z_{[r:n]r}$, $r = 1, 2, \cdots, n$ be the *rss* observations on a study variate Z generated out of ranking made on an auxiliary variate W. Then

$$\sigma_2^* = \frac{6}{5n} \sum_{r=1}^{n} Z_{[r:n]r}$$

is an unbiased estimator of σ_2 and its variance is given by

$$Var[\sigma_2^*] = \frac{\sigma_2^2}{n} \left[\frac{13}{25} - \frac{361}{2500} \frac{\phi^2}{n} \sum_{r=1}^{n} \left(\frac{n - 2r + 1}{n + 1} \right)^2 \right]. \tag{10}$$

Proof By using the definition, we have

$$\begin{aligned} E[\sigma_2^*] &= \frac{6}{5n} \sum_{r=1}^{n} E[Z_{[r:n]r}] \\ &= \frac{6}{5n} \sum_{r=1}^{n} \left[\frac{5}{6} - \frac{19}{60} \phi \frac{(n - 2r + 1)}{(n + 1)} \right] \sigma_2. \end{aligned} \tag{11}$$

Using the result,

$$\sum_{r=1}^{n} (n - 2r + 1) = 0. \tag{12}$$

Applying (12) in (11) we get,

$$E[\sigma_2^*] = \sigma_2.$$

Therefore, σ_2^* is an unbiased estimator of σ_2. The variance of σ_2^* is given by,

$$Var[\sigma_2^*] = \frac{36}{25n^2} \sum_{r=1}^{n} Var[Z_{[r:n]r}]. \tag{13}$$

Applying (9) and (12) in (13), we get

$$Var[\sigma_2^*] = \frac{\sigma_2^2}{n} \left[\frac{13}{25} - \frac{361}{2500} \frac{\phi^2}{n} \sum_{r=1}^{n} \left(\frac{n - 2r + 1}{n + 1} \right)^2 \right].$$

Hence the proof.

As mentioned above, if (W, Z) has the FGMBB distribution as defined in (3), then the marginal distribution of both W and Z are Bilal distributions and the pdf of Z is given in (4). The CRLB for the variance of any unbiased estimator of σ_2 based on a random sample of size n drawn from (4) is obtained as $\frac{13}{25} \frac{\sigma_2^2}{n}$. Now we compare the the variance of σ_2^* with the CRLB for the variance of an unbiased estimator of σ_2 involved in (4). If we write $E_1(\sigma_2^*)$ to denote the ratio of $\frac{13}{25} \frac{\sigma_2^2}{n}$ with $Var(\sigma_2^*)$, then we have,

$$E_1(\sigma_2^*) = \frac{1}{\left[1 - \frac{361}{1300} \frac{\phi^2}{n} \sum_{r=1}^{n} \left(\frac{n-2r+1}{n+1} \right)^2 \right]}. \tag{14}$$

It is easily verified that

$$E_1(\sigma_2^*) \geq 1.$$

Thus we arrive at a conclusion that the estimator σ_2^* based on Stoke's *rss* is more efficient as it assert the statement that *rss* always provide more information than simple random sample even if ranking is imperfect [11]. It is very clear that $Var(\sigma_2^*)$ is a decreasing function of ϕ^2 and hence the gain in efficiency of the estimator σ_2^* increases as $|\phi|$ increases.
Again on simplifying (14) we get,

$$E_1(\sigma_2^*) = \frac{1}{1 - \frac{361\phi^2}{1300}\left[\frac{2}{3}\left(\frac{2+1/n}{1+1/n}\right) - 1\right]}.$$

Then,

$$\begin{aligned}\lim_{n\to\infty} E_1(\sigma_2^*) &= \lim_{n\to\infty} \frac{1}{1 - \frac{361\phi^2}{1300}\left[\frac{2}{3}\left(\frac{2+1/n}{1+1/n}\right) - 1\right]} \\ &= \frac{1}{1 - \frac{361\phi^2}{3900}}.\end{aligned}$$

From the above expression it is clear that the maximum value for $E_1(\sigma_2^*)$ is attained when $|\phi| = 1$ and in this case $E_1(\sigma_2^*)$ tends to 3900/3539.
Next we discuss the efficiency comparison of σ_2^* with the asymptotic variance of MLE of σ_2 involved in the FGMBB distribution. If (W, Z) follows a FGMBB distribution with pdf given in (3), then

$$\begin{aligned}\frac{\partial \log f(x,y)}{\partial \sigma_1} &= \frac{1}{\sigma_1}\left\{-1 + \frac{2w}{\sigma_1} - \frac{we^{-\frac{w}{\sigma_1}}}{\sigma_1(1-e^{-\frac{w}{\sigma_1}})}\right. \\ &\left. + \frac{4\phi we^{-\frac{2w}{\sigma_1}}\left[-3 + 18e^{-\frac{2z}{\sigma_2}} - 12e^{-\frac{3z}{\sigma_2}} + 3e^{-\frac{w}{\sigma_1}} - 18e^{-\frac{w}{\sigma_1}}e^{-\frac{2z}{\sigma_2}} + 12e^{-\frac{w}{\sigma_1}}e^{-\frac{3z}{\sigma_2}}\right]}{\sigma_1\left\{1 + \phi\left[1 - 2e^{-\frac{2w}{\sigma_1}}(3-2)e^{-\frac{w}{\sigma_1}}\right]\left[1 - 2e^{-\frac{2z}{\sigma_2}}(3-2)e^{-\frac{z}{\sigma_2}}\right]\right\}}\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log f(x,y)}{\partial \sigma_2} &= \frac{1}{\sigma_2}\left\{-1 + \frac{2z}{\sigma_2} - \frac{ze^{-\frac{z}{\sigma_2}}}{\sigma_2(1-e^{-\frac{z}{\sigma_2}})}\right. \\ &\left. + \frac{4\phi ze^{-\frac{2z}{\sigma_2}}\left[-3 + 18e^{-\frac{2w}{\sigma_1}} - 12e^{-\frac{3w}{\sigma_1}} + 3e^{-\frac{z}{\sigma_2}} - 18e^{-\frac{z}{\sigma_2}}e^{-\frac{2w}{\sigma_1}} + 12e^{-\frac{z}{\sigma_2}}e^{-\frac{3w}{\sigma_1}}\right]}{\sigma_2\left\{1 + \phi\left[1 - 2e^{-\frac{2w}{\sigma_1}}(3-2)e^{-\frac{w}{\sigma_1}}\right]\left[1 - 2e^{-\frac{2z}{\sigma_2}}(3-2)e^{-\frac{z}{\sigma_2}}\right]\right\}}\right\}.\end{aligned}$$

Then we have,

$$\begin{aligned}I_{\sigma_1}(\phi) &= E\left(\frac{\partial \log f(x,y)}{\partial \sigma_1}\right)^2 \\ &= \frac{36}{\sigma_1^2}\int_0^\infty\int_0^\infty e^{-2u}(1-e^{-u})\Big\{-1 + 4u + ue^{-u}(u - 2 + 2e^{-u})(1-e^{-u})^{-2} \\ &- \frac{12\alpha u^2 e^{-2u}\left[-2 + 12e^{-2v} - 8e^{-3v} + 3e^{-u} - 18e^{-u}e^{-2v} + 12e^{-u}e^{-3v}\right]}{\{1 + \alpha[1 - 2e^{-2u}(3 - 2e^{-u})][1 - 2e^{-2v}(3 - 2e^{-v})]\}} \\ &+ \frac{24\alpha ue^{-2u}\left[-1 + 6e^{-2v} - 4e^{-3v} + e^{-u} - 6e^{-u}e^{-2v} + 4e^{-u}e^{-3v}\right]}{\{1 + \alpha[1 - 2e^{-2u}(3 - 2e^{-u})][1 - 2e^{-2v}(3 - 2e^{-v})]\}^2} \\ &\times\{1 + \alpha[1 - 2e^{-2v}(3 - 2e^{-v})][1 - 6e^{-2u} - 6ue^{-2u} + 4e^{-3u} + 6ue^{-3u}]\} \\ &\Big\}\, e^{-2v}(1-e^{-v})\{1 + \alpha[1 - 2e^{-2u}(3 - 2e^{-u})][1 - 2e^{-2v}(3 - 2e^{-v})]\}dudv,\end{aligned}$$

$$
\begin{aligned}
I_{\sigma_2}(\phi) &= E\left(\frac{\partial \log f(x,y)}{\partial \sigma_2}\right)^2 \\
&= \frac{36}{\sigma_2^2}\int_0^\infty\int_0^\infty e^{-2u}(1-e^{-u})\Big\{-1+4v+ve^{-v}(v-2+2e^{-v})(1-e^{-v})^{-2} \\
&-\frac{12\alpha v^2 e^{-2v}\left[-2+12e^{-2u}-8e^{-3u}+3e^{-v}-18e^{-v}e^{-2u}+12e^{-v}e^{-3u}\right]}{\{1+\alpha[1-2e^{-2u}(3-2e^{-u})][1-2e^{-2v}(3-2e^{-v})]\}} \\
&+\frac{24\alpha ve^{-2v}\left[-1+6e^{-2u}-4e^{-3u}+e^{-v}-6e^{-v}e^{-2u}+4e^{-v}e^{-3u}\right]}{\{1+\alpha[1-2e^{-2u}(3-2e^{-u})][1-2e^{-2v}(3-2e^{-v})]\}^2} \\
&\times\{1+\alpha[1-2e^{-2u}(3-2e^{-u})][1-6e^{-2v}-6ve^{-2v}+4e^{-3v}+6ve^{-3v}]\} \\
&\Big\}\, e^{-2v}(1-e^{-v})\{1+\alpha[1-2e^{-2u}(3-2e^{-u})][1-2e^{-2v}(3-2e^{-v})]\}dudv
\end{aligned}
$$

and

$$
\begin{aligned}
I_{\sigma_1\sigma_2}(\phi) &= E\left(\frac{\partial^2 \log f(x,y)}{\partial \sigma_1 \partial \sigma_2}\right) \\
&= \frac{36}{\sigma_1\sigma_2}\int_0^\infty\int_0^\infty e^{-2u}(1-e^{-u})\Big\{144\alpha uve^{-2u}e^{-2v}\left[1-e^{-u}-e^{-v}+e^{-u}e^{-v}\right] \\
&-\frac{144\alpha^2 uve^{-2u}e^{-2v}\left[-1+6e^{-2v}-4e^{-3v}+e^{-u}-6e^{-u}e^{-2v}+4e^{-u}e^{-3v}\right]}{\{1+\alpha[1-2e^{-2u}(3-2e^{-u})][1-2e^{-2v}(3-2e^{-v})]\}^2} \\
&\times\{[1-2e^{-2u}(3-2e^{-u})][e^{-v}-1]\} \\
&\Big\}\, e^{-2v}(1-e^{-v})\{1+\alpha[1-2e^{-2u}(3-2e^{-u})][1-2e^{-2v}(3-2e^{-v})]\}dudv.
\end{aligned}
$$

Thus the Fisher information matrix associated with the random variable (W, Z) is given by,

$$
I(\phi) = \begin{bmatrix} I_{\sigma_1}(\phi) & -I_{\sigma_1\sigma_2}(\phi) \\ -I_{\sigma_1\sigma_2}(\phi) & I_{\sigma_2}(\phi) \end{bmatrix}. \tag{15}
$$

We have computed the values of $\sigma_1^{-2}I_{\sigma_1}(\phi)$ and $\sigma_1^{-1}\sigma_2^{-1}I_{\sigma_1\sigma_2}(\phi)$ numerically for $\phi = \pm 0.25, \pm 0.50, \pm 0.75, \pm 1$ (clearly $\sigma_1^{-2}I_{\sigma_1}(\phi)$=$\sigma_2^{-2}I_{\sigma_2}(\phi)$) and are given below:

ϕ	$\sigma_1^{-2}I_{\sigma_1}(\phi)$	$\sigma_1^{-1}\sigma_2^{-1}I_{\sigma_1\sigma_2}(\phi)$	ϕ	$\sigma_1^{-2}I_{\sigma_1}(\phi)$	$\sigma_1^{-1}\sigma_2^{-1}I_{\sigma_1\sigma_2}(\phi)$
0.25	1.9381	0.1373	-0.25	1.9381	-0.1373
0.50	1.9795	0.2772	-0.50	1.9795	-0.2773
0.75	2.0530	0.4230	-0.75	2.0530	-0.4236
1.00	2.1705	0.5815	-1.00	2.1705	-0.5841

Thus from (15), the asymptotic variance of the MLE $\hat{\sigma_2}$ of σ_2 involved in the FGMBB distribution under a bivariate sample of size n is obtained as

$$
Var(\hat{\sigma}_2) = \frac{1}{n} I_{\sigma_2}^{-1}(\phi), \tag{16}
$$

where $I_{\sigma_2}^{-1}(\phi)$ is the (2,2)th element of the inverse of $I(\phi)$ given by (15).
We have compute the efficiency $E(\sigma_2^*|\hat{\sigma}_2) = \frac{Var(\hat{\sigma}_2)}{Var(\sigma_2^*)}$ of σ_2^* relative to $\hat{\sigma}_2$ for $n = 2(2)20$; $\phi = \pm 0.25, \pm 0.50, \pm 0.75, \pm 1$ and are given in table 1. From the table, one can infer that the estimator σ_2^* is more efficient than $\hat{\sigma}_2$ and efficiency increases with n and $|\phi|$ for $n \geq 4$.
Remark 2.1. For given value of $\phi \in (0, 1]$, once the variance of σ_2^* is evaluated, then this variance is equal to the variance of σ_2^* for $-\phi$ because the variance given in (10) depends only on ϕ by a term containing ϕ^2 only.

3. Best linear unbiased estimator of σ_2 using Stoke's RSS

In this section we derive the BLUE of σ_2 provided the dependence parameter ϕ is known. Suppose $Z_{[r:n]r}$ for $r = 1, 2, \cdots, n$, are the rss observation generated from (3) as per Stoke's RSS scheme. Let

$$\zeta_{r,n} = \frac{5}{6} - \frac{19}{60}\phi\frac{(n-2r+1)}{(n+1)}, \tag{17}$$

$$\psi_{r,r,n} = \frac{13}{36} - \frac{253}{1800}\phi\frac{(n-2r+1)}{(n+1)} - \frac{361}{3600}\phi^2\frac{(n-2r+1)^2}{(n+1)^2}. \tag{18}$$

Using (17) and (18), we get

$$E[Z_{[r:n]r}] = \sigma_2\zeta_{r,n},\ 1 \leq r \leq n \tag{19}$$

and

$$Var[Z_{[r:n]r}] = \sigma_2^2\psi_{r,r,n},\ 1 \leq r \leq n. \tag{20}$$

Also we have

$$Cov[Z_{[r:n]r}, Z_{[s:n]s}] = 0, r, s = 1, 2, \cdots, n \text{ and } r \neq s. \tag{21}$$

Let $\mathbf{Z}_{[n]} = (Z_{[1:n]1}, Z_{[2:n]2}, \cdots, Z_{[n:n]n})'$ denote the column vector of n rss observations. Then from (19), (20) and (21), we can write,

$$E[\mathbf{Z}_{[n]}] = \sigma_2\zeta \tag{22}$$

and the dispersion matrix of $\mathbf{Z}_{[n]}$,

$$D[\mathbf{Z}_{[n]}] = \sigma_2^2\mathbf{G}, \tag{23}$$

where $\zeta = (\zeta_{1,n}, \zeta_{2,n}, \cdots, \zeta_{n,n})'$ and $\mathbf{G} = diag(\psi_{1,1,n}, \psi_{2,2,n}, \cdots, \psi_{n,n,n})$, where $\zeta_{r,n}$ and $\psi_{r,r,n}$ for $r = 1, 2, \cdots, n$ are respectively given by equations (17) and (18). If ϕ contained in ζ and $\mathbf{G}$ are known then (22) and (23) together defines a generalized Gauss-Markov setup and then the BLUE of σ_2 is given by

$$\tilde{\sigma_2} = (\zeta'\mathbf{G}^{-1}\zeta)^{-1}\zeta'\mathbf{G}^{-1}\mathbf{Z}_{[n]}$$

and the variance of σ_2 is given by

$$Var(\tilde{\sigma_2}) = \frac{\sigma_2^2}{\zeta'\mathbf{G}^{-1}\zeta}.$$

On simplifying, we get

$$\tilde{\sigma_2} = \frac{\sum_{r=1}^{n}\frac{\zeta_{r,n}}{\psi_{r,r,n}}}{\sum_{r=1}^{n}\frac{\zeta_{r,n}^2}{\psi_{r,r,n}}}Z_{[r:n]r} \tag{24}$$

and

$$Var(\tilde{\sigma_2}) = \frac{\sigma_2^2}{\sum_{r=1}^{n}\frac{\zeta_{r,n}^2}{\psi_{r,r,n}}}. \tag{25}$$

From (24), we have $\tilde{\sigma_2}$ is a linear functions of the *rss* observations $Z_{[r:n]r}, r = 1, 2, \cdots, n$ and hence $\tilde{\sigma_2}$ can be written as $\tilde{\sigma_2} = \sum_{r=1}^{n} a_r Z_{[r:n]r}$, where

$$a_r = \frac{\frac{\zeta_{r,n}}{\psi_{r,r,n}}}{\sum_{r=1}^{n}\frac{\zeta_{r,n}^2}{\psi_{r,r,n}}}, r = 1, 2, \cdots, n.$$

We have evaluated the numerical values of means and variances using the expressions (17) and (18) respectively for $\phi = 0.25(0.25)1$ and for $n = 2(2)20$. Using these values we have evaluated the variance of BLUE $\tilde{\sigma_2}$ for $\phi = 0.25(0.25)1$ and for $n = 2(2)20$. Also we have computed the ratio $E(\tilde{\sigma}_2|\hat{\sigma}_2) = \frac{Var(\hat{\sigma}_2)}{Var(\tilde{\sigma}_2)}$ to measure the efficiency of our estimator $\tilde{\sigma}_2$ relative to $\hat{\sigma}_2$ for $n = 2(2)20$ and $\phi = 0.25(0.25)1$ and are presented in table 1. From the table it is clear that, BLUE of σ_2 performs well compared to the MLE of σ_2, namely $\hat{\sigma}_2$.

Remark 3.1. As in the case of variance of an unbiased estimator given in (10), for a given value of $\phi \in (0, 1]$, once the variance of $\tilde{\sigma_2}$ is evaluated, then there is no need to again evaluate the variance of $\tilde{\sigma_2}$ when $\phi = -\phi$. To establish this argument we prove the following theorem.

Theorem 2. Let (W, Z) follows a FGMBB distribution with pdf given by (3). For a given $\phi \in (0, 1]$, $Var[\tilde{\sigma}_2(\phi_0)]$ is the variance of the BLUE $\tilde{\sigma}_2$ of σ_2 involved in the FGMBB distribution, then

$$Var[\tilde{\sigma}_2(-\phi_0)] = Var[\tilde{\sigma}_2(\phi_0)]. \tag{26}$$

Proof The terms $\zeta_{r,n}$ and $\psi_{r,r,n}$ defined by (17) and (18) are functions of ϕ, r and n and hence $\zeta_{r,n}$ and $\psi_{r,r,n}$ can be denoted as $\zeta_{r,n}(\phi)$ and $\psi_{r,r,n}(\phi)$ respectively. From (17) and (18), it is clear that

$$\zeta_{r,n}(\phi) = \zeta_{n-r+1,n}(-\phi),\ 1 \leq r \leq n \tag{27}$$

and

$$\psi_{r,r,n}(\phi) = \psi_{n-r+1,n-r+1,n}(-\phi),\ 1 \leq r \leq n. \tag{28}$$

As a consequence of (27) and (28), we get

$$\begin{aligned} Var[\tilde{\sigma}_2(\phi)] &= \frac{\sigma_2^2}{\sum_{r=1}^{n} \frac{\zeta_{r,n}^2(\phi)}{\psi_{r,r,n}(\phi)}} = \frac{\sigma_2^2}{\sum_{r=1}^{n} \frac{\zeta_{n-r+1,n}^2(-\phi)}{\psi_{n-r+1,n-r+1,n}(-\phi)}} \\ &= Var[\tilde{\sigma}_2(-\phi)]. \end{aligned}$$

Hence the proof.

Remark 3.2. *For FGMBB distribution defined in (3), we have evaluated the correlation coefficient between the two variables and is given by $\rho = \frac{361}{1300}\phi$. But in certain real life situations our assumption that ϕ is known may viewed as unrealistic. Hence if we have a situation with ϕ unknown, we compute the sample correlation τ from $(W_{r:n}, Z_{[r:n]})$ for $r = 1, 2, \cdots, n$ and introduce a moment type estimator $\hat{\phi}$ for ϕ as,*

$$\hat{\phi} = \begin{cases} -1, & if\ \ \tau < \frac{-361}{1300} \\ \frac{1300}{361}\tau, & if\ \ \frac{-361}{1300} \leq \tau \leq \frac{361}{1300} \\ 1, & if\ \ \tau > \frac{361}{1300}. \end{cases}$$

Table 1: Efficiencies of the estimators σ_2^* and $\tilde{\sigma_2}$ relative to $\hat{\sigma}_2$.

n	ϕ	$e(\sigma_2^*\|\hat{\sigma}_2)$	$e(\tilde{\sigma}_2\|\hat{\sigma}_2)$	ϕ	$e(\sigma_2^*\|\hat{\sigma}_2)$	$e(\tilde{\sigma}_2\|\hat{\sigma}_2)$
	0.25	0.9992	0.9992	-0.25	0.9992	0.9992
2	0.50	0.9984	0.9984	-0.50	0.9984	0.9984
	0.75	0.9957	0.9957	-0.75	0.9957	0.9957
	1.00	0.9849	0.9849	-1.00	0.9850	0.9853
	0.25	1.0008	1.0008	-0.25	1.0008	1.0008
4	0.50	1.0047	1.0047	-0.50	1.0029	1.0029
	0.75	1.0103	1.0111	-0.75	1.0047	1.0047
	1.00	1.0106	1.0139	-1.00	1.0110	1.0147
	0.25	1.0012	1.0012	-0.25	1.0012	1.0012
6	0.50	1.0082	1.0082	-0.50	1.0082	1.0082
	0.75	1.0168	1.0180	-0.75	1.0168	1.0180
	1.00	1.0223	1.0273	-1.00	1.0230	1.0286
	0.25	1.0015	1.0015	-0.25	1.0015	1.0015

n	ϕ	$e(\sigma_2^*\|\hat{\sigma}_2)$	$e(\tilde{\sigma}_2\|\hat{\sigma}_2)$	ϕ	$e(\sigma_2^*\|\hat{\sigma}_2)$	$e(\tilde{\sigma}_2\|\hat{\sigma}_2)$
8	0.50	1.0094	1.0094	-0.50	1.0094	1.0094
	0.75	1.0192	1.0209	-0.75	1.0192	1.0209
	1.00	1.0282	1.0351	-1.00	1.0300	1.0367
	0.25	1.0019	1.0019	-0.25	1.0019	1.0019
10	0.50	1.0098	1.0098	-0.50	1.0098	1.0098
	0.75	1.0221	1.0241	-0.75	1.0221	1.0241
	1.00	1.0312	1.0398	-1.00	1.0330	1.0419
	0.25	1.0023	1.0023	-0.25	1.0023	1.0023
12	0.50	1.0094	1.0094	-0.50	1.0094	1.0094
	0.75	1.0242	1.0266	-0.75	1.0242	1.0266
	1.00	1.0376	1.0455	-1.00	1.0376	1.0455
	0.25	1.0000	1.0000	-0.25	1.0000	1.0000
14	0.50	1.0110	1.0110	-0.50	1.0110	1.0110
	0.75	1.0225	1.0254	-0.75	1.0225	1.0254
	1.00	1.0380	1.0472	-1.00	1.0380	1.0472
	0.25	1.0031	1.0031	-0.25	1.0031	1.0031
16	0.50	1.0126	1.0126	-0.50	1.0126	1.0126
	0.75	1.0258	1.0291	-0.75	1.0258	1.0291
	1.00	1.0403	1.0473	-1.00	1.0403	1.0473
	0.25	1.0035	1.0035	-0.25	1.0035	1.0035
18	0.50	1.0106	1.0106	-0.50	1.0106	1.0106
	0.75	1.0291	1.0291	-0.75	1.0291	1.0291
	1.00	1.0415	1.0534	-1.00	1.0415	1.0534
	0.25	1.0000	1.0000	-0.25	1.0000	1.0000
20	0.50	1.0118	1.0157	-0.50	1.0118	1.0157
	0.75	1.0242	1.0283	-0.75	1.0242	1.0283
	1.00	1.0420	1.0508	-1.00	1.0420	1.0508

4. Estimation of σ_2 based on censored ranked set sample

In this section, we obtain some estimators of σ_2 using censored RSS scheme. Suppose k units are censored in the Stoke's RSS scheme, then we may represent the *rss* observations on the study variate Z as $\delta_1 Z_{[1:n]1}, \delta_2 Z_{[2:n]2}, \cdots, \delta_n Z_{[n:n]n}$ where,

$$\delta_i = \begin{cases} 0, & if \text{ the } i^{th} \text{ unit is censored,} \\ 1, & \text{otherwise.} \end{cases}$$

and hence $\sum_{i=1}^{n} \delta_i = n - k$. In this case the usual unbiased estimator of σ_2 is equal to $\frac{6\sum_{i=1}^{n}\delta_i Z_{[i:n]i}}{5(n-k)}$. It may be noted that one need not get $\delta_i = 0$ for $i = 1, 2, \cdots, k$ and $\delta_i = 1$ for $i = k+1, k+2, \cdots, n$. Hence if we write m_i, $i = 1, 2, \cdots, n-k$ as the integers such that $1 \le m_1 < m_2 < \cdots m_{n-k}$ and for which δ_{m_i}=1, then,

$$E\left[\frac{6\sum_{i=1}^{n}\delta_i Z_{[i:n]i}}{5(n-k)}\right] = \sigma_2\left[1 - \frac{19\phi}{50(n+1)(n-k)}\sum_{i=1}^{n-k}(n - 2m_i + 1)\right].$$

Thus it is clear that the in the censored case the usual unbiased estimator is not an unbiased estimator of σ_2. However we can construct an unbiased estimator of σ_2 based on $\frac{6\sum_{i=1}^{n}\delta_i Z_{[i:n]i}}{5(n-k)}$ is given in the following theorem.

Theorem 3. Suppose that the random variable (W, Z) has a FGMBB distribution as defined in (3). Let $Z_{[m_i]m_i}$, $i = 1, 2, \cdots, n-k$ be the *rss* observations on the study variate Z resulting out of censoring applied on the auxiliary variable W. Then an unbiased estimator of σ_2 based on

$\frac{6}{5(n-k)} \sum_{i=1}^{n-k} Z_{[m_i]m_i}$ is given by

$$\sigma_2^*(k) = \frac{60(n+1)}{\left[50(n+1)(n-k) - 19\phi \sum_{r=1}^{n-k}(n-2m_i+1)\right]} \sum_{i=1}^{n-k} Z_{[m_i]m_i}$$

and its variance is given by

$$Var[\sigma_2^*(k)] = \frac{3600(n+1)^2\sigma_2^2}{\left[50(n+1)(n-k) - 19\phi \sum_{r=1}^{n-k}(n-2m_i+1)\right]^2} \sum_{i=1}^{n-k} \psi_{m_i}$$

where ψ_{m_i} is as defined in (18).

Proof We have

$$\begin{aligned} E[\sigma_2^*(k)] &= \frac{60(n+1)}{\left[50(n+1)(n-k) - 19\phi \sum_{r=1}^{n-k}(n-2m_i+1)\right]} \sum_{i=1}^{n-k} E[Z_{[m_i]m_i}] \\ &= \frac{60(n+1)}{\left[50(n+1)(n-k) - 19\phi \sum_{r=1}^{n-k}(n-2m_i+1)\right]} \\ &\quad \times \sum_{i=1}^{n-k} \left[\frac{5}{6} - \frac{19}{60}\phi \frac{(n-2m_i+1)}{(n+1)}\right] \sigma_2 \\ &= \frac{60(n+1)}{\left[50(n+1)(n-k) - 19\phi \sum_{r=1}^{n-k}(n-2m_i+1)\right]} \\ &\quad \times \left[\frac{5(n-k)}{6} - \frac{19\phi}{60(n+1)} \sum_{i=1}^{n-k}(n-2m_i+1)\right] \sigma_2 \\ &= \sigma_2. \end{aligned}$$

Thus $\sigma_2^*(k)$ is an unbiased estimator of σ_2. The variance of $\sigma_2^*(k)$ is given by

$$\begin{aligned} Var[\sigma_2^*(k)] &= \frac{3600(n+1)^2}{\left[50(n+1)(n-k) - 19\phi \sum_{r=1}^{n-k}(n-2m_i+1)\right]^2} \sum_{i=1}^{n-k} Var(Z_{[m_i]m_i}) \\ &= \frac{3600(n+1)^2\sigma_2^2}{\left[50(n+1)(n-k) - 19\phi \sum_{r=1}^{n-k}(n-2m_i+1)\right]^2} \sum_{i=1}^{n-k} \psi_{m_i} \end{aligned}$$

where ψ_{m_i} is as defined in (18). Hence the theorem.
As a competitor of the estimator $\sigma_2^*(k)$, next we propose the BLUE of σ_2 based on the censored *rss*, resulting out of ranking of observations on W.
If $\mathbf{Z}_{[n]}(k) = (Z_{[m_1]m_1}, Z_{[m_2]m_2}, \cdots, Z_{[m_{n-k}]m_{n-k}})'$, then the mean vector and the variance-covariance matrix of $\mathbf{Z}_{[n]}(k)$ are given by

$$E[\mathbf{Z}_{[n]}(k)] = \sigma_2 \zeta(k), \tag{29}$$

$$D[\mathbf{Z}_{[n]}(k)] = \sigma_2 G(k), \tag{30}$$

where $\zeta(k) = (\zeta_{m_1}, \zeta_{m_1}, \cdots, \zeta_{m_{n-k}})'$, $G(k) = diag(\psi_{m_1}, \psi_{m_2}, \cdots, \psi_{m_{n-k}})$.
if the parameter ϕ involved in $\zeta(k)$ and $G(k)$ are known then (29) and (30) together defines a generalized Gauss-Markov setup and hence the BLUE $\tilde{\sigma}_2(k)$ of σ_2 is obtained as,

$$\tilde{\sigma}_2(k) = [(\zeta(k))'(G(k))^{-1}\zeta(k)]^{-1}(\zeta(k))'(G(k))^{-1}\mathbf{Z}_{[n]}(k) \tag{31}$$

and the variance of σ_2 is given by

$$Var(\tilde{\sigma}_2(k)) = [(\zeta(k))'(G(k))^{-1}\zeta(k)]^{-1}\sigma_2^2. \tag{32}$$

On substituting the values of $\zeta(k)$ and $G(k)$ in (31) and (32) and simplifying we get,

$$\tilde{\sigma}_2(k) = \frac{\sum_{i=1}^{n-k}(\zeta_{m_i}/\psi_{m_i})}{\sum_{i=1}^{n-k}(\zeta_{m_i}^2/\psi_{m_i})} Z_{[m_i]m_i} \tag{33}$$

and

$$Var(\tilde{\sigma}_2(k)) = \frac{1}{\sum_{i=1}^{n-k}(\zeta_{m_i}^2/\psi_{m_i})}\sigma_2^2. \tag{34}$$

Remark 4.1. Since both the BLUE $\tilde{\sigma}_2(k)$ and the unbiased estimator $\sigma_2^*(k)$ based on the censored ranked set sample utilize the distributional property of the parent distribution they lose the usual robustness property. Hence in this case the BLUE $\tilde{\sigma}_2(k)$ shall be considered as a more preferable estimator than $\sigma_2^*(k)$.

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