MOMENTS PROPERTIES OF CONCOMITANTS OF GENERALIZED ORDER STATISTICS FROM FGMTBM EXPONENTIAL DISTRIBUTION

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Abstract

Concurrent or induced order statistics are produced when individuals in a random sample are ordered in compliance with the corresponding values of some other random sample. Concomitants are most helpful when k(n) individuals are to be chosen using a selection technique based on their X-values. The relevant Y-values are then used to reflect how well a characteristic has performed. In this paper, concomitants of generalized order statistics (GOS) from Farlie Gumbel Morgenstern type bivariate moment (FGMTBM) exponential distribution are obtained. Additionally, distribution function (df) and probaility density function (pdf) r-th generalized order statistics and a joint pdf of r-th and s-th GOS were also obtained. Furthermore, we provide the minimum variance linear unbiased estimator (MVLUE) of the position and scale parameters of the concomitants of the k-th upper record values and order statistics for the distribution under consideration. Finally, an implementation of the suggested methodology has been taken into account.

Keywords: Generalized order statistics, Concomitants, Record Values, FGMTBM exponential distribution, Single and Product Moments, Minimum variance linear unbiased estimator.
AMS Subject Classification: 62G30, 62E10

1. Introduction

In a wide range of statistical applications, order statistics (OS) and record values are often used in statistical modeling and inference. In both models, random variables are listed in descending order of magnitude. GOS offers an integrated approach to a wide variety of ordered random variable models with various interpretations. The theory of GOS is pioneered by [1]. Since then, a number of writers have incorporated the idea of GOS into their works, including [2-3], and others.

Let $n \in N, n \geq 2, k > 0$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \Re^{n-1}, M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1,2,\dots,n-1\}$. Then $X(r,n,\tilde{m},k), r \in \{1,2,\dots,n\}$ are called gos if their joint pdf is given by

$$k(\prod_{j=1}^{n-1}\gamma_j)(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i}f(x_i))[1-F(x_n)]^{k-1}f(x_n) \quad (1.1)$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\Re^n$.

Now, by selecting the proper values for the parameters, models such as ordinary OS $(\gamma_i = n - i + 1; i = 1,2,\dots,n\ i.e.\ m_1 = m_2 = \cdots = m_{n-1} = 0, k = 1,)$, k^{th} record values $(\gamma_i = k$ i.e. $m_1 = m_2 \dots = m_{n-1} = -1, k \in N,)$, sequential OS $(\gamma_i = (n-i+1)\alpha_i;\ )$ $(\alpha_1, \alpha_2, \dots, \alpha_n > 0)$, OS with non-integral sample size $(\gamma_i = (\alpha - i + 1); \alpha > 0)$, Pfeifer's record values $(\gamma_i = \beta_i;\ \beta_1, \beta_2, \dots, \beta_n > 0)$ and progressive type-II censored OS $(m_i \in N_0, k \in N)$ are obtained.
The pdf of $r-th$ GOS, $X(r,n,m,k)$ is

$$f_{X(r,n,m,k)} = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1}f(x)[g_m(F(x))]^{r-1} \quad (1.2)$$

and joint pdf of $X(s,n,m,k)$ and $X(r,n,m,k)$, $1 \leq r < s \leq n$, is

$$f_{X(r,s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}[\bar{F}(x)]^m f(x)\ [g_m(F(x))]^{r-1}$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1}f(y), \alpha \leq x < y \leq \beta \quad (1.3)$$

where,

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \gamma_i = k + (n-i)(m+1),\ \ g_m(x) = h_m(x) - h_m(0),\ x \in (0,1)$$

$$\text{and } h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & , m \neq -1 \\ -\log(1-x) & , m = -1 \end{cases}$$

Let $F(x,y)$ be the df of some arbitrary bivariate population. Assume also that (X_i, Y_i), $i = 1,2,\dots,n$, are n pairs of independent random variables from the poulation having distrubution $F(x,y)$. Let the ascending order of X variates is $X(1,n,m,k) \leq X(2,n,m,k) \leq, \dots, \leq X(n,n,m,k)$, now, if we arrange the Y variates pairwise (necessarily not in increasing order) with these generalized ordered statistics, then Y variates are called the concomitants of GOS and are generally denoted by $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \dots, Y_{[n,n,m,k]}$. Now, the pdf of $Y_{[r,n,m,k]}$, the r^{th} concomitant of GOS is given as

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_{X(r,n,m,k)}(x)\, dx \quad (1.4)$$

and the joint pdf of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ is

$$g_{[r,s,n,m,k]}(y_1,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f_{Y|X}(y_1|x_1)\, f_{Y|X}(y_2|x_2)\, f_{X(r,s,n,m,k)}(x_1,x_2)\, dx_1\, dx_2 \quad (1.5)$$

The FGM family of bivariate distributions is frequently employed in practice. This family is defined by the given marginal dfs $F_X(x)$ and $F_Y(y)$ of random variables X and Y and a parameter α. As a result, the FGM bivariate distribution function is given as follows:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \alpha(1-F_X(x))(1-F_Y(y))], \quad (1.6)$$

together with associated pdf

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1+\alpha(1-2F_X(x))(1-2F_Y(y))]. \quad (1.7)$$

The marginals of $f_{X,Y}(x,y)$ in this case are $f_X(x)$ and $f_Y(y)$. The two random variables X and Y are independent when the parameter α, also known as the association parameter, is zero. For exponential marginals, such a model was first presented by [5] and examined by [6]. The general form in Eq. (1.6) is credited to [7] and [8]. The acceptable range of the association parameter α is $-1 \leq \alpha \leq 1$, and the maximum value of the Pearson correlation coefficient ρ between X and Y must not exceed the value 1/3. Now, for given X, the conditional df and pdf of Y are:

$$F_{Y|X}(y|x) = F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))] \quad (1.8)$$

and

$$f_{Y|X}(y|x) = f_Y(y)[1+\alpha(1-2F_X(x))(1-2F_Y(y))] \quad (1.9)$$

There have been various studies that discuss the concomitants of OS in the research. Weighted inverse Gaussian distribution used to conduct a comparative study on the concomitant of OS and record values by [9]. Shannon's entropy is calculated by [10] after taking into account the concomitants of distribution of FGM family. Readers might check [11, 12] for some outstanding reviews on the concomitants of GOS. In [13], parameter estimation is discussed. They estimated the parameters of the FGM-type bivariate exponential distribution using the concomitants of GOS. The concomitants of GOS for the bivariate Lomax distribution is investigated by [14]. In addition, [15] obtained the concomitants of GOS for bivariate Pareto distribution. Also, [16] explored the concomitants of m-GOS from the generalized FGM distribution family. For a few examples of recent studies based on the notion of concomitants of OS, readers may refer to [15, 17]. For some applications of OS in reliability and accelerated life testing, readers may refer to [18-22].

In this study, we examined the FGMTBM exponential distribution. For $\theta > 0, 0 < x, y < \infty, -1 \leq \alpha \leq 1$, the pdf, df and the conditional pdf of Y given X of the FGMTBM exponential distribution is described by the following equation [23].

$$f(x,y) = \theta^4 xy\, e^{-\theta(x+y)}\{1+\alpha[1-2\{1-(1+\theta x)e^{-\theta x}\}]\,[1-2\{1-(1+\theta y)e^{-\theta y}\}]\} \quad (1.10)$$

$$F(x,y) = \{1-(1+\theta\, x)e^{-\theta x}\}\{1-(1+\theta\, y)e^{-\theta y}\}[1+\alpha(1+\theta x)(1+\theta y)e^{-\theta(x+y)}] \quad (1.11)$$

$$f(y|x) = \theta^2 y\, e^{-\theta y}\{1+\alpha[1-2\{1-(1+\theta x)e^{-\theta x}\}]\,[1-2\{1-(1+\theta y)e^{-\theta y}\}]\} \quad (1.12)$$

The marginal pdf and df of X are respectively

$$f(x) = \theta^2 x\, e^{-\theta x}, \quad 0 < x < \infty, \; \theta > 0, \quad (1.13)$$

$$F(x) = 1-(1+\theta x)e^{-\theta x}, \; 0 < x < \infty, \; \theta > 0, \quad (1.14)$$

2. PDF of Concomitants

For the FGMTBM exponential distribution as given in Eq. (1.10), using Eq. (1.12) and Eq. (1.2) in Eq. (1.4), the pdf of $r-th$ concomitants $Y_{[r,n,m,k]}$of GOS for $m \neq -1$ is given as:

$$g_{[r,n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\,\theta^2 y e^{-\theta y}$$

$$\times \int_0^\infty \{1+\alpha[1-2\{1-(1+\theta x)e^{-\theta x}\}]\,[1-2\{1-(1+\theta y)e^{-\theta y}\}]$$

$$\times\ [(1+\theta x)e^{-\theta x}]^{\gamma_{r-1}-1}[1-\{(1+\theta x)e^{-\theta x}\}^{m+1}]^{r-1}\,\theta^2 x e^{-\theta x}dx. \quad (2.1)$$

Setting $z=(1+\theta x)e^{-\theta x}$ in Eq. (2.1), we get

$$=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\,\theta^2 y e^{-\theta y}\int_0^1 \{1+\alpha(1-2z)\,[1-2\{1-(1+\theta y)e^{-\theta y}\}]\}\,z^{\gamma_r-1}[1-z^{m+1}]^{r-1}\,dz. \quad (2.2)$$

Making transformation $t = 1-z^{m+1}$, we get

$$=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\,\theta^2 y e^{-\theta y}\int_0^1 t^{r-1}(1-t)^{\frac{\gamma_r}{m+1}-1}\,\{1+\alpha[1-2(1-t)^{\frac{1}{m+1}}]\,[1-2\{1-(1+\theta y)e^{-\theta y}\}]\}\,dt \quad (2.3)$$

$$=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\,\theta^2 y e^{-\theta y}\{B(r,\tfrac{\gamma_r}{m+1})+\alpha[B(r,\tfrac{\gamma_r}{m+1})-2B(r,\tfrac{\gamma_r+1}{m+1})][1-2\{1-(1+\theta y)e^{-\theta y}\}]\}. \quad (2.4)$$

Which after simplification yields

$$g_{[r,n,m,k]}(y)=\theta^2 y e^{-\theta y}[1+\alpha\{1-2\{1-(1+\theta y)e^{-\theta y}\}\}\{1-2\prod_{i=1}^{r}(1+\tfrac{1}{\gamma_i})^{-1}\}]. \quad (2.5)$$

It may be verified that $\int_0^\infty g_{[r,n,m,k]}(y)dy=1$

Remark 2.1: Set $m=0, k=1$ in Eq. (2.5), to get the pdf of $r-th$ concomitants of OS from FGMTBM exponential distribution as

$$g_{[r:n]}(y)=\theta^2 y e^{-\theta y}[1+\alpha\{1-2\{1-(1+\theta y)e^{-\theta y}\}\}\{1-2\tfrac{(n-r+1)}{n+1}\}].$$

Remark 2.2: At $m=-1$ in Eq. (2.5), to get the pdf of $r-th$ concomitants of $k-th$ upper record values from FGMTBM exponential distribution as

$$g_{[r,n,-1,k]}(y)=\theta^2 y e^{-\theta y}[1+\alpha\{1-2\{1-(1+\theta y)e^{-\theta y}\}\}\{1-2(\tfrac{k}{k+1})^r\}].$$

3. Single Moments

Now, by using the results from the previous section, we obtain the moments of $Y_{[r,n,m,k]}$ for the FGMTBM exponential distribution in this section. As a result, in the light of Eq. (2.5), the moments of $Y_{[r,n,m,k]}$ are given as:

$$E[Y_{[r,n,m,k]}^{k_1}]=\theta^2\int_0^\infty y^{k_1+1}e^{-\theta y}[1+\alpha\{1-2[1-(1+\theta y)e^{-\theta y}]\}\{1-2\prod_{i=1}^{r}(1+\tfrac{1}{\gamma_i})^{-1}\}]dy. \quad (3.1)$$

Note that

$$\int_0^\infty x^{\alpha-1}e^{-\beta x}dx=\frac{\Gamma(\alpha)}{\beta^\alpha}. \quad (3.2)$$

Using Eq. (3.2) in Eq. (3.1), we get after simplification

$$E[Y_{[r,n,m,k]}^{k_1}] = \frac{\Gamma k_1+2}{\theta^{k_1}}[1+\alpha\{\frac{k_1+4}{2^{k_1+2}}-1\}\{1-2\prod_{i=1}^{r}(1+\frac{1}{\gamma_i})^{-1}\}]. \quad (3.3)$$

Remark 3.1: Insert $m = 0, k = 1$ in Eq. (3.3) to retrieve the moments of OS concomitants from the FGMTBM exponential distribution as:

$$E[Y_{[r:n]}^{k_1}] = \frac{\Gamma k_1+2}{\theta^{k_1}}[1+\alpha\{\frac{k_1+4}{2^{k_1+2}}-1\}\{1-\frac{2(n-r+1)}{n+1}\}].$$

Remark 3.2: The moments of concomitants of the k-th upper record statistics from the FGMTBM exponential distribution can be obtained as follows by setting $m = -1$ in Eq. (3.3):

$$E[Y_{[r,n,-1,k]}^{k_1}] = \frac{\Gamma k_1+2}{\theta^{k_1}}[1+\alpha\{\frac{k_1+4}{2^{k_1+2}}-1\}\{1-2(\frac{k}{k+1})^{r}\}]$$

Table 3.1: *Mean of the concomitant of OS for FGMTBM exponential distribution with θ = 0.4*

n	r	α = −0.9	α = −0.5	α = −0.1	α = 0.1	α = 0.5	α = 0.9
1	1	5	5	5	5	5	5
2	1	4.437	4.687	4.937	5.062	5.312	5.562
	2	5.562	5.312	5.062	4.937	4.687	4.437
3	1	4.156	4.531	4.906	5.093	5.468	5.843
	2	5	5	5	5	5	5
	3	5.843	5.468	5.093	4.906	4.531	4.156
4	1	3.987	4.437	4.887	5.112	5.562	6.012
	2	4.662	4.812	4.962	5.037	5.187	5.337
	3	5.337	5.187	5.037	4.962	4.812	4.662
	4	6.012	5.562	5.112	4.887	4.437	3.987

Table 3.2: *Mean of the concomitant of record value for FGMTBM exponential distribution with θ = 0.4 and k = 2.*

r	α = −0.9	α = −0.5	α = −0.1	α = 0.1	α = 0.5	α = 0.9
1	4.437	4.687	4.937	5.062	5.312	5.562
2	5.187	5.104	5.02	4.979	4.895	4.812
3	5.687	5.381	5.076	4.923	4.618	4.312
4	6.02	5.567	5.113	4.886	4.432	3.979

4. Joint PDF of Two Concomitants

Now, using Eqs. (1.3) and (1.12) in Eq. (1.5), the joint pdf of the r-th and s-th concomitants of GOS $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ for $m \neq -1$ for the FGMTBM exponential distribution can be derived as:

$$g_{[r,s,n,m,k]}(y_1, y_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\theta^4 y_1 y_2 e^{-\theta y_1} e^{-\theta y_2}$$

$$\times \int_0^{\infty}\int_0^{x_2}\{1+\alpha[1-2\{1-(1+\theta x_1)e^{-\theta x_1}\}][1-2\{1-(1+\theta y_1)e^{-\theta y_1}\}]\}$$

$$\times \{1+\alpha[1-2\{1-(1+\theta x_2)e^{-\theta x_2}\}]\,[1-2\{1-(1+\theta y_2)e^{-\theta y_2}\}]\}$$

$$\times (\theta^4\theta\, e^{-\theta x_1}e^{-\theta x_2})\{\tfrac{1}{m+1}(1-[(1+\theta x_1)e^{-\theta x_1}]^{m+1})\}^{r-1}\{(1+\theta x_2)e^{-\theta x_2}\}^{\gamma_s-1}$$

$$\times \{-\tfrac{1}{m+1}[(1+\theta x_2)e^{-\theta x_2}]^{m+1}+\tfrac{1}{m+1}[(1+\theta x_1)e^{-\theta x_1}]^{m+1}\}^{s-r-1}$$

$$\times [(1+\theta x_1)e^{-\theta x_1}]^{m}\, dx_1 dx_2 \tag{4.1}$$

And utilizing the transformations $U=1-(1+\theta x_1)e^{-\theta x_1}$ and $V=1-(1+\theta x_2)e^{-\theta x_2}$ in Eq. (4.1), we get:

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\,\theta^4 y_1 y_2 e^{-\theta y_1}e^{-\theta y_2}\int_0^1\int_0^{V=U}\{1+\alpha(1-2U)\,[1-2\{1-(1+\theta y_1)e^{-\theta y_1}\}]$$

$$\times \{1+\alpha(1-2V)\,[1-2\{1-(1+\theta y_2)e^{-\theta y_2}\}]\}U^m\,[1-U^{m+1}]^{r-1}\,[U^{m+1}-V^{m+1}]^{s-r-1}\,V^{\gamma_s-1}du\,dv \tag{4.2}$$

Subsequently, using the transformation $V=Ut$ in the preceding equation, we obtained the following results:

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\,\theta^4 y_1 y_2 e^{-\theta y_1}e^{-\theta y_2}\int_0^1\int_0^1\{1+\alpha(1-2U)[1-2\{1-(1+\theta y_1)e^{-\theta y_1}\}]\}$$

$$\times \{1+\alpha(1-2Ut)\,[1-2\{1-(1+\theta y_2)e^{-\theta y_2}\}]\}U^{\gamma_r-1}\,[1-U^{m+1}]^{r-1}\,t^{\gamma_s-1}\,[1-t^{m+1}]^{s-r-1}\,du\,dt \tag{4.3}$$

By setting $p=U^{m+1}$ and $q=t^{m+1}$ in Eq. (4.3), we get after simplification

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s}}\,\theta^4 y_1 y_2 e^{-\theta y_1}e^{-\theta y_2}\int_0^1\int_0^1\{1+\alpha(1-2p^{\frac{1}{m+1}})\,[1-2\{1-(1+\theta y_1)e^{-\theta y_1}\}]\}$$

$$\times\{1+\alpha(1-2p^{\frac{1}{m+1}}q^{\frac{1}{m+1}})\,[1-2\{1-(1+\theta y_2)e^{-\theta y_2}\}]\}p^{\frac{\gamma_r}{m+1}-1}\,(1-p)^{r-1}\,q^{\frac{\gamma_s}{m+1}-1}\,(1-q)^{s-r-1}\,dp\,dq \tag{4.4}$$

Upon further simplification of the preceding result, we obtain

$$g_{[r,s,n,m,k]}(y_1,y_2)=\theta^4 y_1 y_2 e^{-\theta y_1}e^{-\theta y_2}\,[1+\alpha^2\,[1-2\{1-(1+\theta y_1)e^{-\theta y_1}\}]\,[1-2\{1-(1+\theta y_2)e^{-\theta y_2}\}]$$

$$\times\{1-2\prod_{i=1}^{r}(1+\tfrac{1}{\gamma_i})^{-1}-2\prod_{i=1}^{s}(1+\tfrac{1}{\gamma_i})^{-1}+4\prod_{i=1}^{r}(1+\tfrac{2}{\gamma_i})^{-1}\}$$

$$+\alpha\{[1-2\{1-(1+\theta y_1)e^{-\theta y_1}\}]+[1-2\{1-(1+\theta y_2)e^{-\theta y_2}\}]\}\{1-2\prod_{i=1}^{s}(1+\tfrac{1}{\gamma_i})^{-1}\}] \tag{4.5}$$

It may be varified that $\int_0^\infty\int_0^\infty g_{[r,s,n,m,k]}(y_1,y_2)dy_1dy_2=1$.

Remark 4.1: We can obtain the joint pdf of concomitants of OS by putting $m=0, k=1$ in Eq. (4.5), as well as the joint pdf of concomitants of the k-th upper record values for the FGMTBM exponential distribution by setting $m=-1$.

5. Product Moments of Two Concomitants

The product moments of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ are given as:

$$E(Y_{[r,n,m,k]}^{l} Y_{[s,n,m,k]}^{p}) = \int_0^\infty \int_0^\infty y_1^l\, y_2^p\, g_{[r,s,n,m,k]}(y_1, y_2)\, dy_1\, dy_2 \tag{5.1}$$

In view of Eq. (4.5) and Eq. (5.1), we have

$$E(Y_{[r,n,m,k]}^{l} Y_{[s,n,m,k]}^{p}) = \theta^4 \int_0^\infty y_2^{p+1} e^{-\theta y_2} \{\int_0^\infty y_1^{l+1}\, e^{-\theta y_1}[1 + \alpha^2\, [1 - 2\{1 - (1 + \theta y_1)e^{-\theta y_1}\}]$$
$$\times [1 - 2\{1 - (1 + \theta y_2)e^{-\theta y_2}\}]\{1 - 2\prod_{i=1}^{r} (1 + \frac{1}{\gamma_i})^{-1} - 2\prod_{i=1}^{s} (1 + \frac{1}{\gamma_i})^{-1} + 4\prod_{i=1}^{r} (1 + \frac{2}{\gamma_i})^{-1}\}$$
$$+\alpha\{[1 - 2\{1 - (1 + \theta y_1)e^{-\theta y_1}\}] + [1 - 2\{1 - (1 + \theta y_2)e^{-\theta y_2}\}]\}\{1 - 2\prod_{i=1}^{s} (1 + \frac{1}{\gamma_i})^{-1}\}]dy_1\}dy_2 \tag{5.2}$$

$$E(Y_{[r,n,m,k]}^{l} Y_{[s,n,m,k]}^{p}) = \frac{\theta^2\, \Gamma l+2}{\theta^l} \int_0^\infty y_2^{p+1} e^{-\theta y_2}\{[1 + \alpha^2\, (\frac{2^{l+2} - 2^{l+3} + l + 4}{2^{l+2}})\, [1 - 2\{1 - (1 + \theta y_2)e^{-\theta y_2}\}]$$
$$\times [1 - 2\prod_{i=1}^{r} (1 + \frac{1}{\gamma_i})^{-1} - 2\prod_{i=1}^{s} (1 + \frac{1}{\gamma_i})^{-1} + 4\prod_{i=1}^{r} (1 + \frac{2}{\gamma_i})^{-1}] + \alpha[(\frac{2^{l+2} - 2^{l+3} + l + 4}{2^{l+2}})$$
$$+[1 - 2\{1 - (1 + \theta y_2)e^{-\theta y_2}\}]][1 - 2\prod_{i=1}^{s} (1 + \frac{1}{\gamma_i})^{-1}]\}dy_2 \tag{5.3}$$

Solving Eq. (5.2), we get after simplification

$$E(Y_{[r,n,m,k]}^{l} Y_{[s,n,m,k]}^{p}) = \frac{\Gamma(l+2)\,\Gamma(p+2)}{\theta^{l+p}}\{[1 + \alpha^2\, (\frac{2^{l+2} - 2^{l+3} + l + 4}{2^{l+2}})\, (\frac{2^{p+2} - 2^{p+3} + p + 4}{2^{p+2}})]$$
$$\times [1 - 2\prod_{i=1}^{r} (1 + \frac{1}{\gamma_i})^{-1} - 2\prod_{i=1}^{s} (1 + \frac{1}{\gamma_i})^{-1} + 4\prod_{i=1}^{r} (1 + \frac{2}{\gamma_i})^{-1}]$$
$$+\alpha[(\frac{2^{l+2} - 2^{l+3} + l + 4}{2^{l+2}}) + (\frac{2^{p+2} - 2^{p+3} + p + 4}{2^{p+2}})][1 - 2\prod_{i=1}^{s} (1 + \frac{1}{\gamma_i})^{-1}]\} \tag{5.4}$$

Remark 5.1: The product moments of the concomitants of OS from the FGMTBM exponential distribution can be obtained by setting $m = 0, k = 1$ in Eq. (5.4) as follows:

$$E(Y_{[r:n]}^{l} Y_{[s:n]}^{p}) = \frac{\Gamma(l+2)\,\Gamma(p+2)}{\theta^{l+p}}\{[1 + \alpha^2\, (\frac{2^{l+2} - 2^{l+3} + l + 4}{2^{l+2}})\, (\frac{2^{p+2} - 2^{p+3} + p + 4}{2^{p+2}})]$$
$$\times [1 + 4(\frac{(n-s+1)(n-s+2)}{(n+1)(n+2)}) - 2(\frac{n-s+1}{n+1}) - 2(\frac{n-r+1}{n+1})]$$
$$+\alpha[(\frac{2^{l+2} - 2^{l+3} + l + 4}{2^{l+2}}) + (\frac{2^{p+2} - 2^{p+3} + p + 4}{2^{p+2}})][1 - 2(\frac{n-r+1}{n+1})]\}$$

Remark 5.2: The product moments of the concomitants of $k - th$ upper record value from the FGMTBM exponential distribution can be obtained by setting $m = -1$ in Eq. (5.4) as follows:

$$E(Y_{[r,n,-1,k]}^{l} Y_{[s,n,-1,k]}^{j-p}) = \frac{\Gamma(l+2)\,\Gamma(p+2)}{\theta^{l+p}}\{[1 + \alpha^2\, (\frac{2^{l+2} - 2^{l+3} + l + 4}{2^{l+2}})\, (\frac{2^{p+2} - 2^{p+3} + p + 4}{2^{p+2}})]$$

$$\times [1+4(\frac{k}{k+2})^r - 2(\frac{k}{k+1})^r - 2(\frac{k}{k+1})^s]$$

$$+\alpha[(\frac{2^{l+2}-2^{l+3}+l+4}{2^{l+2}}) + (\frac{2^{p+2}-2^{p+3}+p+4}{2^{p+2}})][1-2(\frac{k}{k+1})^s]\}$$

Table 5.1: *Covariance of the concomitant of OS for FGMTBM exponential distribution with θ = 0.4*

n	s	r	α = −0.9	α = −0.5	α = −0.1	α = 0.1	α = 0.5	α = 0.9
4	1	1	30.981	19.839	12.493	10.243	8.589	10.731
	2	1	13.102	13.488	11.939	10.439	5.988	-0.397
		2	11.567	6.996	4.179	3.429	3.246	4.817
	3	1	1.297	9.011	11.46	10.71	5.261	-5.452
		2	-0.692	2.378	3.695	3.695	2.378	-0.692
		3	-2.682	-4.253	-4.07	-3.32	-0.503	4.067
	4	1	-4.431	6.41	11.056	11.056	6.41	-4.431
		2	-6.877	-0.363	3.285	4.035	3.386	-0.127
		3	-9.322	-7.136	-4.485	-2.985	0.363	4.177
		4	-11.768	-13.91	-12.256	-10.006	-2.66	8.481

Table 5.2: *Covariance of the concomitant of record value for FGMTBM exponential distribution with θ=0.4, and k = 2.*

s	r	α = −0.9	α = −0.5	α = −0.1	α = 0.1	α = 0.5	α = 0.9
1	1	7.352	2.287	0.098	0.081	2.204	7.202
2	1	21.549	7.57	0.633	-0.193	3.431	14.099
	2	5.002	1.537	0.059	0.064	1.565	5.052
3	1	31.013	11.092	0.991	-0.377	4.25	18.697
	2	14.091	4.944	0.411	-0.123	2.268	9.275
	3	6.607	2.016	0.072	0.092	2.118	6.79
4	1	37.323	13.44	1.229	-0.499	4.795	21.762
	2	20.151	7.214	0.646	-0.248	4	12.09
	3	12.5	4.236	0.305	-0.034	2.535	9.439
	4	9.298	2.836	0.101	0.131	2.987	9.57

6. Application

Finding the MVLUE of the location and scale parameters is an intriguing application of this work. Although the [24] methodology could be utilized to obtain MVLUE but it is quite challenging to acquire the inverse of the variance-covariance matrix in closed form for this methodology.

Therefore, these estimates are computed using numerical techniques.

Assume that the distribution of the random variables have μ and σ as the location and scale parameters. Now, based on the [24] technique, the MVLUE of θ is given as $\hat{\theta} = (A'V^{-1}A)^{-1}(A'V^{-1}y)$, where, $V = (V_{i,j})$ represents the variance of the $i-th$ and $j-th$ concomitants, V^{-1} represents the inverse of the matrix V, and y' is the observed value of the vector $Y' = Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \dots, Y_{[n,n,m.,k]}$. For $\mu = 0$ and $\sigma = 1$, A and θ can be obtained as:

$$A = \begin{bmatrix} 1, & 1 & \dots & 1 \\ \mu_{d[1,n,m,k]} & \mu_{d[2,n,m,k]} & \dots & \mu_{d[n,n,m,k]} \end{bmatrix}$$

$$\theta' = [\mu \quad \sigma]$$

Table 6.1: *Coefficients of MVLUE of μ and σ for FGMTBM exponential distribution based on OS with n = 4, θ = 0.4*

α	**Estimate**	**Coefficients**			
-0.9	$\hat{\mu}$	-0.0828	-0.1638	1.26766	-0.5523
	$\hat{\sigma}$	-0.5017	-0.9922	7.67812	-3.3452
-0.5	$\hat{\mu}$	5.14576	3.93097	-8.4672	0.39049
	$\hat{\sigma}$	-0.9503	-0.825	1.83408	-0.0588
-0.1	$\hat{\mu}$	14.4297	28.8015	-32.732	-9.4992
	$\hat{\sigma}$	-2.7511	-5.8872	6.69428	1.944
0.1	$\hat{\mu}$	-9.1516	-35.805	33.9053	12.0516
	$\hat{\sigma}$	1.98121	7.01331	-6.6369	-2.3576
0.5	$\hat{\mu}$	-5.0134	-3.5421	10.2922	-0.7368
	$\hat{\sigma}$	1.07724	0.74098	-2.047	0.2288
0.9	$\hat{\mu}$	-1.9202	-0.6639	1.18162	2.40244
	$\hat{\sigma}$	0.40411	0.19224	-0.1153	-0.481

Table 6.2: *Coefficients of MVLUE of μ and σ for FGMTBM exponential distribution based on record values with n = 4, θ = 0.4 and k = 2*

α	**Estimate**	**Coefficients**			
-0.9	$\hat{\mu}$	-9.0704	37.5315	-32.689	5.22765
	$\hat{\sigma}$	2.27034	-8.4608	7.36915	-1.1787
-0.5	$\hat{\mu}$	-9.6243	57.7496	-68.289	21.1632
	$\hat{\sigma}$	2.26517	-12.314	14.5604	-4.5111
-0.1	$\hat{\mu}$	26.7814	-5.3193	24.1669	-44.629
	$\hat{\sigma}$	-5.2229	1.07976	-4.8968	9.04001
0.1	$\hat{\mu}$	-32.758	4.67707	12.0107	17.0699
	$\hat{\sigma}$	6.66864	-0.9234	-2.3731	-3.3721
0.5	$\hat{\mu}$	-19.659	26.1102	4.18588	-9.6375
	$\hat{\sigma}$	3.88893	-4.915	-0.7883	1.81434
0.9	$\hat{\mu}$	-7.5351	10.0486	-1.2653	-0.2481
	$\hat{\sigma}$	1.53447	-1.8065	0.22748	0.04455

7. Conclusions

In this study, FGMTBM exponential distributions was considered and concomitants of GOS were obtained. Additionally, pdf pdf r-th GOS and a joint pdf of r-th and s-th GOS were also obtained. Furthermore, the MVLUE of the location and scale parameters of the concomitants of the k-th upper record values, as well as OS, were obtained for the distribution under consideration. Finally, an example was considered for the purpose of putting the suggested methodology into exercise. When k(<n) individuals are to be picked using a selection method based on their X-values, concomitants are most useful. The performance of a characteristic is then depicted using the pertinent Y-values. When individuals in a random sample are arranged in accordance with the corresponding values of another random sample, concurrent or induced OS are created.

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