A TWO-STATE FEEDBACK RETRIAL QUEUEING SYSTEM HAVING TWO HETEROGENEOUS PARALLEL SERVERS AND IMPATIENT CUSTOMERS

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Abstract

Objective: In the present paper we consider a two-state retrial queueing system with feedback having two heterogeneous parallel servers and impatient customers. Transient state probabilities for exact number of arrivals and departures from the system will be obtained when both, one or none of the servers is busy. Numerical and graphical solutions will also be obtained. Methods: The difference-differential equations governing the system are solved recursively, Laplace transform is then used to obtain the transient state probabilities for exact number of arrivals and departures from the system. Findings: Time dependent probabilities are obtained when both, one and none of the servers is busy. Numerical and Graphical solutions are also obtained using MATLAB programming. Novelty: In past research, models considered arrivals and departures from the orbit whereas in present model arrivals and departures from the system are studied along with the concept of feedback. Applications: This type of model is implemented in computer systems.

1. Introduction

In addition to classical queueing systems, there exists a new class of queueing models, popularly known as retrial queueing models. Recently, a significant contribution has been provided in this direction. The retrial queueing systems are characterized by an arriving customer who is served instantly if it finds the server free else leaves the service area and joins the virtual queue (orbit) and repeats its demand for service after a random amount of time from the orbit. 'Basic problems of telephone traffic and the influence of repeated calls' published by [1] which is the initial work on retrial queues. Books [2], [3] are available as a great source for retrial queues.

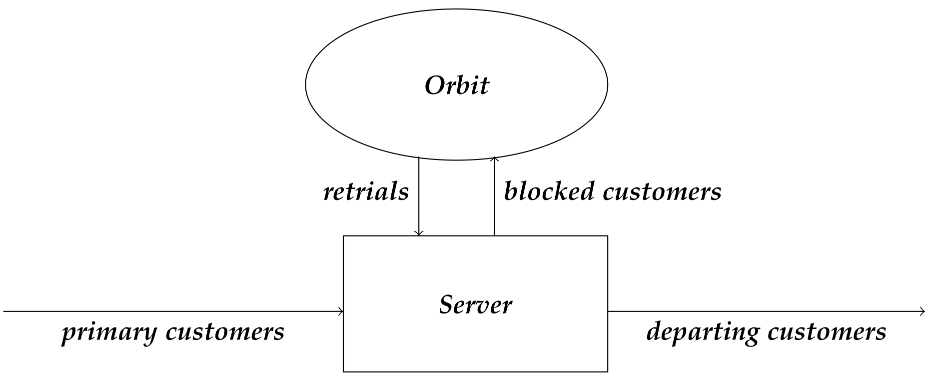

Figure 1: *Basic Structure of a Retrial Queueing System*

In [4] the author analyzed 'Some new results for the $M/M/1$ queue' in which a closed form solution with finite sums is obtained for the probability that exactly i number of arrivals, j services occur over a time interval t. In [5] authors worked on 'Performance analysis of a two-state queueing model with retrials' where they obtained the transient state probabilities for exact number of arrivals and departures from the system.
In some systems various servers posses different service rate depending on the requirements and other reasons, these servers are called heterogeneous or non-identical servers. Here the same type of job is rendered by different servers with different service rates. The expressions for the Laplace transforms of the waiting time as well as arbitrary moments are derived in [6]. 'Retrial queueing model with two heterogeneous server using matrix geometric method' done by [7] where the stationary analysis has been carried out using matrix geometric method.
In case of higher demand for any service, reneged and balked i.e., impatient customers are usually observed. A wide study has been done in this direction for both standard and retrial queueing systems. In [8] blocking probability and the mean number of customers in the orbit was derived. 'A Single Server Retrial Queue with Impatient Customers' is analyzed in [9].
In some situations, customers seek service again as a result of dissatisfaction from the received service. This concept is known as feedback. For instance: when a message faces a failed transmission in multiple access telecommunication systems, it can be sent again. 'On multiserver feedback retrial queues with balking and control retrial rate' published by [10]. The author in [11] analyzed 'On an unreliable-server retrial queue with customer feedback and impatience'. The time dependent probability generating function was obtained using supplementary variable technique in [12].
The section wise description of the paper is as follows:
The model is described theoretically as well as mathematically in section 2. In section 3 the transient state probabilities are obtained. Verification of some results is given in section 4. The numerical and graphical results are obtained in section 5. Section 6 gives the busy period probabilities along with its graphical representation. Finally, the paper is concluded in section 7 and followed by references at the end.

2. Model Description

We considered a two-state retrial queueing model with feedbak having two non-identical parallel servers and impatient customers. The primary or fresh customers arrive at system following Poisson process. On arrival if the entering customer finds any of the servers free it is served immediately else it may balk from the system or may join the orbit and retry for service as a repeated call or secondary customer. The secondary customers also follow Poisson process. The customers retrying from orbit in case getting busy servers may renege from the orbit. Also, if the customer after service feels unsatisfied may join the orbit in order to obtain satisfied service. Service times follow exponential distribution.

- Arrival Process: The primary calls arrive at the system following Poisson process with mean arrival rate λ.
- An arriving customer joins the first server with probability a_1 and second with probability a_2.
- The Retrial Process: On arrival of a customer if any of the servers is free, it is served immediately. Otherwise, the customer joins the orbit and calls repeatedly until any of the servers is free. The retrial customers also follow Poisson process with parameter θ.
- Impatience: The fresh customers on encountering busy server may balk with probability $(1-\beta)$, $\beta > 0$. Also, the customers retring for service from orbit as secondary customers on encountering busy server may renege with probability $(1-\alpha)$,$\alpha > 0$.
- Feedback Rule: After receiving service, the customer joins the orbit with probability γ (i.e., when unsatisfied) and departs from the system with probability 1-γ.

- The Service Process: Service times follow exponential distribution with parameters μ_1 for first server and μ_2 for second server.

The input flow of primary calls, intervals between repetitions, service times are statistically independent.
Laplace Transformation of $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt; \quad Re(s) > 0$$

The Laplace Inverse of

$$\frac{Q(p)}{P(p)} = \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k - l} e^{a_k t}}{(m_k - l)!(l-1)!} \times \left(\frac{d}{dp}\right)^{l-1} \frac{Q(p)}{P(p)} (p - a_k)^{m_k} \; \forall \; p = a_k, a_i \neq a_k \; for \; i \neq k$$

where

$$P(p) = (p - a_1)^{m_1} (p - a_2)^{m_2} ... (p - a_n)^{m_n}$$
$$Q(p) \; is \; a \; polynomial \; of \; degree < m_1 + m_2 + m_3 + ... + m_n - 1$$

The Laplace Inverse of

$$\bar{N}^{a,b,c}_{n_1,n_2,n_3}(s) = \frac{1}{(s+a)^{n_1} + (s+b)^{n_2} + (s+c)^{n_3}} \; is$$

$$N^{a,b,c}_{n_1,n_2,n_3}(t) = \sum_{l=1}^{n_3} \sum_{m=1}^{l} \frac{e^{-at} t^{n_3 - l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_1 + g_1)\right) \left(\prod_{g_2=0}^{m-2} (n_2 + g_2)\right)}{(n_3 - l)!(m-1)!(b-a)^{n_2+m-1} (c-a)^{n_1+l-m}}$$
$$+ \sum_{l=1}^{n_2} \sum_{m=1}^{l} \frac{e^{-bt} t^{n_2 - l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_1 + g_1)\right) \left(\prod_{g_2=0}^{m-2} (n_3 + g_2)\right)}{(n_2 - l)!(m-1)!(a-b)^{n_3+m-1} (c-b)^{n_1+l-m}}$$
$$+ \sum_{l=1}^{n_1} \sum_{m=1}^{l} \frac{e^{-ct} t^{n_2 - l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (n_2 + g_1)\right) ((n_3 + g_2))}{(n_1 - l)!(m-1)!(a-c)^{n_3+m-1} (b-c)^{n_2+l-m}}$$

$$If \; L^{-1}\{f(s)\} = F(t) \; and \; L^{-1}\{g(s)\} = G(t), \; then$$
$$L^{-1}\{f(s)\, g(s)\} = \int_0^1 F(u)G(t-u)du = F * G,$$

F*G is called the convolution of F and G.

2.1. The Two Dimensional State Model

2.1.1 Definitions

$P_{i,j,0}(t)$ = Probability that there are exactly i number of arrivals, j number of departures from the system by time t when both the servers are free.

$P_{i,j,1,k}(t)$ = Probability that there are exactly i number of arrivals,j number of departures from the system by time t and $k^{th}(k = 1 \; or \; 2)$ server is busy .

$P_{i,j,2}(t)$ = Probability that there are exactly i number of arrivals, j number of departures from the system by time t and both the servers are busy.

$P_{i,j}(t)$ = Probability that there are exactly i number of arrivals, j number of departures from the system by time t.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t), \ \forall \ i,j; \ \ i \geq j$$
$$P_{i,j,1}(t) = P_{i,j,1,1}(t) + P_{i,j,1,2}(t)$$

$$P_{i,j,0}(t) = 0, \ i < j; \quad P_{i,j,1,k}(t) = 0, \ (k = 1 \ or \ 2) \ i < j; \quad P_{i,j,2}(t) = 0, \ i < j$$

Initially

$$P_{0,0,0}(0) = 1;$$
$$P_{i,j,0}(0) = 0; \ \ P_{i,j,1,k}(0) = 0 \ (k = 1 \ or \ 2); \ \ P_{i,j,2}(0) = 0; \ i \neq j$$

2.2. Difference-Differential Equations Governing the System

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu_1(1-\gamma)P_{i,j-1,1,1}(t) + \mu_1\gamma P_{i,j,1,1}(t) + \mu_2(1-\gamma)P_{i,j-1,1,2}(t) + \mu_2\gamma P_{i,j,1,2}(t); \qquad i \geq j \geq 0 \tag{1}$$

$$\frac{d}{dt}P_{1,0,1,1}(t) = -(\lambda + \mu_1)P_{1,0,1,1}(t) + \lambda a_1 P_{0,0,0}(t) + \theta a_1 P_{1,0,0}(t) \tag{2}$$

$$\frac{d}{dt}P_{1,0,1,2}(t) = -(\lambda + \mu_2)P_{1,0,1,2}(t) + \lambda a_2 P_{0,0,0}(t) + \theta a_2 P_{1,0,0}(t) \tag{3}$$

$$\frac{d}{dt}P_{i,j,1,1}(t) = -(\lambda + \mu_1 + (i-j-1)\theta)P_{i,j,1,1}(t) + \lambda a_1 P_{i-1,j,0}(t) + (i-j)\theta a_1 P_{i,j,0}(t) + \mu_2(1-\gamma)P_{i,j-1,2}(t) + \mu_2\gamma P_{i,j,2}(t); \qquad i > 1, \ i > j \geq 0 \tag{4}$$

$$\frac{d}{dt}P_{i,j,1,2}(t) = -(\lambda + \mu_2 + (i-j-1)\theta)P_{i,j,1,2}(t) + \lambda a_2 P_{i-1,j,0}(t) + (i-j)\theta a_2 P_{i,j,0}(t) + \mu_1(1-\gamma)P_{i,j-1,2}(t) + \mu_1\gamma P_{i,j,2}(t); \qquad i > 1, \ i > j \geq 0 \tag{5}$$

$$\frac{d}{dt}P_{i,j,2}(t) = -(\lambda\beta + \mu_1 + \mu_2 + (i-j-2)\theta(1-\alpha))P_{i,j,2}(t) + \lambda\left\{P_{i-1,j,1,1}(t) + P_{i-1,j,1,2}(t)\right\} + \lambda\beta(i - \delta_{i-2,j})P_{i-1,j,2}(t) + (i-j-1)\theta\left\{P_{i,j,1,1,}(t) + P_{i,j,1,2}(t)\right\} + (i-j-1)\theta(1-\alpha)P_{i,j-1,2}(t); \qquad i \geq 2, \ i < j \geq 0 \tag{6}$$

Using Laplace Transform $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt; \quad Re(s) > 0$$

and using initial condition in equations (1) to (6), we have

$$(s + \lambda + (i-j)\theta)\bar{P}_{0,0,0}(s) = \mu_1(1-\gamma)\bar{P}_{i,j-1,1,1}(s) + \mu_1\gamma\bar{P}_{i,j,1,1}(s) + \mu_2(1-\gamma)\bar{P}_{i,j-1,1,2}(s) + \mu_2\gamma\bar{P}_{i,j,1,2}(s); \qquad i \geq j \geq 0 \tag{7}$$

$$(s + \lambda + \mu_1)\bar{P}_{1,0,1,1}(s) = \lambda a_1\bar{P}_{0,0,0}(s) + \theta a_1\bar{P}_{1,0,0}(s) \tag{8}$$

$$(s + \lambda + \mu_2)\bar{P}_{1,0,1,2}(s) = \lambda a_2\bar{P}_{0,0,0}(s) + \theta a_2\bar{P}_{1,0,0}(s) \tag{9}$$

$$(s + \lambda + \mu_1 + (i-j-1)\theta)\bar{P}_{i,j,1,1}(s) = \lambda a_1\bar{P}_{i-1,j,0}(s) + (i-j)\theta a_1\bar{P}_{i,j,0}(s) + \mu_2(1-\gamma)\bar{P}_{i,j-1,2}(s) + \mu_2\gamma\bar{P}_{i,j,2}(s); \qquad i > j \geq 0 \tag{10}$$

$$(s + \lambda + \mu_2 + (i-j-1)\theta)\bar{P}_{i,j,1,2}(s) = \lambda a_2\bar{P}_{i-1,j,0}(s) + (i-j)\theta a_2\bar{P}_{i,j,0}(s) + \mu_1(1-\gamma)\bar{P}_{i,j-1,2}(s) + \mu_1\gamma\bar{P}_{i,j,2}(s); \qquad i > j \geq 0 \tag{11}$$

$$(s + \lambda\beta + \mu_1 + \mu_2 + (i-j-2)\theta(1-\alpha))\bar{P}_{i,j,2}(s) = \lambda\left\{\bar{P}_{i-1,j,1,1}(s) + \bar{P}_{i,j,1,2}(s)\right\} + \lambda\beta(1 - \delta_{i-2,j})\bar{P}_{i-1,j,2}(s) + (i-j-1)\theta\left\{\bar{P}_{i,j,1,1}(s) + \bar{P}_{i,j,1,2}(s)\right\} + (i-j-1)\theta(1-\alpha)\bar{P}_{i,j-1,2}(s); \qquad i > j \geq 0 \tag{12}$$

where

$$\delta_{i-2,j} = \begin{cases} 1; & i-2=j \\ 0; & otherwise \end{cases}$$

3. Solution of the Problem

Solving equations (7) -(12) recursively, we get:

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} \tag{13}$$

$$\bar{P}_{i,0,0}(s) = \frac{\mu_1\gamma}{s+\lambda+i\theta}\bar{P}_{i,0,1,1}(s) + \frac{\mu_2\gamma}{s+\lambda+i\theta}\bar{P}_{i,0,1,2}(s); \qquad i \geq 1 \tag{14}$$

$$\bar{P}_{1,0,1,1}(s) = \frac{\lambda a_1}{s+\lambda+\mu_1}\bar{P}_{0,0,0}(s) + \frac{\theta a_1}{s+\lambda+\mu_1}\bar{P}_{1,0,0}(s) \tag{15}$$

$$\bar{P}_{1,0,1,2}(s) = \frac{\lambda a_2}{s+\lambda+\mu_2}\bar{P}_{0,0,0}(s) + \frac{\theta a_2}{s+\lambda+\mu_2}\bar{P}_{1,0,0}(s) \tag{16}$$

$$\begin{aligned} \bar{P}_{i,i,0}(s) = {} & \frac{\lambda}{s+\lambda}\left[\frac{\mu_1(1-\gamma)a_1}{s+\lambda+\mu_1} + \frac{\mu_2(1-\gamma)a_2}{s+\lambda+\mu_2}\right]\bar{P}_{i-1,i-1,0}(s) + \\ & \frac{\theta}{s+\lambda}\left[\frac{\mu_1(1-\gamma)a_1}{s+\lambda+\mu_1} + \frac{\mu_2(1-\gamma)a_2}{s+\lambda+\mu_2}\right]\bar{P}_{i,i-1,0}(s) + \\ & \frac{\mu_1(1-\gamma)\mu_2(1-\gamma)}{s+\lambda}\left[\frac{1}{s+\lambda+\mu_1} + \frac{1}{s+\lambda+\mu_2}\right]\bar{P}_{i,i-2,2}(s); \qquad i \geq 1 \end{aligned} \tag{17}$$

$$\bar{P}_{i,i-1,1,1}(s) = \frac{\lambda a_1}{s+\lambda+\mu_1}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta a_1}{s+\lambda+\mu_1}\bar{P}_{i,i-1,0}(s) + \frac{\mu_2(1-\gamma)}{s+\lambda+\mu_1}\bar{P}_{i,i-2,2}(s); \qquad i \geq 2 \tag{18}$$

$$\bar{P}_{i,i-1,1,2}(s) = \frac{\lambda a_2}{s+\lambda+\mu_2}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta a_2}{s+\lambda+\mu_2}\bar{P}_{i,i-1,0}(s) + \frac{\mu_1(1-\gamma)}{s+\lambda+\mu_2}\bar{P}_{i,i-2,2}(s); \qquad i \geq 2 \tag{19}$$

$$\begin{aligned} \bar{P}_{i,1,1,1}(s) = {} & \frac{\lambda a_1}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i-1,1,0}(s) + \frac{(i-1)\theta a_1}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i,1,0}(s) + \\ & \frac{\mu_2(1-\gamma)}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i,0,2}(s) + \frac{\mu_2\gamma}{s+\lambda+\mu_1+(i-2)\theta}\bar{P}_{i,1,2}(s); \qquad i \geq 3 \end{aligned} \tag{20}$$

$$\begin{aligned} \bar{P}_{i,1,1,2}(s) = {} & \frac{\lambda a_2}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i-1,1,0}(s) + \frac{(i-1)\theta a_2}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i,1,0}(s) + \\ & \frac{\mu_1(1-\gamma)}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i,0,2}(s) + \frac{\mu_1\gamma}{s+\lambda+\mu_2+(i-2)\theta}\bar{P}_{i,1,2}(s); \qquad i \geq 3 \end{aligned} \tag{21}$$

$$\begin{aligned} \bar{P}_{i,0,2}(s) = {} & \frac{\lambda}{s+\lambda\beta+\mu_1+\mu_2+(i-2)\theta(1-\alpha)}\left\{\bar{P}_{i-1,0,1,1}(s) + \bar{P}_{i-1,0,1,2}(s)\right\} + \\ & \frac{\lambda\beta}{s+\lambda\beta+\mu_1+\mu_2+(i-2)\theta(1-\alpha)}\bar{P}_{i-1,0,2}(s) + \\ & \frac{(i-1)\theta}{s+\lambda\beta+\mu_1+\mu_2+(i-2)\theta(1-\alpha)}\left\{\bar{P}_{i,0,1,1}(s) + \bar{P}_{i,0,1,2}(s)\right\}; \qquad i \geq 3 \end{aligned} \tag{22}$$

$$\bar{P}_{i,j,2}(s) = \sum_{k=1}^{i-j}\left\{\prod_{p=k-1}^{i-j-2}\left(\frac{\lambda^{i-j-k}\beta^{i-j-(k+1)}}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\eta_k^{'}(s)\right\}\left\{\bar{P}_{j+k,j,1,1}(s) + \bar{P}_{j+k,j,1,2}(s)\right\}$$

$$+\sum_{k=1}^{i-j-1}\left\{\prod_{p=k-1}^{i-j-2}\left(\frac{(\lambda\beta)^{i-j-k-1}k\theta(1-\alpha)}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\right\}\bar{P}_{j+k+1,j-1,2}(s);$$

$$i\geq j+2,\ j\geq 1 \qquad (23)$$

where

$$\eta_k^{'}=\begin{cases}1 & k=1\\ 1+\dfrac{(k-1)\theta\beta}{s+\lambda\beta+\mu_1+\mu_2+(k-2)\theta(1-\alpha)} & k=2\ to\ i-j-1\\ \dfrac{(k-1)\theta}{s+\lambda\beta+\mu_1+\mu_2+(k-2)\theta(1-\alpha)} & k=i-j\end{cases}$$

$$\bar{P}_{i,j,1,1}(s)=\frac{\lambda a_1}{s+\lambda+\mu_1+(i-j-1)\theta}\bar{P}_{i-1,j,0}(s)+\frac{(i-j)\theta a_1}{s+\lambda+\mu_1+(i-j-1)\theta}\bar{P}_{i,j,0}(s)+$$
$$\frac{\mu_2(1-\gamma)}{s+\lambda+\mu_1+(i-j-1)\theta}\left[\sum_{k=0}^{i-j}\left\{\prod_{p=k}^{i-j-1}\left(\frac{\lambda^{i-j-k}\beta^{i-j-(k+1)}}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\psi_k^{'}(s)\right\}\right.$$
$$\left.\left\{\bar{P}_{j+k,j-1,1,1}(s)+\bar{P}_{j+k,j-1,1,2}(s)\right\}+\sum_{k=1}^{i-j}\left\{\prod_{p=k-1}^{i-j-1}\left(\frac{(\lambda\beta)^{i-j-k}k\theta(1-\alpha)}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\right\}\bar{P}_{j+k,j-2,2}(s)\right]$$
$$+\frac{\mu_2\gamma}{s+\lambda+\mu_1+(i-j-1)\theta}\left[\sum_{k=1}^{i-j}\left\{\prod_{p=k-1}^{i-j-2}\left(\frac{\lambda^{i-j-k}\beta^{i-j-(k+1)}}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\eta_k^{'}(s)\right\}\right.$$
$$\left.\left\{\bar{P}_{j+k,j,1,1}(s)+\bar{P}_{j+k,j,1,2}(s)\right\}+\sum_{k=1}^{i-j-1}\left\{\prod_{p=k-1}^{i-j-2}\left(\frac{(\lambda\beta)^{i-j-k-1}k\theta(1-\alpha)}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\right\}\bar{P}_{j+k+1,j-1,2}(s)\right];$$

$$i\geq j+2,\ j>1 \qquad (24)$$

$$\bar{P}_{i,j,1,2}(s)=\frac{\lambda a_2}{s+\lambda+\mu_2+(i-j-1)\theta}\bar{P}_{i-1,j,0}(s)+\frac{(i-j)\theta a_2}{s+\lambda+\mu_2+(i-j-1)\theta}\bar{P}_{i,j,0}(s)+$$
$$\frac{\mu_1(1-\gamma)}{s+\lambda+\mu_1+(i-j-1)\theta}\left[\sum_{k=0}^{i-j}\left\{\prod_{p=k}^{i-j-1}\left(\frac{\lambda^{i-j-k}\beta^{i-j-(k+1)}}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\psi_k^{'}(s)\right\}\right.$$
$$\left.\left\{\bar{P}_{j+k,j-1,1,1}(s)+\bar{P}_{j+k,j-1,1,2}(s)\right\}+\sum_{k=1}^{i-j}\left\{\prod_{p=k-1}^{i-j-1}\left(\frac{(\lambda\beta)^{i-j-k}k\theta(1-\alpha)}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\right\}\bar{P}_{j+k,j-2,2}(s)\right]$$

$$+\frac{\mu_1\gamma}{s+\lambda+\mu_1+(i-j-1)\theta}\left[\sum_{k=1}^{i-j}\left\{\prod_{p=k-1}^{i-j-2}\left(\frac{\lambda^{i-j-k}\beta^{i-j-(k+1)}}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\eta_k^{'}(s)\right\}\right.$$
$$\left.\left\{\bar{P}_{j+k,j,1,1}(s)+\bar{P}_{j+k,j,1,2}(s)\right\}+\sum_{k=1}^{i-j-1}\left\{\prod_{p=k-1}^{i-j-2}\left(\frac{(\lambda\beta)^{i-j-k-1}k\theta(1-\alpha)}{s+\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha)}\right)\right\}\bar{P}_{j+k+1,j-1,2}(s)\right];$$

$$i\geq j+2,\ j>1 \qquad (25)$$

where

$$\psi_k^{'}=\begin{cases}1 & k=0\\ 1+\dfrac{k\theta\beta}{s+\lambda\beta+\mu_1+\mu_2+(k-1)\theta(1-\alpha)} & k=1\ to\ i-j-1\\ \dfrac{k\theta}{s+\lambda\beta+\mu_1+\mu_2+(k-1)\theta(1-\alpha)} & k=i-j\end{cases}$$

$$\eta_k^{'} = \begin{cases} 1 & k=1 \\ 1+\dfrac{(k-1)\theta\beta}{s+\lambda\beta+\mu_1+\mu_2+(k-2)\theta(1-\alpha)} & k=2 \ to \ i-j-1 \\ \dfrac{(k-1)\theta}{s+\lambda\beta+\mu_1+\mu_2+(k-2)\theta(1-\alpha)} & k=i-j \end{cases}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu_1(1-\gamma)}{s+\lambda+(i-j)\theta}\bar{P}_{i,j-1,1,1}(s) + \frac{\mu_1\gamma}{s+\lambda+(i-j)\theta}\bar{P}_{i,j,1,1}(s) + \frac{\mu_2(1-\gamma)}{s+\lambda+(i-j)\theta}\bar{P}_{i,j-1,1,2}(s) + \frac{\mu_2\gamma}{s+\lambda+(i-j)\theta}\bar{P}_{i,j,1,2}(s); \quad i>j>1 \tag{26}$$

Taking the Laplace inverse of (13)-(26), we get the transient state probabilities as:

$$P_{0,0,0}(t) = e^{-\lambda t} \tag{27}$$

$$P_{i,0,0}(t) = \mu_1\gamma e^{-(\lambda+i\theta)t} * P_{i,0,1,1}(t) + \mu_2\gamma e^{-(\lambda+i\theta)t} * P_{i,0,1,2}(t); \quad i \geq 1 \tag{28}$$

$$P_{1,0,1,1}(t) = \lambda a_1 e^{-(\lambda+\mu_1)t} * P_{0,0,0}(t) + \theta a_1 e^{-(\lambda+\mu_1)t} * P_{1,0,0}(t) \tag{29}$$

$$P_{1,0,1,2}(t) = \lambda a_2 e^{-(\lambda+\mu_2)t} * P_{0,0,0}(t) + \theta a_2 e^{-(\lambda+\mu_2)t} * P_{1,0,0}(t) \tag{30}$$

$$\begin{aligned} P_{i,i,0}(t) = {} & \left[\lambda a_1\mu_1(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + \lambda a_2\mu_2(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\}\right] * P_{i-1,i-1,0}(t) \\ & + \left[\theta a_1\mu_1(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + \theta a_2\mu_2(1-\gamma)e^{-\lambda t}\left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\}\right] * P_{i,i-1,0}(t) \\ & \mu_1(1-\gamma)\mu_2(1-\gamma)e^{-\lambda t}\left[\left\{\frac{1}{\mu_1}-\frac{e^{-\mu_1 t}}{\mu_1}\right\} + \left\{\frac{1}{\mu_2}-\frac{e^{-\mu_2 t}}{\mu_2}\right\}\right] * P_{i,i-2,2}(t); \quad i \geq 1 \end{aligned} \tag{31}$$

$$P_{i,i-1,1,1}(t) = \lambda a_1 e^{-(\lambda+\mu_1)} P_{i-1,i-1,0}(t) + \theta a_1 e^{-(\lambda+\mu_1)} P_{i,i-1,0}(t) + \mu_2(1-\gamma)e^{-(\lambda+\mu_1)} P_{i,i-2,2}(t); \quad i \geq 2 \tag{32}$$

$$P_{i,i-1,1,2}(t) = \lambda a_2 e^{-(\lambda+\mu_2)} P_{i-1,i-1,0}(t) + \theta a_2 e^{-(\lambda+\mu_2)} P_{i,i-1,0}(t) + \mu_1(1-\gamma)e^{-(\lambda+\mu_2)} P_{i,i-2,2}(t); \quad i \geq 2 \tag{33}$$

$$\begin{aligned} P_{i,1,1,1}(t) = {} & \lambda a_1 e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i-1,1,0}(t) + (i-1)\theta a_1 e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i,1,0}(t) + \\ & \mu_2(1-\gamma)e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i,0,2}(t) + \mu_2\gamma e^{-(\lambda+\mu_1+(i-2)\theta)t} * P_{i,1,2}(t); \quad i \geq 3 \end{aligned} \tag{34}$$

$$\begin{aligned} P_{i,1,1,2}(t) = {} & \lambda a_2 e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i-1,1,0}(t) + (i-1)\theta a_2 e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i,1,0}(t) + \\ & \mu_1(1-\gamma)e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i,0,2}(t) + \mu_1\gamma e^{-(\lambda+\mu_2+(i-2)\theta)t} * P_{i,1,2}(t); \quad i \geq 3 \end{aligned} \tag{35}$$

$$\begin{aligned} P_{i,0,2}(t) = {} & \lambda e^{-(\lambda\beta+\mu_1+\mu_2+(i-2)\theta(1-\alpha))t} * \{P_{i-1,0,1,1}(t) + P_{i-1,0,1,2}(t)\} + \lambda\beta \\ & e^{-(\lambda\beta+\mu_1+\mu_2+(i-2)\theta(1-\alpha))t} * P_{i-1,0,2}(t) + (i-1)\theta\lambda e^{-(\lambda\beta+\mu_1+\mu_2+(i-2)\theta(1-\alpha))t} * \\ & \{P_{i,0,1,1}(t) + P_{i,0,1,2}(t)\}; \quad i \geq 3 \end{aligned} \tag{36}$$

$$\begin{aligned} P_{i,j,2}(t) = {} & \lambda^{i-j-1}\beta^{i-j-2}\prod_{p=0}^{i-j-2}\left\{e^{-(\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha))t}\frac{t^p}{p!}\right\} * \{P_{j+1,j,1,1}(t) + P_{j+1,j,1,2}(t)\} \\ & + \sum_{k=2}^{i-j-1}\lambda^{i-j-k}\beta^{i-j-(k+1)}\prod_{p=k-1}^{i-j-2}\left\{e^{-(\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha))t}\frac{t^{p-k+1}}{(p-k+1)!}\right\} * \left\{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\right\} \\ & + \sum_{k=2}^{i-j-1}(\lambda\beta)^{i-j-k}(k-1)\theta\prod_{p=k-2}^{i-j-2}\left\{e^{-(\lambda\beta+\mu_1+\mu_2+p\theta(1-\alpha))t}\frac{t^{p-k+2}}{(p-k+2)!}\right\} * \left\{P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t)\right\} \\ & + (i-j-1)\theta e^{-(\lambda\beta+\mu_1+\mu_2+(i-j-2)\theta(1-\alpha))t} * \{P_{i,j,1,1}(t) + P_{i,j,1,2}(t)\}; \quad i \geq j+2, \ j \geq 1 \end{aligned} \tag{37}$$

$$P_{i,j,1,1}(t) = \lambda a_1 e^{-(\lambda+\mu_1+(i-j-1)\theta)} * P_{i-1,j,0}(t) + (i-j)\theta a_1 e^{-(\lambda+\mu_1+(i-j-1)\theta)} * P_{i,j,0}(t) +$$

$$\mu_2(1-\gamma)\lambda^{i-j-1}\beta^{i-j-2}e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\beta^{i-j-(k+1)}$$

$$\prod_{p=k}^{i-j-1}\left\{\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+1}} - e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p-k}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+1}}\right\} * \left\{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\right\} +$$

$$\mu_2(1-\gamma)e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}(\lambda\beta)^{i-j-k}\prod_{p=k}^{i-j-1}\left\{\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+1}} - e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+2}}\right\} * \left\{P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t)\right\} +$$

$$\mu_2(1-\gamma)e^{-(\lambda+\mu_1+(i-j-1)\theta)t}(i-j)\theta\left\{\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{(i-j-1)\theta(1-\alpha)}{\beta}\right)} - \frac{e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{(i-j-1)\theta(1-\alpha)}{\beta}\right)t}}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{(i-j-1)\theta(1-\alpha)}{\beta}\right)}\right\} * \left\{P_{i,j-1,1,1}(t) + P_{i,j-1,1,2}(t)\right\} + \mu_2(1-\gamma)e^{-(\lambda+\mu_1+(i-j-1)\theta)t}$$

$$\sum_{k=1}^{i-j}(\lambda\beta)^{i-j-k}k\theta(1-\alpha)\prod_{p=k-1}^{i-j-1}\left\{\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+2}} - e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+2}}\right\} * P_{j+k,j-2,2}(t) + \mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\lambda^{i-j}\beta^{i-j-1}$$

$$\prod_{p=0}^{i-j-2}\left\{\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p+1}} - e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-r+1}}\right\}$$

$$* \left\{P_{j+1,j,1,1}(t) + P_{j+1,j,1,2}(t)\right\} + \mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}\lambda^{i-j-k}\beta^{i-j-(k+1)}$$

$$\prod_{p=k-1}^{i-j-2}\left\{\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+2}} - e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+2}}\right\}$$

$$* \left\{ P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t) \right\} + \mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}$$

$$\sum_{k=2}^{i-j-1}(k-1)\theta(\lambda\beta)^{i-j-k}\prod_{p=k-2}^{i-j-2}\left\{ \frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+3}} - e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\right.$$

$$\left.\sum_{r=0}^{p-k+2}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+3}}\right\} * \left\{ P_{j+k,j,1,1}(t) + P_{j+k,j,1,2}(t) \right\} + \mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}$$

$$(i-j-1)\theta\left\{ \frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{(i-j-2)\theta(1-\alpha)}{\beta}\right)} - \frac{e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{(i-j-2)\theta(1-\alpha)}{\beta}\right)t}}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{(i-j-2)\theta(1-\alpha)}{\beta}\right)}\right\}$$

$$* \left\{ P_{i,j,1,1}(t) + P_{i,j,1,2}(t) \right\} + \mu_2\gamma e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}k\theta(1-\alpha)(\lambda\beta)^{i-j-k-1}$$

$$\prod_{p=k-1}^{i-j-2}\left\{ \frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+2}} - e^{-\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\right.$$

$$\left.\sum_{r=0}^{p-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_2}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+2}}\right\} * P_{j+k+1,j-1,2}(t); \qquad i \geq j+2,\ j > 1 \tag{38}$$

$$P_{i,j,1,2}(t) = \lambda a_2 e^{-(\lambda+\mu_2+(i-j-1)\theta)} * P_{i-1,j,0}(t) + (i-j)\theta a_2 e^{-(\lambda+\mu_2+(i-j-1)\theta)} * P_{i,j,0}(t) +$$

$$\mu_1(1-\gamma)\lambda^{i-j-1}\beta^{i-j-2}e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\prod_{p=0}^{i-j-1}\left\{ \frac{1}{\left(\frac{\mu_1}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p+1}} - \right.$$

$$\left. e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-r+1}}\right\} * \left\{ P_{j,j-1,1,1}(t) + P_{j,j-1,1,2}(t) \right\}$$

$$+ \mu_1(1-\gamma)e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\lambda^{i-j-k}\beta^{i-j-(k+1)}\prod_{p=k}^{i-j-1}\left\{ \frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+1}} - \right.$$

$$\left. e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p-k}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+1}}\right\} * \left\{ P_{j+k,j-1,1,1}(t) + P_{j+k,j-1,1,2}(t) \right\}$$

$$+ \mu_1(1-\gamma)e^{-(\lambda+\mu_1+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}(\lambda\beta)^{i-j-k}$$

$$\left\{\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{(i-j-1)\theta(1-\alpha)}{\beta}\right)}-\frac{e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{(i-j-1)\theta(1-\alpha)}{\beta}\right)t}}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{(i-j-1)\theta(1-\alpha)}{\beta}\right)}\right\}*\left\{P_{i,j-1,1,1}(t)+P_{i,j-1,1,2}(t)\right\}$$

$$+\mu_1(1-\gamma)e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=1}^{i-j}(\lambda\beta)^{i-j-k}k\theta(1-\alpha)$$

$$\prod_{p=k-1}^{i-j-1}\left\{\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+2}}-e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+2}}\right\}$$

$$*P_{j+k,j-2,2}(t)+\mu_1\gamma e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\lambda^{i-j}\beta^{i-j-1}$$

$$\prod_{p=0}^{i-j-2}\left\{\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p+1}}-e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\sum_{r=0}^{p}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-r+1}}\right\}$$

$$*\left\{P_{j+1,j,1,1}(t)+P_{j+1,j,1,2}(t)\right\}+\mu_1\gamma e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}\lambda^{i-j-k}\beta^{i-j-(k+1)}$$

$$\prod_{p=k-1}^{i-j-2}\left\{\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+2}}-e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\right.$$

$$\left.\sum_{r=0}^{p-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+2}}\right\}*\left\{P_{j+k,j,1,1}(t)+P_{j+k,j,1,2}(t)\right\}$$

$$+\mu_1\gamma e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}(k-1)\theta(\lambda\beta)^{i-j-k}$$

$$\prod_{p=k-2}^{i-j-2}\left\{\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+3}}-e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_3}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\right.$$

$$\left.\sum_{r=0}^{p-k+2}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+3}}\right\}*\left\{P_{j+k,j,1,1}(t)+P_{j+k,j,1,2}(t)\right\}+\mu_1\gamma e^{-(\lambda+\mu_2+(i-j-1)\theta)t}$$

$$+(i-j-1)\theta\left\{\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{(i-j-2)\theta(1-\alpha)}{\beta}\right)}-\frac{e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{(i-j-2)\theta(1-\alpha)}{\beta}\right)t}}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{(i-j-2)\theta(1-\alpha)}{\beta}\right)}\right\}$$

$$*\left\{P_{i,j,1,1}(t)+P_{i,j,1,2}(t)\right\}$$

$$+\mu_1\gamma e^{-(\lambda+\mu_2+(i-j-1)\theta)t}\sum_{k=2}^{i-j-1}k\theta(1-\alpha)(\lambda\beta)^{i-j-k-1}$$

$$\prod_{p=k-1}^{i-j-2}\left\{\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k+2}}-e^{-\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)t}\right.$$

$$\left.\sum_{r=0}^{p-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu_2}{\beta}+\frac{\mu_1}{\beta}+\frac{p\theta(1-\alpha)}{\beta}\right)^{p-k-r+2}}\right\}*P_{j+k+1,j-1,2}(t);\qquad i\geq j+2,\ j>1 \tag{39}$$

$$P_{i,j,0}(t)=\mu_1(1-\gamma)e^{-(\lambda+(i-j)\theta)t}P_{i,j-1,1,1}(t)+\mu_1\gamma e^{-(\lambda+(i-j)\theta)t}P_{i,j,1,1}(t)+$$
$$\mu_2(1-\gamma)e^{-(\lambda+(i-j)\theta)t}P_{i,j-1,1,2}(t)+\mu_2\gamma e^{-(\lambda+(i-j)\theta)t}P_{i,j,1,2}(t);\qquad i>j>1 \tag{40}$$

4. Verification of Results

1. Summing equations (13)-(26) over i and j we get:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left\{\bar{P}_{i,j,0}(s)+\bar{P}_{i,j,1,1}(s)+\bar{P}_{i,j,1,2}(s)+\bar{P}_{i,j,2}(s)\right\}=\frac{1}{s}$$

and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left\{P_{i,j,0}(t)+P_{i,j,1,1}(t)+P_{i,j,1,2}(t)+P_{i,j,2}(t)\right\}=1$$

which is the verification of our results.

2. Define $Q_{n,m}(t)$ = Probability that there are n customers in the orbit when $m(m=0,1,2)$ servers are busy at time t.
When server is free, it is represented by probability $Q_{n,0}(t)$

$$Q_{n,0}(t)=\sum_{j=0}^{\infty}P_{j+n,j,0}(t)$$

The number of customers in the orbit in this case are calculated by using the following formula:
$n\ =\ (number\ of\ arrivals\ -\ number\ of\ departures)$
When one server is busy($m=1$), it is represented by the probability $Q_{n,m,k}(t)$

$$Q_{n,m,k}(t)=\sum_{j=0}^{\infty}P_{j+n+m,j,m,k}(t);\qquad k=1,2$$

The number of customers in the orbit in this case are calculated by using the following formula:
$n\ =\ (number\ of\ arrivals\ -\ number\ of\ departures\ -\ m\ )$
When both servers are busy ($m=2$), it is represented by the probability $Q_{n,m}(t)$

$$Q_{n,m}(t)=\sum_{j=0}^{\infty}P_{j+n+m,j,m}(t)$$

The number of customers in the orbit in this case are calculated by using the following formula:
$n = (number\ of\ arrivals - number\ of\ departures - m)$
Using above definitions in equations (1) to (6) and let μ_1=μ_2=1, γ=0, β=1 and α=1 and using $Q_{n,1,1} + Q_{n,1,2} = Q_{n,1}$ and let $a_1 = a_2 = \frac{1}{2}$ and adding equations we get:

$$(\lambda + n\theta)Q_{n,0} = Q_{n,1}; \qquad n \geq 0 \qquad (41)$$

$$(\lambda + n\theta + 1)Q_{n,1} = \lambda Q_{n,0} + (n+1)\theta Q_{n+1,0} + 2Q_{n,2}; \qquad n \geq 0 \qquad (42)$$

$$(\lambda + 2)Q_{n,2} = \lambda Q_{n,1} + (n+1)\theta Q_{n+1,1} + \lambda(1 - \delta_{n,0})Q_{n-1,2}; \qquad n \geq 0 \qquad (43)$$

which coincides with the results (1)-(3) of [2].

5. Numerical Solution and Graphical Representation

Using MATLAB programming the following numerical results are generated for the case ρ=0.8, η=0.6, γ=0.7, r_1=0.6 (r_2=1-r_1), a_1=0.5 (a_2=1-a_1), β=0.7 and α=0.6. Observing the tables below for various time instants t it could be concluded that the sum of probabilities approaches to 1.

Table 1: *At t=1*

$P_{0,0,0}$	$P_{1,0,0}$	$P_{1,1,0}$	$P_{1,0,1,1}$	$P_{2,0,1,1}$	$P_{1,0,1,2}$	$P_{2,0,1,2}$	$P_{2,0,2}$	$P_{3,0,2}$	Sum
0.4493	0.0444	0.0232	0.1393	0.0128	0.1525	0.0174	0.1016	0.0222	0.9627

Table 2: *At t=15*

$P_{8,5,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{8,4,1,1}$	$P_{8,5,1,1}$	$P_{8,6,1,1}$	$P_{8,7,1,1}$	$P_{7,4,1,2}$	$P_{7,5,1,2}$	$P_{8,3,1,2}$
0.0192	0.0409	0.0504	0.0272	0.0146	0.0349	0.0502	0.0307	0.0057	0.0057	0.0061

$P_{8,4,1,2}$	$P_{8,5,1,2}$	$P_{8,6,1,2}$	$P_{8,7,1,2}$	$P_{6,2,2}$	$P_{6,3,2}$	$P_{6,4,2}$	$P_{7,2,2}$	$P_{7,3,2}$	$P_{7,4,2}$	$P_{7,5,2}$
0.0221	0.0532	0.0772	0.048	0.0064	0.0111	0.0071	0.0052	0.0137	0.0179	0.0093

$P_{8,2,2}$	$P_{8,3,2}$	$P_{8,4,2}$	$P_{8,5,2}$	$P_{8,6,2}$	Sum
0.0066	0.0288	0.0778	0.1262	0.0963	0.8925

Table 3: *t=25*

$P_{0,0,0}$	$P_{1,0,0}$	$P_{8,5,0}$	$P_{8,6,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{8,6,1,1}$	$P_{8,7,1,1}$	$P_{8,5,1,2}$	$P_{8,6,1,2}$	$P_{8,7,1,2}$
0	0	0.0043	0.0377	0.1635	0.2894	0.0451	0.0966	0.0114	0.07	0.1549

$P_{8,4,2}$	$P_{8,5,2}$	$P_{8,6,2}$	Sum
0.0041	0.0254	0.0843	0.9867

Table 4: *At t=35*

$P_{0,0,0}$	$P_{4,3,0}$	$P_{6,4,0}$	$P_{8,5,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{8,6,1,1}$	$P_{8,7,1,1}$	$P_{8,7,1,2}$	$P_{8,5,2}$	Sum
0	0	0	0.0002	0.1247	0.6225	0.0111	0.0732	0.01188	0.0013	0.9518

The probabilities against time are graphically represented in following figures:

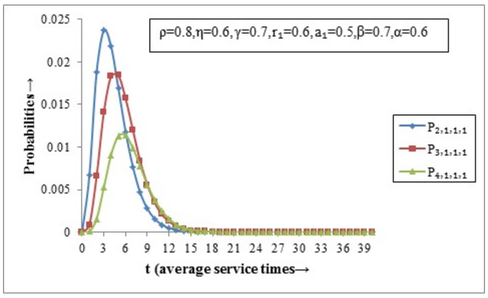

Figure 2

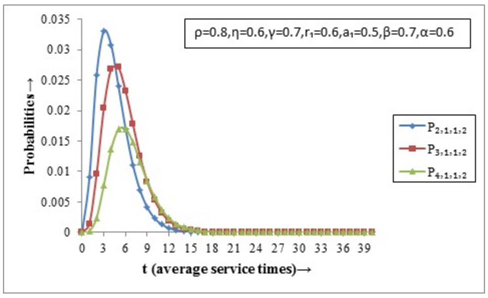

Figure 3

The probabilities $P_{2,1,1}$, $P_{3,1,1}$ and $P_{4,1,1}$ for both the servers 1 and 2 against time t are depicted in figures 2 and 3. From both the figures it is clearly interpreted that probabilities start increasing from 0 at t=0 in the beginning and then start decreasing. Also, the curve peaks are higher for lower number of arrivals. If we compare both the graphs, the probabilities are higher for second server than first because of the difference in r_1 and r_2.

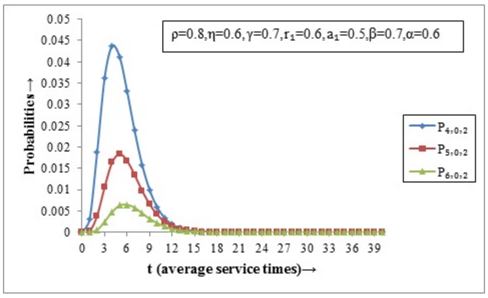

Figure 4

Figure 4 depicts the probabilities $P_{4,0,2}$, $P_{5,0,2}$ and $P_{6,0,2}$ against time t. Beginning with value 0 at t=0 the probabilities increases rapidly to their highest values and then decreases gradually. Also, the probabilities are higher for lower number of arrivals when both the servers are busy.

6. Busy Period Probabilities

The probability that the server is busy is given as:

$$P(Server\ is\ busy) = \sum_{i>j\geq 0} (P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t)) \tag{44}$$

The probability that the system is busy is given as:

$$P(System\ is\ busy) = \sum_{i>j\geq 0} (P_{i,j,0}(t) + P_{i,j,1,1}(t) + P_{i,j,1,2}(t) + P_{i,j,2}(t)) \tag{45}$$

6.1. Numerical and Graphical Representation of Busy Period

Following [13] work and using MATLAB programming the numerical results are generated. The probability for system busy and server busy are studied by varying the value of ρ and keeping other parameters as constant(η=0.6, γ=0.7, r_1=0.6 (r_2=1-r_1), a_1=0.5 (a_2=1-a_1), β=0.7 and α=0.6).

	Probability(System Busy)			Probability(Server Busy)		
t	ρ=0.4	ρ=0.6	ρ=0.8	ρ=0.4	ρ=0.6	ρ=0.8
0	0	0	0	0	0	0
1	0.3122	0.4295	0.5269	0.2772	0.3855	0.4773
2	0.5073	0.6541	0.7571	0.4290	0.5674	0.6713
3	0.6358	0.7801	0.867	0.5286	0.6732	0.771
4	0.7235	0.8542	0.9229	0.5999	0.7413	0.8286
5	0.7848	0.8995	0.9527	0.6532	0.7874	0.864
6	0.8286	0.9281	0.9694	0.6940	0.8196	0.8866
7	0.8606	0.9467	0.979	0.7256	0.8425	0.9009

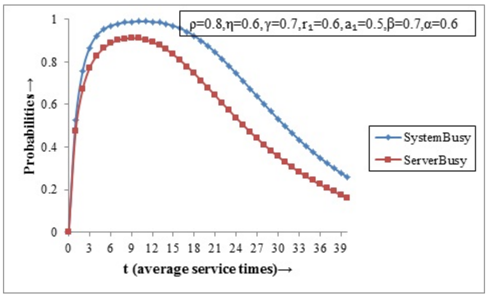

Figure 5

In figure 5 the probabilities for System busy and Server busy are plotted against time t for the case when ρ=0.8, η=0.6, γ=0.7, r_1=0.6 (r_2=1-r_1), a_1=0.5 (a_2=1-a_1), β=0.7 and α=0.6. It is shown that both the probabilities increases rapidly and then decreases gradually Also, System busy remains higher than Server busy as required.

7. Conclusion

We considered a retrial queueing system with non-identical parallel servers having feedback and impatient customers. The transient state probabilities for exact number of arrivals and departures from the system are obtained when both, none or one servers are busy. Various results are also been verified. Numerical and graphical representations are provided in order to study the effect of change of various parameters.

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