RT&A, No 2 (73)
Volume 18, June 2023

BAYESIAN AND CLASSICAL ESTIMATIONS OF TRANSMUTED INVERSE GOMPERTZ DISTRIBUTION

*[1]T. M. Adegoke, [2]K.O. Obisesan, [3]O. M. Oladoja & [4]G.K. Adegoke

•

*[1,2,3]Department of Mathematics and Statistics, First Technical University, Ibadan, Nigeria.
[4]Department of Computer Science, Nasarawa State University,Keffi, Nigeria.
*[1]taiwo.adegoke@tech-u.edu.ng, [2]obisesan.olalekan@tech-u.edu.ng,
[3]oladapo.oladoja@tech-u.edu.ng, [4]gbolagade.adegoke@nsuk.edu.ng

Abstract

In this article, the use of the transmuted inverse Gompertz distribution in modeling lifetime data is investigated particularly in cases where standard probability distributions are not able to properly handle complex datasets. The quadratic rank transmutation map scheme is utilized to obtain the distribution. The study explores several characteristics of the transmuted inverse Gompertz model, including the estimation of parameters through classical approaches such as maximum likelihood estimation, least-squares estimation, Crammér-Von Misses estimation, and maximum product spacing estimation. Additionally, the Bayesian techniques is used under different loss functions, such as the Linex loss function, square error loss function, and general entropy loss function. The estimates obtained from both classical and Bayesian techniques are evaluated using simulation. To illustrate the potential benefits of the transmuted inverse Gompertz model, a dataset on the strength of aircraft window glass is employed. The results obtained from the application of the new distribution to the real-life dataset demonstrate that it yields superior fits in comparison to other well-known distributions. The study's findings suggest that the transmuted inverse Gompertz model can provide a useful alternative for modeling lifetime data. The research offers valuable insights into the distribution's properties and estimation techniques, as well as its superiority over other commonly used distributions. The new model can contribute to the development of more accurate and efficient models for analyzing lifetime dataset. Overall, the study highlights the importance of exploring new statistical models and techniques to improve data analysis and decision-making

Keywords: Classical method, Bayesian method, Posterior distribution, Loss functions, Transmuted inverse Gompertz distribution.

1. Introduction

The Gompertz distribution with two parameters is an expansion of the exponential distribution and was proposed by Gompertz [1]. It is widely used in survival analysis to construct accurate actuarial and human mortality tables. Additionally, it is a valuable tool for modeling survival distributions with increasing hazard rates and for describing the distribution of adult lifespans by demographers and actuaries (Willemse and Koppelaar [2]).

The inverse distribution was developed to model actuarial surveys biological and demography (El-Bassiouny *et al.* [3]). The inverse Gompertz distribution was proposed by Eliwa *et al.* [4] and it is useful in modeling lifetime observations. The cumulative probability density (CDF) and probability density function (PDF) are expressed as

$$G(x;\theta,\gamma) = e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} \qquad \theta > 0, \gamma > 0, x \geq 0 \tag{1}$$

$$g(x;\theta,\gamma) = \frac{\gamma}{x^2} e^{-\frac{\gamma}{\beta}\left(e^{\frac{\theta}{x}}-1\right)+\frac{\theta}{x}} \qquad \theta > 0, \gamma > 0, x \geq 0 \tag{2}$$

where the scale and shape parameters are presented as θ and γ.
Recently in literature the development of new family of lifetime distributions using transmuted method has recently been attempted to estimate model parameters efficiently for the subject model (see Aryal and Tsokos [5] and Khan *et al.* [6]). Using the techniques known as quadratic rank transmutation map, proposed by Shaw and Buckley [7], we are able to propose a new three parameters transmuted inverse Gompertz (TR-IG) distribution.

If a random variable X has a transmuted distribution, then its cumulative probability density function (CDF) and probability density function (PDF) satisfy the following relationship:

$$F(x) = (1-\lambda)G(x) - \lambda G^2(x), \qquad |\lambda| \leq 1 \tag{3}$$

$$f(x) = g(x)\left((1-\lambda) - 2\lambda G(x)\right) \tag{4}$$

where G(x) is the CDF of the baseline model, g(x) and f(x) are the corresponding PDF associated with G(x) and g(x), respectively.

The motivation of this work is to proposed a model that can be used to model complex dataset which the standard probability distributions model can't handle properly. Also we examine the potential use of the TR-IG distribution and determine a number of the mathematical features.

2. Transmuted inverse Gompertz distribution

Consider a positive value of x with three parameters TR-IG distribution having a shape parameter γ and scale parameter θ and the transmuted parameter $|\lambda| \leq 1$, the CDF can be derived by substituting Equation (1) into Equation (3)

$$F(x;\gamma,\theta,\lambda) = (1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2 \qquad \gamma > 0, \theta > 0, x \geq 0, |\lambda| \leq 1 \tag{5}$$

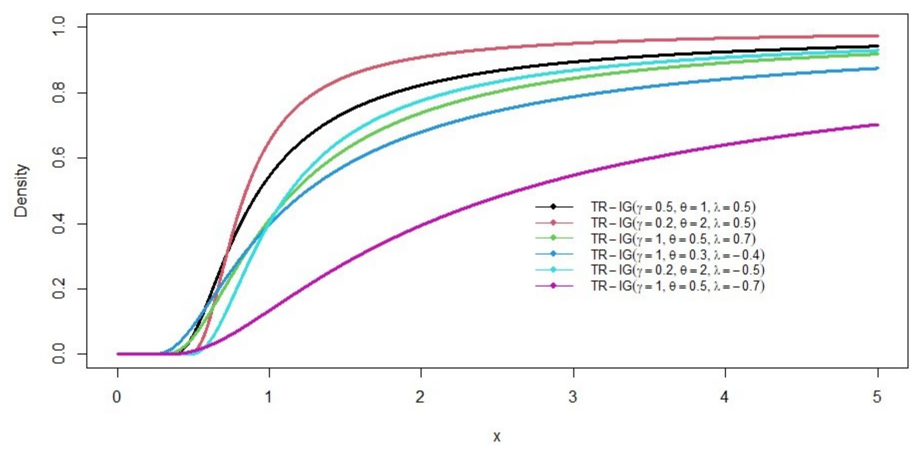

Figure 1: *CDF Plot of TR-IG distribution for different parameter values .*

Figure 1 shows the CDF plots of TR-IG distribution and we deduced that as x increases the CDF increase and remains constant as it tends to 1.
Substituting Equations (1) and (2) into Equation (4) produces the PDF of TR-IG distribution

$$f(x;\gamma,\theta,\lambda) = \frac{\gamma}{x^2} e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)+\frac{\theta}{x}}\left[1+\lambda-2\lambda e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right] \qquad \gamma > 0, \theta > 0, x \geq 0, |\lambda| \leq 1 \tag{6}$$

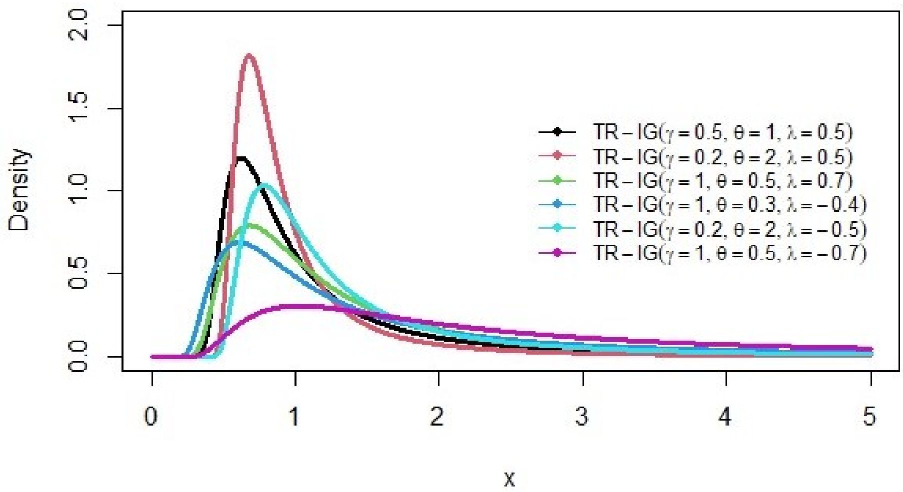

Figure 2: *PDF plot of TR-IG distribution for different parameter values*

Figure 2 shows that the PDF of TR-IG distribution is positively skewed. When $|\lambda| = 0$, the TR-IG distribution CDF and PDF (5) and (6) becomes inverse Gompertz distribution with CDF (1) and PDF (2).

3. Statistical Properties

This section looks into some of the statistical characteristics of $TR - IG(\gamma, \beta, \lambda)$ such survival function, reversed hazard function, odds function, hazard function and moments.

3.1. Survival Function

A survival function of the TR-IG distribution can be expressed as

$$S(x) = 1 - (1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2 \qquad \gamma > 0, \theta > 0, |\lambda| \leq 1, x > 0 \qquad (7)$$

3.2. Hazard Function

The hazard function for the TR-IG distribution can be expressed as

$$h(x) = \frac{\frac{\gamma}{x^2}e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)+\frac{\theta}{x}}\left[1+\lambda-2\lambda e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]}{1-(1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2} \qquad \gamma > 0, \theta > 0, |\lambda| \leq 1, x > 0 \qquad (8)$$

3.3. Reversed Hazard Function (RHF)

The reversed hazard function for the TR-IG distribution can be expressed as

$$\tau(x) = \frac{\frac{\gamma}{x^2} e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)+\frac{\theta}{x}} \left[1+\lambda-2\lambda e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]}{(1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2} \qquad \gamma>0, \theta>0, |\lambda|\leq 1, x>0 \tag{9}$$

3.4. Odd Function

The odd function for the TR-IG distribution can be expressed as

$$O(x) = \frac{(1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2}{1-(1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2} \qquad \gamma>0, \theta>0, |\lambda|\leq 1, x>0 \tag{10}$$

3.5. Cumulative Hazard Function

The cumulative hazard function for the TR-IG distribution can be expressed as

$$H(x) = -\log\left(1-(1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2\right) \gamma>0, \theta>0, |\lambda|\leq 1, x>0 \tag{11}$$

3.6. The Quantile Function

The expression for the quantile function is $Q(u) = F^{-1}(u)$. Consequently, the TR-IG distribution quantile function can be written as

$$Q(u) = \frac{\theta}{\log\left\{1-\frac{\theta}{\gamma}\log\left[\frac{(1+\lambda)-\sqrt{1+(2-4u)\lambda+\lambda^2}}{2\lambda}\right]\right\}} \tag{12}$$

3.7. Moments

Moments can be used to analyze a variety of distributional features, including tendency, skewness, dispersion, and kurtosis. If $X \sim TR-IG(\gamma,\beta,\lambda)$, then the r^{th} moment expression of TR-IG distribution can be expressed as

$$\mu'_r = \int_0^\infty x^r f(x)dx \tag{13}$$

after performing some mathematical expressions, we obtained the moment generating function for the TR-IG distribution

$$\mu'_r = M_x(t) = \sum_{i,j,r=0}^{\infty} {}^iC_j(-1)^{i+j}\frac{1}{i!}\frac{(-t)^r}{r!}\frac{\Gamma(1-r)}{[\theta(j-i-1)]^{1-r}} \tag{14}$$

4. Estimations of Transmuted Inverse Gompertz Distribution

4.1. Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, \ldots x_n$ be random variables (rvs) drawn from $TR-IG(\gamma, \theta, \lambda)$. The log-likelihood function is defined as follows:

$$L(\gamma,\theta,\lambda|x) = \gamma^n \sum_{i=1}^{n} \frac{1}{x} e^{-\frac{\gamma}{\theta}\sum_{i=1}^{n}\left(e^{\frac{\theta}{x}}-1\right)+\theta\sum_{i=1}^{n}\frac{1}{x}} \prod_{i=1}^{n}\left[1+\lambda-2\lambda e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right] \tag{15}$$

taking the logarithm of Equation (15), we obtained the log-likelihood function

$$l = n\log\gamma - \log(x) - \frac{\gamma}{\theta}\sum_{i=1}^{n}\left(e^{\frac{\theta}{x}}-1\right) + \theta\sum_{i=1}^{n}\frac{1}{x} + \sum_{i=1}^{n}\log\left[1+\lambda-2\lambda e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right] \tag{16}$$

Maximizing $logL(\gamma, \theta, \lambda)$ with respect to γ, θ and λ, the system with non-linear equations is obtained as:

$$\frac{\partial l}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n}\left[\frac{\left(\mathrm{e}^{\frac{\theta}{x}}-1\right)}{\theta}\right] + \sum_{i=1}^{n}\left[\frac{2\lambda\mu\left(\mathrm{e}^{\frac{\theta}{x}}-1\right)}{\theta\omega}\right] \tag{17}$$

$$\frac{\partial l}{\partial\theta} = \sum_{i=1}^{n}\left[\frac{\gamma\left(\mathrm{e}^{\frac{\theta}{x}}-1\right)}{\theta^2}\right] - \sum_{i=1}^{n}\left[\frac{\gamma\mathrm{e}^{\frac{\theta}{x}}}{\theta x}\right] + \sum_{i=1}^{n}\left[\frac{1}{x}\right] - \sum_{i=1}^{n}\left[\frac{2\lambda\varrho\mu}{\omega}\right] \tag{18}$$

$$\frac{\partial l}{\partial\lambda} = \sum_{i=1}^{n}\left[\frac{(1-2\mu)}{\omega}\right] \tag{19}$$

where $\varrho = \left(\frac{\gamma\left(\mathrm{e}^{\frac{\theta}{x}}-1\right)}{\theta^2} - \frac{\gamma\mathrm{e}^{\frac{\theta}{x}}}{\theta x}\right)$, $\mu = \mathrm{e}^{-\frac{\gamma\left(\mathrm{e}^{\frac{\theta}{x}}-1\right)}{\theta}}$, $\omega = (1+\lambda-2\lambda\mu)$

To obtain the estimates of $\hat{\gamma}_{MLE}$, $\hat{\theta}_{MLE}$ and $\hat{\lambda}_{MLE}$, we equate expressions (17) - (19) to zero and solve the system of nonlinear equation. It was observed that solution of this system can not be obtained analytically so we employed a numerical approach known as Newton Raphson method.

4.2. Least-squares Method (OLS)

Let the related order statistics (or) of rvs $x_1, x_2, \ldots, x_n$ from the TR-IG distribution sorted in ascending order are denoted as $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. According to Swain *et al.*[8], the OLS estimates of $\hat{\gamma}_{OLS}$, $\hat{\theta}_{OLS}$ and $\hat{\lambda}_{OLS}$ can be obtained by minimizing Equation 20 with respect to γ, θ and λ and equate the non-linear equations to zero.

$$S(\gamma,\theta,\lambda|x) = \sum_{i=1}^{n}\left[\left\{(1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2\right\} - \frac{i}{n+1}\right]^2 \tag{20}$$

4.3. Cramér-von Mises (CVM)

Let the related os of rvs $x_1, x_2, \ldots, x_n$ from the TR-IG distribution sorted in ascending order are denoted as $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. To obtain the CVM estimates of $\hat{\gamma}_{CVM}$, $\hat{\theta}_{CVM}$ and $\hat{\lambda}_{CVM}$, Equation 21 is minimized with respect to γ, θ and λ and equate the non-linear equations to zero.

$$C(\gamma,\theta,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\left\{(1+\lambda)e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} - \lambda\left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)}\right]^2\right\} - \frac{2i-1}{2n}\right]^2 \tag{21}$$

4.4. Maximum Product Spacing Method (MPS)

Let the related os of rvs $x_1, x_2, \ldots, x_n$ from the TR-IG distribution sorted in ascending order are denoted as $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. According to Cheng and Amin [9], the MPS estimates of $\hat{\gamma}_{MPS}$, $\hat{\theta}_{MPS}$ and $\hat{\lambda}_{MPS}$, can be obtained by minimizing Equation 22 with respect to γ, θ and λ and equate the non-linear equations to zero.

$$\begin{aligned} D &= \left\{ (1+\lambda) e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x_i}}-1\right)} - \lambda \left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x_i}}-1\right)} \right]^2 \right\} \\ &\quad - \left\{ (1+\lambda) e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x_{i-1}}}-1\right)} - \lambda \left[e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x_{i-1}}}-1\right)} \right]^2 \right\} \end{aligned} \tag{22}$$

4.5. Bayesian Analysis

Let $x = (x_1, x_2, \ldots x_n)$ be a rv with parameters γ, θ and λ having a size n. The posterior probability density function of the parameters γ, θ and λ given x can be expressed as

$$\Pr(\gamma, \theta, \lambda | x) = \frac{\pi(\gamma, \theta, \lambda) l(\gamma, \theta, \lambda)}{\int\int\int \pi(\gamma, \theta, \lambda) l(\gamma, \theta, \lambda) \partial(\gamma, \theta, \lambda)} \tag{23}$$

where $l(\gamma, \theta, \lambda)$ is the likelihood function expressed in Equation (15) and $\pi(\gamma, \theta, \lambda)$ is the prior probability distribution, which is expressed in Equation (24).

$$\pi(\gamma, \theta, \lambda) = \frac{1}{\gamma \lambda \Gamma(a) b^a} \theta^{a-1} e^{-\frac{\theta}{b}} \qquad \gamma, \theta, \lambda, a, b > 0 \tag{24}$$

where: $\gamma \sim Uniform(0, \gamma)$; $\quad \lambda \sim Uniform(0, \lambda)$; $\quad \theta \sim Gamma(a, b)$
substituting Equations (15) and (24), we obtained the posterior probability distribution

$$\Pr(\gamma, \theta, \lambda | x) = \frac{\frac{1}{\gamma \lambda \Gamma(a) b^a} \theta^{a-1} e^{-\frac{\theta}{b}} \frac{\gamma}{x^2} e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)+\frac{\theta}{x}} \left[1+\lambda - 2\lambda e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} \right]}{\int\int\int \frac{1}{\gamma \lambda \Gamma(a) b^a} \theta^{a-1} e^{-\frac{\theta}{b}} \frac{\gamma}{x^2} e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)+\frac{\theta}{x}} \left[1+\lambda - 2\lambda e^{-\frac{\gamma}{\theta}\left(e^{\frac{\theta}{x}}-1\right)} \right] \partial(\gamma, \theta, \lambda)} \tag{25}$$

4.6. Loss Functions

The Bayesian method was employed for the estimation of TR-IG distribution parameters, utilizing three different types of loss function. The first loss function considered was the LINEX loss function (LLF), also referred to as the linear-exponential loss function, initially proposed by Varian [11]. LLF is an asymmetric loss function that rises exponentially on one side of zero and linearly on the other, as described by Preda and Panaitescu [12].

The second loss function used was the General Entropy loss function (GELF), first introduced by Calabria and Pulcini [13]. GELF is also an asymmetric loss function that has been utilized by several authors, such as Dey and Liu [14], Sule and Adegoke [15], and Ogunsanya *et al.* [17], who used GELF in its original form by setting c to be equal to 1.

The third and final loss function considered was the Squared Error loss function (SELF), which is a common loss function in statistics and machine learning. SELF is also known as the L2 loss function and is used to measure the difference between the estimated and true values squared.

If θ is an estimator required to estimate the parameter $\hat{\theta}$, then the square error loss function can be defined as follows:

$$L_{SELF} \propto (\hat{\theta} - \theta)^2 \tag{26}$$

The LLF can be expressed as

$$L_{LLF} \propto \kappa(e^{c(\hat{\theta}-\theta)} - c(\hat{\theta} - \theta) - 1) \qquad \kappa > 0, c \neq 0 \tag{27}$$

where c and κ are the scale and shape parameters of the LLF. In this study, we assume that κ= 1. Provided that $E_\theta\left[e^{-c\theta}\right]$ exits, Bayes estimator of the LLF is the value $\hat{\theta}$ that minimizes Equation (27) (Zeller [18]).

$$\hat{\theta} = -\frac{1}{c}\ln\left(E_\theta\left[e^{-c\theta}\right]\right) \tag{28}$$

The GELF is defined as

$$L_{GELF}\left(\frac{\hat{\theta}}{\theta}\right)^c \propto \left[\left(\frac{\hat{\theta}}{\theta}\right)^c - c\log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right] \tag{29}$$

where $c > 0$. The minimum occurred at $\hat{\theta} = \theta$. The GELF Bayes estimator is the value $\hat{\theta}$ that minimizes Equation (29) and can be expressed as

$$\hat{\theta} = \left[E\left(\theta^{-c}\right)\right]^{-c} \tag{30}$$

4.7. Lindley Approximation Method

To estimate Equation (25), we will employ an iterative techniques known as Lindley's approximation to estimate the parameter of interest for γ, θ and λ. According to Lindley [19], if n is large enough, any ratio of the integral of the form

$$I(x) = E[u(\gamma,\theta,\lambda)] = \frac{\int\int\int u(\gamma,\theta,\lambda)e^{l(\gamma,\theta,\lambda)+\rho(\gamma,\theta,\lambda)}\partial(\gamma,\theta,\lambda)}{\int\int\int e^{l(\gamma,\theta,\lambda)+\rho(\gamma,\theta,\lambda)}\partial(\gamma,\theta,\lambda)} \tag{31}$$

where $u(\gamma,\theta,\lambda)$ is a function of γ, θ and λ only, $l(\gamma,\theta,\lambda)$ is the log-likelihood and $\rho(\gamma,\theta,\lambda)$ is the log of prior distribution $\pi(\gamma,\theta,\lambda)$. Equation (31) can be evaluated as

$$\begin{aligned} I(x) &= u(\hat{\gamma},\hat{\theta},\hat{\lambda}) + (u_1a_1 + u_2a_2 + u_3 + a_3 + a_4 + a_5) + \frac{1}{2}A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) \\ &+ \frac{1}{2}B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + \frac{1}{2}C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33}) \end{aligned} \tag{32}$$

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3} \tag{33}$$
$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23} \tag{34}$$
$$a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}) \tag{35}$$
$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331} \tag{36}$$
$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332} \tag{37}$$
$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333} \tag{38}$$

$$\rho_i = \frac{\partial\rho}{\partial\theta_i}; \qquad u_i = \frac{\partial u(\theta_1,\theta_2,\theta_3)}{\partial\theta_i} u_{i,j} = \frac{\partial^2 \quad u(\theta_1,\theta_2,\theta_3)}{\partial\theta_i\partial\theta_j}; \qquad L_{i,j,k} = \frac{\partial^3 l(\theta_1,\theta_2,\theta_3)}{\partial\theta_i\partial\theta_j\partial\theta_k} \tag{39}$$

where $\theta_1 = \gamma$, $\theta_2 = \theta$ and $\theta_3 = \lambda$. $\sigma_{i,j}$ is the $(i,j)^{th}$ element of the matrix's inverse $L_{i,j}$ all evaluated at the MLE estimation and $i,j,k = 1,2,3$

$$\frac{d^2 l}{d\gamma^2} = -\frac{n}{\gamma^2} - \sum_{i=1}^{n}\left[\frac{2\lambda\epsilon^2\mu}{\theta^2\omega}\right] - \sum_{i=1}^{n}\left[\frac{4\lambda^2\epsilon^2\mu^2}{\theta^2\omega^2}\right] \tag{40}$$

$$\begin{aligned}\frac{d^2 l}{d\theta^2} &= -\sum_{i=}^{n}\left[\frac{2\gamma\epsilon}{\theta^3}\right] + \sum_{i=}^{n}\left[\frac{2\gamma e^{\frac{\theta}{x}}}{\theta^2 x}\right] - \sum_{i=}^{n}\left[\frac{\gamma e^{\frac{\theta}{x}}}{\theta x^2}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\phi\mu}{\omega}\right] \\ &\quad -\sum_{i=}^{n}\left[\frac{2\lambda\gamma\psi^2\mu}{\omega}\right] - \sum_{i=}^{n}\left[\frac{4\lambda^2\gamma\psi^2\mu^2}{\omega^2}\right] \end{aligned}\tag{41}$$

$$\frac{d^2 l}{d\lambda^2} = -\frac{n\Upsilon^2}{\omega^2}; \qquad \frac{\partial^2 l}{\partial\theta\partial\lambda} = -\sum_{i=}^{n}\left[\frac{2\gamma\psi\mu}{\omega}\right] + \sum_{i=}^{n}\left[\frac{2\lambda\gamma\psi\mu\Upsilon}{\omega^2}\right] \tag{42}$$

$$\frac{\partial^2 l}{\partial\gamma\partial\theta} = \sum_{i=}^{n}\left[\frac{\epsilon}{\theta^2}\right] - \sum_{i=}^{n}\left[\frac{e^{\frac{\theta}{x}}}{\theta x}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\epsilon\mu}{\theta^2\omega}\right] + \sum_{i=}^{n}\left[\frac{2\lambda e^{\frac{\theta}{x}}\mu}{\theta x\omega}\right] \tag{43}$$

$$+\sum_{i=}^{n}\left[\frac{2\lambda\epsilon\gamma\psi\mu}{\theta\omega}\right] + \sum_{i=}^{n}\left[\frac{4\lambda^2\epsilon\mu^2\gamma\psi}{\theta\omega^2}\right] \tag{44}$$

$$\frac{\partial^2 l}{\partial\gamma\partial\lambda} = \sum_{i=}^{n}\left[\frac{2\epsilon\mu}{\theta\omega}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\epsilon\mu\Upsilon}{\theta\omega^2}\right]; \qquad \frac{\partial^3 l}{\partial\lambda^3} = \sum_{i=}^{n}\left[\frac{2\Upsilon^3}{\omega^3}\right] \tag{45}$$

$$\frac{\partial^3 l}{\partial\gamma^3} = \frac{2n}{\gamma^3} + \sum_{i=}^{n}\left[\frac{2\lambda\epsilon^3\mu}{\theta^3\omega}\right] + \sum_{i=}^{n}\left[\frac{12n\lambda^2\epsilon^3\mu^2}{\theta^3\omega^2}\right] + \sum_{i=}^{n}\left[\frac{16n\lambda^3\epsilon^3\mu^3}{\theta^3\omega^3}\right] \tag{46}$$

$$\begin{aligned}\frac{\partial^3 l}{\partial\theta^3} &= \sum_{i=}^{n}\left[\frac{6\gamma\epsilon}{\theta^4}\right] - \sum_{i=}^{n}\left[\frac{6\gamma e^{\frac{\theta}{x}}}{\theta^3 x}\right] + \sum_{i=}^{n}\left[\frac{3\gamma n e^{\frac{\theta}{x}}}{\theta^2 x^2}\right] - \sum_{i=}^{n}\left[\frac{\gamma e^{\frac{\theta}{x}}}{\theta x^3}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\tau\mu}{\omega}\right] - \sum_{i=}^{n}\left[\frac{6\lambda\phi\gamma\psi\mu}{\omega}\right] \\ &\quad -\sum_{i=}^{n}\left[\frac{12\lambda^2\phi\mu^2\gamma\psi}{\omega^2}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\gamma\psi^3\mu}{\omega}\right] - \sum_{i=}^{n}\left[\frac{12\lambda^2\gamma\psi^3\mu^2}{\omega^2}\right] - \sum_{i=}^{n}\left[\frac{16\lambda^3\gamma\psi^3\mu^3}{\omega^3}\right] \end{aligned}\tag{47}$$

$$\begin{aligned}\frac{\partial^3 l}{\partial\gamma^2\partial\theta} &= \sum_{i=}^{n}\left[\frac{4\lambda\epsilon^2\mu}{\theta^3\omega}\right] - \sum_{i=}^{n}\left[\frac{4\lambda\epsilon\mu e^{\frac{\theta}{x}}}{\theta^2\omega x}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\epsilon^2\gamma\psi\mu}{\theta^2\omega}\right] - \sum_{i=}^{n}\left[\frac{12\lambda^2\epsilon^2\mu^2\gamma\psi}{\theta^2\omega^2}\right] \\ &\quad +\sum_{i=}^{n}\left[\frac{8\lambda^2\epsilon^2\mu^2}{\theta^3\omega^2}\right] - \sum_{i=}^{n}\left[\frac{8\lambda^2\epsilon\mu^2 e^{\frac{\theta}{x}}}{\theta^2\omega^2 x}\right] - \sum_{i=}^{n}\left[\frac{16\lambda^3\epsilon^2\mu^3\gamma\psi}{\theta^2\omega^3}\right] \end{aligned}\tag{48}$$

$$\frac{\partial^3 l}{\partial\gamma^2\partial\lambda} = -\sum_{i=}^{n}\left[\frac{2\epsilon^2\mu}{\theta^2\omega}\right] + \sum_{i=}^{n}\left[\frac{2\lambda\epsilon^2\mu\Upsilon}{\theta^2\omega^2}\right] - \sum_{i=}^{n}\left[\frac{8\lambda\epsilon^2\mu^2}{\theta^2\omega^2}\right] + \sum_{i=}^{n}\left[\frac{8\lambda^2\epsilon^2\mu^2\Upsilon}{\theta^2\omega^3}\right] \tag{49}$$

$$\begin{aligned}\frac{\partial^3 l}{\partial\theta^2\partial\lambda} &= -\sum_{i=}^{n}\left[\frac{2\phi\mu}{\omega}\right] + \sum_{i=}^{n}\left[\frac{2\lambda\phi\mu\Upsilon}{\omega^2}\right] - \sum_{i=}^{n}\left[\frac{2n\gamma\psi^2\mu}{\omega}\right] \\ &\quad +\sum_{i=}^{n}\left[\frac{2\lambda\gamma\psi^2\mu\Upsilon}{\omega^2}\right] - \sum_{i=}^{n}\left[\frac{8\lambda\gamma\psi^2\mu^2}{\omega^2}\right] + \sum_{i=}^{n}\left[\frac{8\lambda^2\gamma\psi^2\mu^2\Upsilon}{\omega^3}\right] \end{aligned}\tag{50}$$

$$\begin{aligned}\frac{\partial^3 l}{\partial\theta^2\partial\gamma} &= -\sum_{i=}^{n}\left[\frac{2\epsilon}{\theta^3}\right] + \sum_{i=}^{n}\left[\frac{2e^{\frac{\theta}{x}}}{\theta^2 x}\right] - \sum_{i=}^{n}\left[\frac{e^{\frac{\theta}{x}}}{\theta x^2}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\varphi\mu}{\omega}\right] + \sum_{i=}^{n}\left[\frac{2\lambda\phi\epsilon\mu}{\theta\omega}\right] \\ &\quad +\sum_{i=}^{n}\left[\frac{4n\lambda^2\phi\mu^2\epsilon}{\omega^2\theta}\right] - \sum_{i=}^{n}\left[\frac{4\lambda\gamma\psi\mu\gamma\psi}{\omega}\right] + \sum_{i=}^{n}\left[\frac{2\lambda\gamma\psi^2\epsilon\mu}{\theta\omega}\right] \\ &\quad +\sum_{i=}^{n}\left[\frac{12\lambda^2\gamma\psi^2\mu^2\epsilon}{\omega^2\theta}\right] - \sum_{i=}^{n}\left[\frac{8\lambda^2\gamma\psi\mu^2\gamma\psi}{\omega^2}\right] + \sum_{i=}^{n}\left[\frac{16\lambda^3\gamma\psi^2\mu^3\epsilon}{\omega^3\theta}\right] \end{aligned}\tag{51}$$

$$\frac{\partial^3 l}{\partial\lambda^2\partial\gamma} = -\sum_{i=}^{n}\left[\frac{4\Upsilon\epsilon\mu}{\omega^2\theta}\right] + \sum_{i=}^{n}\left[\frac{4\Upsilon^2\lambda\epsilon\mu}{\omega^3\theta}\right]; \qquad \frac{\partial^3 l}{\partial\lambda^2\partial\theta} = \sum_{i=}^{n}\left[\frac{4\Upsilon\gamma\psi\mu}{\omega^2}\right] - \sum_{i=}^{n}\left[\frac{4\Upsilon^2\lambda\gamma\psi\mu}{\omega^3}\right] \tag{52}$$

$$\begin{aligned}\frac{\partial^3 l}{\partial\lambda\partial\theta\partial\gamma} &= -\sum_{i=}^{n}\left[\frac{2\gamma\psi\mu}{\omega}\right] + \sum_{i=}^{n}\left[\frac{2n\gamma\psi\epsilon\mu}{\theta\omega}\right] + \sum_{i=}^{n}\left[\frac{8\gamma\psi\mu^2\lambda\epsilon}{\omega^2\theta}\right] \\ &+ \sum_{i=}^{n}\left[\frac{2\lambda\gamma\psi\mu\Upsilon}{\omega^2}\right] - \sum_{i=}^{n}\left[\frac{2\lambda\gamma\psi\epsilon\mu\Upsilon}{\theta\omega^2}\right] - \sum_{i=}^{n}\left[\frac{8\lambda^2\gamma\psi\mu^2\Upsilon\epsilon}{\omega^3\theta}\right]\end{aligned} \tag{53}$$

where:

$\Upsilon = (1-2\mu)$, $\Phi = \left(\frac{e^{\frac{\theta}{x}}-1}{\theta^2} - \frac{e^{\frac{\theta}{x}}}{\theta x}\right)$, $\psi = \left(\frac{e^{\frac{\theta}{x}}-1}{\theta^2} - \frac{e^{\frac{\theta}{x}}}{\theta x}\right)$, $\phi = \left(-\frac{2\gamma\left(e^{\frac{\theta}{x}}-1\right)}{\theta^3} + \frac{2\gamma e^{\frac{\theta}{x}}}{\theta^2 x} - \frac{\gamma e^{\frac{\theta}{x}}}{\theta x^2}\right)$,

$\varphi = \left(-\frac{2\left(e^{\frac{\theta}{x}}-1\right)}{\theta^3} + \frac{2e^{\frac{\theta}{x}}}{\theta^2 x} - \frac{e^{\frac{\theta}{x}}}{\theta x^2}\right)$, $\varsigma = \left(\frac{6\gamma\left(e^{\frac{\theta}{x}}-1\right)}{\theta^4} - \frac{6\gamma e^{\frac{\theta}{x}}}{\theta^3 x} + \frac{3\gamma e^{\frac{\theta}{x}}}{\theta^2 x^2} - \frac{\gamma e^{\frac{\theta}{x}}}{\theta x^3}\right)$,$\epsilon = \left(e^{\frac{\theta}{x}} - 1\right)$

From the prior distribution in expression (24),

$$\rho = \log(\pi(\gamma,\theta,\lambda)) = (a-1)\ln\theta - \frac{\theta}{b} - \ln\gamma - \ln\lambda$$

$$\rho_1 = \frac{\partial}{\partial\gamma} = -\frac{1}{\gamma}; \qquad \rho_2 = \frac{\partial}{\partial\theta} = \frac{a-1-b\theta}{\theta}; \qquad \rho_3 = \frac{\partial}{\partial\lambda} = -\frac{1}{\lambda} \tag{54}$$

substituting Equations ((40)-(54)) into Equation (31) reduces the Lindley integral, therefore the Bayes estimates using SELF are thus

i. If $u(\gamma,\theta,\lambda) = \hat{\gamma}$
$\hat{\gamma}_{BS} = \hat{\gamma} - \frac{1}{\hat{\gamma}}\sigma_{11} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{12} - \frac{1}{\hat{\gamma}}\sigma_{13} + \frac{1}{2}\left(A\sigma_{11} + B\sigma_{12} + C\sigma_{13}\right)$

ii. If $u(\gamma,\theta,\lambda) = \hat{\theta}$ then
$\hat{\theta}_{BS} = \hat{\theta} - \frac{1}{\hat{\gamma}}\sigma_{21} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{22} - \frac{1}{\hat{\gamma}}\sigma_{23} + \frac{1}{2}\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right)$

iii. If $u(\gamma,\theta,\lambda) = \hat{\lambda}$ then
$\hat{\lambda}_{BS} = \hat{\lambda} - \frac{1}{\hat{\gamma}}\sigma_{31} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{32} - \frac{1}{\hat{\gamma}}\sigma_{33} + \frac{1}{2}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right)$

Also, the Bayes estimate using LINEX are thus

i. If $u(\gamma,\theta,\lambda) = e^{-c\hat{\gamma}}$ then
$\hat{\gamma}_{BL} = \hat{\gamma} + \log\left(1 - c\left(-\frac{1}{\hat{\gamma}}\sigma_{11} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{12} - \frac{1}{\hat{\gamma}}\sigma_{13} - \frac{c}{2}\sigma_{11} + \frac{1}{2}\left(A\sigma_{11} + B\sigma_{12} + C\sigma_{13}\right)\right)\right)$

ii. If $u(\gamma,\theta,\lambda) = e^{-c\hat{\theta}}$ then
$\hat{\theta}_{BL} = \hat{\theta} + \log\left(1 - c\left(-\frac{1}{\hat{\gamma}}\sigma_{21} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{22} - \frac{1}{\hat{\gamma}}\sigma_{23} - \frac{c}{2}\sigma_{22} + \frac{1}{2}\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right)\right)\right)$

iii. If $u(\gamma,\theta,\lambda) = e^{-c\hat{\lambda}}$ then
$\hat{\lambda}_{BL} = \hat{\lambda} + \log\left(1 - c\left(-\frac{1}{\hat{\gamma}}\sigma_{31} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{32} - \frac{1}{\hat{\gamma}}\sigma_{33} - \frac{c}{2}\sigma_{33} + \frac{1}{2}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right)\right)\right)$

Finally, the Bayes estimate using the GELF are thus

i. If $u(\gamma,\theta,\lambda) = \hat{\gamma}^{-c}$ then
$\hat{\gamma}_{BG} = \left[\hat{\gamma}^{-c}\left[1 - \frac{c}{\hat{\gamma}}\left(-\frac{1}{\hat{\gamma}}\sigma_{11} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{12} - \frac{1}{\hat{\gamma}}\sigma_{13} - \frac{c+1}{2\hat{\gamma}} + \frac{1}{2}\left(A\sigma_{11} + B\sigma_{12} + C\sigma_{13}\right)\right)\right]\right]^{-\frac{1}{c}}$

ii. If $u(\gamma,\theta,\lambda) = \hat{\theta}^{-c}$ then
$\hat{\theta}_{BG} = \left[\hat{\theta}^{-c}\left[1 - \frac{c}{\hat{\theta}}\left(-\frac{1}{\hat{\gamma}}\sigma_{21} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{22} - \frac{1}{\hat{\gamma}}\sigma_{23} - \frac{c+1}{2\hat{\theta}} + \frac{1}{2}\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right)\right)\right]\right]^{-\frac{1}{c}}$

iii. If $u(\gamma,\theta,\lambda) = \hat{\lambda}^{-c}$ then
$\hat{\lambda}_{BG} = \left[\hat{\gamma}^{-c}\left[1 - \frac{c}{\hat{\lambda}}\left(-\frac{1}{\hat{\gamma}}\sigma_{31} + \frac{a-1-b\hat{\theta}}{\hat{\theta}}\sigma_{32} - \frac{1}{\hat{\gamma}}\sigma_{33} - \frac{c+1}{2\hat{\lambda}} + \frac{1}{2}\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right)\right)\right]\right]^{-\frac{1}{c}}$

5. Results

5.1. Simulation Techniques

In this section, we consider a monte carlo simulation study to evaluate the performance of all the estimators using the MSE and biases with respect to different sample sizes n =(20, 50, 100,200, 500) for different parameters $TR - IG(\gamma, \theta, \lambda)$ = [(1,1,1), (0.7, 1 0.5), (0.5, 1, -0.5) and (0.5, 0.5 ,-0.2)] respectively. The results obtained from the analysis are displayed in Tables (1 - 7) and the results shown that the estimates using both the classical techniques and Bayesian methods performed excellently in estimating model parameters of the TR-IG distribution since the estimated results are closed to the true parameter values with small MSE and bias as the sample sizes increases for all the estimation techniques considered in this study.

Table 1: *The MLE estimates, Bias and MSE for different parameter values.*

n	Values	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}_{Bias}$	$\hat{\theta}_{Bias}$	$\hat{\lambda}_{Bias}$	$\hat{\gamma}_{MSE}$	$\hat{\theta}_{MSE}$	$\hat{\lambda}_{MSE}$
20		1.3933	1.7840	-2.170	0.3933	0.7840	-3.1707	0.1547	0.6146	0.6146
50	$\gamma = 1$	1.0849	1.0267	2.0412	0.0849	0.0267	1.0412	0.0072	0.0007	0.0007
100	$\theta = 1$	1.2748	1.4741	1.5233	0.2748	0.4741	0.5233	0.0755	0.2248	0.2248
200	$\lambda = 1$	1.0842	0.9210	1.0594	0.0842	-0.0789	0.0594	0.0071	0.0062	0.0062
500		1.0406	0.9230	1.0752	0.0406	-0.0769	0.0752	0.0016	0.0059	0.0059
20		0.8490	0.9426	0.9628	0.1490	-0.0573	0.4628	0.0222	0.0032	0.0032
50	$\gamma = 0.7$	0.8012	0.4173	2.1917	0.1012	-0.5826	1.6917	0.0102	0.3395	0.3395
100	$\theta = 1$	0.9900	0.6209	1.5020	0.2900	-0.3790	1.0020	0.0841	0.1436	0.1436
200	$\lambda = 0.5$	0.8170	1.0175	0.5633	0.1170	0.01757	0.0633	0.0137	0.0003	0.0003
500		0.7696	0.9641	0.5986	0.0696	-0.0358	0.0986	0.0048	0.0012	0.0012
20		0.6706	1.2368	-1.7885	0.1706	0.2368	-1.2885	0.0291	0.0561	0.0561
50	$\gamma = 0.5$	0.4520	0.5452	-0.3392	-0.0479	-0.4547	0.1607	0.0023	0.2068	0.2068
100	$\theta = 1$	0.5724	0.9028	-0.5957	0.0724	-0.0971	-0.0957	0.0052	0.0094	0.0094
200	$\lambda = -0.5$	0.6274	0.9089	-0.3868	0.1274	-0.0910	0.1131	0.0162	0.0082	0.0082
500		0.5189	0.9966	-0.4976	0.0189	-0.0033	0.0023	0.0003	1.1E-05	1.1E-05
20		0.0679	-0.0188	-0.1817	-0.4320	-0.5188	0.0182	0.1866	0.2691	0.2691
50	$\gamma = 0.5$	0.2330	0.0920	0.2068	-0.2667	-0.4079	0.4068	0.0712	0.1664	0.1664
100	$\theta = 0.5$	0.3784	0.3681	-0.2509	-0.1215	-0.1318	-0.0509	0.0147	0.0173	0.0173
200	$\lambda = -0.2$	0.4593	0.4362	-0.1983	-0.0406	-0.0637	0.0016	0.0016	0.0040	0.0040
500		0.4179	0.5230	-0.3623	-0.0820	0.0230	-0.1623	0.0067	0.0005	0.0005

Table 2: *The Bayesian estimate using GELF, Bias and MSE for different parameter values.*

n	Values	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}_{Bias}$	$\hat{\theta}_{Bias}$	$\hat{\lambda}_{Bias}$	$\hat{\gamma}_{MSE}$	$\hat{\theta}_{MSE}$	$\hat{\lambda}_{MSE}$
20	$\gamma = 1$	1.5700	1.9749	0.2172	-0.5700	-0.9749	0.7828	0.3249	0.9504	0.9504
50	$\theta = 1$	1.0552	0.9894	1.6972	-0.0552	0.0106	0.6972	0.0030	0.0001	0.0001
100	$\lambda = 1$	1.2573	2.1781	1.8270	-0.2573	-1.1781	-0.8270	0.0662	1.3878	1.3878
200	c = 1	1.0766	0.9160	1.0860	-0.0766	0.0840	-0.0860	0.0059	0.0071	0.0071
500		1.0377	0.9208	1.0854	-0.0377	0.0792	-0.0854	0.0014	0.0063	0.0063
20	$\gamma = 0.7$	0.7795	0.8400	-2.7292	-0.0795	0.1600	1.2292	0.0063	0.0256	0.0256
50	$\theta = 1$	0.8106	0.4566	-0.8339	-0.1106	0.5434	1.3339	0.0122	0.2953	0.2953
100	$\lambda = 0.5$	0.9805	0.6114	8.3239	-0.2805	0.3886	-7.8239	0.0787	0.1510	0.1510
200	c =1	0.8090	1.0061	0.6476	-0.1090	-0.0061	-0.1476	0.0119	0.0000	0.0000
500		0.7667	0.9596	0.6238	-0.0667	0.0404	-0.1238	0.0044	0.0016	0.0016
20	$\gamma = 0.5$	0.6499	1.1989	-1.7308	-0.1499	-0.1989	1.2308	0.0225	0.0395	0.0395
50	$\theta = 1$	0.4390	0.5267	-0.2967	0.0610	0.4733	-0.2033	0.0037	0.2240	0.2240
100	$\lambda = -0.5$	0.5669	0.8942	-0.5799	-0.0669	0.1058	0.0799	0.0045	0.0112	0.0112
200	c = -1	0.6246	0.9049	-0.3776	-0.1246	0.0951	-0.1224	0.0155	0.0090	0.0090
500		0.5181	0.9953	-0.4949	-0.0181	0.0047	-0.0051	0.0003	0.0000	0.0000
20	$\gamma = 0.5$	0.0362	-0.0674	-0.0715	0.4638	0.5674	-0.1285	0.2151	0.3219	0.3219
50	$\theta = 0.5$	0.2153	0.0708	0.2833	0.2847	0.4292	-0.4833	0.0811	0.1842	0.1842
100	$\lambda = -0.2$	0.3703	0.3577	-0.2202	0.1297	0.1423	0.0202	0.0168	0.0202	0.0202
200	c = -1	0.4553	0.4314	-0.1808	0.0447	0.0686	-0.0192	0.0020	0.0047	0.0047
500		0.4167	0.5214	-0.3567	0.0833	-0.0214	0.1567	0.0069	0.0005	0.0005

Table 3: *The Bayesian estimate using LINEX, Bias and MSE for different parameter values*

n	Values	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}_{Bias}$	$\hat{\theta}_{Bias}$	$\hat{\lambda}_{Bias}$	$\hat{\gamma}_{MSE}$	$\hat{\theta}_{MSE}$	$\hat{\lambda}_{MSE}$
20	$\gamma = 1$	0.3809	0.4126	2.2994	0.6191	0.5874	-1.2994	0.3833	0.3451	0.3451
50	$\theta = 1$	1.2102	0.6444	-2.5123	-0.2102	0.3556	3.5123	0.0442	0.1264	0.1264
100	$\lambda = 1$	1.1813	0.1028	0.6599	-0.1813	0.8972	0.3401	0.0329	0.8050	0.8050
200	c =1	1.1288	1.0870	0.9989	-0.1288	-0.0870	0.0011	0.0166	0.0076	0.0076
500		1.0672	1.0210	1.0269	-0.0672	-0.0210	-0.0269	0.0045	0.0004	0.0004
20	$\gamma = 0.7$	0.9130	1.0132	0.2642	-0.2130	-0.0132	0.2358	0.0454	0.0002	0.0002
50	$\theta = 1$	1.2158	0.9125	-0.3400	-0.5158	0.0875	0.8400	0.2661	0.0077	0.1685
100	$\lambda = 0.5$	1.1905	0.8456	-0.2682	-0.4905	0.1544	0.7682	0.2406	0.0239	0.0239
200	c =1	0.7941	0.9342	0.4847	-0.0941	0.0658	0.0153	0.0089	0.0043	0.0043
500		0.7636	0.9384	0.5599	-0.0636	0.0616	-0.0599	0.0040	0.0038	0.0038
20	$\gamma = 0.5$	0.6660	1.2612	-0.7200	-0.1660	-0.2612	-0.2200	0.0276	0.0682	0.4852
50	$\theta = 1$	0.4446	0.5059	-0.3653	0.0554	0.4941	-0.1347	0.0031	0.2442	0.2442
100	$\lambda = -0.5$	0.5743	0.9084	-0.6158	-0.0743	0.0916	0.1158	0.0055	0.0084	0.0084
200	c=-1	0.6285	0.9109	-0.3955	-0.1285	0.0891	-0.1045	0.0165	0.0079	0.0079
500		0.5193	0.9980	-0.5008	-0.0193	0.0020	0.0008	0.0004	0.0000	0.0000
20	$\gamma = 0.5$	0.0007	-0.3024	-0.2992	0.4993	0.8024	0.0992	0.2493	0.6438	0.6438
50	$\theta = 0.5$	0.2015	0.0137	0.0638	0.2985	0.4863	-0.2638	0.0891	0.2365	0.2365
100	$\lambda = -0.2$	0.3741	0.3568	-0.2791	0.1259	0.1432	0.0791	0.0159	0.0205	0.0205
200	c=-1	0.4584	0.4332	-0.2103	0.0416	0.0668	0.0103	0.0017	0.0045	0.0045
500		0.4181	0.5233	-0.3653	0.0819	-0.0233	0.1653	0.0067	0.0005	0.0005

Table 4: *The Bayesian estimate using SELF, Bias and MSE for different parameter values*

n		$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}_{Bias}$	$\hat{\theta}_{Bias}$	$\hat{\lambda}_{Bias}$	$\hat{\gamma}_{MSE}$	$\hat{\theta}_{MSE}$	$\hat{\lambda}_{MSE}$
20		0.9476	1.2083	0.9019	0.0524	-0.2083	0.0981	0.0027	0.0434	0.0434
50	$\gamma = 1$	1.1452	0.8396	1.0749	-0.1452	0.1604	-0.0749	0.0211	0.0257	0.0257
100	$\theta = 1$	1.2268	0.9133	1.1424	-0.2268	0.0867	-0.1424	0.0514	0.0075	0.0075
200	$\lambda = 1$	1.1058	1.0044	1.0302	-0.1058	-0.0044	-0.0302	0.0112	0.0000	0.0000
500		1.0537	0.9721	1.0515	-0.0537	0.0279	-0.0515	0.0029	0.0008	0.0008
20		0.8749	0.9668	0.6708	-0.1749	0.0332	-0.1708	0.0306	0.0011	0.0011
50	$\gamma = 0.7$	1.0160	0.6759	0.8044	-0.3160	0.3241	-0.3044	0.0998	0.1051	0.1051
100	$\theta = 1$	1.0910	0.7341	0.8397	-0.3910	0.2659	-0.3397	0.1529	0.0707	0.0707
200	$\lambda = 0.5$	0.8050	0.9751	0.5277	-0.1050	0.0249	-0.0277	0.0110	0.0006	0.0006
500		0.7664	0.9509	0.5806	-0.0664	0.0491	-0.0806	0.0044	0.0024	0.0024
20		0.5120	1.0705	-0.6552	-0.0120	-0.0705	0.1552	0.0001	0.0050	0.0050
50	$\gamma = 0.5$	0.6015	0.8504	-0.4043	-0.1015	0.1496	-0.0957	0.0103	0.0224	0.0224
100	$\theta = 1$	0.5988	0.9835	-0.4885	-0.0988	0.0165	-0.0115	0.0098	0.0003	0.0003
200	$\lambda = -0.5$	0.6511	0.9742	-0.3346	-0.1511	0.0258	-0.1654	0.0228	0.0007	0.0007
500		0.5214	1.0081	-0.4678	-0.0214	-0.0081	-0.0322	0.0005	0.0001	0.0001
20		0.4064	0.5934	-0.6487	0.0936	-0.0934	0.4487	0.0088	0.0087	0.0087
50	$\gamma = 0.5$	0.4748	0.4553	-0.3847	0.0252	0.0447	0.1847	0.0006	0.0020	0.0020
100	$\theta = 0.5$	0.4836	0.5361	-0.4501	0.0164	-0.0361	0.2501	0.0003	0.0013	0.0013
200	$\lambda = -0.2$	0.5183	0.5333	-0.3103	-0.0183	-0.0333	0.1103	0.0003	0.0011	0.0011
500		0.4336	0.5511	-0.3879	0.0664	-0.0511	0.1879	0.0044	0.0026	0.0026

Table 5: *OLS estimates, Bias and MSE for the simulated values*

n	Values	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}_{Bias}$	$\hat{\theta}_{Bias}$	$\hat{\lambda}_{Bias}$	$\hat{\gamma}_{MSE}$	$\hat{\theta}_{MSE}$	$\hat{\lambda}_{MSE}$
20		1.0444	0.7986	1.1393	-0.0444	0.2014	-0.1393	0.0020	0.0405	0.0405
50	$\gamma = 1$	0.9979	1.0074	1.0142	0.0021	-0.0074	-0.0142	0.0000	0.0001	0.0001
100	$\theta = 1$	1.0355	1.0912	0.9270	-0.0355	-0.0912	0.0730	0.0013	0.0083	0.0083
200	$\lambda = 1$	1.0684	1.1656	0.8362	-0.0684	-0.1656	0.1638	0.0047	0.0274	0.0274
500		1.0309	1.0772	0.9979	-0.0309	-0.0772	0.0021	0.0010	0.0060	0.0060
20		0.5445	1.0569	0.6307	0.1555	-0.0569	-0.1307	0.0242	0.0032	0.0032
50	$\gamma = 0.7$	0.7043	1.0029	0.5329	-0.0043	-0.0029	-0.0329	0.0000	0.0000	0.0000
100	$\theta = 1$	0.7956	1.0147	0.4648	-0.0956	-0.0147	0.0352	0.0091	0.0002	0.0002
200	$\lambda = 0.5$	0.8361	1.0624	0.3252	-0.1361	-0.0624	0.1748	0.0185	0.0039	0.0039
500		0.7327	1.0585	0.4770	-0.0327	-0.0585	0.0230	0.0011	0.0034	0.0034
20		0.4300	0.9754	-0.3656	0.0700	0.0246	-0.1344	0.0049	0.0006	0.0006
50	$\gamma = 0.5$	0.5046	1.0022	-0.4892	-0.0046	-0.0022	-0.0108	0.0000	0.0000	0.0000
100	$\theta = 1$	0.5923	1.0111	-0.5143	-0.0923	-0.0111	0.0143	0.0085	0.0001	0.0001
200	$\lambda = -0.5$	0.6208	1.0654	-0.5974	-0.1208	-0.0654	0.0974	0.0146	0.0043	0.0043
500		0.5374	1.0534	-0.5178	-0.0374	-0.0534	0.0178	0.0014	0.0029	0.0029
20		0.5315	0.3582	-0.1056	-0.0315	0.1418	-0.0944	0.0010	0.0201	0.0201
50	$\gamma = 0.5$	0.5138	0.4900	-0.1926	-0.0138	0.0100	-0.0074	0.0002	0.0001	0.0001
100	$\theta = 0.5$	0.5237	0.5591	-0.2425	-0.0237	-0.0591	0.0425	0.0006	0.0035	0.0035
200	$\lambda = -0.2$	0.5523	0.5970	-0.3090	-0.0523	-0.0970	0.1090	0.0027	0.0094	0.0094
500		0.5197	0.5442	-0.2383	-0.0197	-0.0442	0.0383	0.0004	0.0020	0.0020

Table 6: *MPS estimates, Bias and MSE for the simulated values*

n	Values	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}_{Bias}$	$\hat{\theta}_{Bias}$	$\hat{\lambda}_{Bias}$	$\hat{\gamma}_{MSE}$	$\hat{\theta}_{MSE}$	$\hat{\lambda}_{MSE}$
20		1.0230	0.9286	0.7827	-0.0230	0.0714	0.2173	0.0005	0.0051	0.0051
50	$\gamma = 1$	1.0807	0.8271	0.9403	-0.0807	0.1729	0.0597	0.0065	0.0299	0.0299
100	$\theta = 1$	1.1303	0.9557	1.0101	-0.1303	0.0443	-0.0101	0.0170	0.0020	0.0020
200	$\lambda = 1$	1.0660	1.0193	0.9686	-0.0660	-0.0193	0.0314	0.0044	0.0004	0.0004
500		1.0181	0.9949	1.0038	-0.0181	0.0051	-0.0038	0.0003	0.0000	0.0000
20		0.9815	0.6772	0.6045	-0.2815	0.3228	-0.1045	0.0792	0.1042	0.1042
50	$\gamma = 0.7$	1.0213	0.6075	0.7410	-0.3213	0.3925	-0.2410	0.1032	0.1540	0.1540
100	$\theta = 1$	1.0915	0.6921	0.7970	-0.3915	0.3079	-0.2970	0.1532	0.0948	0.0948
200	$\lambda = 0.5$	0.8248	0.9325	0.5298	-0.1248	0.0675	-0.0298	0.0156	0.0046	0.0046
500		0.7810	0.9256	0.5875	-0.0810	0.0744	-0.0875	0.0066	0.0055	0.0055
20		1.6379	0.1379	0.5668	-1.1379	0.8621	-1.0668	1.2947	0.7432	0.7432
50	$\gamma = 0.5$	1.5484	0.1713	0.6225	-1.0484	0.8287	-1.1225	1.0991	0.6868	0.6868
100	$\theta = 1$	1.6676	0.2513	0.6615	-1.1676	0.7487	-1.1615	1.3633	0.5606	0.5606
200	λ = -0.5	0.6542	0.9326	-0.3554	-0.1542	0.0674	-0.1446	0.0238	0.0045	0.0045
500		0.9890	0.6552	0.2423	-0.4890	0.3448	-0.7423	0.2391	0.1189	0.1189
20		0.8555	1.1375	0.4301	-0.3555	-0.6375	-0.6301	0.1264	0.4065	0.4065
50	$\gamma = 0.5$	1.1167	0.0400	0.5593	-0.6167	0.4600	-0.7593	0.3803	0.2116	0.2116
100	$\theta = 0.5$	1.2114	0.0889	0.5987	-0.7114	0.4111	-0.7987	0.5062	0.1690	0.1690
200	λ = -0.2	0.5270	0.5020	-0.3156	-0.0270	-0.0020	0.1156	0.0007	0.0000	0.0000
500		1.1111	0.0000	0.6778	-0.6111	0.5000	-0.8778	0.3735	0.2500	0.2500

Table 7: *The CVM estimates, Bias and MSE for the simulated values*

n	Values	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}_{Bias}$	$\hat{\theta}_{Bias}$	$\hat{\lambda}_{Bias}$	$\hat{\gamma}_{MSE}$	$\hat{\theta}_{MSE}$	$\hat{\lambda}_{MSE}$
20		1.0209	0.9660	1.1022	-0.0209	0.0340	-0.1022	0.0004	0.0012	0.0012
50	$\gamma = 1$	1.0513	1.0769	0.9379	-0.0513	-0.0769	0.0621	0.0026	0.0059	0.0059
100	$\theta = 1$	1.0598	1.1720	0.8364	-0.0598	-0.1720	0.1636	0.0036	0.0296	0.0296
200	$\lambda = 1$	1.1118	1.2235	0.7814	-0.1118	-0.2235	0.2186	0.0125	0.0500	0.0500
500		1.0926	1.0991	0.8669	-0.0926	-0.0991	0.1331	0.0086	0.0098	0.0098
20		0.6705	1.0208	0.5809	0.0295	-0.0208	-0.0809	0.0009	0.0004	0.0004
50	$\gamma = 0.7$	0.7955	1.0136	0.4675	-0.0955	-0.0136	0.0325	0.0091	0.0002	0.0002
100	$\theta = 1$	0.8104	1.0894	0.3628	-0.1104	-0.0894	0.1372	0.0122	0.0080	0.0080
200	$\lambda = 0.5$	0.8899	1.1010	0.2976	-0.1899	-0.1010	0.2024	0.0361	0.0102	0.0102
500		0.7668	1.0906	0.4191	-0.0668	-0.0906	0.0809	0.0045	0.0082	0.0082
20		0.4851	1.0183	-0.4424	0.0149	-0.0183	-0.0576	0.0002	0.0003	0.0003
50	$\gamma = 0.5$	0.5593	1.0397	-0.5468	-0.0593	-0.0397	0.0468	0.0035	0.0016	0.0016
100	$\theta = 1$	0.6335	1.0521	-0.5911	-0.1335	-0.0521	0.0911	0.0178	0.0027	0.0027
200	λ = -0.5	0.6754	1.0881	-0.6524	-0.1754	-0.0881	0.1524	0.0308	0.0078	0.0078
500		0.5966	1.0494	-0.5740	-0.0966	-0.0494	0.0740	0.0093	0.0024	0.0024
20		0.5134	0.4739	-0.1671	-0.0134	0.0261	-0.0329	0.0002	0.0007	0.0007
50	$\gamma = 0.5$	0.5278	0.5544	-0.2396	-0.0278	-0.0544	0.0396	0.0008	0.0030	0.0030
100	$\theta = 0.5$	0.5601	0.5904	-0.3026	-0.0601	-0.0904	0.1026	0.0036	0.0082	0.0082
200	λ = -0.2	0.6012	0.6143	-0.3492	-0.1012	-0.1143	0.1492	0.0102	0.0131	0.0131
500		0.5453	0.5665	-0.2794	-0.0453	-0.0665	0.0794	0.0021	0.0044	0.0044

5.2. Application

In this section, we analyzed the strength of glass of aircraft window datasets adopted by Fuller *et al.* [20] (whose dataset is displayed in Table (8) to ascertain that the TR-IG distribution is a good lifetime model, when compared with three known distribution like Inverse Gompertz (IGD), inverse Rayleigh (IR) and inverse Exponential distribution (IE). To assess the TR-IG distribution's goodness-of-fit with these distributions, some criteria (such as the Kolmogorov-Smirnov test statistic (KS), log-likelihood (L) values, Akaike information criterion (AIC), Bayesian information criterion (BIC), and Cramér-von Mises statistic (W^*) and Anderson-Darling statistic (A^*)) were used to fit all of the above-mentioned distributions.

Figure 3 displayed the estimated PDFs, estimated CDFs of all tested distributions and the empirical and theoretical CDF. From the results displayed in Table 9, we deduced that the TR-IG distribution fits the data better than the other four model. Table 10, shows the classical and Bayesian estimates for the strength of glass of aircraft windows.

Table 8: *The strength of glass of aircraft window.*

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52
26.77	26.78	27.05	27.67	29.9	31.11	33.2	33.73	33.76
35.75	35.91	36.98	37.08	37.09	39.58	44.045	45.29	45.381

Table 9: *The estimates and goodness-of-fit measurements for glass strength data set.*

Statistics	Model			
	IE	IR	IG	TR-IG
γ	29.215	810.504	1.249	0.6563
θ	–	- 0.6411	119.762	126.5584
λ	–	–		
KS	0.477	0.325	0.139	0.1349
-L	137.262	118.201	107.884	94.5072
AIC	241.2363	208.4237	195.9174	193.0145
BIC	242.5321	209.7196	198.5090	196.902
A*	7.2108	3.5316	0.6034	0.5349
W*	1.5051	0.6627	0.0841	0.0748

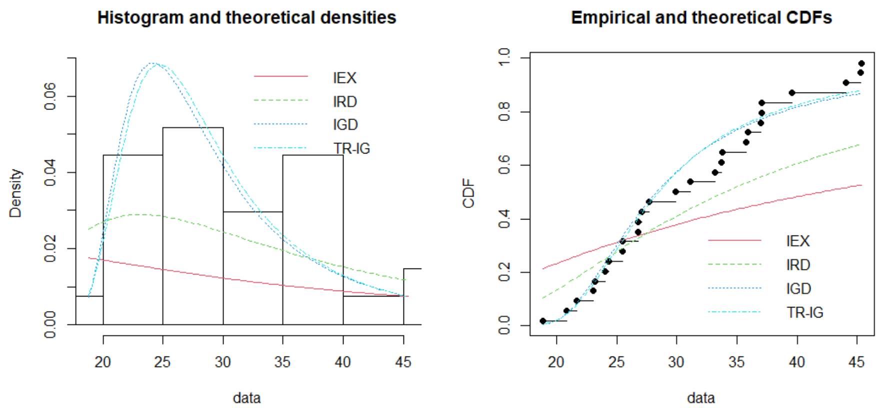

Figure 3: *The Histogram, empirical and theoretical densities for Glass strength dataset .*

Table 10: *Estimated values of the strength of glass for aircraft window under different estimation techniques*

Estimation Techniques	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$
CVM	0.4574	124.1316	-0.7821
MPS	0.9403	112.6889	-0.6803
SELF	0.5428	121.5032	-0.8790
LINEX (c =-1)	0.5467	129.2076	-0.9585
OLS	0.5663	122.5142	-0.5257
GELF (c =1)	0.7698	129.7899	-0.5152

6. Conclusion

We introduced a novel model called the TR-IG distribution, which expands upon the inverse Gompertz distribution for analyzing data with a real support. One clear motivation for extending a standard distribution is to increase the flexibility in modeling complex dataset. We derived some properties such as hazard function, survival function and etc. The parameters were estimated using both the classical and Bayesian estimation techniques.The utilization of the TR-IG distribution on actual data demonstrates that this new distribution can be employed with great effectiveness to yield superior fits in comparison IE, IR and IG distributions.

References

[1] Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality and on a new mode of determining the value of life contingencies. . *Philosophical transactions of the royal society of London*, 115, 513–583. https://doi.org/10.1098/rstl.1825.0026.

[2] Willemse, W. and Koppelaar, H. (2000). Knowledge elicitation of Gompertz law of mortality. *Scandinavian actuarial journal*,2,168–179. https://doi.org/10.1080/034612300750066845.

[3] El-Bassiouny, A. H., El-Damcese, M. Mustafa, A. and Eliwa, M. S. (2017). Exponentiated generalized Weibull-Gompertz distribution with application in survival analysis. *Journal of statistics applications and probability*, 6(1), 7–16. https://dx.doi.org/10.18576/jsap/060102.

[4] Eliwa, M., El-Morshedy, M. and Ibrahim, M. (2019). Inverse Gompertz Distribution: Properties and Different Estimation Methods with Application to Complete and Censored Data. *Annals Of Data Science*. 6, 321-339. https://doi.org/10.1007/s40745-018-0173-0

[5] Aryal, G. and Tsokos, C. (2011). Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution. *European Journal Of Pure And Applied Mathematics*. 4, 89-102.

[6] Khan, M., King, R. and Hudson, I. (2016) Transmuted gompertz distribution: properties and estimation. *Pakistan Journal of Statistics*. 32, 161-182 .

[7] Shaw, W. and Buckley, I. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *ArXiv*.

[8] Swain, J., Venkatraman, S. and Wilson, J. (1988). Least squares Estimation of Distribution Function in Johnson's Translation System. *Journal of Statistical Computation And Simulation* , 29, 271-297.

[9] Cheng, R. and Amin, N. (1983). Estimating Parameters in Continuous Univariate Distributions with a Shifted Origin. *Journal of The Royal Statistical Society: Series B (Methodological)*. 45, 394-403.

[10] Kao, J. (1959). A Graphical Estimation of Mixed Weibull Parameters in Life Testing Electron Tube. *Technometrics*. 1, 389-407.

[11] Varian, H. A. Bayesian Approach to Real Estate Assessment. North Holland, Amsterdam, 1975.

[12] Preda, V. and Panaitescu, E. (2010). Bayes estimators of Modified-Weibull distribution parameters using Lindley's approximation. *WSEAS Transactions On Mathematics*. 9, 539-549.

[13] Calabria, R. and Pulcini, G. (1994). An engineering approach to Bayes estimation for the Weibull distribution. *Microelectronics Reliability*. 34, 789-802, 1994.

[14] Dey, D. and Liu, P. (1992) . On Comparison of Estimators in a Generalized Life Model. *Micro-electron. Reliab*. 45, 207-221.

[15] Sule, B.O and Adegoke, T.M. (2020). Bayesian approach in estimation of shape parameter of an Exponential Inverse Exponential Distribution. *Asian Journal Of Probability And Statistics*. 9, 13 - 27.

[16] Adegoke, T. M., Yahya, W. B and Adegoke, G. K (2018) A Study of the Bayes Estimator of the Parameters in an Inverted Generalized Exponential Distribution with Application. *Annals Of Statistical Theory And Applications*, 1, 1-10.

[17] Ogunsanya, A. S, Yahya, W. B, Adegoke, T. M, Iluno, C., Aderele, O. and Iwada, E. A. (2021) New Three-Parameter Weibull Inverse Rayleigh Distribution: Theoretical Development and Applications. *Mathematics And Statistics*. 9, 249-272.

[18] Zellner, A. (1986). Bayesian Estimation and Prediction using Asymmetric Loss Functions. *Jour. Amer. Statist. Assoc.*. 446 - 451.

[19] Lindley, D. Approximate Bayes Methods. Bayesian Statistics. Valency, 1980.

[20] Fuller, R., Freiman, S., Quinn, J., Quinn, G. and Carter, W. (1994). Fracture Mechanics Approach to the Design of Glass Aircraft Windows: A Case Study. *Proceedings Of SPIE - The International Society For Optical Engineering*.7, 419-430.