ANALYSIS OF $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ RETRIAL QUEUE WITH PRIORITY SERVICES, DIFFERENTIATE BREAKDOWN,REPAIR, SYNCHRONIZED RENEGING AND OPTIONAL VACATION

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Abstract

This study deals with the steady-state analysis of single-server retrial non-preemptive priority queue with differentiate breakdown, repair, synchronized reneging and optional vacation. For this purpose, two categories of customers are considered, priority and ordinary customers, who arrive as per Poisson arrival process. The server consistently affords single service for these customers based on general distribution. The server randomly fails while providing service to the customer. Hard failure and soft failure are the two kinds of system failure. Hard failure is defined as an equipment failure that requires a repairman with specialized knowledge to be physically present, which is a time-consuming process. Whereas soft failure is defined as failure caused by events rather than physical condition and is usually resolved rebooting the system. Ordinary customers may renege the orbit if the server is engaged or unavailable. Furthermore, once the service of all priority customers is completed by the server, the server goes for a vacation or becomes idle. In this study, we used probability generating function and supplementary variable technique to solve the Laplace transforms of time-dependent probabilities of system states. Finally, we evaluated performance measures and expressed the results in numerical values.

Keywords: Batch arrivals; Priority queues; Differentiate Breakdown; Optional Vacation; Synchronized Reneging
AMS Subject Classification (2010): 60K25, 68M30, 90B22.

1. Introduction

A significant feature of a queuing situation (e.g., telecommunication system) is that when all servers are busy, an arriving customer is forced to exit the area of service and return to the retrial group after a particular period. These scenarios can be overcome by using retrial queues; for example if a server is found unavailable by the customers who arrives, then they will join the orbit to try their requests in random orders and at random moments. In real word, retrial queues are widely utilized as models for stochastic phenomena such as telecommunication networks, telephone switching systems, and computer systems so as to gain service from central processing unit. Recently, the literature on retrial queues has grown rapidly. Many researchers have investigated single-server retrial queues with two classes of customers. Wua and Lian [16] analyzed an $M^{[1]}, M^{[2]}/G/1$ G-queueing system with retrial customers and a server subject to breakdown and repair. Choudhury et al. [10] extensively analyzed an $M^{[X]}/G/1$ retrial queue with service interruption and optional service . Ammar and Rajadurai [2] studied a preemptive priority queueing system with disaster and the server working at the lower speed. Ayyappan and Udayageetha [4] discussed a priority retrial queueing system with collisions, working breakdown, reneging, two-way communication and immediate feedback . Arivudainambi and Godhandaraman [3] investigated single-server retrial queueing system with second optional service, balking and single vacation.

In the history of queueing analysis, priority queueing system has gained crucial attention. Preemptive and non-preemptive disciplines are the two types of priority disciplines. Priority customers, in case of non-preemptive discipline, have to wait until the service to ordinary customers completed. However, in case of preemptive discipline, priority customers will always interrupt the service provided to ordinary customers. Dhas et al. [11] described a preemptive priority queue with general bulk service and heterogeneous arrivals. Brandwajn and Begin [8] examined preemptive priority system with general inter arrival and service times. Kim et al. [17] explained a non-preemptive priority queue with two classes of customers and multiple vacations. Krishnamoorthy et al. [18] analyzed non-preemptive priority queue with priorities and service time that are self generated and follows PH-distribution. Dimitriou [12] investigated a retrial queue with mixed priorities, unreliable server, negative arrivals and multiple vacations.

A server in a queuing system would be unavailable for some time due to different reasons, for example, being under maintenance work, busy at another queue, or simply taking a break. Baruah et al. [7] explained a two-stage of service with reneging during breakdown and vacation periods. Choudhury and Kalita [9] analyzed the steady-state behaviour of a $M/G/1$ queue with optional repeated service and two types of general heterogeneous service subject to server breakdown that randomly occurs at any point in time while serving the customers and during delayed repair. Maragathasundari and Srinivasan [20] investigated a non-Markovian queueing model with multistage of services, in which the authors considered reneging to prevail in case of unavailability of the server during system breakdown or vacation periods. Jain et al. [14] studied a general retrial $M^{[X]}/G/1$ queue with Bernoulli vacation and second optional service. In this model, breakdowns are observed at random intervals at any point in time during the provision of service to the customers. Janani [15] described transient analysis of single-server queueing model with differentiated breakdown.

The server is assumes to take vacations at random intervals. However, even if the system has a single priority customer, no vacation is allowed. Therefore, the server may take an optional random length vacation after the last service of the last priority customer is served. Gupta et al. [22] generalized impatient customers in queueing system with optional vacation policies. Madan and Rawwash [19] determined $M^{[X]}/G/1$ queueing system with feedback and optional server vacations based on single vacation. Laxmi et al. [24] investigated $M/M/1$ queueing system with second optional service, correlated reneging and working vacations. Ayyappan et al. [5] analysed $M^{[X]}/G/1$ queueing system with optional server vacation and two phases of service. Ordinary customers become impatient if the server is busy or unavailable. However, they execute synchronized abandonments motivated by remote systems. Adan et al. [1] have performed a detailed analysis of queueing models with impatient customers and vacations. Economou and Kapodistria [13] explained single-server queueing system with synchronized reneging customers. A single server queueing model with reneging, feedback and balking was examined by Rakesh Kumar and Soodam [23]

Two different types of customers, priority and ordinary customers are to be considered in this work with differentiate breakdown, repair, synchronized reneging optional vacation and followed by non preemptive priority discipline. The server provide service to the customer but randomly fails. Hard failure is defined as an equipment failure that usually necessitates the physical presence of a repairman with specialized knowledge, which is a time-consuming process. Soft failure, on the other hand, is defined as failure caused by events rather than physical condition and usually resolved rebooting the system. Ordinary customers may go back from the orbit if the server is unavailable. Furthermore, the server becomes idle or goes for a vacation on completing the service of all priority customers.

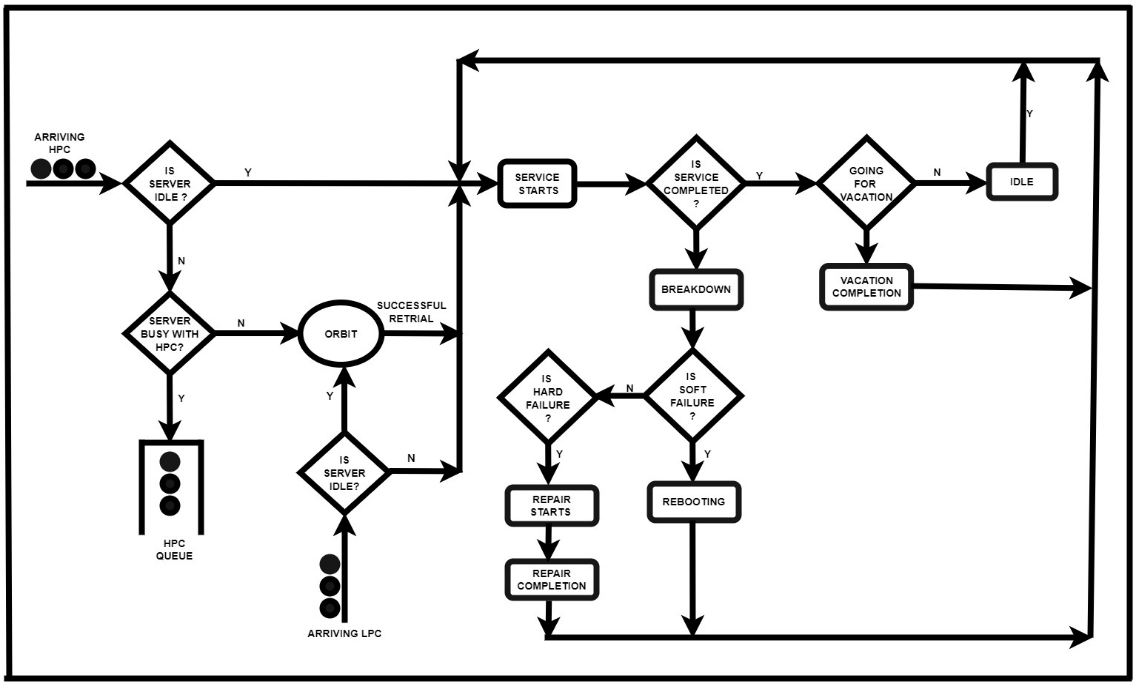

Figure 1: *Schematic representation*

This article is organized as follows. Mathematical model is described in Section 2 and queue size distribution is analyzed in Section 3. An explicit expression for governing equation is enlisted in Section 4. Steady state analysis is discussed in Section 5. Particular cases are obtained in Section 6. The effect of system performance measures is illustrated in Section 7. Numerical and graphical results are derived and conclusion is obtained in Section 8 and 9.

2. Description of the Model

- **Arrival Process :** Two different types of units arrive in batches with independent Poisson compound process. Let $\lambda_1, \lambda_2 > 0$ be the arrival rate for priority and ordinary customers, respectively. Assume that the first order probabilities for priority and ordinary units $\lambda_1 c_i dt$ $(i = 1, 2, 3, ...)$ and $\lambda_2 c_j dt$ $(j = 1, 2, 3, ...)$ with batch size i and j units arrive at the system during a short interval of time $(t, t + dt)$. Here, $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, $0 \leq c_j \leq 1$, $\sum_{j=1}^{\infty} c_j = 1$.
- **Retrial Service Process :** Ordinary customers are known as retrial customers. These customers will go back to the orbit and will request repeatedly for their service after some time if the server is busy or unavailable.
- **Regular Service Process :** Ordinary and priority customers ordinate in batches with distinct queues. Service rate follows general distribution and server renders single service for priority customers and ordinary customers with service rate $\mu_i(\nu)$, $i = 1, 2$ respectively. The service for ordinary customers starts when the priority queue is empty.
- **Differentiate Breakdown and repair :** The rates of hard and soft failure are exponentially distributed with rate α_1 & α_2 respectively. For soft failure, the repair time follows exponential distribution with rate η_1 and for hard failure, the repair time follows general distribution with rate $\eta_2(\nu)$.
- **synchronized Reneging :** If the server is not available in the system, ordinary customer either exit the orbit with probability ξ or join the orbit with probability $1 - \xi$.
- **Vacation:** The server may take a vacation with probability θ or it may remain idle with probability $1 - \theta$ after serving all priority customers. The random variable for vacation time

V, with rate $\gamma(\nu)$ follows general distribution.

3. Analysis of queue size distribution

This section deals with the derivation of governing equations. On account of non-Markovian queueing system, probability generating function and supplementary variable have been used to solve this model.
Let,
$N_1(t)$ = Number of priority customers in the queue at time t,
$N_2(t)$ = Number of ordinary customers in the queue at time t,
$Y(t)$ = State of the server at time t.
Here $M^0(t)$, $B_i^0(t)$ for $i = 1, 2.$, $V^0(t)$, $R^0(t)$, indicates elapsed service time for retrial, service for priority and ordinary customers, vacation and repair at time t.

To obtain a bivariate Markov process $\{N_1(t), N_2(t), Y(t), t > 0\}$, $Y(t)$ denotes the server state. Here $Y(t)$ = (0,1,2,3,4,5), which mean as follows: 0, the server is idle; 1, server is in retrial state; 2, busy with priority customers; 3, busy with ordinary customers; 4, on vacation and 5, repair.

Let us assume that, $M(0) = 0$, $M(\infty) = 1$, $B_i(0) = 0$, $B_i(\infty) = 1$, $V(0) = 0$, $V(\infty) = 1$ and $R^{(2)}(0) = 0$, $R^{(2)}(\infty) = 1$ be continuous at $\nu = 0$ for $i = 1, 2$.

The functions $\beta(\nu)$, $\mu_1(\nu)$, $\mu_2(\nu)$, $\gamma(\nu)$ and $\eta_2(\nu)$ be the hazard rate for retrial, priority and ordinary customers service rate, vacation and repair.

$$\beta(\nu) = \frac{dM(\nu)}{1 - M(\nu)}; \quad \mu_i(\nu) = \frac{dB_i(\nu)}{1 - B_i(\nu)}, i = 1, 2 \quad \gamma(\nu) = \frac{dV(\nu)}{1 - V(\nu)}; \quad \eta_2(\nu) = \frac{dR^{(2)}(\nu)}{1 - R^{(2)}(\nu)}.$$

The probability $I_{0,n}(\nu, t) = PrN_1(t) = 0, N_2(t) = 0, Y(t) = 0$ and probability densities are as follows:

$$\begin{aligned}
I_{0,n}(\nu, t)d\nu &= \Pr\{N_1(t) = 0, N_2(t) = n, Y(t) = 1; \nu \le I^0(t) \le \nu + d\nu\}, n \ge 1 \\
P_{m,n}(\nu, t)d\nu &= \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 2; \nu \le B_1^0(t) \le \nu + d\nu\}, \\
Q_{m,n}(\nu, t)d\nu &= \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 3; \nu \le B_2^0(t) \le \nu + d\nu\}, \\
V_{m,n}(\nu, t)d\nu &= \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 4; \nu \le V^0(t) \le \nu + d\nu\}, \\
R_{m,n}(\nu, t)d\nu &= \Pr\{N_1(t) = m, N_2(t) = n, Y(t) = 5; \nu \le R^0(t) \le \nu + d\nu\},
\end{aligned}$$

for $\nu \ge 0$, $t \ge 0$, $m \ge 0$ and $n \ge 0$.

4. Equation Governing the System

$$\begin{aligned}
\frac{d}{dt} I_{0,0}(t) = &-(\lambda_1 + \lambda_2) I_{0,0}(t) + (1 - \theta) \int_0^\infty P_{0,0}(\nu, t)\mu_1(\nu)d\nu \\
&+ (1 - \theta) \int_0^\infty Q_{0,0}(\nu, t)\mu_2(\nu)d\nu + \int_0^\infty R_{0,0}^{(2)}(\nu, t)\eta_2(\nu)d\nu \\
&+ R_{0,0}^{(1)}(t)\eta_1 + \int_0^\infty V_{0,0}(\nu, t)\gamma(\nu)d\nu.
\end{aligned} \tag{1}$$

$$\frac{\partial}{\partial t} I_{0,n}(\nu, t) + \frac{\partial}{\partial \nu} I_{0,n}(\nu, t) = -(\lambda_1 + \lambda_2 + \beta(\nu)) I_{0,n}(\nu, t) \quad \text{for } n \ge 1. \tag{2}$$

$$\begin{aligned}\frac{\partial}{\partial t}P_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}P_{m,n}(\nu,t) =& -(\lambda_1+\lambda_2+\alpha_1+\alpha_2+\mu_1(\nu))P_{m,n}(\nu,t)\\ &+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iP_{m-i,n}(\nu,t)\\ &+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jP_{m,n-j}(\nu,t) \quad \text{for } m,n\geq 1.\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial}{\partial t}Q_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}Q_{m,n}(\nu,t) =& -(\lambda_1+\lambda_2+\alpha_1+\alpha_2+\mu_2(\nu))Q_{m,n}(\nu,t)\\ &+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iQ_{m-i,n}(\nu,t)\\ &+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jQ_{m,n-i}(\nu,t) \quad \text{for } m,n\geq 1.\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\partial}{\partial t}V_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}V_{m,n}(\nu,t) =& -(\lambda_1+\lambda_2+\xi+\gamma(\nu))V_{m,n}(\nu,t)+\xi V_{m,n+1}(\nu,t)\\ &+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iV_{m-i,n}(\nu,t)\\ &+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jV_{m,n-i}(\nu,t) \quad \text{for } m,n\geq 1.\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{d}{dt}R^{(1)}_{m,n}(\nu,t)+\frac{d}{d\nu}R^{(1)}_{m,n}(\nu,t) =& -(\lambda_1+\lambda_2+\xi+\eta_1)R^{(1)}_{m,n}(t)+\xi R^{(1)}_{m,n+1}(t)\\ &\alpha_1\int_0^\infty (P_{m,n}(\nu,t)+Q_{m,n}(\nu,t))d\nu\\ &+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iR^{(1)}_{m-i,n}(t)\\ &+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jR^{(1)}_{m,n-i}(t) \quad \text{for } m,n\geq 1.\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial}{\partial t}R^{(2)}_{m,n}(\nu,t)+\frac{\partial}{\partial \nu}R^{(2)}_{m,n}(\nu,t) =& -(\lambda_1+\lambda_2+\xi+\eta_2(\nu))R^{(2)}_{m,n}(\nu,t)+\xi R^{(2)}_{m,n+1}(\nu,t)\\ &+\lambda_1(1-\delta_{0m})\sum_{i=1}^{m}c_iR^{(2)}_{m-i,n}(\nu,t)\\ &+\lambda_2(1-\delta_{0n})\sum_{j=1}^{n}c_jR^{(2)}_{m,n-i}(\nu,t) \quad \text{for } m,n\geq 1.\end{aligned} \tag{7}$$

Define, the boundary conditions at $\nu=0$

$$\begin{aligned}P_{m,n}(0,t) =& \int_0^\infty P_{m+1,n}(\nu,t)\mu_1(\nu)d\nu+\int_0^\infty Q_{m+1,n}(\nu,t)\mu_2(\nu)d\nu\\ &+\int_0^\infty R^{(2)}_{m+1,n}(\nu,t)\eta(\nu)d\nu+R^{(1)}_{m+1,n}(t)\eta_1+\lambda_1c_{m+1}I_{0,n}(t),\end{aligned} \tag{8}$$

$$Q_{0,0}(0,t)=\lambda_2c_1I_{0,0}(t)+\int_0^\infty I_{0,1}(\nu,t)\beta(\nu)d\nu \tag{9}$$

$$\begin{aligned}Q_{0,n}(0,t) =& \lambda_2c_{n+1}I_{0,0}(t)+\int_0^\infty I_{0,n+1}(\nu,t)\beta(\nu)d\nu+\sum_{i=1}^{n}\lambda_2C_i(\nu,t)\\ &+\int_0^\infty I_{0,n+1-i}(\nu,t)d\nu \quad \text{for } n\geq 1.\end{aligned} \tag{10}$$

$$V_{0,n}(0,t) = \theta \int_0^\infty P_{0,n}(\nu,t)\mu_1(\nu)d\nu + \theta \int_0^\infty Q_{0,n}(\nu,t)\mu_2(\nu)d\nu, \quad \text{for } n \geq 0. \tag{11}$$

$$R^{(2)}_{m,n}(0,t) = \alpha_2 \int_0^\infty P_{m-1,n}(\nu,t)\mu_1(\nu)d\nu + \alpha_2 \int_0^\infty Q_{m,n}(\nu,t)\mu_2(\nu)d\nu, \text{ for } m,n \geq 0. \tag{12}$$

$$\begin{aligned} I_{0,n}(0,t) = (1-\theta) \int_0^\infty P_{0,n}(\nu,t)\mu_1(\nu)d\nu + \int_0^\infty R^{(2)}_{0,n}(\nu,t)\eta_2(\nu)d\nu + R^{(1)}_{0,n}(t)\eta_1 \\ + (1-\theta) \int_0^\infty Q_{0,n}(\nu,t)\mu_2(\nu)d\nu + \int_0^\infty V_{0,n}(\nu,t)\gamma(\nu)d\nu, \end{aligned} \tag{13}$$

$$\begin{aligned} &P_{m,n}(0) = Q_{m,n}(0) = R^{(1)}_{m,n}(0) = R^{(2)}_{m,n}(0) = V_{m,n}(0) = 0, \text{ for } m,n \geq 0 \text{ and } I_{0,0} = 1, \\ &I_{0,n}(0) = 0, \quad \text{for } n \geq 1 \text{ are the initial conditions.} \end{aligned} \tag{14}$$

Now, we define the Probability Generating Function (PGF),

$$\begin{aligned} I(\nu,t,z_0) = \sum_{n=1}^{\infty} z_2^m I_{0,n}(\nu,t); \quad A(\nu,t,z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n A_{m,n}(\nu,t); \\ A(\nu,t,z_1) = \sum_{m=0}^{\infty} z_1^m A_m(\nu,t); \quad A(\nu,t,z_2) = \sum_{n=0}^{\infty} z_2^n A_n(\nu,t); \end{aligned} \tag{15}$$

here $A = P,\ Q,\ V,\ R^{(1)},\ R^{(2)}$.
By applying Laplace transforms to equations (1) to (13) and by using (14) and (15) , we obtain the following equations:

$$\overline{I}_0(\nu,s,z_2) = \overline{I}_0(0,s,z_2)e^{-(s+\lambda_1+\lambda_2)\nu - \int_0^\nu \beta(t)dt}, \tag{16}$$

$$\overline{P}(\nu,s,z_1,z_2) = \overline{P}(0,s,z_1,z_2)e^{-\phi_1(s,z)\nu - \int_0^\nu \mu_1(t)dt}, \tag{17}$$

$$\overline{Q}(\nu,s,z_1,z_2) = \overline{Q}(0,s,z_1,z_2)e^{-\phi_1(s,z)\nu - \int_0^\nu \mu_2(t)dt}, \tag{18}$$

$$\overline{V}(\nu,s,z_1,z_2) = \overline{V}(0,s,z_1,z_2)e^{-\phi_2(s,z)\nu - \int_0^\nu \gamma(t)dt}, \tag{19}$$

$$\overline{R}^{(2)}(\nu,s,z_1,z_2) = \overline{R}^{(2)}(0,s,z_1,z_2)e^{-\phi_2(s,z)\nu - \int_0^\nu \eta_2(t)dt}. \tag{20}$$

where,

$$\phi_1(s,z) = s + \lambda_1(1-C(z_1)) + \lambda_2(1-C(z_2)) + \alpha_1 + \alpha_2, \tag{21}$$

$$\phi 2(s,z) = s + \lambda_1(1-C(z_1)) + \lambda_2(1-C(z_2)). \tag{22}$$

$$\overline{Q}(0,s,z_2) = \frac{\begin{Bmatrix} 1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s)E_1(s,z) + C(z_2)\lambda_2\overline{I}_{0,0}(s) \\ \dfrac{1}{z_2}[A(s,z)D(s,z) + B(s,z)E_1(s,z)]F(s,z) + C(z_2)\lambda_2\overline{I}_{0,0}(s) \end{Bmatrix}}{\begin{Bmatrix} z_2[E_1(s,z) - A(s,z)C(g(z_2))\lambda_1\Big[\dfrac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big] \\ -[A(s,z)D(s,z) + B(s,z)E_1(s,z)]F(s,z) \end{Bmatrix}} \tag{23}$$

$$\overline{I}_0(0,s,z_2) = \frac{\begin{Bmatrix} 1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s)E_1(s,z) + C(z_2)\lambda_2\overline{I}_{0,0}(s) \\ \dfrac{1}{z_2}[A(s,z)D(s,z) + B(s,z)E_1(s,z)] \end{Bmatrix}}{\begin{Bmatrix} z_2[E_1(s,z) - A(s,z)C(g(z_2))\lambda_1\Big[\dfrac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big] \\ -[A(s,z)D(s,z) + B(s,z)E_1(s,z)] \end{Bmatrix}} \tag{24}$$

$$\overline{P}(0,s,z_1,z_2) = \frac{\begin{Bmatrix} \lambda_1\Big[\dfrac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big]\overline{I}_0(0,s,z_2)[E_1(s,z)C(z_1) - C(g(z_2))E(s,z)] \\ \overline{Q}(0,s,z_2)[E_1(s,z)(d_1(s,z) - d_2(s,z)) - E(s,z)(d_1^*(s,z) - d_2^*(s,z))] \end{Bmatrix}}{\{(z_1 - C(s,z))E_1(s,z) \quad \}} \tag{25}$$

$$\overline{V}_0(0,s,z_2) = \frac{\left\{\begin{aligned} &\theta\overline{B}_1\psi_1(s,z)C(g(z_2))\lambda_1\Big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big]\overline{I}_0(0,s,z_2) \\ &+\theta[E_1(s,z)\overline{B}_2\psi_1(s,z)+(d_1^*(s,z)-d_2^*(s,z))\overline{B}_1\psi_1(s,z)]\overline{Q}(0,s,z_2)\end{aligned}\right\}}{\{E_1(s,z)\quad\}} \tag{26}$$

$$\overline{R}^{(1)}(0,s,z_1,z_2) = \frac{\alpha_1}{\phi(s,z)}\Big[\overline{P}(0,s,z_1,z_2)\Big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\Big] + \overline{Q}(0,s,z_2)\Big[\frac{1-\overline{B}_2(\phi_1(s,z))}{\phi_1(s,z)}\Big]\Big] \tag{27}$$

$$\overline{R}^{(2)}(0,s,z_1,z_2) = \alpha_2 z_1\overline{P}(0,s,z_1,z_2)\Big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\Big] + \alpha_2\overline{Q}(0,s,z_2)\Big[\frac{1-\overline{B}_2(\phi_1(s,z))}{\phi_1(s,z)}\Big]\Big]. \tag{28}$$

Theorem.1 When the system is in regular service, breakdown, repair and vacation by using the Laplace transforms the probability generating function of the number of customers in the respective queue is given by

$$\overline{I}_0(s,z_2) = \overline{I}_0(0,s,z_2)\Big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big], \tag{29}$$

$$\overline{P}(s,z_1,z_2) = \overline{P}(0,s,z_1,z_2)\Big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\Big], \tag{30}$$

$$\overline{Q}(s,z_1,z_2) = \overline{Q}(0,s,z_2)\Big[\frac{1-\overline{B}_2(\phi_1(s,z))}{\phi_1(s,z)}\Big], \tag{31}$$

$$\overline{V}(s,z_1,z_2) = \overline{V}(0,s,z_1,z_2)\Big[\frac{1-\overline{V}(\phi_2(s,z))}{\phi_2(s,z)}\Big], \tag{32}$$

$$\overline{R}^{(2)}(s,z_1,z_2) = \overline{R}^{(2)}(0,s,z_1,z_2)\Big[\frac{1-\overline{R}^{(2)}(\phi_2(s,z))}{\phi_2(s,z)}\Big]. \tag{33}$$

Proof: Integrating the preceding equations (29) to (33) with respect to ν and applying the solution of renewal theory we obtain the following

$$\int_0^\infty \big[1-H(\nu)\big]e^{-s\nu}d\nu = \frac{1-\overline{h(s)}}{s}. \tag{34}$$

Here, the LST of the distribution function of a random variable H(ν) is denoted as $\overline{h(s)}$. The absolute outcomes of the probability generating functions for the successive states, $\overline{P}(s,z_1,z_2)$, $\overline{Q}(s,z_1,z_2)$, $\overline{V}(s,z_1,z_2)$, and $\overline{R}^{(2)}(s,z_1,z_2)$ are obtained by using equation (29) to (33).

5. Steady State Analysis

According to Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t).$$

Despite of the state of the system, the probability generating function of the queue size is as follows:

$$W_q(z_1,z_2) = \frac{Nr(z_1,z_2)}{Dr(z_1,z_2)}, \tag{35}$$

where

$$\begin{aligned}
Nr(z_1,z_2) = {} & N_2(z)D_3(z)\phi_1(z)\Big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big]\Big[\phi_2+\frac{\theta\overline{B}_1\psi_1(z)C(g(z_2))\lambda_1(1-\overline{V}\phi_2)}{E_1}\Big] \\
& + N_4D_2(1-\overline{B}_1\phi_1(z))\Big[(1+\frac{\alpha_1}{\phi(z)})\phi_2+\alpha_2 z_1(1-\overline{R}^{(2)}\phi_2(z))\Big] \\
& + N_3D_3\Big[(1+\frac{\alpha_1}{\phi(z)})\phi_2+\alpha_2(1-\overline{R}^2\phi_2(z))\Big](1-\overline{B}_2\phi_1(z)) \\
& + (1-\overline{V}\phi_2)\Big[\frac{\theta E_1(z)\overline{B}_2\psi_1(z)+\overline{B}_1\psi_1(z)(d_1^*(z)-d_2^*(z))}{E_1}\Big]
\end{aligned}$$

$$Dr(z_1,z_2) = D_2(z)D_3(z)\phi_1(z)\phi_2(z),$$

where,

$$N_2 = z_2(-\lambda_1+\lambda_2)\overline{I}_{0,0}E_1(s,z)+C(z_2)\lambda_2\overline{I}_{0,0}[A(s,z)(d_1^*(s,z)-d_2^*(s,z))+B(s,z)E_1(s,z)]$$

$$\begin{aligned}
N_3 = {} & [(-\lambda_1+\lambda_2)\overline{I}_{0,0}E_1(s,z)+\frac{1}{z_2}C(z_2)\lambda_2\overline{I}_{0,0}[A(s,z)(d_1^*(s,z)-d_2^*(s,z)) \\
& + B(s,z)E_1(s,z)]]F(s,z)+\frac{1}{z_2}C(z_2)\lambda_2\overline{I}_{0,0}
\end{aligned}$$

$$\begin{aligned}
N_4 = {} & N_2\lambda_1\Big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big][E_1(s,z)C(z_1)-C(g(z_2))E(s,z)] \\
& + N_3[E_1(s,z)(d_1(s,z)-d_2(s,z))-E(s,z)(d_1^*(s,z)-d_2^*(s,z))]
\end{aligned}$$

$$\begin{aligned}
D_2 = {} & z_2[E_1(s,z)-A(s,z)C(g(z_2)\lambda_1\Big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big] \\
& - [A(s,z)(d_1^*(s,z)-d_2^*(s,z))+B(s,z)E_1(s,z)]F(s,z)
\end{aligned}$$

$$D_3 = (z_1-C(s,z))E_1(s,z)D_2(s,z)$$

$$A(s,z) = (1-\theta+\theta\overline{V}\psi_2(z))\overline{B}_1\psi_1(z)+\Big[\frac{\alpha_1\eta_1}{\psi(z)}+\alpha_2 z_1\overline{R}^{(2)}\psi_2(z)\Big]\Big[\frac{1-\overline{B}_1\psi_1(z)}{\psi_1(z)}\Big]$$

$$B(s,z) = (1-\theta+\theta\overline{V}\psi_2(z))\overline{B}_2\psi_1(z)+\Big[\frac{\alpha_1\eta_1}{\psi(z)}+\alpha_2 z_1\overline{R}^{(2)}\psi_2(z)\Big]\Big[\frac{1-\overline{B}_2\psi_1(z)}{\psi_1(z)}\Big]$$

$$C(s,z) = \overline{B}_1\phi_1(z)+\Big[\frac{\alpha_1\eta_1}{\phi(z)}+\alpha_2 z_1\overline{R}^{(2)}\phi_2(z)\Big]\Big[\frac{1-\overline{B}_1\phi_1(z)}{\phi_1(z)}\Big]$$

$$d_1(s,z) = \overline{B}_2\phi_1(z)+\Big[\frac{\alpha_1\eta_1}{\phi(z)}+\alpha_2 z_1\overline{R}^2\phi_2(z)\Big]\Big[\frac{1-\overline{B}_2\phi_1(z)}{\phi_1(z)}\Big]$$

$$d_2(s,z) = (1+\theta\overline{V}\psi_2(z)-\theta\overline{V}\phi_2(z))\overline{B}_2\psi_1(z)+\Big[\frac{\alpha_1\eta_1}{\psi(z)}+\alpha_2 z_1\overline{R}^2\psi_2(z)\Big]\Big[\frac{1-\overline{B}_2\psi_1(z)}{\psi_1(z)}\Big]$$

$$d_1^*(s,z) = \overline{B}_2\sigma_1(z)+\Big[\frac{\alpha_1\eta_1}{\sigma(z)}+\alpha_2 z_1\overline{R}^{(2)}\sigma_2(z)\Big]\Big[\frac{1-\overline{B}_2\sigma_1(z)}{\sigma_1(z)}\Big]$$

$$d_2^*(s,z) = (1+\theta\overline{V}\psi_2(z)-\theta\overline{V}\sigma_2(z))\overline{B}_2\psi_1(z)+\Big[\frac{\alpha_1\eta_1}{\psi(z)}+\alpha_2 z_1\overline{R}^2\psi_2(z)\Big]\Big[\frac{1-\overline{B}_2\psi_1(z)}{\psi_1(z)}\Big]$$

$$E(s,z) = (1+\theta\overline{V}\psi_2(z)-\theta\overline{V}\phi_2(z))\overline{B}_1\psi_1(z)+\Big[\frac{\alpha_1\eta_1}{\psi(z)}+\alpha_2 z_1\overline{R}^2\psi_2(z)\Big]\Big[\frac{1-\overline{B}_1\psi_1(z)}{\psi_1(z)}\Big]$$

$$E^*(s,z) = (1+\theta\overline{V}\psi_2(z)-\theta\overline{V}\sigma_2(z))\overline{B}_1\psi_1(z)+\Big[\frac{\alpha_1\eta_1}{\psi(z)}+\alpha_2 z_1\overline{R}^2\psi_2(z)\Big]\Big[\frac{1-\overline{B}_1\psi_1(z)}{\psi_1(z)}\Big]$$

$$F(z) = \overline{M}(\lambda_1+\lambda_2) + c(z_2)\lambda_2\Big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big]$$

$$f_1(z_1,z_2) = \phi_2(z) + \frac{\theta\overline{B}_1\psi_1(z)C(g(z_2))\lambda_1(1-\overline{V}\phi_2(z))}{E^*(z_1,z_2)}$$

$$f_2(z_1,z_2) = (1+\frac{\alpha_1}{\phi(z)}\phi_2(z) + \alpha_2 z_1(1-\overline{R}^{(2)}\phi_2)$$

$$\begin{aligned} f_3(z_1,z_2) = {} & [(1+\frac{\alpha_1}{\phi(z)}\phi_2(z) + \alpha_2 z_1(1-\overline{R}^{(2)}\phi_2)](1-\overline{B}_2\phi_1(z)) \\ & + \Big[\frac{\theta(E^*(z)\overline{B}_2\psi_1(z) + \overline{B}_1\psi_1(z)(d_1^*(z)-d_2^*(z)))}{E^*(z)}\Big](1-\overline{V}\phi_2(z)) \end{aligned}$$

$$\phi(z) = \lambda_1(1-C(z_1) + \lambda_2(1-C(z_2) + \eta_1 + \xi(1-\frac{1}{z_2})$$

$$\sigma(z) = \lambda_1(1-C(g(z_2)) + \lambda_2(1-C(z_2) + \eta_1 + \xi(1-\frac{1}{z_2})$$

$$\sigma_1(z) = \lambda_1(1-C(g(z_2)) + \lambda_2(1-C(z_2) + \alpha_1 + \alpha_2$$

$$\sigma_2(z) = \lambda_1(1-C(g(z_2)) + \lambda_2(1-C(z_2) + \xi(1-\frac{1}{z_2})$$

$$\psi(z) = \lambda_1 + \lambda_2(1-C(z_2) + \eta_1 + \xi(1-\frac{1}{z_2})$$

$$\psi_1(z) = \lambda_1 + \lambda_2(1-C(z_2) + \alpha_1 + \alpha_2$$

$$\psi_2(z) = \lambda_1 + \lambda_2(1-C(z_2) + \xi(1-\frac{1}{z_2}).$$

Using normalization condition $W_q(1,1) + I_{0,0} = 1$, we get

$$I_{0,0} = \frac{\left\{\begin{array}{l}[D_2(1,1)D_3'(1,1)(\alpha_1+\alpha_2)(\xi-(\lambda_1+\lambda_2))] \\ -[N_3(1,1)D_3'(1,1)f'_3(1,1) + N_2'(1,1)D_3'(1,1)f_1'(1,1)(\alpha_1+\alpha_2) \\ \Big[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}\Big] + N_4'(1,1)D_2(1,1)f_2'(1,1)(1-B_1(\alpha_1+\alpha_2))]\end{array}\right\}}{\Big\{D_2(1,1)D_3'(1,1)(\alpha_1+\alpha_2)(\xi-(\lambda_1+\lambda_2))\Big\}} \tag{36}$$

and the utilization factor is given by

$$\rho = \frac{\left\{\begin{array}{l}[N_3(1,1)D_3'(1,1)f'_3(1,1) + N_2'(1,1)D_3'(1,1)f_1'(1,1)(\alpha_1+\alpha_2) \\ \Big[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}\Big] + N_4'(1,1)D_2(1,1)f_2'(1,1)(1-B_1(\alpha_1+\alpha_2))]\end{array}\right\}}{\Big\{D_2(1,1)D_3'(1,1)(\alpha_1+\alpha_2)(\xi-(\lambda_1+\lambda_2))\Big\}}. \tag{37}$$

The stability condition for the model under which steady state exists is $\rho < 1$.

6. Performance Measures

The expected queue size for priority customer is as follows:

$$L_{q_1} = \frac{d}{dz_1}W_q(z_1,1)|_{z_1=1} \tag{38}$$

The expected orbit size for ordinary customer is as follows:

$$L_{q_2} = \frac{d}{dz_2}W_q(1,z_2)|_{z_2=1} \tag{39}$$

then

$$L_{q_1} = \frac{Dr''(1)Nr^{(''')}(1) - Dr^{''')}(1)Nr''(1)}{3(Dr''(1))^2}, \tag{40}$$

$$L_{q_2} = \frac{Dr'''(1)Nr^{(iv)}(1) - Dr^{(iv)}(1)Nr'''(1)}{4(Dr'''(1))^2}. \tag{41}$$

The expected waiting time for priority queue is as follows:

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1} \tag{42}$$

The expected waiting time for orbit is as follows:

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2}. \tag{43}$$

7. Particular Cases

Case 1:
In the absence of priority queue, when there is no breakdown, no reneging, no vacation and no retrial then the above model becomes

$$W_q(z) = \frac{-(1-\overline{B_2})(\lambda - \lambda C(z_2))I_{0,0}}{z_2 - \overline{B_2}(\lambda_2 - \lambda_2 C(z_2))}$$

which is the PGF of Medhi [21].
Case 2:
In the absence of priority queue, when there is no breakdown, no reneging and no retrial then the above model becomes

$$W_q(z) = \frac{\left\{\begin{array}{l} [-\phi_1(z,s)I_{0,0}(1-\theta\overline{V}(\phi_1(z,s)) + \theta\overline{V}(\psi_1(z,s)))(1-\overline{B_1}(\phi(z,s))) \\ +\theta\lambda_1 C(g(z_2))I_{0,0}1 - \theta\overline{V}(\phi_1(z,s))z_1 - \overline{B_1}(\phi(z,s))] \\ +[Q_0(0,z_2)[\theta(\overline{B_2}(\sigma(z,s)) - \overline{B_2}(\psi(z,s)))(1-\overline{V}(\phi(z,s))) \\ (z_1 - \overline{B_1}(\phi(z,s))) - (z_2 - \overline{B_2}(\phi_1(z,s)))(1-\theta\overline{V}(\phi(z,s)) \\ +\theta\overline{V}(\psi_1(z,s)))(1-\overline{B_1}(\phi(z,s))) + (z_1 - \overline{B_1}(\phi(z,s))) \\ (1-\overline{B_2}(\phi(z,s)))(1-\overline{V}(\sigma(z,s))) + \theta\overline{V}(\psi(z,s)))]] \end{array}\right\}}{\{[z_1 - \overline{B_1}(\phi(z,s))(1-\theta\overline{V}(\sigma(z,s)) + \theta\overline{V}(\psi(z,s)))\phi(z,s)] \quad \}}$$

which is the PGF of Ayyappan and Thamizhselvi [6].

8. Numerical Results

This section deals with the numerical and graphical studies of this model. We assume that the service time, breakdown, repair and vacation time are distributed exponentially.

Table 1: *Effect of priority arrival rate (λ_1)*

λ_1	$I_{0,0}$	ρ	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
0.1	0.2806	0.7194	0.2082	2.0816	0.2463	0.1231
0.2	0.2776	0.7224	0.4758	2.3789	0.3580	0.1790
0.3	0.2754	0.7246	0.7650	2.5501	0.4246	0.2123
0.4	0.2743	0.7257	1.0720	2.6801	0.4554	0.2277
0.5	0.2742	0.7258	1.3930	2.7859	0.4616	0.2308

Table 1 shows that when the arrival rate (λ_1) of priority customers for priority queue increases, then the probability of the server being idle decreases. However, average queue lengths busy period and average waiting time for customers in the queues all increase: we assume the values as $\lambda_2 = 2$, $\alpha_1 = 0.1$, $\alpha_2 = 0.1$ $\mu = 5$, $\eta_1 = 10$,$\eta_2 = 15$, $\gamma = 10$, $\theta = 0.9$, $\beta = 20$, $\xi = 0.9$ and $\lambda_1 = 0.1$ to 0.5.

Table 2: *Effect of service rate (μ)*

μ	$I_{0,0}$	ρ	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
5	0.3058	0.6942	0.4275	1.4250	0.6219	0.3110
6	0.4823	0.5177	0.4270	1.4233	0.5472	0.2736
7	0.5950	0.4050	0.4200	1.4001	0.4856	0.2428
8	0.6728	0.3272	0.4109	1.3695	0.4348	0.2174
9	0.7296	0.2704	0.4012	1.3374	0.3926	0.1963
10	0.7727	0.2273	0.3918	1.3060	0.3576	0.1788

Table 2 shows that when the service rate (μ) increases then the probability of the server being idle increases. However, average queue lengths, busy period, and average waiting time for customers in the queues all decrease: we assume the values as $\lambda_1 = 0.3$, $\lambda_2 = 2$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $\eta_2 = 10$,$\eta_1 = 15$, $\gamma = 10$, $\theta = 0.8$, $\beta = 20$, $\xi = 0.9$ and $\mu = 5$ to 10.

Table 3: *Effect of ordinary customers arrival rate (λ_2*

λ_2	$I_{0,0}$	ρ	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
1.5	0.6275	0.3275	0.4029	0.8058	0.0409	0.0273
1.6	0.6406	0.3594	0.4347	0.8694	0.2939	0.1837
1.7	0.6073	0.3927	0.4663	0.9326	0.4609	0.2711
1.8	0.5725	0.4275	0.4974	0.9949	0.5752	0.3196
1.9	0.5361	0.4639	0.5279	1.0557	0.6556	0.3451
2.0	0.4981	0.5019	0.5573	1.1145	0.7133	0.3567

Table 3 shows that when the arrival rate (λ_2) of ordinary customers increases, then the probability server being idle decreases. Busy period, average queue lengths, and average waiting time for customers in the queues all increase: we assume the values as $\lambda_1 = 0.5$, $\mu = 6$, $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\mu = 5$, $\eta_1 = 6$, $\eta_2 = 15$, $\gamma = 10$, $\theta = 0.8$, $\beta = 18$, $\xi = 0.9$ and $\lambda_2 = 1.5$ to 2.

In graphical representations, we assume that the service time, breakdown, repair, and vacation time are follows Erlang-2 distribution. The two-dimensional graphs are shown in Figure 2 - 4. Figure 2 exhibits that the expected length of the queue (L_{q_1}, L_{q_2})rises, the expected length of the queue extends together with the priority arrival rate (λ_1). The behaviour of the queue sizes (L_{q_1}, L_{q_2}), which depends on the service rate (α), is shown in Figure 3, the length of the queue as the service rate rises. Figure 4 shows the expected queue length (L_{q_1}, L_{q_2}), which depends on the ordinary customers arrival rate (λ_2), the expected queue length grows together with the repair rate.

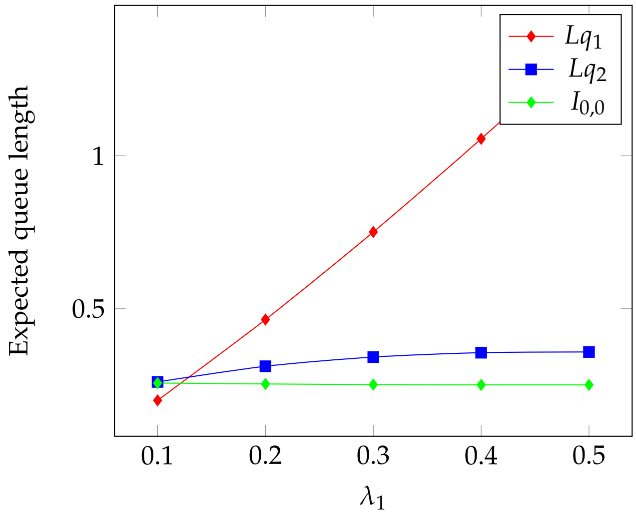

Figure 2: *Expected queue length vs priority arrival rate* λ_1

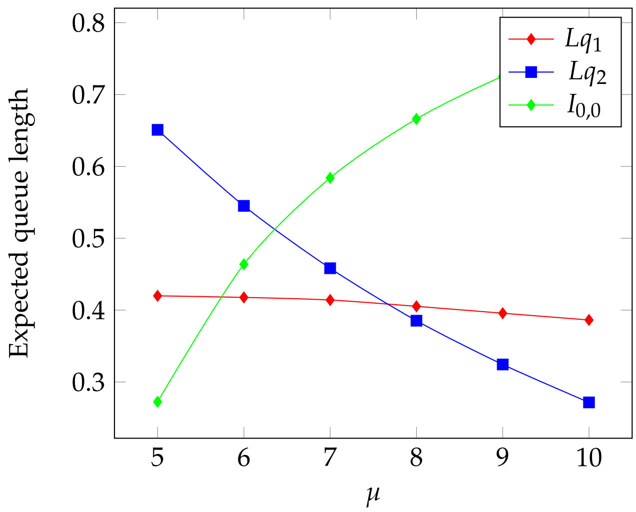

Figure 3: *Expected queue length vs service rate* μ

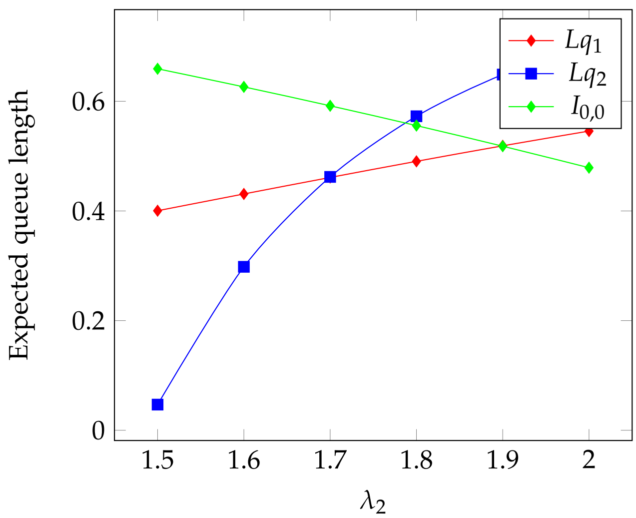

Figure 4: *Expected queue length vs ordinary arrival rate* λ_2

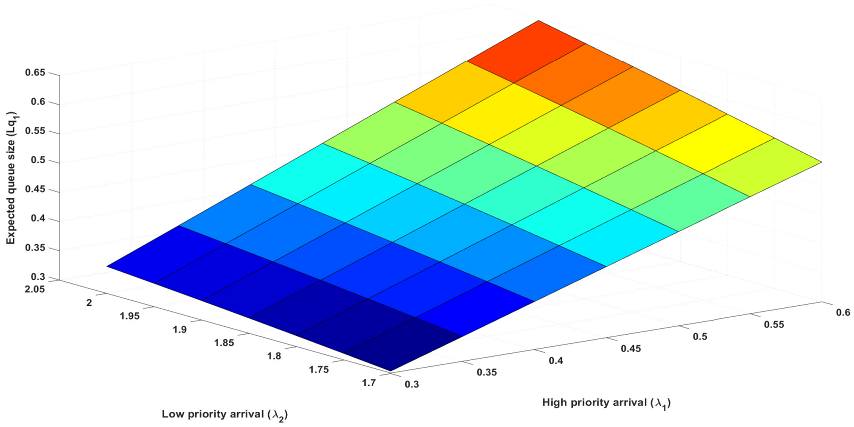

Figure 5: *Lq_1 Vs λ_1 and λ_2*

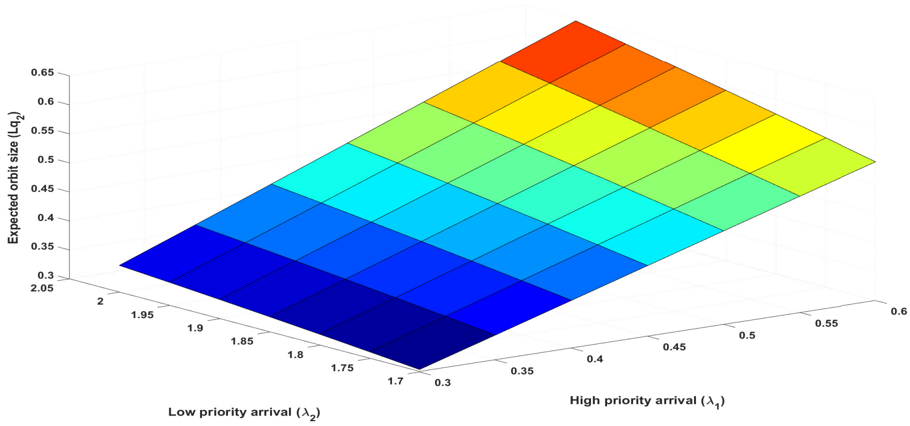

Figure 6: *Lq_2 Vs λ_1 and λ_2*

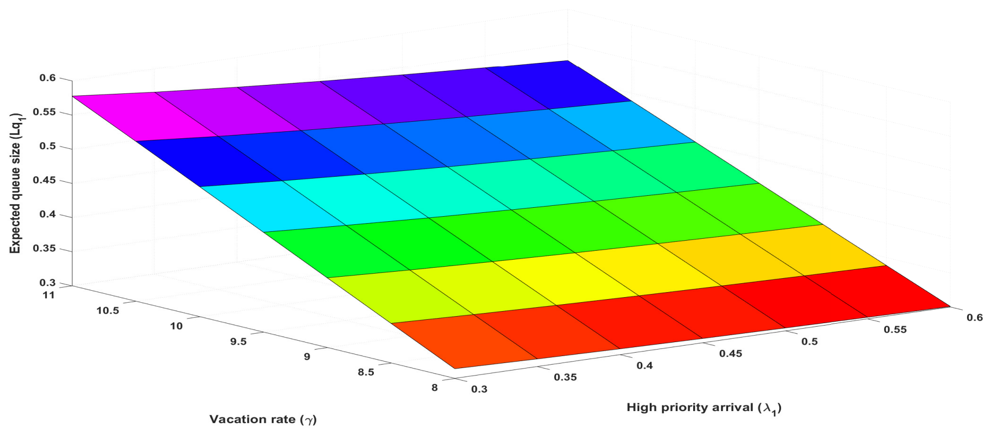

Figure 7: *Lq_1 Vs γ and λ_1*

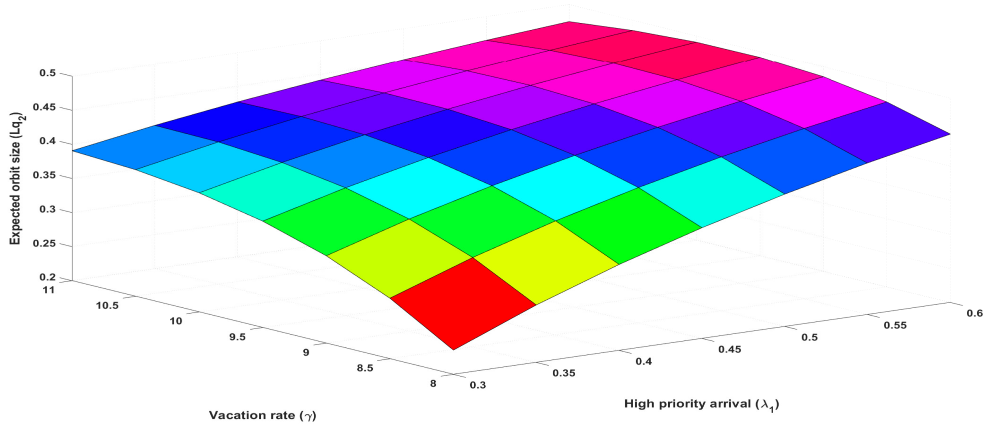

Figure 8: *Lq_2 Vs γ and λ_1*

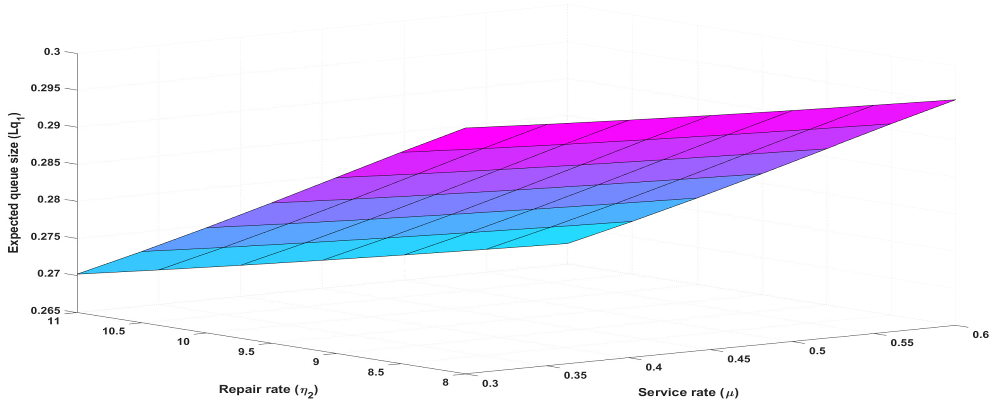

Figure 9: *Lq_2 Vs η_2 and μ*

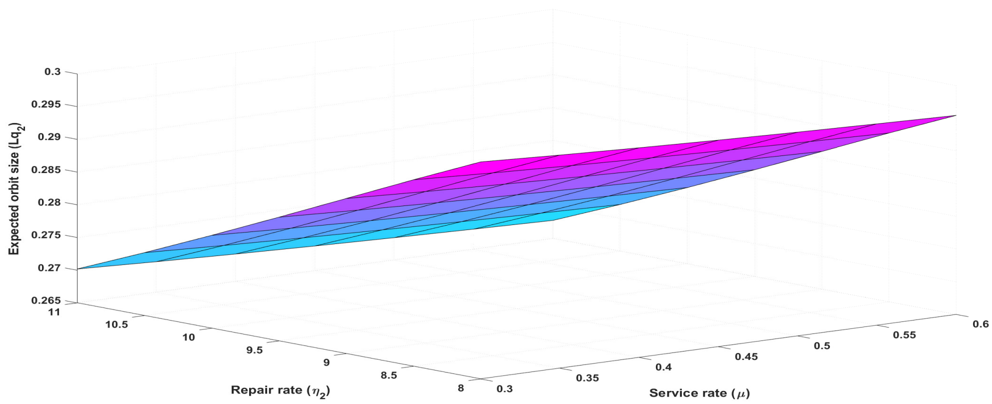

Figure 10: *Lq_2 Vs η_2 and μ*

Graphs in three dimensions can be found in Figures 5 - 10. Figures 5 and 6 in the reference indicate that as the priority arrival rate (λ_1) and ordinary arrival rate (λ_2) increase, the expected queue size (Lq_1) and orbit size (Lq_2) increase as well. The behaviour of queue size (Lq_1) and orbit size (Lq_2) rises for increasing priority arrival rate(λ_1) and vacation rate (γ) as shown in Figure 7 and 8. The behaviour of queue size (Lq_1) and orbit size (Lq_2) increases with lowering repair rate (η_2) and service rate (μ) in Figure 9 and 10 .

9. Conclusion

In this study, we examined a non-preemptive priority retrial queue on a single server with distinct breakdown, repair, synchronised reneging, and optional server vacation. Random failures of servers are common. The breakdown of the model proposed in this study differs from traditional breakdown, which can be either a long breakdown (hard failure) or a short breakdown (soft failure). For example, if a computer system fails, a simple rebooting of the system (soft failure) or repair by a skilled personnel (hard failure) can fix the problem. Ordinary customers may exit the orbit if the server is either down or busy with priority queue. Furthermore, on completing the priority service, the server may become idle or may go for a vacation. Probability generating functions for the system size and its orbit were found by using supplementary variable technique. System characteristics such as steady-state probabilities and mean system size were also obtained. The results obtained analytically were confirmed with numerical illustrations. The model proposed in this study finds significant practical applications in computer processing systems.

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