MUlTI-COMPONENT CONDITIONAL STRESS-STRENGTH PARAMETER

Kavoos Khorshidian[1], Morteza Taheri Saif Abad [2]

•

[1]Department of Statistics, Faculty of Science, Shiraz University, Shiraz, Iran
khorshidian@shirazu.ac.ir
[2]Department of Statistics, Faculty of Science, Shiraz University, Shiraz, Iran
taherisaifmorteza@gmail.com

Abstract

There are situations in which the experimenter has some information about the components of the operating system and he/she wants to use this information for better assessment or operating of the underlying system. In such cases the notion of conditional probability may help the operator to use that information and improve his/her task. In the present study this notion has been examined, and some conditional stress-strength parameters have been introduced for s of k systems. The multi-component conditional stress-strength parameter (MCCSSP) and its maximum likelihood estimator have been calculated when the strength and stress random variables are exponentially distributed. In the case of having extra information about the parameters, a closed form has been derived for the Bayes estimator of MCCSSP and has been calculated by using an algorithm together with Monte Carlo method. For the case of non-exponential stress or strengths, the nonparametric estimator of the defined parameter has also been derived. Finally, some simulation study on the MLE and Bayes estimator, as well as real data analysis for nonparametric estimators have been done to verify the analytic results.

Keywords: Conditional Reliability, Exponential Distribution, Maximum Likelihood Estimator, Multi-Component Systems, Stress-Strength Parameter

1. Introduction

The effects of resistance and shocks which enter to a system are usually studied via a stress-strength model. The term stress-strength was first introduced by [1]. Since then the stress-strength models have been inspected by many researchers due to their applicability in different fields, such as engineering, economics, psychology, medicine and so on. In such models, when the stress that experienced by the system have been represented by a random variable (RV) X and the strength of system by a RV Y, the stress-strength parameter is denoted by $R = P(X > Y)$, it measures the chance that the system fails. It should be mentioned that $1 - R$ is the chance that the considered system operates well and is known as the reliability function or parameter of the system. For the majority of the well-known distributions, including Normal, Exponential, Pareto, Uniform, Weibull, Gamma, Beta, logistic, and Laplace, R has been studied by [2]. Some of the recent studies about R can be seen in [3], [4], [5], [6] and [7]

There are situations that one have some information about the stress and strength RV's and knows that they are greater than some pre-specified values, or one wants to know how much a system can be reliable when stress and strength increase or decrease. Considering conditions like these, the conditional stress-strength parameter was introduced by [8] as:

$$R^{|a,b} = P(X > Y \mid X > a, Y > b). \tag{1}$$

Nowadays in the real life and industries most of the operating systems have become complex with more than one active component, i.e., a lot of working systems are multi-component rather than simple and uni-component. The reliability of a multi-component stress-strength model was first developed by [9]. Afterwards, applications and studies on different characteristics of multi-component stress-strength models grow up rapidly. Some of the recent studies can be seen in [10],[11], [12], [13], [14] and [15].
By developments in most technologies, in many situations there are a lot of information about the working mechanisms which will be precise and helpful, if they have been employed corrected, e.g. in the case of second hand and used devices. For example, consider a large drilling machine in a mine. This machine uses several gears or drills simultaneously for drilling, which are the most important parts of this machine and are often iteratively replaced by another one. Therefore, a lot of information about the amount of stress and strength experienced by this part of the machine can be collected . In this article, we have focussed on the notion of conditional stress-strength parameter to extend, generalize and employ such information in multi-component systems. In order to prepare a complete pack about MCCSSP, it has been calculated and estimated by using different methods for employing it in different real situations of practice. For exponential distribution as the first and most exploited candidate of the lifetimes of components in operating systems, the MCCSSP has been calculated, its MLE has been estimated through samples and its asymptotic behaviors has been studied, as well. For the circumstances that we have extra information about the varying structure of exponentially distributed stress and strengths random variables, the Bayes estimators of MCCSSP has been also derived based on the information included in samples of stress and strength. For the case of non-exponential or unknown life time distributions the non-parametric estimators have been also derived.
The structure of this article is as follows: A general formula for computing MCCSSP will have been provided in Section 2. In Section 3, the MCCSSP has been computed in the case of exponential distributions as well as its maximum likelihood estimator and asymptotic distribution of the later. The Bayes estimator of this parameter has been obtained in Section 4, by adopting an algorithm and using the Monte Carlo method. The corresponding nonparametric estimator of this parameter has been obtained in Section 5. Section 6 is devoted to the presentation of some simulation studies on the MLE, Bayesian and nonparametric estimators and their comparison. Some numerical results for a real data-set have been presented in Section 7. Finally in Section 8, some concluding remarks have been given.

2. The MCCSSP

In this section, the MCCSSP will have been introduced and a general formula have been presented to compute it.

Definition 1. Consider the independent RV's $X_1, ..., X_k$ with common continuous distribution function $F(\cdot)$, independent of continuous RV Y with distribution function $G(\cdot)$. The MCCSSP is defined as:

$$R_{s,k}^{|a,b} = P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y \mid X_1 > a, ..., X_k > a, Y > b). \tag{2}$$

The particular cases $s = 1$ and $s = k$ correspond to parallel and series systems, respectively. Note that a special case of this quantity for $a = b = -\infty$ is

$$R_{s,k} = P(\text{ at least s of } X_1, ..., X_k \text{ exceed } Y) = \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} (1 - F(y))^i (F(y))^{k-i} \mathrm{d}G(y) \tag{3}$$

which is introduced by [9] as the multi-component stress-strength parameter.
Suppose that there a lot of information about one of the stress RV's X_z, some specified z, $1 \leq z \leq k$,. For example, in some systems, one of the parts wears out more and is replaced more often, such as drilling machines, where the drill bit is very important and is replaced a lot, and

the other parts are replaced less often. Therefore, there are more information about the lifetime of a specified part than the other parts. For this case $R_{s,k}^{|(z),a,b}$ as the MCCSSP when $X_z > a$ is defined as:

Definition 2.

$$R_{s,k}^{|(z),a,b} = P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y \mid X_z > a, Y > b) \quad (4)$$

Note that (3) is again a special case of (4). A formula for computing (2) has been presented in the following theorem.

Theorem 1. If $R_{a,b}^{|s,k}$ is defined by (2), then

$$R_{s,k}^{|a,b} = \begin{cases} \frac{\sum_{i=s}^{k}\binom{k}{i}\int_b^{\infty}[1-F(y)]^i[F(y)-F(b)]^{k-i}dG(y)}{[1-F(a)]^k[1-G(b)]} & a \le b \\ \frac{\sum_{i=s}^{k}\binom{k}{i}(\int_a^{\infty}[G(x)-G(b)]dF(x))^i(\int_a^{\infty}[1-G(x)]dF(x))^{k-i}}{[1-F(a)]^k[1-G(b)]} & a > b \end{cases} \quad (5)$$

Proof. First, we write (2) as follows:

$$R_{s,k}^{|a,b} = \frac{P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y,\, X_1 > a, ..., X_k > a, Y > b)}{P(X_1 > a, ..., X_k > a, Y > b)}.$$

Since $X_1, ..., X_k$ and Y are independent, the dominator is $(1 - F(a))^k(1 - G(b))$. To compute the numerator, first we write it as follows:

$$\begin{aligned} P(\text{at least s of } X_i \text{ exceed } Y,\, X_1 > a, ..., X_k > a, Y > b) &= P((X_1, ..., X_k, Y) \in A) \\ &= \int ... \int_A dF(x_1)...dF(x_k)dG(y), \end{aligned}$$

where $A = \{(x_1, ..., x_k, y) \mid \text{at least s of } x_1, ..., x_k \text{ exceed } y,\, x_1 > a, ..., x_k > a, y > b\}$. To compute this integral, partition A into two regions A_1 and A_2 for the cases $a \le b$ and $a > b$, where:

$$\begin{aligned} A_1 &= \{(x_1, ..., x_k, y) \mid \text{at least s of } x_1, ..., x_k \text{ exceed } y,\, x_1 > a, ..., x_k > a, y > b, a \le b\} \\ &= \{(x_1, ..., x_k, y) \mid \text{at least s of } x_1, ..., x_k \text{ exceed } y,\, a < x_1 < b, ..., a < x_k < b, y > b, a \le b\} \\ &\quad \bigcup \{(x_1, ..., x_k, y) \mid \text{at least s of } x_1, ..., x_k \text{ exceed } y,\, x_1 > b, ..., x_k > b, y > b, a \le b\} \\ &= B_1 \bigcup B_2, \end{aligned}$$

and

$$\begin{aligned} A_2 &= \{(x_1, ..., x_k, y) \mid \text{at least s of } x_1, ..., x_k \text{ exceed } y, y > b, x_1 > a, ..., x_k > a, a > b\} \\ &= \{(x_1, ..., x_k, y) \mid \text{at least s of } (b, x_1), ..., (b, x_k) \text{ contain } y, y > b, x_1 > a, ..., x_k > a, a > b\}. \end{aligned}$$

where

$$\begin{aligned} B_1 &= \{(x_1, ..., x_k, y) \mid \text{at least s of } x_1, ..., x_k \text{ exceed } y,\, a < x_1 < b, ..., a < x_k < b, y > b, a \le b\}, \\ B_2 &= \{(x_1, ..., x_k, y) \mid \text{at least s of } x_1, ..., x_k \text{ exceed } y,\, x_1 > b, ..., x_k > b, y > b, a \le b\}. \end{aligned}$$

Let

$$R_1 = \int_{A_1} dF(x_1)...dF(x_k)dG(y), \quad (6)$$

and

$$R_2 = \int_{A_2} dG(y)dF(x_1)...dF(x_k) \quad (7)$$

then

$$\begin{aligned} R_1 &= \int_{B_1} dF(x_1)...dF(x_k)dG(y) + \int_{B_2} dF(x_1)...dF(x_k)dG(y) \\ &= \int_{B_2} dF(x_1)...dF(x_k)dG(y) \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_b^{\infty}[1 - F(y)]^i[F(y) - F(b)]^{k-i}dG(y) \end{aligned}$$

The first integral becomes zero because $P(X_i > Y, a < X_i < b, Y > b) = 0$ for $i = 1, ..., k$, and

$$\begin{aligned} R_2 &= P(\text{at least s of } (b, X_1), ..., (b, X_k) \text{ contain } Y, Y > b, X_1 > a, ..., X_k > a, a > b) \\ &= \sum_{i=s}^{k} \binom{k}{i} \Big[\int_a^\infty [G(x) - G(b)]dF(x)\Big]^i \Big[\int_a^\infty [1 - G(x)]dF(x)\Big]^{k-i}, \end{aligned}$$

This completes the proof. ∎

Remark 1. Consider an s of k multi-component system, which their strengths are denoted by iid RV's $X_1, X_2, \ldots, X_k$ with common continuous distribution function $F(\cdot)$. Also suppose that each component experiences a random stress Y with continuous distribution function $G(\cdot)$, independent of the strengths. Note that the system stays alive only if at least s of k strengths be greater than the stress. Then the conditional reliability of the multi-component system has the following form:

$$R_{s,k}^{|a,b} = P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y \mid X_1 > a, ..., X_k > a, Y > b). \tag{8}$$

In this model, the conditional reliability of the system is represented by (5).

Remark 2. In practice the information in hand and given condition may not have exactly the form $\{x_1 > a, ..., X_k > a, Y > b\}$, but be as $\{X_1 \in \mathcal{A}_1, ..., X_k \in \mathcal{A}_k, Y \in \mathcal{B}\}$ where $\mathcal{A}_1, ..., \mathcal{A}_k$ and $\mathcal{B}$ are linear Borel sets on $(0, \infty)$. In this case, by applying some procedure similar to the approach of Theorem 1, one can compute this generalized MCCSSP. Based on the structure of $\mathcal{A}_1, ..., \mathcal{A}_k$ and $\mathcal{B}$, it is expected that the analytic derivations may be complicated. In this situation and more general case some non-parametric method similar to that given in section 5 as well as Monte Carlo simulation may be applied.

Remark 3. By formula (5), one may show that for the case $a \leq b$, the MCCSSP $R_{s,k}^{|a,b}$ is an increasing function of a, which is expected trivially. Note that in this case:

$$\begin{aligned} \frac{\partial R_{s,k}^{|a,b}}{\partial a} &= \frac{kf(a)(1 - F(a))^{k-1} \sum_{i=s}^{k} \binom{k}{i} \int_b^\infty [1 - F(y)]^i [F(y) - F(b)]^{k-i} dG(y)}{[1 - F(a)]^{2k} [1 - G(b)]} \\ &= kR_{s,k}^{|a,b} \frac{f(a)}{1 - F(a)} \geq 0. \end{aligned}$$

According to the calculations resulting in the formula (5), it can be seen that if $X_1, \ldots, X_k$ have different distributions, it is not easy to calculate the analogous of this formula. In what follows, the formula (5) has been calculated when $X_1, \ldots, X_k$ and Y have the same distributions.

Corollary 1. Suppose that the continuous RV's $X_1, ..., X_k$ and Y are independent and identically distributed with probability density function(pdf) $f(.)$ and cumulative distribution function(cdf) $F(.)$. Then,

$$R_{s,k}^{|a,b} = \begin{cases} \frac{\sum_{i=s}^{k} \binom{k}{i} \int_{F(b)}^{1} [1-y]^i [y-F(b)]^{k-i} dy}{[1-F(a)]^k [1-F(b)]}, & a \leq b \\ (\frac{1}{2})^k \frac{\sum_{i=s}^{k} \binom{k}{i} [1-2F(b)+F(a)]^i [1-F(a)]^{k-i}}{[1-F(b)]}, & a > b. \end{cases} \tag{9}$$

Remark 4. Put $R_{s,k}^{|(z)1,a,b} = P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y,\ X_z \geq Y \mid X_z > a, Y > b)$ and $R_{s,k}^{|(z)2,a,b} = P(\text{at least s of } X_1, ..., X_k \text{ exceed } Y,\ X_z < Y \mid X_z > a, Y > b)$ for some z, $1 \leq z \leq k$. According to the approach of the proof for Theorem 1, after some computation, we have:

$$R_{s,k}^{|(z)1,a,b} = \begin{cases} \frac{\sum_{i=s}^{k} \binom{k}{i} \int_b^\infty [1-F(y)]^i F(y)^{k-i} dG(y)}{[1-F(a)][1-G(b)]} & a \leq b \\ \frac{\sum_{i=s}^{k} \binom{k}{i-1,k-i} \left[\int_0^\infty [G(x)-G(b)]dF(x)\right]^{i-1} \left[\int_a^\infty [G(x)-G(b)]dF(x)\right] \left[\int_0^\infty [1-G(x)]dF(x)\right]^{k-i}}{[1-F(a)][1-G(b)]} & a > b, \end{cases} \tag{10}$$

$$R_{s,k}^{|(z)2,a,b} = \begin{cases} \frac{\sum_{i=s}^{k-1} \binom{k}{i,k-i-1} \int_b^\infty [1-F(y)]^i [F(y)-F(b)] F(y)^{k-(i+1)} dG(y)}{[1-F(a)][1-G(b)]} & a \leq b \\ \frac{\sum_{i=s}^{k-1} \binom{k}{i,k-i-1} [\int_0^\infty [G(x)-G(b)]dF(x)]^i \left[\int_a^\infty [1-G(x)]dF(x)\right] \left[\int_0^\infty [1-G(x)]dF(x)\right]^{k-(i+1)}}{[1-F(a)][1-G(b)]} & a > b. \end{cases} \tag{11}$$

Therefore

$$R_{s,k}^{|(z),a,b} = \begin{cases} R_{s,k}^{|(z)1,a,b} & X_z \geq Y \\ R_{s,k}^{|(z)2,a,b} & X_z < Y. \end{cases} \tag{12}$$

3. Estimation for Exponential Distribution

In this section, the measure (5) has been evaluated for the Exponentially distributed stresses and strength RV's with different parameters. The probability density and cumulative distribution functions of a random variable $X \sim E(\alpha)$ are denoted by: $f(x) = \alpha e^{-\alpha x}$, and $F(x) = 1 - e^{-\alpha x}$ where $x \geq 0, \alpha > 0$. Suppose that $X_i \sim E(\lambda_1)$ for $i = 1, \ldots, k$ and $Y \sim E(\lambda_2)$ are independent, we have:

$$R_1 = \lambda_2 e^{-b(\lambda_1 k + \lambda_2)} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i,j} \frac{(-1)^j}{\lambda_1(i+j) + \lambda_2},$$

and

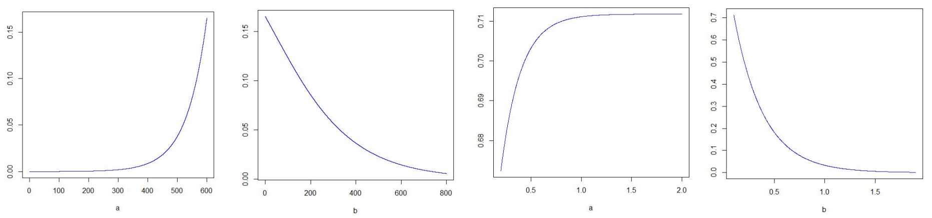
$$R_2 = e^{-ak(\lambda_1+\lambda_2)} \left[\frac{\lambda_1}{\lambda_1+\lambda_2}\right]^k \sum_{i=s}^{k} \binom{k}{i} \left[\frac{\lambda_1+\lambda_2}{\lambda_1} e^{-\lambda_2(b-a)} - 1\right]^i,$$

by dividing the above equations by $[1 - F(a)]^k [1 - G(b)] = e^{-(ak\lambda_1 + b\lambda_2)}$ we have:

$$R_{s,k}^{|a,b} = \begin{cases} \lambda_2 e^{-\lambda_1 k(b-a)} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i,j} \frac{(-1)^j}{\lambda_1(i+j)+\lambda_2} & a \leq b \\ e^{-\lambda_2(ak-b)} [\frac{\lambda_1}{\lambda_1+\lambda_2}]^k \sum_{i=s}^{k} \binom{k}{i} [\frac{\lambda_1+\lambda_2}{\lambda_1} e^{-\lambda_2(b-a)} - 1]^i & a > b. \end{cases} \tag{13}$$

Remark 5. From (13), we conclude that $R_{s,k}^{|a,b}$ for $a \leq b$ in Exponential distribution depends only on the difference between a and b. In other words , if $b_1 - a_1 = b_2 - a_2$ then $R_{s,k}^{|a_1,b_1} = R_{s,k}^{|a_2,b_2}$ for $a_1 \leq b_1$ and $a_2 \leq b_2$.

Figure 1 show the effect of changes in the values a and b in (13). These figures show what happens when the values a and b increase or decrease, in all Figures $(s, k) = (1, 3)$.



(a) *b=600*, $(\lambda_1, \lambda_2) = (0.0049, 0.0005)$ **(b)** $a = 0.1$, $(\lambda_1, \lambda_2) = (0.0049, 0.0005)$ **(c)** $b = 0.1$, $(\lambda_1, \lambda_2) = (1.4, 1.7)$ **(d)** $a = 2$, $(\lambda_1, \lambda_2) = (1.4, 1.7)$

Figure 1: *MCCSSP*

By assuming the Exponential distributions for stresses and strength, from (10) and (12), after some calculation it follows that for the case $X_z \geq Y$, we have:

$$R_{s,k}^{|(z),a,b} = \begin{cases} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i,j} \frac{\lambda_2(-1)^j e^{-\lambda_1(b(k-j)-a)}}{\lambda_1(k-j)+\lambda_2} & a \leq b \\ e^{-\lambda_2(a-b)} [\frac{\lambda_1}{\lambda_1+\lambda_2}]^k \sum_{i=s}^{k} \binom{k}{i} [\frac{\lambda_1+\lambda_2}{\lambda_1} e^{-\lambda_2 b} - 1]^{i-1} [\frac{\lambda_1+\lambda_2}{\lambda_1} e^{-\lambda_2(b-a)} - 1] & a > b, \end{cases} \tag{14}$$

and for the case $X_z < Y$:

$$R_{s,k}^{|(z),a,b} = \begin{cases} \lambda_2 e^{-\lambda_1(b-a)} [\sum_{i=s}^{k-1} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j e^{-\lambda_1(b(i+j))} [\frac{1}{\lambda_1(i+j)+\lambda_2} - \frac{e^{-\lambda_1(b-a)}}{\lambda_1(i+j+1)+\lambda_2}]] & a \leq b \\ e^{-\lambda_2(a-b)} [\frac{\lambda_1}{\lambda_1+\lambda_2}]^k [\sum_{i=s}^{k-1} \binom{k}{i} [\frac{\lambda_1+\lambda_2}{\lambda_1} e^{-\lambda_2 b} - 1]^i] & a > b. \end{cases} \tag{15}$$

Indeed, as in (13) to (15), the stress-strength parameters $R_{s,k}^{|a,b}$ and $R_{s,k}^{|(z),a,b}$ are functions of λ_1 and λ_2. Therefore, it is rational that for evaluating the maximum likelihood estimators of MCCSSP, the first step to be calculating the MLE's of λ_1 and λ_2.

3.1. Maximum Likelihood Estimation

Suppose that $X_1, ..., X_n$ and $Y_1, ..., Y_m$ are two independent random samples from $E(\lambda_1)$ and $E(\lambda_2)$. Then the likelihood function is

$$L(\lambda_1, \lambda_2) = \lambda_1^n \lambda_2^m e^{-\lambda_1 \sum_1^n x_i} e^{-\lambda_2 \sum_1^m y_j} \tag{16}$$

and the MLE's of the parameters λ_1 and λ_2 are $\hat{\lambda_1} = \frac{1}{\bar{X}}$ and $\hat{\lambda}_2 = \frac{1}{\bar{Y}}$, respectively. Therefore, by using the invariance property for MLE's, and substituting $\hat{\lambda_1}$ and $\hat{\lambda_2}$ instead of λ_1 and λ_2 in (14) and (15), one may write the MLE of (13) by:

$$\hat{R}_{s,k}^{|a,b} = \begin{cases} \hat{\lambda}_2 e^{-\hat{\lambda}_1 k(b-a)} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i,j} \frac{(-1)^j}{\hat{\lambda}_1 (i+j) + \hat{\lambda}_2} & a \leq b \\ e^{-\hat{\lambda}_2 (ak-b)} [\frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \hat{\lambda}_2}]^k \sum_{i=s}^{k} \binom{k}{i} [\frac{\hat{\lambda}_1 + \hat{\lambda}_2}{\hat{\lambda}_1} e^{-\hat{\lambda}_2 (b-a)} - 1]^i & a > b. \end{cases} \tag{17}$$

3.2. Asymptotic Distribution

In this subsection the asymptotic distribution of $\hat{R}_{s,k}^{|a,b}$ will have been obtained by using the asymptotic normality of the MLE's and the multivariate delta method. By the fact that $\hat{\boldsymbol{\lambda}} \rightarrow N_2(\boldsymbol{\lambda}, \boldsymbol{\Sigma})$ as n, m tend to infinity, $\frac{n}{m} \rightarrow d$ for some $0 < d < \infty$, where $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \hat{\lambda}_2)^T$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T$ and $\boldsymbol{\Sigma}$ is the inverse of Fisher's information matrix $I(\boldsymbol{\lambda})$, it is easy to see that
$I(\boldsymbol{\lambda}) = \begin{bmatrix} \frac{n}{\lambda_1^2} & 0 \\ 0 & \frac{m}{\lambda_2^2} \end{bmatrix}$, and so $\boldsymbol{\Sigma} = \begin{bmatrix} \frac{\lambda_1^2}{n} & 0 \\ 0 & \frac{\lambda_2^2}{m} \end{bmatrix}$.

The well-known delta method enables us to derive the asymptotic behaviour of functions of an estimator, whenever the estimator is itself asymptotically normal. The delta method have been present and applied in different forms, we have used the following presentation.

Proposition 1. Let $g(.)$ be a mapping $g(.) : \mathbb{R}^d \rightarrow \mathbb{R}$, such that $g(.)$ is continuous in a neighborhood of $\boldsymbol{\mu} \in \mathbb{R}^d$. If $\mathbf{X}_n$ is a sequence of d-dimensional random vectors such that $\mathbf{X}_n \rightarrow N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in distribution, then $\frac{g(\mathbf{X}_n) - g(\boldsymbol{\mu})}{\tau} \rightarrow N(0, 1)$ in distribution, where $\tau^2 = \boldsymbol{\nabla}^T \boldsymbol{\Sigma} \boldsymbol{\nabla} > 0$ and $\boldsymbol{\nabla} = \frac{\partial g(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}$.

We will apply Proposition 1 to $\mathbf{X}_n = \hat{\boldsymbol{\lambda}}$ and

$$g(x_1, x_2) = \begin{cases} x_2 e^{-x_1 k(b-a)} \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i,j} \frac{(-1)^j}{x_1 (i+j) + x_2} & a \leq b \\ e^{-x_2 (ak-b)} [\frac{x_1}{x_1 + x_2}]^k \sum_{i=s}^{k} \binom{k}{i} [\frac{x_1 + x_2}{x_1} e^{-x_2 (b-a)} - 1]^i & a > b. \end{cases}$$

The asymptotic distribution of $\hat{R}_{s,k}^{|ab}$ may be obtained as below:

$$(\hat{R}_{s,k}^{|a,b} - R_{s,k}^{|a,b}) \rightarrow N(0, \boldsymbol{\nabla}^T \boldsymbol{\Sigma} \boldsymbol{\nabla}), \tag{18}$$

where

$$\boldsymbol{\nabla} = (\frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_1}, \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_2})^T,$$

$$\boldsymbol{\nabla}^T \boldsymbol{\Sigma} \boldsymbol{\nabla} = [\frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_1}]^2 \frac{\lambda_1^2}{n} + [\frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_2}]^2 \frac{\lambda_2^2}{m}, \tag{19}$$

For the cases $a \leq b$ and $a > b$, denote (19) by σ_1^2 and σ_2^2 respectively. Put $\psi = \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_1} |_{a \leq b}$, $\nu = \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_1} |_{a > b}$, $\theta = \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_2} |_{a \leq b}$, $\kappa = \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_2} |_{a > b}$, we arrive at:

$$\sigma_1^2 = \psi^2 \frac{\lambda_1^2}{n} + \theta^2 \frac{\lambda_2^2}{m}, \tag{20}$$

$$\sigma_2^2 = \nu^2 \frac{\lambda_1^2}{n} + \kappa^2 \frac{\lambda_2^2}{m}. \tag{21}$$

Therefore, the asymptotic normalized distribution of $\hat{R}_{s,k}^{|a,b}$ for different values of a and b are as follow:

$$\frac{\hat{R}_{s,k}^{|a,b} - R_{s,k}^{|a,b}}{\sigma_i} \rightarrow N(0,1) \qquad i = 1,2, \tag{22}$$

where σ_1 and σ_2 stands for the cases $a \leq b$ and $a > b$ respectively. The above statistics can be used for constructing confidence intervals for $R_{s,k}^{|a,b}$. By employing a similar approach and performing some steps like the above, using lemma 1 and asymptotic normality of $\hat{R}_{s,k}^{|(z),a,b}$, one may arrive at the asymptotic distribution of $R_{s,k}^{|(z),a,b}$.

4. Bayes Estimation

In this section, the Bayesian estimation of the reliability parameter (13) has been considered. Suppose that the parameters λ_1 and λ_2 are RV's, and have independent Gamma prior distributions with parameters $(\alpha_i, \beta_i), i = 1,2$ respectively. The pdf of a random variable $X \sim Gamma(\alpha_i, \beta_i)$ is denoted by

$$\pi(x) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\beta_i x} \qquad x > 0,\ \alpha_i > 0,\ \beta_i > 0. \tag{23}$$

The joint posterior density function of the parameters based on this prior density and the likelihood function can be written as follows:

$$\pi^*(\lambda_1, \lambda_2 \mid \mathbf{x}, \mathbf{y}) = \frac{\pi(\lambda_1, \lambda_2, \mathbf{x}, \mathbf{y})}{\int_0^\infty \int_0^\infty \pi(\lambda_1, \lambda_2, \mathbf{x}, \mathbf{y}) d\lambda_1 d\lambda_2} \tag{24}$$

where

$$\pi(\lambda_1, \lambda_2, \mathbf{x}, \mathbf{y}) = \pi(\lambda_1)\pi(\lambda_2)L(\lambda_1, \lambda_2) \propto \lambda_1^{\alpha_1+n-1} e^{-\lambda_1(\beta_1 + \sum_{i=1}^n x_i)} \lambda_2^{\alpha_2+m-1} e^{-\lambda_2(\beta_2 + \sum_{j=1}^m y_j)}.$$

It is easily seen that the posterior density functions of λ_1 and λ_2 are respectively

$$\pi^*(\lambda_1 | \lambda_2, \mathbf{x}, \mathbf{y}) \propto \Gamma(\alpha_1 + n, \beta_1 + \sum_{i=1}^n x_i), \tag{25}$$

$$\pi^*(\lambda_2 | \lambda_1, \mathbf{x}, \mathbf{y}) \propto \Gamma(\alpha_2 + m, \beta_2 + \sum_{j=1}^m y_j). \tag{26}$$

The Bayes estimator of $R_{s,k}^{|a,b}$ under the squared error loss (SEL) is obtained as

$$\tilde{R}_{s,k}^{|a,b} = E(R_{s,k}^{|a,b} | \mathbf{x}, \mathbf{y}) = \int_0^\infty \int_0^\infty R_{s,k}^{|a,b} \pi^*(\lambda_1, \lambda_2 \mid \mathbf{x}, \mathbf{y}) d\lambda_1 d\lambda_2. \tag{27}$$

It is not possible to calculate equation (27) analytically. Therefore, to compute the Bayes estimate of reliability parameter $R_{s,k}^{|a,b}$, a Monte Carlo (MC) method has been adopted as follows:

Step 1: Set l=1.
Step 2: Generate $X_1, \ldots, X_n$ from $Exp(\lambda_1)$.
Step 3: Generate $Y_1, \ldots, Y_m$ from $Exp(\lambda_2)$
Step 4: Generate λ_1^l from $Gamma(\alpha_1 + n, \beta_1 + \sum_{i=1}^n x_i)$.
Step 5: Generate λ_2^l from $Gamma(\alpha_2 + m, \beta_2 + \sum_{j=1}^m y_j)$.
Step 6: Compute $R_{s,k}^{l|a,b}$ at $(\lambda_1^l, \lambda_2^l)$.
Step 7: l=l+1.

Step 8: Repeat Steps 2 to 7, M times and obtain the posterior sample $R_{s,k}^{l|a,b}$ for $l = 1, ..., M$.
Now the Bayes estimate of $R_{s,k}^{|a,b}$ with respect to SEL will be obtained as follows:

$$\tilde{R}_{s,k}^{|a,b} = \frac{1}{M} \sum_{l=1}^{M} R_{s,k}^{l|a,b}. \tag{28}$$

5. Nonparametric Estimation

In this section a nonparametric method for estimating $R_{s,k}^{|a,b}$ has been presented. In many situations, we may have no information about the distribution of data or computing $R_{s,k}^{|a,b}$ via 1 may require complex computations, or even may not have a definite answer. Therefore, employing the nonparametric method, in which the structure of the model may have been determined from data, can lead us to better results or at least be more applicable. Let $n(.)$ be the counting measure. For the sample space $\mathbf{S}$ and the event $\mathbf{D}$ as a subset of $\mathbf{S}$ the nonparametric estimator of $P(\mathbf{D})$ is defined as $\hat{P}(\mathbf{D}) = \frac{n(\mathbf{D})}{n(\mathbf{S})}$. To obtain the nonparametric estimator of MCCSSP, one may write (2) in the form:

$$R_{s,k}^{|a,b} = \frac{P(\text{at least } s \text{ of } X_1, \ldots, X_k \text{ exceed } Y,\, X_1 > a, \ldots, X_k > a, Y > b)}{P(X_1 > a, \ldots, X_k > a, Y > b)} \tag{29}$$

where $P(X_1 > a, \ldots, X_k > a, Y > b) > 0$. Since $X_1, \ldots, X_k$ and Y are independent, equation (29) can be written as follows:

$$R_{s,k}^{|a,b} = \frac{P(\text{at least } s \text{ of } X_1, \ldots, X_k \text{ exceed } Y,\, X_1 > a, \ldots, X_k > a,\ Y > b)}{P(X_1 > a, \ldots, X_k > a)P(Y > b)}, \tag{30}$$

where $P(X_1 > a, \ldots, X_k > a)P(Y > b) > 0$.
Let $\mathbf{A} = \{(x_1, \ldots, x_k, y) \mid \text{at least s of } x_1, \ldots, x_k \text{ exceed } y,\ x_1 > a, \ldots, x_k > a,\ y > b\}$, $\mathbf{B} = \{(x_1, \ldots, x_k) \mid x_1 > a, \ldots, x_k > a\}$ and $\mathbf{C} = \{y \mid y > b\}$. The nonparametric estimator of (30) can be written as follows:

$$R_{s,k}^{Np|a,b} = \frac{n(\mathbf{A})}{n(\mathbf{B})n(\mathbf{C})}. \tag{31}$$

Let $X_{1i}, ..., X_{ki} \sim X$ for $i = 1, ..., n$ and $Y_1, ..., Y_m \sim Y$ be independent random samples. Also, let $I(E)$ be the indicator function of the event E, that is a RV that takes value 1 when the event E happens and 0 when the event does not happen. By assuming $n(.)$ as the counting measure, we have:

$$n(\mathbf{B}) = \sum_{i=1}^{n} \mathrm{I}(X_{1i} > a, \ldots, X_{ki} > a), \tag{32}$$

$$n(\mathbf{C}) = \sum_{j=1}^{m} \mathrm{I}(Y_j > b), \tag{33}$$

and by the properties of the indicator function:

$$\begin{aligned} n(\mathbf{A}) = & \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{s of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j)\mathrm{I}(X_{1i} > a, \ldots, X_{ki} > a)\mathrm{I}(Y_j > b) + \ldots \\ & + \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{I}(\text{k of } X_{1i}, ..., X_{ki} \text{ exceed } Y_j)\mathrm{I}(X_{1i} > a, \ldots, X_{ki} > a)\mathrm{I}(Y_j > b). \end{aligned} \tag{34}$$

Let $\mathbf{X}_i = (X_{1i}, \ldots, X_{ki})$ for $i = 1, \ldots, n$. Those observations $\mathbf{X}_i$ and Y_j for them both $\mathbf{X}_i \leq a$ and $Y_j \leq b$ simultaneously, have been removed in calculating $n(\mathbf{A})$, since in details of calculating $P(\mathbf{A})$ or $R_{s,k}^{NP|a,b} = \frac{n(\mathbf{A})}{n(\mathbf{B})n(\mathbf{C})}$, the numerator is an strict subset of denominator. Note that in this case the values of the second and third indicators will automatically equal one in $n(\mathbf{A})$, (34). It is worth noting that the number of reminded samples of $\mathbf{X}_i$ and Y_j are $n(\mathbf{B})$ and $n(\mathbf{C})$, so $n(\mathbf{A})$ can be written as follows:

$$n(\mathbf{A}) = \sum_{i=1}^{n(\mathbf{B})}\sum_{j=1}^{n(\mathbf{C})} \mathrm{I}(\text{s of } X_{1i},...,X_{ki} \text{ exceed } Y_j) + \cdots + \sum_{i=1}^{n(\mathbf{B})}\sum_{j=1}^{n(\mathbf{C})} \mathrm{I}(\text{k of } X_{1i},...,X_{ki} \text{ exceed } Y_j)$$

In the case of $n = m$, the formula (31) may have simpler form and computations, since we only keep those $(X_{1i},...,X_{ki},Y_i)$ $i = 1,...,n$ which for them $(X_{1i} > a,...,X_{ki} > a, Y_i > b)$ and remove the rest and also, $n(\mathbf{B}) = n(\mathbf{C})$. In what follows, we introduce a definition and representation for non-parametric estimator of multi-component stress-strength parameter. To the best of our knowledge, interestingly this estimator has not been defined till now.

Definition 3. The nonparametric estimator of $R_{s,k}$ is defined as follows:

$$R_{s,k}^{NP} = \frac{n(\mathbf{A})}{n(\mathbf{B})n(\mathbf{C})} \tag{35}$$

where $n(\mathbf{B}) = n$, $n(\mathbf{C}) = m$ and

$$n(\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{I}(\text{s of } X_{1i},...,X_{ki} \text{ exceed } Y_j) + \cdots + \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{I}(\text{k of } X_{1i},...,X_{ki} \text{ exceed } Y_j).$$

Note that (35) can be obtained from (31) by assuming $a = b = 0$.

Remark 6. (i): By (31), and according to the definitions of $n(\mathbf{A}), n(\mathbf{B})$ and $n(\mathbf{C})$, it can be concluded that for fixed values of a, $a \le b$, the estimator $R_{s,k}^{NP|a,b}$ is a decreasing function of b.
(ii): By (31), and according to the definitions of $n(\mathbf{A}), n(\mathbf{B})$ and $n(\mathbf{C})$, it can be concluded that for fixed values of b, $a > b$, the estimator $R_{s,k}^{NP|a,b}$ is an increasing function of a.

In applications, the data observed for different stresses may differ greatly in their values. Therefore, selecting a minimum value of a, w.r.t. it all stresses in MCCSSP through definition 1, satisfy the corresponding condition $X_i > a$, may be not useful. So, in what follows, the MCCSSP has been defined in some general way to be more realistic and applicable.

Definition 4. The generalized conditional multi-component stress-strength parameter is defined as follows:

$$R_{s,k}^{|a_1,...,a_k,b} = P(\text{at least s of } X_1,...,X_k \text{ exceed } Y \mid X_1 > a_1,...,X_k > a_k, Y > b) \tag{36}$$

where the RV's $Y, X_1,...,X_k$ are independent, $G(\cdot)$ is the continuous distribution function of Y and $F(\cdot)$ is the common continuous distribution function of $X_1,...,X_k$.

Theorem 2. If $X_{ri} > max(a_1,\ldots,a_k)$ for $r = 1,\ldots,k$; $i = 1,\ldots,n$ and $Y_j > b$ for $j = 1,\ldots,m$ then $R_{s,k}^{NP|a_1,...,a_k,b} = R_{s,k}^{NP}$.

Proof. Replace $\mathrm{I}(X_{1i} > a_1,\ldots,X_{ki} > a_k)$ with $\mathrm{I}(X_{1i} > a,\ldots,X_{ki} > a)$ in (32) and (34). Since $\mathrm{I}(X_{1i} > a_1,\ldots,X_{ki} > a_k) = 1$ and $\mathrm{I}(Y_j > b) = 1$ we have $n(\mathbf{B}) = n$, $n(\mathbf{C}) = m$ and $n(\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{I}(\text{s of } X_{1i},...,X_{ki} \text{ exceed } Y_j) + \cdots + \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{I}(\text{k of } X_{1i},...,X_{ki} \text{ exceed } Y_j)$. ■

Of course, a special case of (36) is (2). In parametric case (MLE method) when $a_1,\ldots,a_k$ are closed in values, a can be considered as the minimum or maximum of $a_1,\ldots,a_k$ and approximate (36) through (4). In some situations, $a_1,\ldots,a_k$ are very different, and using (36) is not very helpful or may not be accurate. In these cases, the non-parametric method is more practical and it is enough to consider $\mathbf{A} = \{(x_1,\ldots,x_k,y) \mid \text{at least s of } x_1,\ldots,x_k \text{ exceed y}, x_1 > a_1,\ldots,x_k > a_k, y > b\}$, and $\mathbf{B} = \{(x_1,\ldots,x_k) \mid x_1 > a_1,\ldots,x_k > a_k\}$ in (31). It is easy to see that the results of nonparametric estimation of (29) can also be used for nonparametric estimation of (36), where a_i is substituted instead of a for $i = 1,\ldots,k$. Note that in this case, one advantage of the nonparametric method is that the assumption of common distribution for stress RV's may be relaxed. The later makes this method much more practical. If $\mathbf{B} = \{(x_1,\ldots,x_k,y) \mid x_1 > a_1,\ldots,x_k > a_k, y > b\}$, then the nonparametric estimator of the generalized MCCSSP where stresses and strength RV's are not independent, can also be easily computed through the same method.

6. Simulation

In this section, a simulation study has been done to assess the quality and the efficiency of performance of $R_{s,k}^{|ab}$, its MLE, Bayes and nonparametric estimators. The performances of the MLE, Bayes and nonparametric estimators have been studied by using their biases. The performances of the confidence intervals for MLE are studied by using average confidence lengths (ACL's) and coverage probabilities (CP's). It would be mentioned that, the proportion of the times that the intervals contain the true value of interest is called the coverage probability of a confidence interval. The simulations have been only done for $a \neq b$ since for $a = b$ the conditional and unconditional cases have the same results.
The results for $R_{s,k}^{|ab}$, MLE's, Biases, MSE's, ACL's and CP's and different values of m and n where the other parameters are fixed, have been shown in the Tables 1 for $\lambda_1 = 1$, $\lambda_2 = 2$ and shown in 2 for $\lambda_1 = 1.5$, $\lambda_2 = 0.7$. According to these tables larger sample sizes have more reliable results. A comparison among MLE, $R_{1,3}^{NP|a,b}$ and $\tilde{R}_{1,3}^{|a,b}$ assuming $\alpha_1 = 2$, $\beta_1 = 3$, $\alpha_2 = 5$, $\beta_2 = 4$ for different values of a and b, $n = m = 100$, $\lambda_1 = 0.0003$ and $\lambda_2 = 0.0005$ has been done and the results presented in Tables 3 and 4. A comparison among $\hat{R}_{2,4}^{|a,b}$ and $\tilde{R}_{2,4}^{|a,b}$ assuming $\lambda_1 = 3$, $\lambda_2 = 2$, $\alpha_1 = 5$, $\beta_1 = 0.8$, $\alpha_2 = 4$, $\beta_2 = 0.2$ for different sample sizes has been done and the results presented in Table 5. A nonparametric simulation for different values of a_1, a_2, a_3, $\lambda_1 = 0.004$ and $\lambda_2 = 0.002$ has been done and the results are presented in Table 6.

Table 1: *Comparison of estimators,* $R_{1,3}^{|0.7,1} = 0.3659$, $R_{2,4}^{|0.7,1} = 0.2409$

	n	15	20	35	50	85	100
(s,k)	m	15	25	35	50	75	100
	$\hat{R}_{1,3}^{|0.7,1}$	0.3556	0.3577	0.3612	0.3628	0.3632	0.3646
	MSE	0.0439	0.0256	0.0180	0.0125	0.0083	0.0062
(1,3)	Bias	-0.0102	-0.0082	-0.0046	-0.0030	-0.0026	-0.0013
	ACL	0.6889	0.5202	0.4425	0.3674	0.2990	0.2588
	CP	0.9384	0.9498	0.9736	0.9802	0.9818	0.9844
	$\hat{R}_{2,4}^{|0.7,1}$	0.2372	0.2357	0.2390	0.2393	0.2400	0.2400
	MSE	0.0243	0.0139	0.0100	0.0069	0.0046	0.0034
(2,4)	Bias	-0.0036	-0.0052	-0.0018	-0.0015	-0.0008	-0.0008
	ACL	0.5145	0.3879	0.3296	0.2741	0.2228	0.1921
	CP	0.8988	0.9012	0.938	0.9518	0.9640	0.9690

Table 2: *Comparison of estimators,* $R_{1,3}^{|1.2,0.5} = 0.4603$, $R_{2,4}^{|1.2,0.5} = 0.2798$

	n	15	20	35	50	75	100
(s,k)	m	15	25	35	50	75	100
	$\hat{R}_{1,3}^{|1.2,0.5}$	0.4404	0.4481	0.4551	0.4537	0.4562	0.4567
	MSE	0.0391	0.0225	0.0097	0.0096	0.0060	0.0046
(1,3)	Bias	-0.0199	-0.0122	-0.0052	-0.0065	-0.0041	-0.0035
	ACL	0.5655	0.4581	0.3110	0.3085	0.2481	0.2183
	CP	0.9374	0.9696	0.9826	0.9838	0.9876	0.9932
	$\hat{R}_{2,4}^{|1.2,0.5}$	0.2608	0.2679	0.2718	0.2742	0.2758	0.2771
	MSE	0060	0.0045	0.0025	0.0017	0.0010	0.0008
(2,4)	Bias	-0.0197	-0.0118	-0.0079	-0.0056	-0.0040	-0.0027
	ACl	0.2349	0.2079	0.1600	0.1351	0.0.0104	0.0964
	CP	0.9156	0.9548	0.9620	0.9690	0.9712	0.9837

Table 3: *Comparison of* $\hat{R}_{1,3}^{|a,b}$, $R_{1,3}^{NP|a,b}$, $\tilde{R}_{1,3}^{|a,b}$ *for* $a \le b$

a	10	25	70	78	170	215	300
b	20	40	74	120	190	260	310
$R_{1,3}^{\|a,b}$	0.8607	0.8568	0.8653	0.8362	0.8530	0.8340	0.0.8607
$\hat{R}_{1,3}^{\|a,b}$	0.8555	0.8516	0.8602	0.8311	0.8478	0.8327	0.8555
$\tilde{R}_{1,3}^{\|a,b}$	0.8598	0.8559	0.8646	0.8349	0.8519	0.8321	0.8598
$R_{1,3}^{NP\|a,b}$	0.8657	0.8655	0.8697	0.8681	0.8691	0.8646	0.8668
Bias($\hat{R}_{1,3}^{\|a,b}$)	-0.0051	-0.0051	-0.0051	-0.0051	-0.0051	-0.0013	-0.0051
Bias($\tilde{R}_{1,3}^{\|a,b}$)	-0.0008	-0.0009	-0.0007	-0.0013	-0.0010	-0.0019	-0.0008
Bias($R_{1,3}^{NP\|a,b}$)	0.0049	0.0087	0.0043	0.0618	0.0161	0.0305	0.0061

Table 4: *Comparison of* $\hat{R}_{1,3}^{|a,b}$, $R_{1,3}^{NP|a,b}$, $\tilde{R}_{1,3}^{|a,b}$ *for* $a > b$

a	7	22	45	67	100	120	240
b	4	11	38	65	90	70	230
$R_{1,3}^{\|a,b}$	0.9437	0.9377	0.9124	0.8877	0.8664	0.8867	0.7532
$\hat{R}_{1,3}^{\|a,b}$	0.9397	0.9338	0.9088	0.8844	0.8633	0.8836	0.7514
$\tilde{R}_{1,3}^{\|a,b}$	0.9372	0.9313	0.90559	0.8812	0.8598	0.8805	0.7466
$R_{1,3}^{NP\|a,b}$	0.8654	0.8658	0.8658	0.8653	0.8674	0.8686	0.8680
Bias($\hat{R}_{1,3}^{\|a,b}$)	-0.0039	-0.0038	-0.0036	-0.0033	-0.0030	-0.0030	-0.0017
Bias($\tilde{R}_{1,3}^{\|a,b}$)	-0.0064	-0.0063	-0.0064	-0.0065	-0.0065	-0.0062	-0.0065
Bias($R_{1,3}^{NP\|a,b}$)	-0.0782	-0.0719	-0.0466	-0.0224	0.0097	-0.0181	0.1147

Table 5: *Comparison of* $\hat{R}_{2,4}^{|a,b}$, $\tilde{R}_{2,4}^{|a,b}$, *exact values* $R_{2,4}^{|0.6,0.8} = 0.0430$, $R_{2,4}^{|0.9,0.6} = 0.0243$

n	10	20	30	60	95	100	150
m	10	22	30	58	100	120	150
$\hat{R}_{2,4}^{\|0.6,0.8}$	0.0505	0.0475	0.0457	0.0443	0.0437	0.0436	0.0434
$\tilde{R}_{2,4}^{\|0.6,0.8}$	0.0360	0.0407	0.0405	0.0418	0.0419	0.0421	0.0424
$Bias(\hat{R}_{2,4}^{\|0.6,0.8})$	-0.0075	-0.0045	-0.0027	-0.0013	-0.0007	-0.0006	-0.0004
$Bias(\tilde{R}_{2,4}^{\|0.6,0.8})$	0.0069	0.0022	0.0024	0.0011	0.0010	0.0008	0.0005
$\hat{R}_{2,4}^{\|0.9,0.6}$	0.0269	0.0260	0.0254	0.0249	0.0246	0.0246	0.0245
$\tilde{R}_{2,4}^{\|0.9,0.6}$	0.0342	0.0301	0.0290	0.0271	0.0261	0.0257	0.0252
$Bias(\hat{R}_{2,4}^{\|0.9,0.6})$	-0.0026	-0.0016	-0.0010	-0.0006	-0.0003	-0.0002	-0.0001
$Bias(\tilde{R}_{2,4}^{\|0.9,0.6})$	-0.0098	-0.0057	-0.0046	-0.0027	-0.0017	-0.0013	-0.0008

Table 6: *Values of* $R_{1,3}^{NP|a_1,a_2,a_3,b}$ *for* $\lambda_1 = 0.004$ *and* $\lambda_2 = 0.002$

a_1	1	1	8	27	40	40	95	100	100
a_2	3	5	14	60	40	42	98	100	100
a_3	7	7	28	90	40	47	100	100	100
b	5	3	19	43	30	30	110	110	180
$R_{1,3}^{NP|a_1,a_2,a_3,b}$	0.532	0.535	0.532	0.540	0.546	0.549	0.510	0.509	0.468

7. Real Data Analysis

In this section the numerical results of the parameters estimation for a real data set with Exponential distribution have been presented. This data set was used for the first time by [16] and can be find in it. Also, it have been used by many other authors, e.g., [17], [18] and [19].These data present the tensile properties of the jute fibres at different gauge lengths 5, 10, 15 and 20 mm which measured in MPa. The data sets corresponding to the breaking strength of jute fibres with 10mm and 15mm gauge lengths have been considered as the stresses measurement and 20mm in gauge lengths, which represents the strength measurement.
Each data has been separately fitted to the some Exponential distribution and examined by using the Kolmogorov-Smirnov goodness-of-fit test, the results have been reported in Table 7. The Kolmogorov-Smirnov statistics and the corresponding P-values indicate that the Exponential distribution fits the data sets. The estimation of MCCSSP for different values of a and b by MLE, nonparametric methods and Bayesian approach assuming $\alpha_1 = 2$, $\beta_1 = 3$, $\alpha_2 = 5$, $\beta_2 = 4$ for parameters of prior distributions have been presented in Table 8. The estimation of MCCSSP for different values of a_1, a_2 and b by nonparametric methods have been presented in Table 9. The estimation of (4) for $a_1 = 0$ or $a_2 = 0$ by nonparametric methods have been presented in Table 10. The data set consisting of the breaking strength of jute fiber 5 mm in gauge length have been fitted with the Normal distribution with mean 384.37 and standard deviation 188.77 using the Kolmogorov-Smirnov goodness-of-fit test. For this data, the Lilliforce significance correction criteria (modified Kolmogorov-Smirnov test to check the normality of the data) and the P-value are 0.143 and 0.122. Note that by adding this length to the model, the assumption of exponentially for all stresses fails and the MLE method may not be employed. The nonparametric estimators of MCCSSP for real data and different values of $a_1,\ a_2,\ a_3$ and b have been presented in Table 11 where X_1 has Normal distribution, X_2 and X_3 have Exponential distribution.

Table 7: *Estimate of parameters, K-S test for strength of jute fiber data*

data	Mean	$\hat{\lambda}$	K-S	p-value
10 mm	365.72	0.0027	0.958	0.317
15 mm	367.87	0.0027	0.999	0.271
20 mm	340.74	0.0029	0.727	0.666

Table 8: *Values of estimates of MCCSSP for real data*

a	30	45	45	78	85	100	220
b	25	50	40	90	75	80	245
$\hat{R}_{1,2}^{|a,b}$	0.7200	0.6680	0.6893	0.6432	0.6280	0.6288	0.5996
$\tilde{R}_{1,2}^{|a,b}$	0.7431	0.6737	0.6458	0.6207	0.6477	0.5970	0.6151
$R_{1,2}^{NP|a,b}$	0.6744	0.6760	0.6886	0.6462	0.6485	0.6485	0.6944

Table 9: *Values of* $R_{1,2}^{NP|a_1,a_2,b}$ *for real data*

a_1	20	30	90	180	202	200	300
a_2	40	100	70	170	200	250	280
b	30	70	80	175	201	225	290
$R_{1,2}^{NP\|a_1,a_2,b}$	0.674	0.640	0.654	0.755	0.755	0.694	0.760

Table 10: *Values of* $R_{1,2}^{NP|(z),a,b}$ *for real data*

a_1	0	0	90	160	0	190	0
a_2	30	50	0	0	150	0	280
b	45	35	40	145	160	255	290
$R_{1,2}^{NP\|a_1,a_2,b}$	0.663	0.670	0.679	0.608	0.663	0.617	0.640

Table 11: *Values of* $R_{1,3}^{NP|a_1,a_2,a_3,b}$ *for real data*

a_1	10	42	80	111	150	215	300
a_2	30	58	90	121	160	221	400
a_3	60	71	100	171	170	240	100
b	34	54	85	154	165	220	340
$R_{1,3}^{NP\|a_1,a_2,a_3,b}$	0.730	0.736	0.705	0.750	0.859	0.625	0.750

8. Conclusion

The MCCSSP ($R_{s,k}^{|a,b}$) as an appropriate extension of multi-component stress-strength parameter has been introduced. A general formula for computing $R_{s,k}^{|a,b}$ in the case of continuous RV's has been presented. The maximum likelihood estimator of $R_{s,k}^{|a,b}$ for Exponential distribution has been estimated. The asymptotic distribution of maximum likelihood estimator has been obtained and been used to obtain asymptotic confidence intervals of $R_{s,k}^{|a,b}$. A Formula for estimating the MCCSSP by nonparametric method has also been presented. Some numerical computation and simulation studies have been done for illustrating the inferential procedures.
In the past decades, a lot of researches have been done for studying the behavior of reliability function in multi-component stress-strength models, many of similar works can be done for the conditional case. As an specific idea, $R_{s,k}^{|a,b}$ can be obtained and estimated for other distributions. As another idea, one may interested in the amounts of information which are measurable, lost, unpredictable, etc.

Declarations

- **Authors Contribution**: All parts of this study has been done jointly by both authors, unless the simulation and graphical study which has been compiled by M.Taheri and the final edition which has been done by K.Khorshidian.
- **Competing Interest**: There is no conflict of interest between authors.
- **Availability of Data and Materials**: Please contact M.Taheri, taherisaifmorteza@gmail.com in order to request any data corresponding to the simulation or graphical subsection.

- **Funding**: No funding was obtained for this study.

References

[1] Church, J. D. and Harris, B. (1970). The estimation of reliability from stress-strength relationships. *Technometrics*, 12(1): 49–54.

[2] Kotz, S., Lumelski, Y. and Pensky, M. The Stress-Strength Model and its Generalizations: Theory and Applications, World Scientific, Singapore, 2003.

[3] Jose, J. K. (2022). Estimation of stress-strength reliability using discrete phase type distribution. *Communications in Statistics-Theory and Methods*, 51(2): 368–386.

[4] Khan, A. H., Jan, T. R. (2020). Estimation of stress-strength reliability model using finite mixture of M-transformed Exponential distributions. *Reliability: Theory and Applications*, 15(2): 90–103.

[5] Varghese, A. K., Chacko, V. M. (2022). Estimation of stress-strength reliability for Akash distribution. *Reliability: Theory and Applications*, 17(3): 52–58.

[6] Chauhan, K. S. (2022). Estimation and testing procedures of P (Y< X) for the inverse distributions family under type-II censoring. *Reliability: Theory and Applications*, 17(3): 328–339.

[7] Xavier, T., Jose, J. K. (2021). A study of stress-strength reliability using a generalization of power transformed half-logistic distribution. *Communications in Statistics-Theory and Methods*, 50(18): 4335–4351.

[8] Saber, M. M. and Khorshidian, K. (2021). Introduction to reliability for conditional stress-strength parametter. *Journal of sciencees, Islamic Republic of Iran*, 32(4):349–357.

[9] Bhattacharya, G. k. and Johnson, R. A. (1974). Estimation of reliability in a multicomponent stress-strngth model. *Journal of American statistical association*, 69(348): 966–970.

[10] Cetinkaya C. (2021). Reliability estimation of the stress-strength model with non-identical jointly type-II censored Weibull component strengths. *Statistical Computation and Simulations*, 91(14): 2917–2936.

[11] Goswami, A., Seal, B. (2022). Stress-strength Reliability for Equi-correlated Multivariate Normal and its estimation. *Reliability: Theory and Applications*, 17(4): 249–267.

[12] Awodutire, P. O., Xavier, T., Jose, J. K. (2022). Inferences on stress-strength reliability in multi-component system for type I generalized half-Logistic distribution. *Reliability: Theory and Applications*, 17(1): 18–32.

[13] Jana, N. and Bera, N. (2022). Interval estimation of multi-component stress-strength reliability based on inverse Weibull distribution. *Mathematics and Computers in Simulation*, 92(4):667–704.

[14] Kotb, M. S. and Raqab, M. Z. (2021). Estimation of multi-component stress-strength model based on modified Weibull distribution. *Statistical Papers*, 62(6): 2763–2797.

[15] Saini, S., Tomer, S. and Garg, R. (2022). On the reliability estimation of multi-component stress-strength model for Burr XII distribution using progressively first-failure censored samples. *Statistical Computation and Simulation*, 191: 95–119.

[16] Xia, Z. P., Yu, J. Y., Cheng, L. D., Liu, L. F. and Wang, W. M. (2009). Study on the breaking strength of Jute Fibers using modified Weibull distribution. *Compus-A*, 40:54–59.

[17] Chaturvedi, A., Kumar, N. and Kumar, K. (2018). Statistical inference for the reliability functions of a family of lifetime distributions based on progressive type II right censoring. *Statistica*, 78(1):81–101.

[18] Hassan, A. S., Naqy, H. F., Muhammad, H. Z. and Saad, M. S. (2020). Estimation of multi-component stress-strengt reliability following weibull distribution based on upper record values. *Journal of Taibah University for science*, 1(14):244–253.

[19] Rao, G. S., Mbwambo, S. and Pak, A. (2018). Estimation of multi-component stress-strength reliability from exponentiated inverse Rayleigh distribution. *Journal of Statistics and Management Systems*, 1–20.