MOMENTS OF GENERALIZED RECORD VALUES FROM MODIFIED FRÉCHET DISTRIBUTION AND ITS CHARACTERIZATION

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Abstract

The aim of this paper is to introduce the relations for moments and characterizing results for the newly introduced modified Fréchet distribution based on generalized record values. Here, we used an ordered random variable approach like generalized record values for generating the results. We have established the recurrence relations for single and product moments of generalized record values from modified Fréchet distribution. These relations are also deduced for the lower record values and some specific distributions, which are the special cases of modified Fréchet distribution. Further, the characterization results for this distribution have been established by using recurrence relations for single and product moments and conditional expectation of a function of generalized record values and truncated moments.

Keywords: Order statistics, generalized record values, modified Fréchet distribution, single moments, product moments, recurrence relations and characterization.

1. Introduction

The modified Fréchet distribution is an extension of the Fréchet distribution which was introduced by Tablada and Cordeiro [23] and pointed out that this distribution is quite effective to provide the best fits for real data sets. Since the results on real life data compared with other known distributions such as Fréchet, exponentiated Fréchet, Marshall–Olkin Fréchet, exponentiated Weibull, revealed that modified Fréchet distribution provides a better fit for modeling real life data.

A random variable X follows modified Fréchet distribution, if it's probability density function pdf is of the form

$$f(x)=\frac{(\lambda+\alpha x)}{x}\left(\frac{\beta}{x}\right)^{\lambda}\exp\left[-\alpha x-\left(\frac{\beta}{x}\right)^{\lambda}e^{-\alpha x}\right],\quad x\geq 0,\quad \alpha,\ \beta,\ \lambda>0 \tag{1}$$

with the distribution function (*df*)

$$F(x) = \exp\left[-\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x}\right], \quad x \ge 0, \quad \alpha, \ \beta, \quad \lambda > 0. \tag{2}$$

Where , β and λ are shape parameters.

Note that $f(x)$ and $F(x)$ satisfy the relation.

$$f(x) = \frac{(\lambda + \alpha x)}{x}[-\ln F(x)]F(x). \tag{3}$$

The Fréchet and standard Gumbel distributions are the special cases of the modified Fréchet distribution, when $\lambda = 0$ and $\lambda = 0$, $\alpha = 1$ respectively.

Initially, Chandler [8] was the first who laid down the concept of record values inspired by the extreme weather conditions. As a result, he designed the model for successive extremes values in a sequence of identically independently distributed (*iid*) continuous random variables. Dziubdziela and Kopociński [9] have generalized the concept of record values by choosing random variables of more generalized nature and these random variables are called the k −th record values. Later, the record values defined by Dziubdziela and Kopociński [9] have been called as generalized record values by Minimol and Thomas [15], since the r − th member of the sequence of the ordinary record values is also known as the r − th record value. Setting $k = 1$, we obtain ordinary record statistics.

Generally, the record values means the values which are not acquired before, e.g., fastest century in the one day cricket match, the longest winning streak in basketball, the world record in high jumping, the lowest time to cover a fixed distance in freestyle swimming and so on. The observation which is greater (or less) than the previous all observations is known as the record value. Record values arise naturally in many real life applications involving data relating to weather, sports, economics and life-tests.

For more details on the applications of record values, see Ahsanullah [1], Ahsanullah and Nevzorov [2], Arnold et. al. [5].

Let $\{X_n, n \ge 1\}$ be a sequence of independently identically distributed (*iid*) random variables with *df* $F(x)$ and *pdf* $f(x)$. The r − th order statistics of a random sample $X_1, X_2, ..., X_n$ is denoted by $X_{r:n}$. For fixed $k \ge 1$, we define the sequence $\{L_k(n), n \ge 1\}$ of k − th record times of $\{X_n, n \ge 1\}$ as follows:

$$L_k(1) = 1$$

$$L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}.$$

The sequence $\{Z_n^{(k)}, \ n \ge 1\}$ with $Z_n^{(k)} = X_{k:L_k(n)+k-1}$, $n = 1, 2, \ldots$ is called the sequence of k − th lower record values of $\{X_n, n \ge 1\}$. For convenience, we shall also take and $Z_0^{(k)} = 0$. Note that for $k = 1$ we have $Z_n^{(1)} = X_{L(n)}$, $n \ge 1$. Then *pdf* of $Z_n^{(k)}$ and the joint *pdf* $Z_m^{(k)}$ and $Z_n^{(k)}$ are as follows:

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x), \quad n \ge 1 \tag{4}$$

$$f_{Z_m^{(k)}, Z_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}[-\ln F(x)]^{m-1} \frac{f(x)}{F(x)}[\ln F(x) - \ln F(y)]^{n-m-1}$$

$$\times [F(y)]^{k-1} f(y), \quad x > y, \ 1 \le m < n, \ n \ge 2. \tag{5}$$

The conditional *pdf* of $Z_n^{(k)}$ given $Z_m^{(k)} = x$ as given

$$f_{Z_n^{(k)} \mid Z_m^{(k)}}(y \mid x) = \frac{k^{n-m}}{(n-m-1)!}[\ln F(x) - \ln F(y)]^{n-m-1}\left(\frac{F(y)}{F(x)}\right)^{k-1}\frac{f(y)}{F(x)}, \quad y < x. \tag{6}$$

For some recent developments on generalized record values with special reference to those arising from NH, exponentiated Rayleigh, Kappa distribution, additive-Weibull lifetime, Power function, extended Erlang-truncated exponential, Kumaraswamy-log-logistic, Weibull-Rayleigh, Weibull-power function, Fréchet distributions see, Alam et al. [4], Khan et al. ([12], [13]), Khan et al. [14], MirMostafaee et al. [16], Paul [17], Singh and Khan [19], Singh et al. ([20], [21], [22]) Thomas and Paul [24], etc. In this paper we mainly studied the generalized lower record values arising from the modified Fréchet distribution.

The plots represent the shapes of the *pdf* of lower record values, arises from the modified Fréchet distribution.

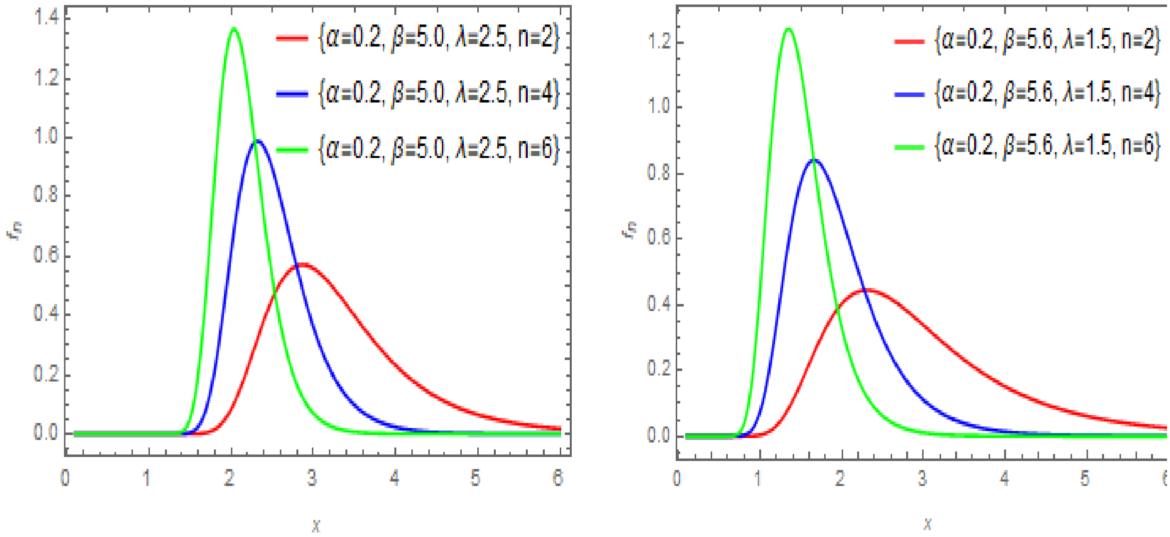

Figure. Plots of the *pdf* of lower record values from modified Fréchet distribution for selected values of parameters.

2. Relations for single moments

Theorem 2.1. For the modified Fréchet distribution given in (2) and $1 \le k \le n$, $j = 0, 1, \ldots$

$$E(Z_{n+1}^{(k)})^j - E(Z_n^{(k)})^j = \frac{j\alpha}{(j+1)\lambda}\{E(Z_n^{(k)})^{j+1} - E(Z_{n+1}^{(k)})^{j+1}\} - \frac{j}{n\lambda}E(Z_n^{(k)})^{j+1} \tag{7}$$

Consequently for $n \ge 1$, $1 \le k \le n$ and $j = 0, 1, \ldots$

$$jE(Z_{n+1}^{(k)})^{j+1} = jE(Z_1^{(k)})^{j+1} + \frac{j+1}{\alpha}\sum_{p=1}^{n}\frac{(p\lambda - j)}{p}E(Z_p^{(k)})^j - \frac{\lambda(j+1)}{\alpha}\sum_{p=2}^{n+1}E(Z_p^{(k)})^j. \tag{8}$$

Proof. From (3) and (4), we get

$$E(Z_n^{(k)})^j = \frac{\lambda k^n}{(n-1)!}\int_0^\infty x^{j-1}[-\ln F(x)]^n [F(x)]^k\, dx + \frac{\alpha k^n}{(n-1)!}\int_0^\infty x^j[-\ln F(x)]^n [F(x)]^k\, dx. \tag{9}$$

In view of Bieniek and Szynal [7], note that

$$E(Z_n^{(k)})^j - E(Z_{n-1}^{(k)})^j = -j\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^k\, dx.$$

Therefore,

$$E(Z_{n+1}^{(k)})^j - E(Z_n^{(k)})^j = -j\frac{k^n}{n!}\int_0^\infty x^{j-1}[-\ln F(x)]^n [F(x)]^k \, dx .$$

On substituting in (9), we get

$$E(Z_n^{(k)})^j = \frac{n\lambda}{j}\{E(Z_n^{(k)})^j - E(Z_{n+1}^{(k)})^j\} + \frac{n\alpha}{(j+1)}\{E(Z_n^{(k)})^{j+1} - E(Z_{n+1}^{(k)})^{j+1}\} .$$

On rewriting above expression, we derive the recurrence relation in (7). Then, by repeatedly applying the recurrence relation in (7), we simply derive the recurrence relation in (8).

Remark 2.1. For $k=1$ in (7), the recurrence relation for single moments of lower record values from the modified Fréchet distribution given as

$$E(X_{L(n+1)})^j - E(X_{L(n)})^j\} = \frac{j\alpha}{(j+1)\lambda}\{E(X_{L(n)})^j - E(X_{L(n+1)})^j\} - \frac{j}{n\lambda}E(X_{L(n)})^j .$$

Remark 2.2. Putting $k=1$, $\lambda=0$, $\alpha=1$ in (7), we deduced the recurrence relation for single moments of lower record values from the standard Gumbel distribution as obtained by Balakrishnan et al. [6].

Remark 2.3. Setting $\alpha=0$ in (7), we deduced the recurrence relation for generalized record values from inverse Weibull distribution as established by Pawlas and Szynal [18] for replacing n by $n-1$.

Table I Moments of lower record values

n	$\alpha=1.5,\ \beta=4$			$\alpha=2,\ \beta=5$		
	$\lambda=1.5$	$\lambda=2.5$	$\lambda=3.5$	$\lambda=1.5$	$\lambda=2.5$	$\lambda=3.5$
1	1.45394	1.78177	2.02126	1.32077	1.68029	1.95621
2	1.06968	1.44242	1.71475	1.00841	1.39679	1.69422
3	0.90020	1.28891	1.57416	0.86681	1.26561	1.57166
4	0.79618	1.19248	1.48489	0.77828	1.18208	1.49298
5	0.72301	1.12323	1.42020	0.71510	1.12151	1.43553

Table II Variances of lower record values

n	$\alpha=1.5,\ \beta=4$			$\alpha=2,\ \beta=5$		
	$\lambda=1.5$	$\lambda=2.5$	$\lambda=3.5$	$\lambda=1.5$	$\lambda=2.5$	$\lambda=3.5$
1	0.28211	0.21581	0.17375	0.18045	0.14615	0.12349
2	0.08176	0.06541	0.05419	0.05577	0.04695	0.04059
3	0.04147	0.03480	0.02952	0.02950	0.02579	0.02271
4	0.02588	0.02267	0.01960	0.01900	0.01718	0.01535
5	0.01798	0.01638	0.01440	0.01355	0.01263	0.01142

3. Relations for product moments

Theorem 3.1. For the modified Fréchet distribution given in (2) and $n \ge 1$, $m \ge k$, $i, j = 0,1,\ldots$

$$E[(Z_{m+1}^{(k)})^{i+j}] - E(Z_m^{(k)})^i (Z_{m+1}^{(k)})^j$$

$$= \frac{i\alpha}{(i+1)\lambda}\{E[(Z_m^{(k)})^{i+1}(Z_{m+1}^{(k)})^j] - E(Z_{m+1}^{(k)})^{i+j+1}\} - \frac{i}{m\lambda}E[(Z_m^{(k)})^{i+j}] . \quad (10)$$

and for $1 \le m < n$, $n \ge 2$, $i, j = 0,1,\ldots$

$$E[(Z_{m+1}^{(k)})^i (Z_n^{(k)})^j] - E(Z_m^{(k)})^i (Z_n^{(k)})^j$$

$$= \frac{i\alpha}{(i+1)\lambda} \{E[(Z_m^{(k)})^{i+1} (Z_n^{(k)})^j] - E(Z_{m+1}^{(k)})^{i+1} (Z_n^{(k)})^j\} - \frac{i}{m\lambda} E[(Z_m^{(k)})^i (Z_n^{(k)})^j]. \quad (11)$$

Proof. From (3) and (5), we have

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = \frac{\lambda\, k^n}{(m-1)!(n-m-1)!} \int_0^\infty \int_y^\infty x^{i-1} y^j [-\ln F(x)]^m [\ln F(x) - \ln F(y)]^{n-m-1}$$

$$\times [F(y)]^{k-1} f(y) dx\, dy + \frac{\alpha\, k^n}{(m-1)!(n-m-1)!} \times \int_0^\infty \int_y^\infty x^i y^j [-\ln F(x)]^m$$

$$\times [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y) dx\, dy,$$

$$= \frac{\lambda\, k^n}{(m-1)!(n-m-1)!} \int_0^\infty y^j [F(y)]^{k-1} f(y) I_{i-1}(x,y) dy$$

$$+ \frac{\alpha\, k^n}{(m-1)!(n-m-1)!} \int_0^\infty y^j [F(y)]^{k-1} f(y) I_i(x,y) dy, \quad (12)$$

where

$$I_b(x,y) = \int_y^\infty x^b [-\ln F(x)]^m [\ln F(x) - \ln F(y)]^{n-m-1} dx.$$

Integrating by part, taking x^b for integration and rest of the part for differentiation, we get

$$I_b(x,y) = \frac{m}{b+1} \int_y^\infty x^{b+1} [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-1} dx$$

$$- \frac{(n-m-1)}{b+1} \int_y^\infty x^{b+1} [-\ln F(x)]^m \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-2} dx.$$

Substituting in (12), we get

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = \frac{m\lambda\, k^n}{i\,(m-1)!(n-m-1)!} \int_0^\infty \int_y^\infty x^i y^j [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)}$$

$$\times [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y) dx\, dy - \frac{\lambda\, k^n}{i\,(m-1)!(n-m-2)!}$$

$$\times \int_0^\infty \int_y^\infty x^i y^j [-\ln F(x)]^m \frac{f(x)}{F(y)} [\ln F(x) - \ln F(y)]^{n-m-2} [F(y)]^{k-1} f(y) dx\, dy$$

$$+ \frac{m\alpha\, k^n}{(i+1)(m-1)!(n-m-1)!} \int_0^\infty \int_y^\infty x^{i+1} y^j [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)}$$

$$\times [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y) dx\, dy$$

$$- \frac{\alpha\, k^n}{(i+1)(m-1)!(n-m-2)!} \int_0^\infty \int_y^\infty x^{i+1} y^j [-\ln F(x)]^m \frac{f(x)}{F(x)}$$

$$\times [\ln F(x) - \ln F(y)]^{n-m-2} [F(y)]^{k-1} f(y) dx\, dy.$$

After simplification, we obtain the required result as given in (11).

Proceeding in a similar manner for the case $n=m+1$, the recurrence relation given in (10) can easily be established.

On can also note that Theorem 2.1. can be deduced from Theorem 3.1. by putting $j=0$.

Remark 3.1. Putting $k=1$ in (11), the recurrence relations for product moments of lower record values is deduced for the modified Fréchet distribution in the form

$$E[(X_{L(m+1)})^i (X_{L(n)})^j] - E[(X_{L(m)})^i (X_{L(n)})^j] = -\frac{i}{m\lambda} E[(X_{L(m)})^i (X_{L(n)})^j]$$

$$+\frac{i\alpha}{(i+1)\lambda}\{E[(X_{L(m)})^{i+1} (X_{L(n)})^j] - E[(X_{L(m+1)})^{i+1} (X_{L(n)})^j]\}.$$

Remark 3.2. Setting $k=1$, $\lambda=0$, $\alpha=1$ in (11), we get the recurrence relations for product moments of lower record values from the standard Gumbel distribution as obtained by Balakrishnan et al. [6].

Remark 3.3. Assuming $\alpha=0$ in (11), the recurrence relations for product moments of generalized record values is deduced for inverse Weibull distribution as established by Pawlas and Szynal [18].

4. Characterizations

Theorem 4.1. If k and j be are positive integers. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (2) is that

$$E(Z_{n+1}^{(k)})^j - E(Z_n^{(k)})^j = \frac{j\alpha}{(j+1)\lambda}\{E(Z_n^{(k)})^{j+1} - E(Z_{n+1}^{(k)})^{j+1}\} - \frac{j}{n\lambda} E(Z_n^{(k)})^j . \quad (13)$$

Proof. The necessary part follows from Theorem 2.1. On the other hand if the recurrence relation (13) is satisfied, then on using Bieniek and Szynal [7], we have

$$\frac{k^n}{(n-1)!}\int_0^\infty x^j [-\ln F(x)]^{n-1} [F(x)]^{k-1} f(x)dx = \frac{\lambda k^n}{(n-1)!}\int_0^\infty x^{j-1} [-\ln F(x)]^n [F(x)]^k dx$$

$$+\frac{\alpha k^n}{(n-1)!}\int_0^\infty x^j [-\ln F(x)]^n [F(x)]^k dx ,$$

which implies

$$\int_0^\infty x^j [-\ln F(x)]^{n-1} [F(x)]^k \left\{\frac{f(x)}{F(x)} - \frac{(\lambda+\alpha x)}{x}[-\ln F(x)]\right\} dx = 0 .$$

Now applying a generalization of the Müntz-Szász theorem (see for example Hwang and Lin [11]) to above expression, we get

$$f(x) = \frac{(\lambda+\alpha x)}{x}[-\ln F(x)]F(x) ,$$

which proves the sufficiency part.

Theorem 4.2. For a positive integer $k \ge 1$ and let i, j are non-negative integers, a necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$E[(Z_{m+1}^{(k)})^i (Z_n^{(k)})^j] - E(Z_m^{(k)})^i (Z_n^{(k)})^j = -\frac{i}{m\lambda} E[(Z_m^{(k)})^i (Z_n^{(k)})^j]$$

$$+\frac{i\alpha}{(i+1)\lambda}\{E[(Z_m^{(k)})^{i+1} (Z_n^{(k)})^j] - E(Z_{m+1}^{(k)})^{i+1} (Z_n^{(k)})^j\} . \quad (14)$$

Proof. The necessary part follows from Theorem 3.1. On the other hand if the relation in (14) is satisfied, then (14) can be written as

$$\frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_y^\infty x^i y^j[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[\ln F(x)-\ln F(y)]^{n-m-1}$$
$$\times[F(y)]^{k-1}f(y)dx\,dy=\frac{\lambda m k^n}{i(m-1)!(n-m-1)!}\int_0^\infty\int_y^\infty x^i y^j[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}$$
$$\times[\ln F(x)-\ln F(y)]^{n-m-2}[F(y)]^{k-1}f(y)\left\{\ln F(x)-\ln F(y)-\frac{(n-m-1)[-\ln F(x)]}{m}\right\}dxdy$$
$$+\frac{\alpha m k^n}{(i+1)(m-1)!(n-m-1)!}\int_0^\infty\int_y^\infty x^{i+1}y^j[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[\ln F(x)-\ln F(y)]^{n-m-2}$$
$$\times[F(y)]^{k-1}f(y)\left\{\ln F(x)-\ln F(y)-\frac{(n-m-1)[-\ln F(x)]}{m}\right\}dxdy\,. \tag{15}$$

Let

$$h(x,y)=-\frac{1}{m}[-\ln F(x)]^m[\ln F(x)-\ln F(y)]^{n-m-1}$$

Differentiating both sides with respect to x, we get

$$\frac{\partial}{\partial x}h(x,y)=[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[\ln F(x)-\ln F(y)]^{n-m-2}$$
$$\times\left\{\ln F(x)-\ln F(y)-\frac{(n-m-1)[-\ln F(x)]}{m}\right\}.$$

Thus, (15) can be written as

$$\frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_y^\infty x^i y^j[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y)dx\,dy$$
$$=\frac{\lambda m k^n}{i(m-1)!(n-m-1)!}\int_0^\infty y^j[F(y)]^{k-1}f(y)I_i(x,y)\,dy$$
$$+\frac{\alpha m k^n}{(i+1)(m-1)!(n-m-1)!}\int_0^\infty y^j[F(y)]^{k-1}f(y)I_{i-1}(x,y)\,dy\,, \tag{16}$$

where

$$I_b(x,y)=\int_y^\infty x^b\frac{\partial}{\partial x}h(x,y)\,dx\,.$$

Integrating by parts, treating x^b for differentiation and rest of the part for integration, we get

$$I_b(x,y)=-b\int_y^\infty x^{b-1}h(x,y)\,dx\,,$$

On substituting $I_b(x,y)$ in (16), yields

$$\frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_y^\infty x^i y^j[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[\ln F(x)-\ln F(y)]^{n-m-1}$$
$$\times[F(y)]^{k-1}f(y)\left\{\frac{f(x)}{F(x)}-\frac{(\lambda+\alpha x)}{x}[-\ln F(x)]\right\}dx\,dy=0\,.$$

Now applying a generalization of the Müntz-Szász theorem (see for example Hwang and Lin [11]), we get

$$f(x)=\frac{(\lambda+\alpha x)}{x}[-\ln F(x)]F(x)\,.$$

Hence the sufficiency part proved.

Theorem 4.3. Let X be an absolutely continuous non-negative random variable having df $F(x)$, with $F(0)=0$ and $0 \le F(x) \le 1 \;\; \forall \;\; 0 < x < \infty$, then

$$E[\xi(Z_n^{(k)}) \mid Z_m^{(k)} = x] = -\ln F(x) + \frac{n-m}{k} \tag{17}$$

if and only if

$$F(x) = \exp\left[-\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x}\right], \quad x > 0\,, \quad \alpha, \;\; \beta, \;\; \lambda > 0\,,$$

where

$$\xi(y) = \left(\frac{\beta}{y}\right)^{\lambda} e^{-\alpha y}\,.$$

Proof. From (6), we have

$$E[\xi(Z_n^{(k)}) \mid (Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}\int_0^x \left(\frac{\beta}{y}\right)^{\lambda} e^{-\alpha y}[\ln F(x) - \ln F(y)]^{n-m-1}\left(\frac{F(y)}{F(x)}\right)\frac{f(y)}{F(x)}dy\,.$$

By setting $z = \dfrac{F(y)}{F(x)}$, we have

$$E[\xi(Z_n^{(k)}) \mid (Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}\int_0^1 [-\ln F(x) - \ln z][-\ln z]^{n-m-1} z^{k-1} dz\,. \tag{18}$$

Now (17) can be seen in view of (Gradshteyn and Ryzhik [10], p-551)

$$\int_0^1 (-\ln x)^{\mu-1} x^{\upsilon-1} dx = \frac{\Gamma\mu}{\upsilon^{\mu}}, \quad \mu > 0,\; \upsilon > 0\,.$$

To prove the sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!}\int_0^x \left(\frac{\beta}{y}\right)^{\lambda} e^{-\alpha y}[\ln F(x) - \ln F(y)]^{n-m-1}[F(y)]^{k-1} f(y)dy = [F(x)]^k g_{n\mid m}(x)\,, \tag{19}$$

where

$$g_{n\mid m}(x) = -\ln F(x) + \frac{n-m}{k}\,.$$

Differentiating both sides of (19) with respect to x, we get

$$\frac{k^{n-m} f(x)}{(n-m-2)!F(x)}\int_0^x \left(\frac{\beta}{y}\right)^{\lambda} e^{-\alpha y}[\ln F(x) - \ln F(y)]^{n-m-2}$$

$$\times [F(y)]^{k-1} f(y)dy = g'_{n\mid m}(x)[F(x)]^k + k\, g_{n\mid m}(x)[F(x)]^{k-1} f(x)$$

or

$$k\, g_{n\mid m+1}(x)[F(x)]^{k-1} f(x) = g'_{n\mid m}(x)[F(x)]^k + k\, g_{n\mid m}(x)[F(x)]^{k-1} f(x)\,.$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{g'_{n\mid m}(x)}{k[g_{n\mid m+1}(x) - g_{n\mid m}(x)]} = \frac{(\lambda + \alpha x)}{x}\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x}\,,$$

where

$$g'_{n\,|m}(x) = -\frac{(\lambda+\alpha x)}{x}\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x}$$

$$g_{n\,|m+1}(x) - g_{n|m}(x) = -\frac{1}{k}.$$

Now integrating both the sides with respect to x, we get

$$\ln F(x) = -\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} + \log C$$

which implies

$$F(x) = C\exp\left[-\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x}\right].$$

Since , $F(x)=0$ as $x \to 0$ and $F(x)=C$ as $x \to \infty$.

Thus, by definition of *df* $F(x)=1$ as $x \to \infty$, this implies that $C=1$.

Hence the sufficiency part proved.

Remark 4.3. If $k=1$ in (17), we get the following characterization of lower record values for modified Fréchet distribution

$$E[\xi(X_{L(n)} \mid X_{L(m)} = x] = -\ln F(x) + n - m.$$

Theorem 4.4. Suppose X be an absolutely continuous (with respect to Lebesque measure) random variable with the *df* $F(x)$ and *pdf* $f(x)$ $\forall$ $0<x<\infty$, such that $f'(x)$ and $E(X \mid X \le x)$, exist for all x, $0<x<\infty$, then

$$E(X \mid X \le x) = g(x)\eta(x), \tag{20}$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = \frac{x^2 e^{\alpha x}}{(\lambda+\alpha x)\left(\frac{\beta}{x}\right)^{\lambda}} - \frac{x e^{\alpha x}\exp\left[\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x}\right]}{(\lambda+\alpha x)\left(\frac{\beta}{x}\right)^{\lambda}}\int_0^x \exp\left[-\left(\frac{\beta}{u}\right)^{\lambda} e^{-\alpha u}\right]du$$

if and only if

$$f(x) = \frac{(\lambda+\alpha x)}{x}\left(\frac{\beta}{x}\right)^{\lambda}\exp\left[-\alpha x - \left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x}\right], \quad x \ge 0, \quad \alpha, \beta, \lambda > 0.$$

Proof. From (1), we have

$$E(X \mid X \le x) = \frac{1}{F(x)}\int_0^x u\frac{(\lambda+\alpha u)}{u}\left(\frac{\beta}{u}\right)^{\lambda} e^{-\alpha u}\exp\left[-\left(\frac{\beta}{u}\right)^{\lambda} e^{-\alpha u}\right]du.$$

Integrating by parts, taking $\frac{(\lambda+\alpha u)}{u}\left(\frac{\beta}{u}\right)^{\lambda} e^{-\alpha u}\exp\left[-\left(\frac{\beta}{u}\right)^{\lambda} e^{-\alpha u}\right]$ for integration and rest of the integrand for differentiation, we get

$E(X \mid X \le x)$

$$= \frac{1}{F(x)} \left\{ x \exp\left[-\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} \right] - \int_0^x \exp\left[-\left(\frac{\beta}{u}\right)^{\lambda} e^{-\alpha u} \right] du \right\}$$

Now dividing and multiplying by $f(x)$, we obtain the result as given in (20).

For proving sufficient part, we have from (20)

$$\int_0^x u\, f(u) du = g(x) f(x) .$$

Differentiating on both sides with respect to x, we find that

$$x f(x) = g'(x) f(x) + g(x) f'(x) .$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} \qquad \text{Ahsanullah } et.\ al \text{ [24]}$$

$$\frac{f'(x)}{f(x)} = -\frac{\lambda}{x} - \frac{\lambda}{x(\lambda + \alpha x)} + \frac{(\lambda + \alpha x)}{x} \left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} - \alpha , \tag{21}$$

where

$$g'(x) = x - g(x) \left(-\frac{\lambda}{x} - \frac{\lambda}{x(\lambda + \alpha x)} + \frac{(\lambda + \alpha x)}{x} \left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} - \alpha \right) .$$

On integrating (21) both sides with respect to x, we get

$$f(x) = C \frac{(\lambda + \alpha x)}{x} \left(\frac{1}{x}\right)^{\lambda} e^{-\alpha x} \exp\left[-\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} \right] .$$

Further, To obtain the value of C (constant of integration), we have used the property of *pdf* that is

$$\int_0^{\infty} f(x) dx = 1 .$$

Thus,

$$\frac{1}{C} = \int_0^{\alpha} \frac{(\lambda + \alpha x)}{x} \left(\frac{1}{x}\right)^{\lambda} e^{-\alpha x} \exp\left[-\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} \right] dx = \frac{1}{\beta^{\lambda}} ,$$

which proves that

$$f(x) = \frac{(\lambda + \alpha x)}{x} \left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} \exp\left[-\left(\frac{\beta}{x}\right)^{\lambda} e^{-\alpha x} \right], \quad x > 0, \quad \alpha, \beta, \lambda > 0 .$$

Remark 4.5. Setting $\alpha = 1$ and $\lambda = 0$, Theorem 4.4 gives characterizing result for standard Gumbel distribution and for $\alpha = 0$, it gives the characterizing result for inverse Weibull distribution.

Conclusion: In this paper, we have presented the new results for the single and product moments of modified Fréchet distribution based on generalized lower record values. These results include some well-known results for standard Gumbel and inverse Weibull distributions as obtained by

Balakrishnan et al. [6] and Pawlas and Szynal [18]. Later, we established the characterizing results for this distribution by utilizing the relations for single and product moments and conditional expectation of a function of generalized lower record value, and using truncated moments.

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