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Analysis of projected profit in an M/M/K Encouraged Arrival Queueing model using Chi-Square Test

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Abstract

Nowadays, queues be seen in fast food restaurants and in all service-based businesses. This study is a mathematical analysis of such business firms with the help of Queueing Theory . The discounts and promotions entice customers to the firm and in this study such attracted customers are referred to as Encouraged Arrivals. The Chi-square test is used to determine the kind of encouraged arrival pattern that adheres to the data observed from a fastfood outlet. We introduce the encouraged arrivals in an M/M/k queueing model for the analysis of performance metrics. The performance metrics of the various encouraged arrival patterns are compared and the ideal one is chosen for the firm. The economic analysis shows that with encouraged arrivals, the cost associated with the time lost due to waiting is reduced gradually with increasing number of servers. Thus the firm increases its projected profit with encouraged arrivals. This study helps the entrepreneurs to decide the kind of discounts that would attract the customers simultaneously improving the firm's profit. Little's law is also verified.

Keywords: Encouraged Arrivals (eaη), Promotions, Expenses, Chi-square test, Frequency, Degrees of freedom (Dof), Little's formula (LF).

1. Introduction

Queuing theory is one of the earliest and most used quantitative analytic methodology. It involves the study of waiting queues. Everyday activities such as grocery shopping, gas purchases, bank deposits, and many organisations are affected by waiting lines. The Latin word cauda, which means "tail," is where the word queue first originated. Any service system will inevitably have customers wait in line to get services, making queue management a significant task. It is commonly referred to as the theory of congestion, which comes under the branch of operational research examines the connection between the level of demand for a system of services and the delays experienced by its users. The goal of the research of queues is to quantify the phenomenon of standing in queues by employing benchmark performance indicators like avereage length of the queue, queue's average waiting time and utilization.

Poor service patterns, queue management issues, unhelpful service staff attitudes, subpar

amenities and delivery are widespread in most service-based businesses including restaurants. All of these elements have an impact on customer relationships and overall efficiency. This study aims to analyse the data to increase the efficacy and economy of the organization. The analysis is carried out based on the primary data collected from a fastfood outlet [1].
The objective of this study is to reduce the amount of time consumers spend waiting in lines by using the M/M/k model in a fast food outlet. First in, first out (FIFO) queue discipline is utilised and we have taken into account eaη.

A comparison of network tools was performed to figure out the ideal values of performance metrics for flexible manufacturing systems in [2]. Tsarouhas [3] presented a theory to estimate the overall waiting times for processing each pizza at a workstation in a food manufacturing outlet. A model describing the psychological mechanisms that influence the connection between satisfaction and perceived wait time was suggested and tested by McGuire [4]. In [5] according to Mahmoud and Lu, numerous branches of research and engineering use queuing theory and Markov chains as common analysis, evaluation, and decision-making methods. For analysis of the steady-state and transient behaviour, real-world systems could be modelled. According to stochastic replacement intervals, special discounts are taken into account for a specific item from the supplier in [6]. A demand-satisfaction dilemma involving two products that are interchangeable was examined in [7]. The goal was to obtain the order quantity for each product that optimises the combined profit function. Location-inventory models were also taken into account in [8]. They employed a bi-level Markov process(MP) to create a stochastic inventory model. Som et al. [9] studied a multi- server queuing model with limited capacity for any organization encountering eaη and reverse reneging. Customers who are drawn into an organisation as a result of special offers (known as Encouraged Arrivals (eaη), a term coined by Som and Seth [10]. Jain et al. [11] described the idea of customer mobilisation and stated that a system attracts a new consumer by taking a look at its sizable customer base. eaη deals with the percentage change in clients as a result of promotions and discounts. Som et al. [12] studied a multi- server queuing model with limited capacity for any organization encountering eaη and reverse reneging. The Banking sector has become the most inevitable part of public units. Most banks make use of common queuing models. An M/M/1 queueing model is used to analyse the ATM's performance in [13]. The performance metrics of a toll plaza is analysed to find the traffic flow and to set up the system in an efficient way in [14]. A review on bulk arrivals [15] heps the researchers to model problems without congestion.

The introduction of the paper is given in Section 1. Section 2 provides the mathematical notation. The proposed mathematical model is given in section 3. In section 4, chi-square test is performed to check the fit of various encouraged arrivals. Analysis of performance metrics and economic analysis is given in section 5. Little's law is verified in section 6. Section 7 wraps up the paper with remarks and conclusion.

2. Mathematical Notation

The proposed queueing model uses the following notations.

- Encouraged arrivals are denoted by 'eaη' and frequency of arrivals is denoted by 'fa'. The arrivals happen sequentially according to a Poisson process with the parameter λ(1+eaη), where "eaη" denotes the change of percentage in the total count of clients estimated from observed data. For instance, if a firm previously offered discounts and a percentage change in the total count of clients was noticed of +10%, +30% or +50%, then eaη= 0.1, 0.3 or eaη=0.5, respectively.
- The model follows an exponentially distributed service times with parameter μ.
- Customers are served in the order of their arrival i.e., FCFS.
- The system has k parallel servers and there is no limit placed on the waiting space in the

system at any time 't'.
- Probability that there is no clients in the system is given by Pb_0.

3. Mathematical Model

The following diagram depicts the proposed model. We construct an M/M/k model to analyse the performance measures of the encouraged arrivals. The arrival rate is denoted by the parameter λ_n,

$$\lambda_n = \lambda(1+ea\eta), \text{ for all n}$$

Where n is the number of customers in the system.

The model follows an exponentially distributed service times with parameter μ and there are k servers in the model. If there are k or more clients, then it is understood that all the k servers are busy.

Hence, the service rate is given by $\mu_n = \begin{cases} n\mu & (1 \le n \le k) \\ k\mu & (n \ge k) \end{cases}$

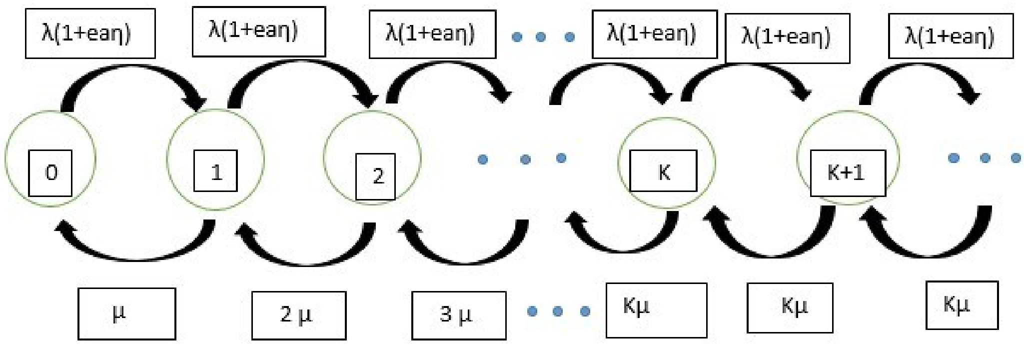

Figure 1. *Rate transition diagram of the proposed model*

4. Chi-square test to check the goodness of fit

In this study, Chi-square test is employed to determine the encouraged arrival pattern that adheres to the data observed from a fastfood outlet. This particular statistical test was created to examine the consistency between a set of actual frequencies and expected frequencies under the presumption of a hypothesis for the phenomenon under study. The test is employed to determine if two classification attributes are dependent on one another or not. The chi-square formula provides a measurement for the gap between reported and expected frequency which is given by

$$\chi^2 = \sum \frac{(fa-ea)^2}{ea} \tag{1}$$

Here 'fa' denotes the frequency of arrivals and 'ea' denotes the expected arrivals, in our case we consider ea as follows:
i.e., ea = λ (1+ eaη) when 20 % , 30% or 40% eaη are offered .

Case 1:
Consider the following hypothesis to check whether the 20% eaη fits the observed data [1]

Null hypothesis: The 20% eaη fits the distribution
Alternative hypothesis: The 20% eaη does not fits the distribution.

Table 1 : *Chi-square test to check the goodness of fit of 20% eaη*

Number of arrivals	Frequency of arrivals (fa = λ)	ea =λ (1+ 0.2)	$\frac{(fa-ea)^2}{ea}$
0	6	7.2	0.2
1	14	16.8	0.467
2	26	31.2	0.867
3	18	21.6	0.6
4	6	7.2	0.2
5	2	2.4	0.067

The ea should be greater than or equal to 5 for good approximation in a chi-square test. If any of the ea does not staisfy this condition then it has to be combined with some other ea until the condition is satisfied.

Table 1 thus becomes

Table : 1.1

Number of arrivals	Frequency of arrivals (fa = λ)	ea =λ (1+ 0.2)	$\frac{(fa-ea)^2}{ea}$
0	6	7.2	0.2
1	14	16.8	0.467
2	26	31.2	0.867
3	18	21.6	0.6
4 or 5	8	9.6	0.267
			$\sum\frac{(fa-ea)^2}{ea}$ = 2.401

$$\chi^2 = \sum\frac{(fa-ea)^2}{ea} = 2.401$$

The Dof is given by (number of observed data) – (number of parameters to be estimated) -1
Hence, Dof = 3
The value of chi-square at 5% level of significance for 3 Dof from the chi-square distribution table = 7.8147

The calculated value is lesser than table value therefore null hypothesis is accepted.
(i.e.,) The 20% eaη fits the distribution.

Table 2 : *To find arrival rate from the 20% eaη*

Number of arrivals	Frequency ofarrivals (fa = λ)	ea =λ (1+ 0.2)	Number of arrivals × ea
0	6	7.2	0
1	14	16.8	16.8
2	26	31.2	62.4
3	18	21.6	64.8
4	6	7.2	28.8
5	2	2.4	12
			Total = 184.8

Hence the system has the capacity N= 184.8
The interarrival time for the N clients is observed as 714 minutes [1]
The time taken to serve the N clients is observed as 1012 minutes [1]
Then $\lambda = \frac{N}{inter\ arrival\ Time\ for\ N\ clients} = \frac{184.8}{714} = 0.258$
$\mu = \frac{N}{Time\ taken\ for\ N\ clients\ to\ be\ served} = \frac{184.8}{1012} = 0.183$

Case 2:
Consider the following hypothesis to check whether the 30% eaη fits the observed data [1]

Null hypothesis: The 30% eaη fits the distribution
Alternative hypothesis: The 30% eaη does not fits the distribution

Table 3 : *Chi-square test to check the goodness of fit of 30% eaη*

Number of arrivals	Frequency of arrivals (fa = λ)	ea = λ (1+ 0.3)	$\frac{(fa-ea)^2}{ea}$
0	6	7.8	0.4153
1	14	18.2	0.9692
2	26	33.8	1.8
3	18	23.4	1.246
4 or 5	8	10.4	0.553
			$\sum\frac{(fa-ea)^2}{ea}$ = 4.9835

$$\chi^2 = \sum\frac{(fa-ea)^2}{ea} = 4.9835$$

The Dof is given by (number of observed data) – (number of parameters to be estimated) -1
Hence, Dof = 3
The value of chi-square at 5% level of significance for 3 Dof from the chi-square distribution table= 7.8147
The calculated value is lesser than table value therefore null hypothesis is accepted.
(i.e.,) The 30% eaη fits the distribution.

Table 4 : *To find arrival rate from the 30% eaη*

Number of arrivals	Frequency of arrivals (fa = λ)	ea =λ (1+ 0.3)	Number of arrivals × ea
0	6	7.8	0
1	14	18.2	18.2
2	26	33.8	67.6
3	18	23.4	70.2
4	6	7.8	31.2
5	2	2.6	13
			Total = 200.2

Let the system has the capacity N= 200.2
The interarrival time for the N clients is observed as 714 minutes [1]
The time taken to serve the N clients is observed as 1012 minutes [1]
Then $\lambda = \frac{N}{inter\ arrival\ Time\ for\ N\ clients} = \frac{200.2}{714} = 0.28$
$\mu = \frac{N}{Time\ taken\ for\ N\ clients\ to\ be\ served} = \frac{200.2}{1012} = 0.197$

Case 3:
Consider the following hypothesis to check whether the 40% eaη fits the observed data [1]

Null hypothesis: The 40% eaη fits the distribution
Alternative hypothesis: The 40% eaη does not fits the distribution.

Table 5: *Chi-square test to check the goodness of fit of 40% eaη*

Number of arrivals	Frequency of arrivals (fa = λ)	ea = λ (1+ 0.4)	$\frac{(fa-ea)^2}{ea}$
0	6	8.4	0.685
1	14	19.6	1.6
2	26	36.4	2.97
3	18	25.2	2.057
4 or 5	8	11.2	0.9142
			$\sum\frac{(fa-ea)^2}{ea}$ = 8.2262

$$\chi^2 = \sum\frac{(fa-ea)^2}{ea} = 8.2262$$

The Dof is given by (number of observed data) – (number of parameters to be estimated) -1
Hence, Dof = 3
The value of chi-square at 5% level of significance for 3 Dof from the chi-square distribution table= 7.8147
The calculated value is gretaer than table value therefore null hypothesis is not accepted.
(i.e.,) The 40% eaη does not fits the distribution.

Remarks :

- Using the chi-square test it is very easy to find the kind of eaη that fits the observed data[1].
- Thus we can infer from the chi-square test that 20% and 30% eaη fits the observed data with which we can further investigate the performance metrics to analyse which is more effective for the firm to increse the projected profit.

5. Analysis of performance metrics

Probability that there are no clients in the system is given by Pb_0

$$\sum_{j=0}^{\infty} Pb_j = 1$$

$$\sum_{j=0}^{k-1} Pb_n + \sum_{j=k}^{\infty} Pb_n = 1$$

$$\sum_{j=0}^{k-1}\frac{1}{j!}\left(\frac{\lambda(1+ea\eta)}{\mu}\right)^j Pb_0 + \frac{\left(\frac{\lambda(1+ea\eta)}{\mu}\right)^k}{k!\left(1-\frac{\lambda(1+ea\eta)}{k\mu}\right)} Pb_0 = 1$$

$$Pb_0 = \left(\sum_{j=0}^{k-1}\frac{1}{j!}\left(\frac{\lambda(1+ea\eta)}{\mu}\right)^j + \frac{\left(\frac{\lambda(1+ea\eta)}{\mu}\right)^k}{k!\left(1-\frac{\lambda(1+ea\eta)}{k\mu}\right)}\right)^{-1}$$

$$Pb_0 = \left(\frac{m^k}{k!(1-r)} + \sum_{n=0}^{k-1} \frac{m^n}{n!} \right)^{-1} \text{ where } m = \frac{\lambda(1+ea\eta)}{\mu}, \; r = \frac{m}{k} \tag{2}$$

Provided that $\frac{m}{k} < 1$

The expected number of clients in the queue is given by L_q

$$L_q = \sum_{j=k+1}^{\infty} (j-k) Pb_j$$

$$L_q = \left(\frac{m^k r}{k!(1-r)^2} \right) Pb_0 \tag{3}$$

Using LF, $L_q = \lambda W_q$

Where W_q is the expected waiting time in queue

$$w_q = \frac{\lambda}{L_q}$$

$$W_q = \frac{\lambda(1+ea\eta)}{\left(\frac{m^k r}{k!(1-r)^2} \right) Pb_0} \tag{4}$$

The expected number of clients in the system is given by $L_s = m + L_q$

The expected waiting time in the system is given by $W_s = \frac{1}{\mu} + W_q$

Expected time lost per day due to waiting = λ (1+ eaη) × W_q × 8 hours
(The working hours per day is taken as 8 hours) [1]

Expected Cost associated with lost time = W_q × Rs. 50
(The cost associated with time lost by waiting is taken as Rs. 50) [1]

Table 6: *Comparison of performance measures between 20% and 30% eaη*

Number of servers	2	3	4
L_q for 20% eaη	1.97	0.3862	0.0948
L_q for 30% eaη	2.033	0.3968	0.0978
W_q for 20% eaη	7.635	1.496	0.367
W_q for 30% eaη	7.26	1.417	0.3492
L_s for 20% eaη	3.379	1.7952	1.503
L_s for 30% eaη	3.454	1.817	1.5188
W_s for 20% eaη	13.099	6.96	5.831
W_s for 30% eaη	12.33	6.493	5.425

Comparing all the performance metrics of eaη with respect to number of servers in Table 6.

Remarks:

- From comparing all the performance metrics of 20% and 30% eaη we observe the following
- In case of expected total count of clients both in queue as well as system (i.e.,) L_q and L_s respectively, 30% eaη increases the count when compared with 20% eaη with varying number of servers.
- In case of waiting time in queue as well as system (i.e.,) W_q and W_s 30% eaη reduces the time spent in waiting when compared with 20% eaη with varying number of servers.

- Therefore we come to a conclusion that in case of performance metrics (i.e.,) expected count of clients and expected waiting time 30% eaη increases the size and at the same time reduces the waiting time.

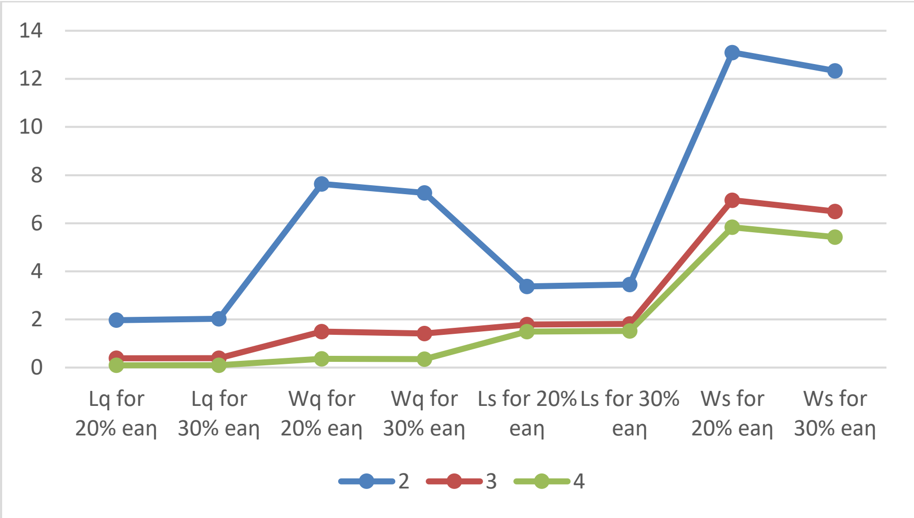

Figure 2. *Comparing performance metrics between 20% and 30% eaη with respect to number of servers*

5.1. Comparison of expected time lost per day with respect to number of servers

Now, the expected time lost per day for poisson, 20% eaη and 30% eaη with respect to number of servers are compared.

Expected time lost per day due to waiting = λ (1+ eaη) × W_q × 8 hours
(The working hours per day is taken as 8 hours) [1]

Table 7 : *Calculating Expected lost time for poisson arrival*

Number of servers	Expected lost time for poisson arrival	Expected lost time for 20% eaη	Expected lost time for 30% eaη
2	18.7719	15.758	16.2624
3	3.1612	3.0877	3.174
4	0.775	0.7574	0.7822

Calculating the Expected lost time for poisson arrival with respect to number of servers in Table 7.

Remarks:
We infer from the Table 7 that the lost time for poisson is more than that of 20% and 30% eaη. Any firm's aim is to reduce the waiting time thus administering eaη helps us to reduce the amount of time lost in waiting for service.

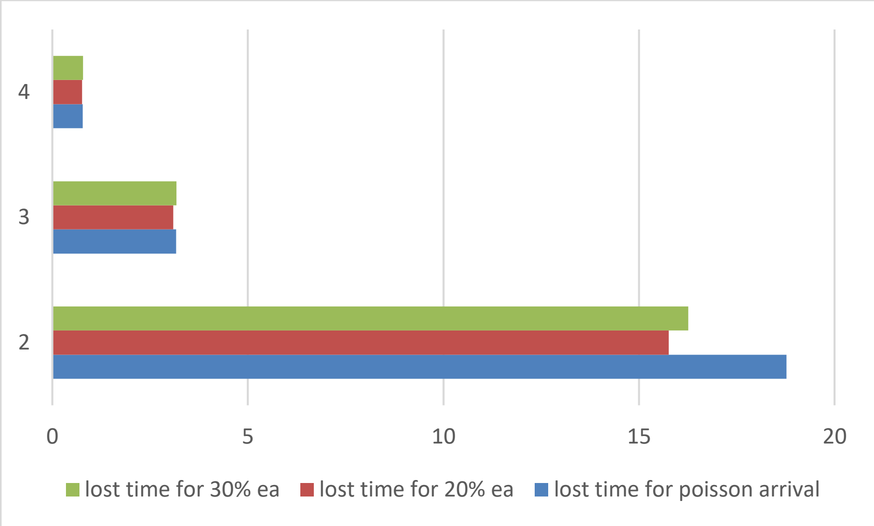

Figure 3. *Variation in Expected lost time per day with respect to number of servers*

5.2. Comparison of cost associated with lost time per day

Comparing the cost associated with lost time per day for poisson, 20% eaη and 30% eaη with respect to number of servers.

Expected Cost associated with lost time = $W_q \times$ Rs. 50

(The cost associated with time lost by waiting is taken as Rs. 50) [1]

Table 8 : *Calculating cost associated with lost time for 20% eaη*

Number of servers	λ (1+ 0.2)	Cost associated with lost time for poisson arrival (in Rs.)	Cost associated with lost time for 20% eaη (in Rs.)	Savings with 20% eaη (in Rs.)
2	0.258	938.597	787.932	150.665
3	0.258	158.06	154.387	3.63
4	0.258	38.875	37.87	1.005

Calculating the cost associated with lost time for 20% eaη and also the cost saved with 20% eaη when compared with λ [1] with respect to number of servers in Table 8.

Table 9 : *Calculating cost associated with lost time for 30% eaη*

Number of servers	λ (1+ 0.3)	Cost associated with lost time for poisson arrival (in Rs.)	Cost associated with lost time for 30% eaη (in Rs.)	Savings with 30% eaη (in Rs.)
2	0.28	938.597	813.12	125.477
3	0.28	158.06	158.704	-0.644
4	0.28	38.875	39.11	-0.235

Calculating the cost associated with lost time for 30% eaη and also the cost saved with 30% eaη when compared with λ [1] with respect to number of servers in Table 9.

Remarks:
We infer from the Tables 7,8 and 9 that the cost associated with lost time due to waiting is more for λ [1] than 20% and 30% eaη. While comparing the cost saved with eaη and λ, we observe that among the eaη 20% yields better gain than 30% for the firm which is our primary goal. Thus we conclude that 20% eaη is the best to be offered by the firm as it would increase the organisation's profit.

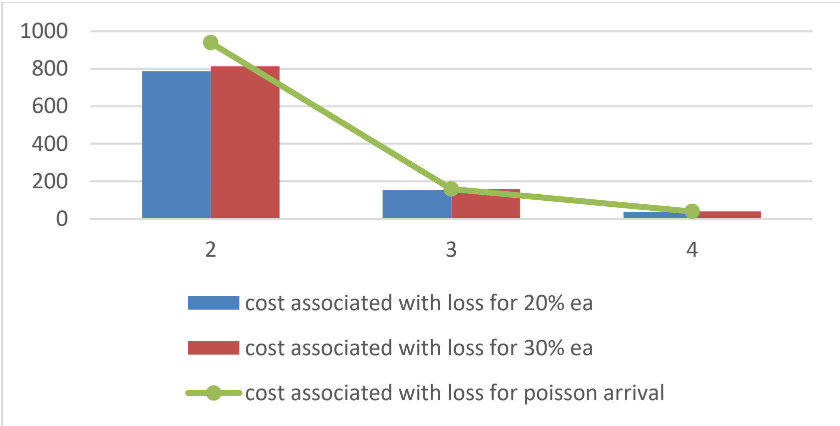

Figure 4. *Variation in cost associated with lost time with respect to number of servers*

6. Verification of Little Formula(LF)

Using LF, $L_s = \lambda W_s$

Table 10: *Verification of LF for 20% eaη with respect to system*

Number of servers	λ(1+ eaη)	L_s	W_s	$L_s = \lambda W_s$
2	0.258	3.379	13.099	3.379 = (0.258)(13.099) verified
3	0.258	1.7952	6.96	1.7952 = (0.258)(6.96) verified
4	0.258	1.503	5.831	1.503 = (0.258)(5.831) verified

Table 11: *Verification of LF for 30% eaη with respect to system*

Number of servers	λ(1+ eaη)	L_s	W_s	$L_s = \lambda W_s$
2	0.28	3.454	12.33	3.454 = (0.28)(12.33) Verified
3	0.28	1.817	6.493	1.817 = (0.28)(6.493) Verified
4	0.28	1.5188	5.425	1.5188 = (0.28)(5.425) verified

The LF is verified for both 20% and 30% eaη in terms of queue as well as system size which shows that the system is well balanced.

7. Conclusion

- Using the chi-square test, it is very easy to find the kind of eaη that fits the observed data[1].
- We infer from the chi-square test that, 20% and 30% eaη fits the observed data with which we can further investigate the performance metrics to analyse and is found to be more effective for the firm to increse the profit.
- While comparing the expenses with eaη and λ, we observe that among the eaη 20% yields better gain than 30% for the firm which is our primary goal.
- We conclude that 20% eaη is the best to be offered by the firm as it would increase the organisation's projected profit.

We infer from the model created that adding one additional server will assist shorten the time clients wait in line and lower the cost associated with it. In order to decrease the time customers must wait to receive services and to lower the expense associated with waiting, we advise the organisation to raise the number of servers to at least three.

With encouraged arrivals the results show that the cost associated with lost time reduces gradually with increasing number of servers than poisson arrivals [1] and we see that 20% eaη is ideal for the proposed model as 20% eaη yields better projected profit . Therefore by using the chi-square test we analysed the kind of encouraged arrival pattern that adheres to the firm simultaneously increasing the firm's projected profit. Thus this study helps the entrepreneurs to decide the kind of discounts that would attract the customers simultaneously improving the firm's profit.

Declaration of conflicting interest: The authors declare that there is no conflict of interest.

References

[1] Igbinoba, J.O., Sule, O.K., Ugboya, A.P. and Akinwunmi, M. (2019). Application Of Queueing Theory In Optimization Of Service Process, A Case Study Of Gt Plaza Fast Food, *Journal of Multidisciplinary Engineering Science and Technology (JMEST)*, 6(1):2458-9403.

[2] Ullah, H. (2011). Petri net versus queuing theory for evaluation of FMS, *Assembly Automation*, 31(1):29-37.

[3] Tsarouhas, P.H. (2011). A comparative study of performance evaluation based on field failure data for food production lines, *Journal of Quality in Maintenance Engineering*, 17(1):26-39.

[4] McGuire, K.A., Kimes, S.E., Lynn, M., Pullman, M.E. and Lloyd, R.C. (2010). A framework for evaluating the customer wait experience, *Journal of Service Management*, 21(3):269-290.

[5] Kunfeng Lu., Jiqing Qiu. and Mahmoud, S. (2011). Robust passive filter design for uncertain singular stochastic Markov jump systems with mode-dependent time delays, 2nd International Conference on Intelligent Control and Information Processing, *IEEE*.

[6] Karimi-Nasab, M. and Konstantaras, I. (2013). An inventory control model with stochastic replenishment interval and special sale offer, *European Journal of Operational Research*, 227(1):81-87.

[7] Krommyda, I.P., Skouril, K. and Konstantaras, I. (2015).Optimal ordering quantities for substitutable products with stock-dependent demand,*Applied Mathematical Modelling*, 39(1):147-164.

[8] Rashid, R., Hoseini, S.F. and Gholamian, M.R. (2015). Application of queuing theory in production-inventory optimization. *Journal of Industrial Engineering International*, 11:485–494

[9] Som, B. K. and Seth, S. (2017). An M/M/1/N Queuing system with Encouraged Arrivals, *Global Journal of Pure and Applied Mathematics*, 17:3443-3453.

[10] Som, B.K. (2020). Multi-server Finite Waiting-space Encouraged Arrival Queuing System with Reverse Reneging, *Jagannath University Research Journal*, 1(1):2582-6263.

[11] Jain, N.K., Kumar, R. and Som, B.K. (2014). An M/M/1/N Queuing system with reverse balking, *American Journal of Operational Research*, 4(2):17-20.

[12] Patel, B. and Bhathawala, P. (2012). Case study for bank ATM queuing model. International Journal of Engineering Research and Applications (IJERA), 2(5).

[13] Kim, B.J.(2011). Conceptualization of traffic flow for designing Toll Plaza configuration: A case study using simulation with estimated traffic, *International Journal of Industrial Engineering*, 18(1):51-57.

[14] Niranjan, S.P. and Indhira, K. (2016). A review on classical bulk arrival and batch service queueing model, *International Journal of Pure and Applied Mathematics* , 106(8):45-51.

[15] Donald Gross, John F. Shortle, James M. Thompson. Fundamentals of Queueing Theory, Wiley Series in Probability and Statistics, 2018.

[16] Medhi J. Stochastic models in Queueing theory, 2003.

[17] Veerarajan T. Probability, statistics and random processes with Queueing Theory and Queueing Networks, 2009.

[18] Taha A. Operations research an introduction, 2002.

[19] Ajay kumar Sharma . Queuing theory approach with queuing model: A study, 2013.

[20] Daniel A. – Introduction to Queuing theory, 1995.