# Optimal spare-switching times in series systems under a general framework

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**Abstract**

*Spares are commonly used to improve system performances. They are allocated to original components during system missions. The optimal allocations depend on system configurations and lifetimes of components and spares. Various methods for finding optimal allocations have been proposed in the literature. For sake of brevity, lifetimes of components are commonly assumed to be independent. This paper deals with series systems, a common configuration, under a general setting, i.e. component lifetimes are dependent and heterogeneous. Moreover, the spare is also allowed to switch among original components to impose more flexibility for spare managements. This allowance occurred usually in network servers and electrical generators which manage by a dispatching center. Explicit expressions for system reliability functions are derived in detail. Since system lifetimes are random phenomena, stochastic orders are utilized for comparison purposes. Various illustrative examples are also given.*

**Keywords:** Allocation; Reliability; Stochastic orders; Redundancy; Switching.

## 1. Introduction

Implementing spare (redundant) is an effective and common method to attain high reliability systems. The redundancy allocation problem (RAP) has been extensively used in real-world applications such as circuit design, power plant, electrical power systems, transportation, safety, telecommunication, satellite, consumer-electronics industry, etc. Specifically redundants are also used in computer science and especially in network systems (servers), to guard the primary system against random failure as a backup system. In this case, redundant components can include both hardware elements of a system such as disk drives, peripherals, servers, switches, routers and software elements such as operating systems, applications, and databases. In the world of information technology (IT), redundancy is the "duplication of critical components or functions of a system with the intention of increasing reliability of the system, usually in the form of a backup or fail-safe, or to improve actual system performance, such as in the case of GNSS receivers (GNSS antennas), or multi-threaded computer processing" (See, e.g. [7]). Issues such as a hardware failure, network problems, or application faults could cause the primary servers to stop functioning correctly. This can leave users unable to access services, which poses a real barrier to productivity. Server redundancy helps businesses by protecting critical data by ensuring it exists in more than just one place. This means that the business can recover data if something happens to a live server. For applications where data integrity and access are vital, redundant servers are very important.

Additional components (spares) are used to improve engineering system performances. For more details, see Barlow and Proschan [2], Nakagawa [15]. There are many papers deal with redundancy allocation problem in reliability systems. Boland et al. [5] applied stochastic orders to consider this problem for series and parallel systems. Zhao et al. [24, 23] studied optimal allocation of redundancies with exponential components in the sense of various stochastic orders. Xie et al. [20] investigated the redundancy allocation problem in $k$-out-of-$n$ hot standby systems to maximize the operational availability. But in the case of dependent components, there are not many works. Among a few works, Navarro et al. [17] studied the performance of a system composed by various kinds of units have dependent lifetimes to evaluate reliability. Navarro and Durante [16] studied the behaviour of the residual lifetimes of coherent systems with possibly dependent components. Belzunce et al. [3, 4] used the concept of "*joint stochastic orders*" and Jeddi and Doostparast [9, 10] studied this problems for series and parallel systems. Redundants are allocated to original components during system missions. Commonly, spares do not switch among the original components. Adding redundants to a system may be done with respect to some limitations such as cost, weight and volume. Switching a spare between original (primary) components is a possible way to overtake these limitations. For example, communication networks, managers may be able to control and switch spares (servers) among original components (or severs) to achieve more reliable connections among customers. This management is usually done by a dispatching center. In other words, redundants can change dynamically their respective original components. For more examples and recent developments, see Kim et al. [12], Li et al. [14], Jia et al. [11] and references therein. Notice that, the spare can not switch if the corresponding original component fails.

In the sequel, let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space and $\mathbf{X} = (X_1, \cdots, X_k) : \Omega \to \mathbb{R}^k_+$, for $k \geq 1$, be an absolutely continuous random vector with the joint distribution function and joint survival function

$$F_{X_1,\cdots,X_k}(a_1, \cdots, a_k) = P(X_1 \leq a_1, \cdots, X_k \leq a_k), \;\; \forall (a_1, \cdots, a_k) \in \mathbb{R}^k_+,$$

and

$$\overline{F}_{X_1,\cdots,X_k}(a_1, \cdots, a_k) = P(X_1 > a_1, \cdots, X_k > a_k), \;\; \forall (a_1, \cdots, a_k) \in \mathbb{R}^k_+,$$

respectively. Here, $\mathbb{R}^k_+$ stands for the $k$-dimensional Euclidean space. Then, the density function of $\mathbf{X}$ is given by $f_{X_1,\cdots,X_k}(a_1, \cdots, a_k) = \partial F_{X_1,\cdots,X_k}(a_1, \cdots, a_k)/(\partial a_1 \cdots \partial a_k)$. The marginal distribution of $X_i (1 \leq i \leq k)$ is denoted by $F_{X_i}(x) = P(X_i \leq x), \forall x \in \mathbb{R}^{+k}$. The random variables $X_i$ is said to be smaller than $X_j (j \neq i)$ in usual stochastic order denoted by $X_i \leq_{st} X_j$, if $F_{X_i}(x) \geq F_{X_j}(x), \forall x \in \mathbb{R}$. Equivalently, $\overline{F}_{X_i}(x) \leq \overline{F}_{X_j}(x)$ where $\overline{F}_{X_i}(x) = 1 - F_{X_i}(x), \forall x$ and for $1 \leq i \leq k$; See Shaked and Shanthikumar [19].

This paper is organized as follows. In Section 2, two possible spare allocations are described. Then, the improved system lifetimes by the two schemes for allocating the spare are also derived. In Section 3, a general form for the system reliability function is presented. Section 4 deals with comparing the two schemes. Indeed, The main result holds for systems with heterogeneous and dependent component lifetimes. Also, it provides the interval time for switching the spare among original components when the components and spare are independent. In Section 5, two general classes of lifetimes and some well known lifetimes are analysed in detail. Section 6 concludes the paper and provides further topics for future research.

## 2. System lifetime with the spare

Consider a 2-component series system consisting of a spare which can be added to the system configuration. The spare can switch only one-time. Let $\tau > 0$ be a preassigned deterministic constant and $T_i^{[0,\tau]} (i = 1, 2)$ denote the system lifetime when the spare is allocated (in parallel) to Component $i$ during interval $[0, \tau]$ and then to Component $j (\neq i)$ beyond $\tau (> 0)$ ; See Figure 1. In Figure 1, the spare is allocated to Component 1 during the time interval $[0, \tau]$. Then, the spare

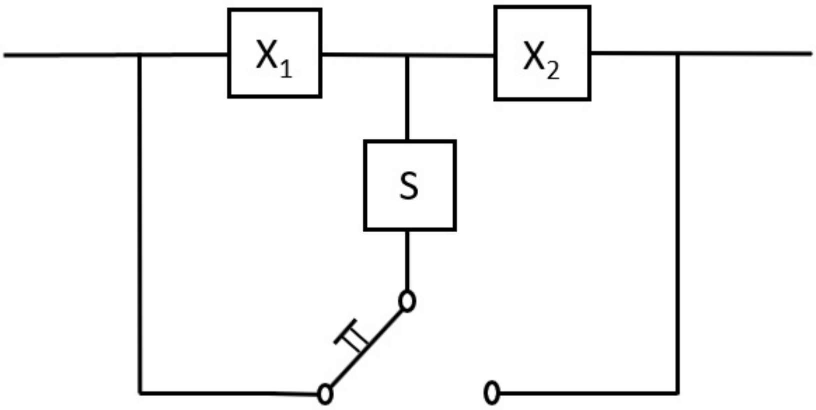


**Figure 1:** *Allocation a spare under a switching scheme.*

is allocated to Component 2. We call this allocation strategy as Scheme 1. Similarly, Scheme 2 is defined. Figure 2 pictures two possible schemes for allocating the spare.

Now, we obtain the system lifetime when the spare is added to the original components. To do this, let $X_1$ and $X_2$ denote the original component lifetimes, and $S$ stands for the spare lifetime. Therefore, it can be seen that the improved system lifetime under Scheme I is

$$\begin{aligned} T_1^{[0,\tau]} &= \begin{cases} \wedge(\vee(X_1,S),X_2), & \text{if } X_1 \leq \tau, \\ X_2, & \text{if } X_1 > \tau, X_2 \leq \tau, \\ \tau + \wedge(X_1-\tau, X_2-\tau), & \text{if } X_1 > \tau, X_2 > \tau, S \leq \tau, \\ \tau + \wedge(X_1-\tau, \vee(X_2-\tau, S-\tau)), & \text{if } X_1 > \tau, X_2 > \tau, S > \tau, \end{cases} \\ &= \begin{cases} \wedge(\vee(X_1,S),X_2), & \text{if } X_1 \leq \tau, \\ X_2, & \text{if } X_1 > \tau, X_2 \leq \tau, \\ \tau + \wedge(X_1-\tau, \vee(X_2-\tau, S-\tau)), & \text{if } X_1 > \tau, X_2 > \tau, \end{cases} \\ &= \begin{cases} \wedge(\vee(X_1,S),X_2), & \text{if } X_1 \leq \tau, \\ X_2, & \text{if } X_1 > \tau, X_2 \leq \tau, \\ \wedge(X_1, \vee(X_2,S)), & \text{if } X_1 > \tau, X_2 > \tau, \end{cases} \end{aligned} \tag{1}$$

where $\vee(a_1,a_2) = \max\{a_1,a_2\}$ and $\wedge(a_1,a_2) = \min\{a_1,a_2\}$. Similarly, the improved system lifetime under Scheme II is

$$T_2^{[0,\tau]} = \begin{cases} \wedge(X_1, \vee(X_2,S)), & \text{if } X_2 \leq \tau, \\ X_1, & \text{if } X_2 > \tau, X_1 \leq \tau, \\ \wedge(\vee(X_1,S),X_2)), & \text{if } X_2 > \tau, X_1 > \tau. \end{cases} \tag{2}$$

Equations (1) and (2) can be unified as

$$\begin{aligned} T_1^{[0,\tau]} &= \wedge(\vee(X_1,S),X_2)I(X_1 \leq \tau) + X_2 I(X_1 > \tau, X_2 \leq \tau) \\ &+ \wedge(X_1,\vee(X_2,S))I(X_1 > \tau, X_2 > \tau), \end{aligned} \tag{3}$$

and

$$\begin{aligned} T_2^{[0,\tau]} &= \wedge(X_1,\vee(X_2,S))I(X_2 \leq \tau) + X_1 I(X_2 > \tau, X_1 \leq \tau) \\ &+ \wedge(\vee(X_1,S),X_2))I(X_2 > \tau, X_1 > \tau), \end{aligned} \tag{4}$$

where $I_A(t)$ denotes the indicator function of the set $A$, i.e., $I_A(t) = 1$ for $t \in A$, and $I_A(t) = 0$ otherwise.

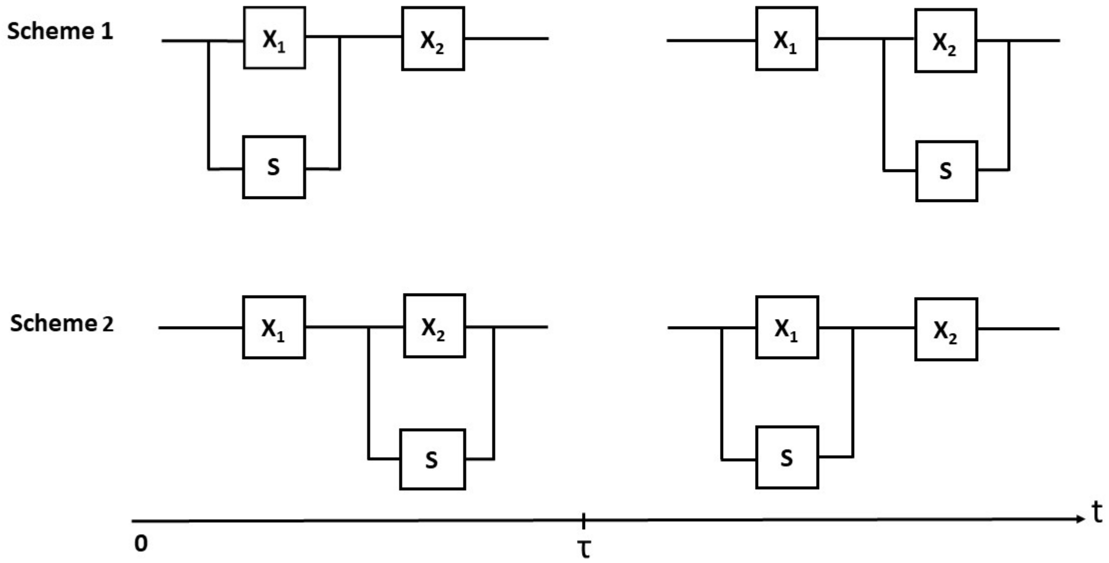


**Figure 2:** *Scheme 1 with system lifetime $T_1^{[0,\tau]}$ and Scheme 2 with system lifetime $T_2^{[0,\tau]}$.*

## 3. System reliability with the spare

In this section, the reliability function of the system is derived when the spare allows to switch among the system components one and only one time. To end this, from Equation (3), we have for $0 < t < \tau$,

$$\begin{aligned}
P(T_1^{[0,\tau]} > t) &= P(T_1^{[0,\tau]} > t, X_1 \leq \tau) + P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 \leq \tau) \\
&+ P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 > \tau) \\
&= P(\wedge(\vee(X_1, S), X_2) > t, X_1 \leq \tau) + P(X_1 > \tau, X_2 \leq \tau, X_2 > t) \\
&+ P(\wedge(X_1, \vee(X_2, S)) > t, X_1 > \tau, X_2 > \tau) \\
&= P(\vee(X_1, S) > t, X_1 \leq \tau, X_2 > t) + P(X_1 > \tau, t < X_2 \leq \tau) \\
&+ P(X_1 > t, \vee(X_2, S) > t, X_1 > \tau, X_2 > \tau) \\
&= P(X_1 \leq \tau, X_2 > t) - P(X_1 \leq \tau, X_2 > t, X_1 \leq t, S \leq t) \\
&+ P(X_1 > \tau, X_2 > t) - P(X_1 > \tau, X_2 > \tau) \\
&+ P(X_1 > \tau, X_2 > \tau) - P(X_1 > \tau, X_2 > \tau, X_2 \leq t, S \leq t) \\
&= P(X_1 \leq \tau, X_2 > t) - P(X_1 \leq t, X_2 > t, S \leq t) \\
&+ P(X_1 > \tau, X_2 > t) \\
&= P(X_2 > t) - P(X_1 \leq t, S \leq t) + P(X_1 \leq t, X_2 \leq t, S \leq t) \\
&= 1 - F_{X_2}(t) - F_{X_1,S}(t,t) + F_{X_1,X_2,S}(t,t,t). \qquad (5)
\end{aligned}$$

Similarly, for $t \geq \tau$, we have

$$
\begin{aligned}
P(T_1^{[0,\tau]} > t) &= P(T_1^{[0,\tau]} > t, X_1 \leq \tau) + P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 \leq \tau) \\
&+ P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 > \tau) \\
&= P(\wedge(\vee(X_1, S), X_2) > t, X_1 \leq \tau) + P(X_1 > \tau, X_2 \leq \tau, X_2 > t) \\
&+ P(\wedge(X_1, \vee(X_2, S)) > t, X_1 > \tau, X_2 > \tau) \\
&= P(\vee(X_1, S) > t, X_1 \leq \tau, X_2 > t) \\
&+ P(X_1 > t, \vee(X_2, S) > t, X_1 > \tau, X_2 > \tau) \\
&= P(X_1 \leq \tau, X_2 > t) - P(X_1 \leq \tau, X_2 > t, X_1 \leq t, S \leq t) \\
&+ P(X_1 > t, X_2 > \tau) - P(X_1 > t, X_2 > \tau, X_2 \leq t, S \leq t) \\
&= P(X_1 \leq \tau, X_2 > t) - P(X_1 \leq \tau, X_2 > t, S \leq t) \\
&+ P(X_1 > t, X_2 > \tau) - P(X_1 > t, \tau < X_2 \leq t, S \leq t) \\
&= P(X_1 \leq \tau, X_2 > t, S > t) + P(X_1 > t, X_2 > \tau) \\
&\quad -P(X_1 > t, X_2 > \tau, S \leq t) + P(X_1 > t, X_2 > t, S \leq t) \\
&= P(X_2 > t, S > t) - P(X_1 > \tau, X_2 > t, S > t) \\
&\quad +P(X_1 > t, X_2 > \tau, S > t) + P(X_1 > t, X_2 > t) - P(X_1 > t, X_2 > t, S > t) \\
&= \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,X_2,S}(\tau,t,t) + \bar{F}_{X_1,X_2,S}(t,\tau,t) + \bar{F}_{X_1,X_2}(t,t) \\
&\quad -\bar{F}_{X_1,X_2,S}(t,t,t). \qquad (6)
\end{aligned}
$$

Finally, from Equations (5) and (6), the next proposition is obtained.

**Proposition 1.** The system reliability functions of $T_1^{[0,\tau]}$ and $T_2^{[0,\tau]}$ are $\overline{F}_{T_1^{[0,\tau]}}(t) = y_1(t)I_{[0,\tau)}(t) + y_2(t)I_{[\tau,\infty)}(t)$, and $\overline{F}_{T_2^{[0,\tau]}}(t) = z_1(t)I_{[0,\tau)}(t) + z_2(t)I_{[\tau,\infty)}(t)$, respectively, where $y_i(t)$ and $z_i(t) (i = 1,2)$ are defined by

$$
\begin{aligned}
y_1(t) &= \bar{F}_{X_2}(t) - F_{X_1,S}(t,t) + F_{X_1,X_2,S}(t,t,t), \qquad (7)\\
y_2(t) &= \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,X_2,S}(\tau,t,t) + \bar{F}_{X_1,X_2,S}(t,\tau,t) + \bar{F}_{X_1,X_2}(t,t) - \bar{F}_{X_1,X_2,S}(t,t,t), \qquad (8)\\
z_1(t) &= \bar{F}_{X_1}(t) - F_{X_2,S}(t,t) + F_{X_1,X_2,S}(t,t,t), \qquad (9)\\
z_2(t) &= \bar{F}_{X_1,S}(t,t) - \bar{F}_{X_1,X_2,S}(t,\tau,t) + \bar{F}_{X_1,X_2,S}(\tau,t,t) + \bar{F}_{X_1,X_2}(t,t) - \bar{F}_{X_1,X_2,S}(t,t,t),
\end{aligned}
$$

for all $t > 0$.

Notice that $\lim_{t\to\tau^-} y_1(t) = \lim_{t\to\tau^+} y_2(t)$ and $\lim_{t\to\tau^-} z_1(t) = \lim_{t\to\tau^+} z_2(t)$. Therefore, the next corollary follows.

**Corollary 1.** The reliability functions of the lifetimes $T_1^{[0,\tau]}$ and $T_2^{[0,\tau]}$ are continuous in $t \in (0, +\infty)$.

## 4. Comparison and optimal time to switch

System lifetimes are random variables and then partially orders should be considered for comparison purposes. Among various partially orders, stochastic orders are commonly used in reliability analyses. See, e.g. Boland et al. [5], Navarro et al. [17] and Belzunce et al. [3]. In this section, the main result of this paper is presented under a general setting for component and spare lifetimes. In the rest of this paper and for lifetimes $U_1, U_2$ and $U_3$, let $\bar{F}_{U_1|(U_2,U_3)}(u_1|u_2,u_3) := P(U_1 > u_1|U_2 > u_2, U_3 > u_3)$.

**Proposition 2.** Suppose that $X_1$, $X_2$ and $S$ be dependent random variables and $[X_1|S = s] \leq_{st} [X_2|S = s]$ for all $s \geq 0$. If $\bar{F}_{X_1|(X_2,S)}(\tau|t,t) \leq 1/2$ and $\bar{F}_{X_2|(X_1,S)}(\tau|t,t) \geq 1/2$ for $t > \tau$, then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$.

**Proof.** [i] For $0 < t \leq \tau$, Equations (7) and (9) conclude

$$\begin{aligned}\overline{F}_{T_1^{[0,\tau]}}(t) - \overline{F}_{T_2^{[0,\tau]}}(t) &= \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,S}(t,t) \\ &= \int_t^{+\infty} \Big( P(X_2 > t|S=s) - P(X_1 > t|S=s) \Big) dF_S(s) \geq 0, \quad (10)\end{aligned}$$

since $[X_1|S=s] \leq_{st} [X_2|S=s]$ for all $s > 0$. For $t > \tau$, Equations (8) and (10) imply

$$\begin{aligned}\overline{F}_{T_1^{[0,\tau]}}(t) - \overline{F}_{T_2^{[0,\tau]}}(t) &= \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,S}(t,t) + 2\bar{F}_{X_1,X_2,S}(t,\tau,t) - 2\bar{F}_{X_1,X_2,S}(\tau,t,t) \\ &= \bar{F}_{X_2,S}(t,t)(1 - 2\bar{F}_{X_1|X_2,S}(\tau|t,t)) + \bar{F}_{X_1,S}(t,t)(2\bar{F}_{X_2|X_1,S}(\tau|t,t) - 1) \\ &\geq 0. \quad (11)\end{aligned}$$

Since $\bar{F}_{X_1|(X_2,S)}(\tau|t,t) \leq 1/2$ and $\bar{F}_{X_2|(X_1,S)}(\tau|t,t) \geq 1/2$ for $t > \tau$, the desired result follows. ■
There are situations in which components and spares are remote and hence they are approximately statistically independent. For example, suppose that there are two main servers in a network and the system administrator (in a dispatching center) wishes to improve system reliability by adding an extra server. Therefore, the three servers would be independent. In sequel, some conditions are assumed which simplify the main result is given in Proposition 2. First, assume that the spare is independent of original component lifetimes $X_1$ and $X_2$ while the original components may be dependent.

**Corollary 2.** Let $S$ be independent of $(X_1, X_2)$. If $X_1 \leq_{st} X_2$ and $\bar{F}_{X_1|X_2}(\tau|t) \leq 1/2$ and $\bar{F}_{X_2|X_1}(\tau|t) \geq 1/2$ for $t > \tau$, then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$.

The next Proposition states that if the original components and the spare are independent, and the switch time lies between medians of the original component DFs, then Scheme 1 dominates Scheme 2 in st-order.

**Proposition 3.** Let $X_1$, $X_2$ and $S$ be independent. If $X_1 \leq_{st} X_2$ and $m_1 < \tau < m_2$, where $m_1$ and $m_2$ stand for medians of $X_1$ and $X_2$, respectively. Then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$.

**Proof.** For $0 < t \leq \tau$, Equations (7) and (9) conclude

$$\begin{aligned}\overline{F}_{T_1^{[0,\tau]}}(t) - \overline{F}_{T_2^{[0,\tau]}}(t) &= g_1(t) - z_1(t) \\ &= F_{X_1}(t) - F_{X_2}(t) + F_{X_2}(t)F_S(t) - F_{X_2}(t)F_S(t) \\ &= \bar{F}_S(t)(F_{X_1}(t) - F_{X_2}(t)) \geq 0, \quad (12)\end{aligned}$$

since $X_1 \leq_{st} X_2$. For $t > \tau$, Equations (8) and (10) imply

$$\begin{aligned}\overline{F}_{T_1^{[0,\tau]}}(t) - \overline{F}_{T_2^{[0,\tau]}}(t) &= g_2(t) - z_2(t) \\ &= \bar{F}_{X_2}(t)\bar{F}_S(t) - \bar{F}_{X_1}(t)\bar{F}_S(t) \\ &\quad +2\big(\bar{F}_{X_1}(t)\bar{F}_{X_2}(\tau)\bar{F}_S(t) - \bar{F}_{X_1}(\tau)\bar{F}_{X_2}(t)\bar{F}_S(t)\big) \\ &= \bar{F}_{X_2}(t)\bar{F}_S(t)(1 - 2\bar{F}_{X_1}(\tau)) + \bar{F}_{X_1}(t)\bar{F}_S(t)(2\bar{F}_{X_2}(\tau) - 1) \\ &\geq 0, \quad (13)\end{aligned}$$

since $\tau > m_1$ and $\tau < m_2$. and the desired result follows. ■ Proposition 3 says that if component and spare lifetimes are independent, the spare should allocate to the weaker component at least up to its median lifetime and then before reaching to the median lifetime of the other component, the spare must switch.

**Remark 1.** The distribution of $S$ in Proposition 3 is free and the given conditions do not rely on the DF of $S$.

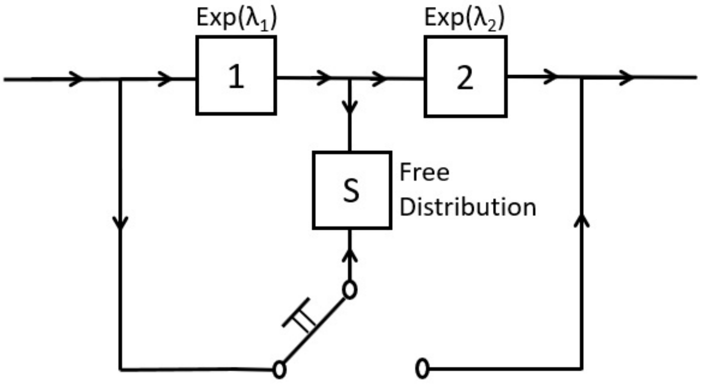


**Figure 3:** *Allocation a spare under a switching scheme with independent exponential random variables.*

## 5. Examples

In this section, some examples are analyzed to derived the optimal switching times for the spare.

**Example 4.** Let $X_1$ and $X_2$, be independent exponential random variables with means $1/\lambda_1$ and $1/\lambda_2$, respectively. If $\lambda_1 > \lambda_2$, then Proposition 3 implies that $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ provided that the switching occurs after $\frac{\ln 2}{\lambda_1}$ but before $\frac{\ln 2}{\lambda_2}$, that is $\frac{\ln 2}{\lambda_1} < \tau < \frac{\ln 2}{\lambda_2}$; See Figure 3. □

**Example 5.** Let $X_1$, $X_2$ and $S$ be independent and $X_i \sim Pa(\alpha_i, 1), i = 1,2$ where $\alpha_i > 0$ and $Pa(a,b)$ stands for the Pareto distribution Type I with density $f(x) = \dfrac{a}{x^{a+1}}, x \geq 1$. It is easy to see that the medians of $X_1$ and $X_2$, respectively, are given by $m_1 = \sqrt[\alpha_1]{2}$ and $m_2 = \sqrt[\alpha_2]{2}$. If $\alpha_1 > \alpha_2$ and $\sqrt[\alpha_1]{2} < \tau < \sqrt[\alpha_2]{2}$ then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ from Proposition 3. □

**Example 6.** The class of *newsboy distributions*, introduced by Braden and Freimer[6], includes absolutely continues distribution functions of the form $F_\theta(x) = 1 - e^{-\theta l(x)}$, where $l(x)$ is non-negative, increasing, differentiable and unbounded function with $l(0) = 0$. The newsboy distributions are used extensively in the modelling of excess demand of the inventory level that is lost and thus unobserved. Now assume that $X_1$, $X_2$ and $S$ be independent and $X_i \sim F_{\theta_i}(x) = 1 - e^{-\theta_i l(x)}, i = 1,2$ where $\theta_i > 0$. If $\theta_1 > \theta_2$ and $l^{-1}\left(\dfrac{\ln 2}{\theta_1}\right) < \tau < l^{-1}\left(\dfrac{\ln 2}{\theta_2}\right)$ then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ by Proposition 3; See Table 1. Here, $l^{-1}(.)$ denotes the inverse of the function $l(.)$.

**Example 7.** Let $\Phi$ be the class of absolutely continues distribution function $F_\theta$ of the form $F_\theta(x) = 1 - e^{-K_\theta(x)}, x > 0$, where $K_\theta(x)$ is increasing in $x$ and positive function $\theta \in \Theta$. Then the probability of density function is given by $f_\theta(x) = k_\theta(x)e^{-K_\theta(x)}, x > 0$, where $k_\theta(x) = \dfrac{\partial}{\partial x}K_\theta(x)$. This class include several important distribution such as exponential, Pareto, Weibull and has been studied in literature; See e.g, Al-Hussaini [1] for more details. Let $X_1 \sim F_{\theta_1}(x)$ and $X_2 \sim F_{\theta_2}(x)$. Then medians $X_1$ and $X_2$ are $m_1 = K_{\theta_1}^{-1}(\ln 2)$ and $m_2 = K_{\theta_2}^{-1}(\ln 2)$ respectively. If $K_{\theta_1}(x) \geq K_{\theta_2}(x)$ and $K_{\theta_1}^{-1}(\ln 2) \leq \tau \leq K_{\theta_2}^{-1}(\ln 2)$, then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ by Proposition 3. For example, let $K_\theta(x) = \lambda x^\alpha$ and $\Theta = (\alpha, \lambda), \alpha, \lambda > 0$. Thus $X_i$, $i = 1,2$, has the Weibull distribution with density function $f_{\alpha_i,\lambda}(x) = \alpha_i \lambda x^{\alpha_i - 1} e^{-\lambda x^{\alpha_i}}, \alpha_i, \lambda > 0$, therefore $m_i = K_{\theta_i}^{-1}(\ln 2) = \sqrt[\alpha_i]{(\dfrac{\ln 2}{\lambda})}$ where `$_{\mathbf{i}}$ $= (\alpha_i, \lambda)$, $i = 1,2$. If $\alpha_1 > \alpha_2$ and $\sqrt[\alpha_1]{(\dfrac{\ln 2}{\lambda})} < \tau < \sqrt[\alpha_2]{(\dfrac{\ln 2}{\lambda})}$ then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ by Proposition 3. Table 1 presents some selected members in the class $\Phi$ and corresponding optimal switching times.

□

**Table 1:** *Some well known members of Class $\Phi$ in Example 7*

| Distribution | Vector parameter \`$_i$ | $K_{\theta_i}(x)(i=1,2)$ | optimal switching time $K_{\theta_1}^{-1}(\ln 2) \leq \tau \leq K_{\theta_2}^{-1}(\ln 2)$ |
|---|---|---|---|
| Exponential | $\lambda_i$ | $\lambda_i x$ | $\frac{\ln 2}{\lambda_1} < \tau < \frac{\ln 2}{\lambda_2}$ |
| Pareto | $(\alpha_i, \beta)$ | $\alpha_i \ln\left(\frac{x}{\beta}\right)$ | $\beta \sqrt[\alpha_1]{2} < \tau < \beta \sqrt[\alpha_2]{2}$ |
| Weibull | $(\alpha_i, \lambda)$ | $\lambda x^{\alpha_i}$ | $\sqrt[\alpha_1]{\left(\frac{\ln 2}{\lambda}\right)} < \tau < \sqrt[\alpha_2]{\left(\frac{\ln 2}{\lambda}\right)}$ |
| Compound Weibull (Burr type XII) | $(\alpha_i, \gamma)$ | $\gamma \ln(1 + x^{\alpha_i})$ | $\sqrt[\alpha_1]{e^{\frac{\ln 2}{\gamma}} - 1} < \tau < \sqrt[\alpha_2]{e^{\frac{\ln 2}{\gamma}} - 1}$ |
| Rayleigh | $\sigma_i$ | $\frac{x^2}{2\sigma_i^2}$ | $\sqrt{2}\sigma_1 \ln 2 < \tau < \sqrt{2}\sigma_1 \ln 2$ |
| Newsboy | $\theta_i$ | $\theta_i l(x)$ | $l^{-1}\left(\frac{\ln 2}{\theta_1}\right) < \tau < l^{-1}\left(\frac{\ln 2}{\theta_2}\right)$ |

## 6. Conclusions

This paper derived the system reliability function consisting of two original components and a spare. The spare can switch among the original components. The finding of this paper hold under a general setting. The optimal scheme for switching was also provided. Some special cases which have also practical applications were studied in detail. It was shown that the best time for switching the spare falls between the median lifetimes of the original component lifetimes provided that all component and spare lifetimes be independent. This result does not depend on the distribution of spare lifetimes. The results of this paper may be extended in various directions. For example, one can study the system behaviour under some parametric conditions such as multivariate distribution functions for component lifetimes. The optimal switching times for the case of two and multiple redundancies may also be considered. Engineering systems including parallel-series, series-parallel and mixed systems are worth for consideration in details. Another problem is that the spare allows to switch at possible times $\tau_1, \cdots, \tau_k (k \geq 1)$. This means that the spare can switch for $k$ times. Finding optimal switching times $\tau_1, \cdots, \tau_k$ is essential in practice. Works on these topics are under consideration and we hope to report findings soon.

## References

[1] AL-Hussaini, E.K., 1999. Predicting Observables from a General Class of Distributions, Journal of Statistical Planning and Inference, Vol. **79**, pp.79-91.

[2] Barlow, R. E., Proschan, F., 1975. Statistical theory of reliability and life testing. Holt, Rinehart and Winston Inc. New York.

[3] Belzunce, F., Martinez-Puertas, H. Ruiz, J. M., 2011. On optimal allocation of redundant components for series and parallel systems of two dependent components, Journal of Statistical Planning and Inference, Vol. **141**, pp. 3094-3104.

[4] Belzunce, F., Martinez-Puertas, H. Ruiz, J. M., 2013. On allocation of redundant components for systems with dependent components, European Journal of Operational Research, Vol. **230**, pp. 573-580.

[5] Boland, P. J., EI-Neweihi, E., Porschan, F., 1992. Stochastic order for redundancy allocations in series and parallel systems, Advances in Applied Probability, Vol. **24**, pp.161-171.

[6] D.J. Braden, M. Freimer, 1991. Informational dynamics of censored observations, Management Science, Vol. **37**, pp.1390-1404.

[7] Flammini, F., Marrone, S., Mazzocca, N., Vittorini, V., 2009. A new modeling approach to the safety evaluation of N-modular redundant computer systems in presence of imperfect maintenance, Reliability Engineering & System Safety, Vol. **94**, pp.1422-1432.

[8] Hollander, M., Proschan, F., Sethuraman, J., 1977. Functions decreasing in transposition and their applications in ranking problems, Annals of Statistics, Vol. **5**, pp.722-733.

[9] Jeddi, H., Doostparast, M., 2016. Optimal redundancy allocation problems in engineering systems with dependent component lifetime, Applied Stochastic Models in Business and industry, Vol. **32**, pp.199-208.

[10] Jeddi, H., Doostparast, M., 2022. Allocation of redundancies in systems: A general dependency-base framework, Annals of Operations Research, Vol. **312**, pp.259-273.

[11] Jia, X., Chen, H., Cheng, Z., Gou, B., 2016. A comparison between two switching policies for two-unit standby system, Reliability Engineering & System Safety, Vol. **148**, pp.109-118.

[12] Kim, T. H., Lee, S. H., Lim, J. T., 2001. Stability analysis of switched systems via redundancy factors. *International Journal of Systems Science*, Vol. **32**, pp.1199-1204.

[13] Lehmann, E. L., 1966. Some concepts of dependence. The Annals of Mathematical Statistics, Vol. **37**, pp.1137-1153.

[14] Li, X., Fang, R., Mi, J., 2015. On the timing to switch on the standby in $k$-out-of-$n$: $G$ redundant systems, Statistics and Probability Letters, Vol. **96**, pp.10-20.

[15] Nakagawa, T., 2005. Maintenance theory of reliability. Springer, New York.

[16] J. Navarro, F. Durante, 2017. Copula-based representations for the reliability of the residual lifetimes of coherent systems with dependent components, Journal of Multivariate Analysis, Vol. **158**, pp.87-102.

[17] J. Navarro, F. Pellerey, A. Di Crescenzo, 2015. Orderings of coherent systems with randomized dependent components, European Journal of Operational Research, Vol. **240**, pp.127-139.

[18] Nelsen, R. B., 2006. An introduction to copulas. Springer, New York.

[19] Shaked, M., Shanthikumar, J. G., 2007. Stochastic orders. Springer-Verlag, New York.

[20] Wei Xie, Haitao Liao, Tongdan Jin, (2014). Maximizing system availability through joint decision on component redundancy and spares inventory. European Journal of Operational Research, **237**, 164-176.

[21] Zhang, Y., E Amini-Seresht, Ding W., (2017). Component and system active redundancies for coherent systems with dependent components. Applied Stochastic Models in Business and Industry, **33**, 409-421.

[22] Zhang, X., Zhang, Y., Fang, R., (2020). Allocations of cold standbys to series and parallel systems with dependent components. Applied Stochastic Models in Business and Industry, **36**, 432-451.

[23] Zhao, P., Chan, P.S., Li, L., Ng, H.K.T., (2013). Allocation of two redundancies in two-component series systems. Naval Research Logistics, **60**, 588-598.

[24] Zhao, P., Chan, P.S., Ng, H.K.T., (2012). Optimal allocation of redundancies in series systems. European Journal of Operational Research, **220**, 673-683.