ANALYSIS OF A NON-IDENTICAL COMPONENT-STRENGTHS SYSTEM BASED ON LOWER RECORD DATA

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Abstract

In engineering applications and reliability literature, stress-strength models play a crucial role. The goal of this study is to develop more accurate stress-strength models by addressing the reliability estimation in multi-component systems with non-identical component strengths and stress. In the context of lower record values, the system's reliability is assessed using both classical and Bayesian approaches. In classical estimation, the maximum likelihood estimator of the reliability function is constructed, and a simulation study based on measurements of precision is used to assess the behavior of various estimates. The Bayesian estimators of reliability under general entropy, logarithmic and precautionary loss functions are computed. The suggested Bayesian estimates are calculated using the Markov Chain Monte Carlo method through a simulation study because there is no one particular way to do it. We found through simulated research that the accuracy of measurements decreases as the number of records rises. The theoretical results are validated using an example from actual data sets.

Keywords: Exponentiated Pareto model; Lower record data; Bayesian inference; General entropy loss function.

MSC: 62N05; 62D99; 62F15; 62F40; 94A20

1. Introduction

Record values are crucial when collecting observations is challenging or when they are lost during experimental operations. Real-world problems needing destructive stress testing, industrial quality control trials, and statistics on the weather, the economy, and sports all depend critically on record values. Only observations that exceed or fall below the most recent extreme values are recorded in this case. The total number of observations is frequently much lower than the overall sample size and only successive severe items are measured. Suppose that $\{U_i, i=1,2,...\}$ is an unlimited sequence of independent and identically distributed (iid). An observation U_i is called a lower record value (**LRV**) if $U_i < U_j$ for each $i < j$.

The stress-strength (SS) model is a fundamental of reliability testing. When a stress Y surpasses strength X, the SS framework $\Re = P(X > Y)$ indicates the possibility of failure. In other words, the system keeps functioning as long as the stress does not outweigh its ability. Reference [1] was the author who initially presented this idea, and [2] later developed it. Many studies have addressed inferences based on various methods and distributional presumptions; for some recent works, see [3-10].

To build a system with two or more components, the basic idea of $\Re = P(X > Y)$ can be changed. Reliability in a multi-component SS (MSS) model was first created in [11], who investigated the MSS model under the assumption that c out of t system components, where, $(1 \le c \le t)$ components survive a common random stress Y. The MSS is applicable to several fields, including communications, logistics, military, and manufacturing operations. For illustration, if only four of a car's eight cylinders are burning, it might be possible to drive the vehicle. It can therefore be expressed as 4 out of 8: G system.

Assume that if c $(1 \le c \le t)$ or more of the components cooperate, a system with t similar components will function. In its operational setting, the system is subjected to a stress Y that is a random variable with cumulative distribution function (cdf) $G(y)$. The component strengths, or the minimal stresses necessary to manufacture failure, are iid random variables with cdf $F(x)$, then the reliability of the c-out-of-t system is represented by $\Re_{c,t}$ which is developed in [12], is given by:

$$\Re_{c,t} = P\left[at\,least\,c\,of\,(X_1, X_2, ..., X_t)\;\; exceed\,Y\right]$$
$$= \sum_{i=c}^{t} \binom{t}{i} \int_0^\infty [1 - F(x)]^i F(x)^{t-i}\, dG(x).$$

For various SS distributions and sampling procedures, many authors addressed the estimation of reliability in MSS models based on different sampling scheme, for example, see [13-19].

Due to the different topologies of system components, the assumption of comparable strength distributions may not be feasible in many actual circumstances. With systems that have backup components, this is frequently the situation. The strengths of different items, even those constructed of the same material, can vary. For instance, heat treating metals to acquire desired mechanical properties in the field of mechanics can lead to various types of breaking when the metal is quenched or cooled. As a result, the strengths of the components vary. Another example, if two different ropes are used to consolidate a rope, the tensile strengths of both ropes may not be evenly distributed. A model that at least incorporates non-identical random strengths for system components appears to be more realistic, see [20].

Assume a system has t components, of which t_1 are of kind 1, t_2 are of kind 2,…, and the remainder $t_n = t - \sum_{i=1}^{n-1} t_i$ components are of kind n. Let $F_i(.)$, i =1, 2,…, n, be the cdf of the random strengths for components of the i^{th} kind. Assume that Y is a common stress with cdf $Q(.)$ that all components are subjected to. The system will function as long as the c-out-of-the-t components can resist the stress. Reference [21] presented the system reliability $\Re_{c_1,...,c_n,t_1,...,t_n}$ with non-identical component strengths as follows:

$$\Re_{c_1,...,c_n,t_1,...,t_n} = \sum_{j_1=c_1}^{t_1} \cdots \sum_{j_n=c_n}^{t_n} \left(\prod_{i=1}^{n} \binom{t_i}{j_i} \right) \int_0^\infty \prod_{i=1}^{n} (1 - F_i(x))^{j_i} (F_i(x))^{t_i - j_i}\, dQ(x), \tag{1}$$

where summation ranges over all possible combinations $(j_1, j_2, \ldots, j_n)$ with $0 \le j_i \le t_i$ for i = 1, 2, … , n such $c \le \sum_{i=1}^{n} j_i \le t$. Each c_i indicates the minimal number of components of the ith type that the system needs to function.

Considering the investigation of a system with two different sorts of components, the model (1) can be expressed as follows:

$$\Re_{c_1,c_2,t_1,t_2}=\sum_{j_1=c_1}^{t_1}\sum_{j_2=c_2}^{t_2}\binom{t_1}{j_1}\binom{t_2}{j_2}\int_0^{\infty}(1-F_1(x))^{j_1}(F_1(x))^{t_1-j_1}(1-F_2(x))^{j_2}(F_2(x))^{t_2-j_2}dQ(x). \quad (2)$$

In order to construct more realistic models, The Bayesian estimation of $\Re_{c_1,c_2,t_1,t_2}$ assuming the Weibull and exponential distributions on the strength and stress variates, respectively, was taken into consideration in [22] and [23]. The exponentiated Pareto distribution (EPD) was used to estimate $\Re_{c_1,c_2,t_1,t_2}$ for non-identical MSS in [24]. Recently, [25] studied the case of non-identical component-strengths from the family of Kumaraswamy generalized distributions under upper record data. Reference [26] examined the estimation of $\Re_{c_1,c_2,t_1,t_2}$ when component strengths and stress follow inverse Lomax distribution based on complete sample. Reference [27] proposed the estimation of $\Re_{c_1,c_2,t_1,t_2}$ when component strengths and stress follow Weibull distributions under generalized progressive hybrid censoring scheme.

It is important to note that the majority of the work on the estimate of the SS reliability conducted to date requires to employ complete or censored samples and that record values are rarely used. Particularly in the estimation of MSS systems of non-identical component-strengths, we are interested in developing MSS models within the record scheme in the case of non-identical component-strengths, where component strengths and stress follow an EPD. A maximum likelihood estimator (MLE) of $\Re_{c_1,c_2,t_1,t_2}$ is derived under LRV and a simulation study is investigated. The general entropy loss function (GELF), the logarithmic loss function (LLF), and the precautionary loss function (PLF) are used to derive the Bayesian estimator of $\Re_{c_1,c_2,t_1,t_2}$. Since these estimators are incapable of being reduced to simple closed forms, we use the Markov Chain Monte Carlo (MCMCO) approach for Bayesian estimates of $\Re_{c_1,c_2,t_1,t_2}$. To show the relevance of our work, we also examined real data sets.

The following is how the rest of the essay is presented. The formulation of $\Re_{c_1,c_2,t_1,t_2}$, and its MLE under LRV along with a numerical analysis is provided in Section 2. Section 3 discusses the Bayesian estimators of $\Re_{c_1,c_2,t_1,t_2}$, through GELF, LLF, and PLF. The MCMCO technique is presented in Section 4. For the purposes of illustration, Section 5 includes real data sets. Final remarks are included in Section 6.

2. Model Description and Classical Estimation of $\Re_{c_1,c_2,t_1,t_2}$

In this section, a model description of $\Re_{c_1,c_2,t_1,t_2}$ is provided. The MLE of $\Re_{c_1,c_2,t_1,t_2}$ is obtained in the presence of LRV. Numerical analysis is also carried out.

2.1. Expression of $\Re_{c_1,c_2,t_1,t_2}$

Here, expression of $\Re_{c_1,c_2,t_1,t_2}$ is provided when the strength and stress random variables follow the EPD.

The EPD may be successfully used to assess numerous lifetime data sets, as argued in [28]. The EPD has a very flexible structure as a result of its decreasing or upside-down bathtub shape failure rates depending on shape parameters. This property provides advantages for modeling extreme events, particularly in hydrology. Furthermore, the EPD is a reasonable equivalent to the exponential distribution because of its heavier or lighter tail features. A variety of lifetime data might seem nice in the EPD. The EPD 's cdf is expressed by

$$F(x)=\left[1-\left(1+x\right)^{-\gamma}\right]^{\delta},\ x>0,\ \gamma>0,\delta>0,$$

where δ and γ are the shape parameters. The associated probability density function (pdf) is given by:

$$f(x)=\delta\gamma\left[1-(1+x)^{-\gamma}\right]^{\delta-1}(1+x)^{-(\gamma+1)},\ x>0,\ \gamma>0,\delta>0.$$

Several scholars addressed the EP's research and applications, for instance see [29].

From the total of t system components in the model (2), let the first t_1 of first kind component strengths follow EPD(γ,δ_1), while the remaining $t_2=t-t_1$ of kind 2 component strengths follow EPD(γ,δ_2). Also, suppose that Y follows EPD(γ,δ_3) independently. The respective distribution functions are as below:

$$F_i(x)=\left[1-(1+x)^{-\gamma}\right]^{\delta_i},\ x,\gamma,\delta_i>0, i=1,2, \tag{3}$$

$$Q(y)=\left[1-(1+y)^{-\gamma}\right]^{\delta_3},\ y,\gamma,\delta_3>0. \tag{4}$$

By replacing F_1, F_2, Q given in (2) by (3) and (4), the formula of $\Re_{c_1,c_2,t_1,t_2}$ for such a system is as follows:

$$\Re_{c_1,c_2,t_1,t_2}=\sum_{j_1=c_1}^{t_1}\sum_{j_2=c_2}^{t_2}\binom{t_1}{j_1}\binom{t_2}{j_2}\int_0^\infty(1-(1-(1+x)^{-\gamma})^{\delta_1})^{j_1}(1-(1+x)^{-\gamma})^{\delta_1(t_1-j_1)}(1-(1-(1+x)^{-\gamma})^{\delta_2})^{j_2}$$

$$\times(1-(1+x)^{-\gamma})^{\delta_2(t_2-j_2)}\delta_3\gamma\left[1-(1+x)^{-\gamma}\right]^{\delta_3-1}(1+x)^{-(\gamma+1)}\,dx.$$

Let $z=1-(1+x)^{-\gamma}$, $dz=\gamma(1+x)^{-\gamma-1}$, then $\Re_{c_1,c_2,t_1,t_2}$ is as follows

$$\Re_{c_1,c_2,t_1,t_2}=\sum_{j_1=c_1}^{t_1}\sum_{j_2=c_2}^{t_2}\binom{t_1}{j_1}\binom{t_2}{j_2}\delta_3\int_0^1(1-z^{\delta_1})^{j_1}(1-z^{\delta_2})^{j_2}z^{\delta_1(t_1-j_1)+\delta_2(t_2-j_2)+\delta_3-1}dz.$$

Using the binomial expansion for $(1-z^{\delta_1})^{j_1}$ and $(1-z^{\delta_2})^{j_2}$ leads to the following

$$\begin{aligned}\Re_{c_1,c_2,t_1,t_2}&=E_{j_1,j_2,m,n}\,\delta_3\int_0^1 z^{\delta_1(m+t_1-j_1)+\delta_2(n+t_2-j_2)+\delta_3-1}dz\\&=\frac{E_{j_1,j_2,m,n}\delta_3}{\delta_1(m+t_1-j_1)+\delta_2(n+t_2-j_2)+\delta_3},\end{aligned} \tag{5}$$

where $E_{j_1,j_2,m,n}=\sum_{j_1=c_1}^{t_1}\sum_{j_2=c_2}^{t_2}\sum_{m=0}^{j_1}\sum_{n=0}^{j_2}\binom{t_1}{j_1}\binom{t_2}{j_2}\binom{j_1}{m}\binom{j_2}{n}(-1)^{m+n}$.

Note that expression (5) depends on the parameters δ_1, δ_2 and δ_3.

2.2. The MLE of $\Re_{c_1,c_2,t_1,t_2}$ under LRV

In order to find the MLE of $\Re_{c_1,c_2,t_1,t_2}$ based on LRV, we first need to obtain the MLE of the parameters δ_1, δ_2 and δ_3 assuming γ is given known. So, let $\underline{r}=\{r_1,...,r_n\}$ $\underline{p}=\{p_1,...,p_m\}$ and $\underline{s}=\{s_1,...,s_w\}$ be three independent sets of LRV of sizes n, m, and w from EPD(γ,δ_1), EPD(γ,δ_2) and EPD(γ,δ_3) respectively. The likelihood function of the observed records, according to [30], is defined by:

$$L(\underline{u}\mid\eta)=h(u_d)\prod_{i=1}^{d-1}\frac{h(u_i;\eta)}{H(u_i;\eta)},\quad -\infty<u_d<...<u_1<\infty. \tag{6}$$

Hence, the observed LRV data $\underline{r},\underline{p}$ and $\underline{s}$, given η, based on (6), are given as below:

$$L(\underline{r},\underline{p},\underline{s}\mid\eta)=L_1(\underline{r}\mid\gamma,\delta_1)\,L_2(\underline{p}\mid\gamma,\delta_2)\,L_3(\underline{s}\mid\gamma,\delta_3)$$
$$=\delta_1^{\,n}\delta_2^{\,m}\delta_3^{\,w}(\gamma)^{n+m+w}(1+r_n)^{-(\gamma+1)}(1+p_m)^{-(\gamma+1)}(1+s_w)^{-(\gamma+1)}[\tau_n]^{\delta_1-1}$$
$$\times[\varphi_m]^{\delta_2-1}[\zeta_w]^{\delta_3-1}\prod_{i=1}^{n-1}(1+r_i)^{-(\gamma+1)}(\tau_i)^{-1}\prod_{j=1}^{m-1}(1+p_j)^{-(\gamma+1)}(\varphi_j)^{-1} \tag{7}$$
$$\times\prod_{u=1}^{w-1}(1+s_u)^{-(\gamma+1)}(\zeta_u)^{-1},$$

where $\tau_i=1-(1+r_i)^{-\gamma}$, $\varphi_j=1-(1+p_j)^{-\gamma}$, and $\zeta_u=1-(1+s_u)^{-\gamma}$, $i=1,\dots,n,\ j=1,\dots,m,$ $u=1,\dots,w.$

Consequently, the joint log-likelihood function, denoted by $\ln\ell,$ is derived as:

$$\ln\ell=n\ln(\delta_1)+m\ln(\delta_2)+w\ln(\delta_3)+(n+m+w)\ln(\gamma)-(\gamma+1)\ln(1+r_n)+\ln(1+p_m)+\ln(1+s_w)$$
$$+(\delta_1-1)\ln[\tau_n]+(\delta_2-1)\ln[\varphi_m]+(\delta_3-1)\ln[\zeta_w]-\sum_{i=1}^{n-1}\left[(\gamma+1)\ln(1+r_i)+\ln(\tau_i)\right]$$
$$-\sum_{j=1}^{m-1}\left[(\gamma+1)\ln(1+p_j)+\ln(\varphi_j)\right]-\sum_{u=1}^{w-1}\left[(\gamma+1)\ln(1+s_u)+\ln(\zeta_u)\right].$$

Given that γ is a known, the following are the partial derivatives of $\ln\ell$ with respect to δ_1,δ_2 and δ_3 respectively

$$\frac{\partial\ln\ell}{\partial\delta_1}=\frac{n}{\delta_1}+\ln[\tau_n],\ \frac{\partial\ln\ell}{\partial\delta_2}=\frac{m}{\delta_2}+\ln[\varphi_m],\ \frac{\partial\ln\ell}{\partial\delta_3}=\frac{w}{\delta_3}+\ln[\zeta_w].$$

Then, the MLEs of δ_1,δ_2 and δ_3 denoted by $\hat{\delta}_1$, $\hat{\delta}_2$ and $\hat{\delta}_3$ are obtained by setting $\partial\ln\ell/\partial\delta_1$, $\partial\ln\ell/\partial\delta_2$ and $\partial\ln\ell/\partial\delta_3$ to be zero. Hence $\hat{\delta}_1,\hat{\delta}_2$ and $\hat{\delta}_3$ are obtained as

$$\hat{\delta}_1=\frac{-n}{\ln[\tau_n]},\hat{\delta}_2=\frac{-m}{\ln[\varphi_m]},\hat{\delta}_3=\frac{-w}{\ln[\zeta_w]}. \tag{8}$$

Therefore, based on invariance property, we obtain the MLE of $\Re_{c_1,c_2,t_1,t_2}$ by inserting $\hat{\delta}_1$, $\hat{\delta}_2$ and $\hat{\delta}_3$ in (5) as follows

$$\hat{\Re}_{c_1,c_2,t_1,t_2}=\frac{E_{j_1,j_2,m,n}\,\hat{\delta}_3}{\hat{\delta}_1(m+t_1-j_1)+\hat{\delta}_2(n+t_2-j_2)+\hat{\delta}_3}. \tag{9}$$

2.3. Numerical Study

The MLE for the MSS variables is thoroughly numerically analysed in this subsection. In order to assess the accuracy of estimates for various parameter values and record numbers, two criteria are used: absolute biases (ABs) and mean squared errors (MSEs). The numerical research is performed in the following way:

- Create LRV samples based on the parameter values provided.
- The parameters values are selected as $(\delta_1,\delta_2,\delta_3)$ = (0.5,0.5,0.2), (1.1,0.5,0.2), (0.5,0.5,0.5) and (1.1,1,0.2) for $\gamma=1$ in all situations. The specified values for c-out-of-t systems are $(c_1,\ c_2,\ t_1,\ t_2)$ = (1,1,2,2), (1,2,2,2), (2,1,2,2) and (2,2,2,2).
- The true values at $(c_1,\ c_2,\ t_1,\ t_2)$= (1,1,2,2) are 0.533, 0.871, 0.758 and 0.809, at $(c_1,\ c_2,\ t_1,\ t_2)$= (1,2,2,2) are 0.3, 0.746,0.573 and 0.591, at $(c_1,\ c_2,\ t_1,\ t_2)$=(2,1,2,2) are 0.301, 0.760, 0.573 and 0.724, and at $(c_1,\ c_2,\ t_1,\ t_2)$ = (2,2,2,2) are 0.2, 0.687, 0.477 and 0.562.
- The sample sizes of LRV samples $(n,\ m,\ w)$ are selected to be (2,2,2), (5,5,5), (7,7,7), (10,10,10), (2,2,3), (5,5,6), (7,7,8) and (10,10,11).
- 5000 repetitions are used to evaluate the ABs and MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$.

- The simulated outcomes are shown in Table 1 and are illustrated in Figures 1–6.
- As the number of records increases, the MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ for all values of $(c_1,\ c_2,\ t_1,\ t_2)$ decrease (Figure 1). For all true value of parameters, the MSE of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ decreases at $(c_1,\ c_2,\ t_1,\ t_2)$ = (2,2,2,2) when the number of records $n = m$ (Figure 2).

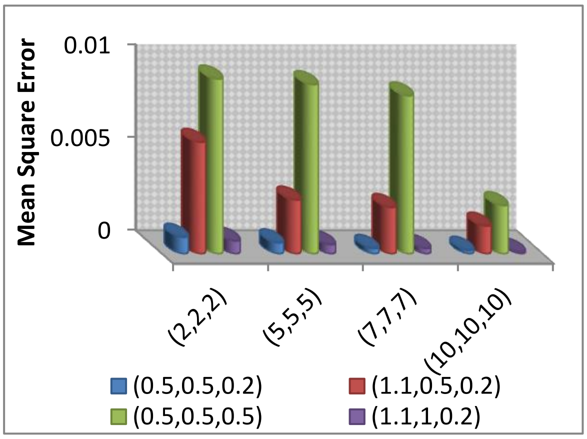


Figure1: *MSEs of* $\hat{\Re}_{c_1,c_2,t_1,t_2}$ *for different* $(\delta_1,\delta_2,\delta_3)$ *values at* $(c_1,c_2,t_1,t_2)=(1,1,2,2)$ *and* $n = m = w$

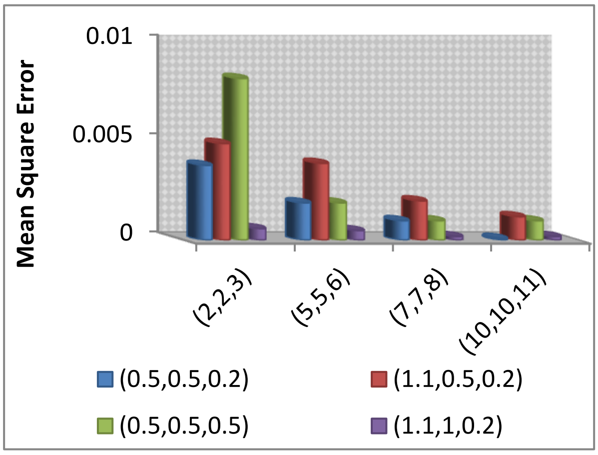


Figure 2: *MSEs of* $\hat{\Re}_{c_1,c_2,t_1,t_2}$ *for different* $(\delta_1,\delta_2,\delta_3)$ *values at* $(c_1,c_2,t_1,t_2)=(2,2,2,2)$ *and n=m*

- Figure 3 demonstrates that as the number of *n*, *m* and *w* increases, the ABs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ for all actual values of $(\delta_1,\delta_2,\delta_3)$ are decreasing.
- Figure 4 illustrates that the MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ at $(c_1,\ c_2,\ t_1,\ t_2)$ = (2,2,2,2) are larger than the MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ for others values of $(c_1,\ c_2,\ t_1,\ t_2)$.

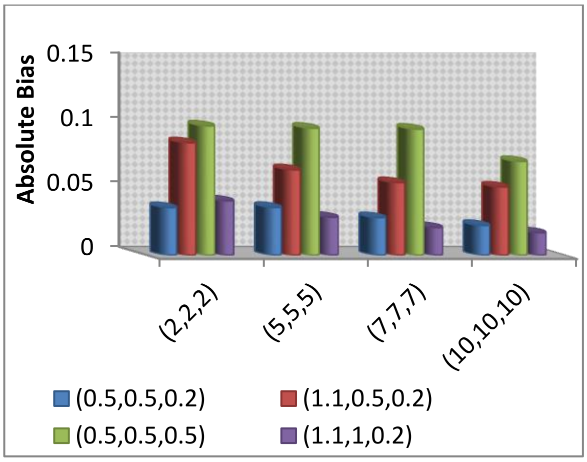


Figure 3: *ABs of* $\hat{\Re}_{c_1,c_2,t_1,t_2}$ *for different* $(\delta_1,\delta_2,\delta_3)$ *values at* $(c_1,c_2,t_1,t_2)=(1,2,2,2)$ *and* $n = m = w$

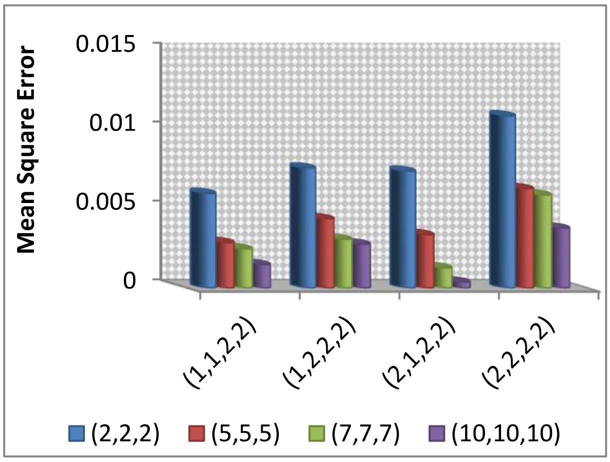


Figure 4: *MSEs of* $\hat{\Re}_{c_1,c_2,t_1,t_2}$ *for different* (c_1,c_2,t_1,t_2) *values at* $n = m = w$ *and* $(\delta_1,\delta_2,\delta_3)=(1.1,0.5,0.2)$

- Figure 5 illustrates that the MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ at $(c_1,\ c_2,\ t_1,\ t_2)$ = (1, 1, 2, 2) are smaller than the MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ for others values of $(c_1,\ c_2,\ t_1,\ t_2)$.
- Figure 6 illustrates that the MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ decrease when the true value of $\Re_{c_1,c_2,t_1,t_2}$ increases.

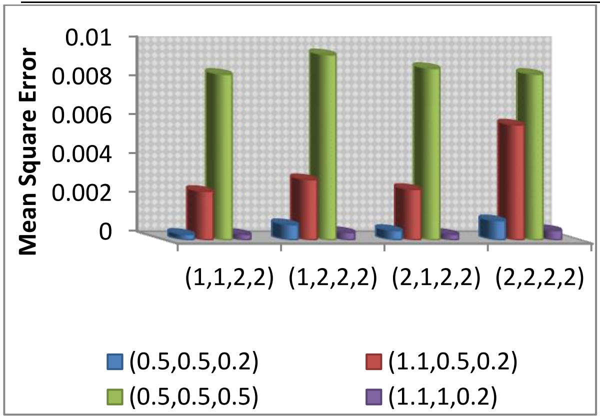


Figure 5: *MSEs of* $\hat{\Re}_{c_1,c_2,t_1,t_2}$ *for different* $(\delta_1,\delta_2,\delta_3)$ *and* (c_1,c_2,t_1,t_2) *values at n = m = w =7*

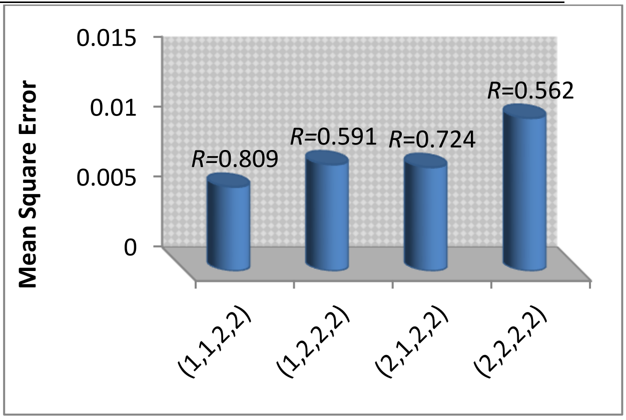


Figure 6: *MSEs of* $\hat{\Re}_{c_1,c_2,t_1,t_2}$ *for* $(\delta_1,\delta_2,\delta_3)$ *= (1.1, 0.5, 0.2) at n = m = w =2*

Table1: *Numerical results of* $\hat{\Re}_{c_1,c_2,t_1,t_2}$ *for different values of* $(\delta_1,\delta_2,\delta_3)$

$(\delta_1,\delta_2,\delta_3)=(0.5,0.5,0.2)$					$(\delta_1,\delta_2,\delta_3)=(1.1,0.5,0.2)$				
(c_1,c_2,t_1,t_2)	Real $\Re_{c_1,c_2,t_1,t_2}$	(n,m,w)	AB	MSE	(c_1,c_2,t_1,t_2)	Real $\Re_{c_1,c_2,t_1,t_2}$	(n,m,w)	AB	MSE
(1,1,2,2)	**0.758**	(2,2,2)	0.0303	0.0009	**(1,1,2,2)**	**0.809**	(2,2,2)	0.0772	0.0060
		(5,5,5)	0.0255	0.0006			(5,5,5)	0.0546	0.0029
		(7,7,7)	0.0197	0.0003			(7,7,7)	0.0506	0.0025
		(10,10,10)	0.0182	0.0002			(10,10,10)	0.0394	0.0015
		(2,2,3)	0.0244	0.0005			(2,2,3)	0.0284	0.0008
		(5,5,6)	0.0137	0.0002			(5,5,6)	0.0282	0.0007
		(7,7,8)	0.0075	0.0001			(7,7,8)	0.0205	0.0004
		(10,10,11)	0.0066	0.0001			(10,10,11)	0.0212	0.0004
(1,2,2,2)	**0.573**	(2,2,2)	0.0371	0.0013	**(1,2,2,2)**	**0.591**	(2,2,2)	0.0872	0.0076
		(5,5,5)	0.0372	0.0013			(5,5,5)	0.0664	0.0044
		(7,7,7)	0.0293	0.0008			(7,7,7)	0.0564	0.0031
		(10,10,10)	0.0233	0.0004			(10,10,10)	0.0530	0.0028
		(2,2,3)	0.0345	0.0011			(2,2,3)	0.0628	0.0039
		(5,5,6)	0.0307	0.0009			(5,5,6)	0.0484	0.0023
		(7,7,8)	0.0281	0.0007			(7,7,8)	0.0302	0.0009
		(10,10,11)	0.0118	0.0001			(10,10,11)	0.0305	0.0009
(2,1,2,2)	**0.573**	(2,2,2)	0.0306	0.0009	**(2,1,2,2)**	**0.724**	(2,2,2)	0.0863	0.0074
		(5,5,5)	0.0276	0.0007			(5,5,5)	0.0583	0.0034
		(7,7,7)	0.0238	0.0005			(7,7,7)	0.0364	0.0026
		(10,10,10)	0.0210	0.0003			(10,10,10)	0.0202	0.0004
		(2,2,3)	0.0233	0.0005			(2,2,3)	0.0303	0.0009
		(5,5,6)	0.0162	0.0003			(5,5,6)	0.0285	0.0008
		(7,7,8)	0.0128	0.0002			(7,7,8)	0.0272	0.0007
		(10,10,11)	0.0115	0.0001			(10,10,11)	0.0264	0.0007
(2,2,2,2)	**0.477**	(2,2,2)	0.0978	0.0095	**(2,2,2,2)**	**0.562**	(2,2,2)	0.1046	0.0109
		(5,5,5)	0.0530	0.0028			(5,5,5)	0.0794	0.0063
		(7,7,7)	0.0317	0.0010			(7,7,7)	0.0770	0.0059
		(10,10,10)	0.0305	0.0008			(10,10,10)	0.0619	0.0038
		(2,2,3)	0.0622	0.0038			(2,2,3)	0.0705	0.0049
		(5,5,6)	0.0437	0.0019			(5,5,6)	0.0629	0.0039
		(7,7,8)	0.0323	0.0010			(7,7,8)	0.0450	0.0020
		(10,10,11)	0.0211	0.0001			(10,10,11)	0.0346	0.0012

	$(\delta_1,\delta_2,\delta_3)=(0.5,0.5,0.5)$					$(\delta_1,\delta_2,\delta_3)=(1.1,1,0.2)$			
(c_1,c_2,t_1,t_2)	**Real** $\Re_{c_1,c_2,t_1,t_2}$	(n,m,w)	**AB**	**MSE**	(c_1,c_2,t_1,t_2)	**Real** $\Re_{c_1,c_2,t_1,t_2}$	(n,m,w)	**AB**	**MSE**
(1,1,2,2)	**0.533**	(2,2,2)	0.0972	0.0094	**(1,1,2,2)**	**0.871**	(2,2,2)	0.0267	0.0007
		(5,5,5)	0.0958	0.0091			(5,5,5)	0.0222	0.0005
		(7,7,7)	0.0922	0.0085			(7,7,7)	0.0190	0.0003
		(10,10,10)	0.0515	0.0026			(10,10,10)	0.0183	0.0001
		(2,2,3)	0.0707	0.0050			(2,2,3)	0.0173	0.0003
		(5,5,6)	0.0628	0.0039			(5,5,6)	0.0130	0.0002
		(7,7,8)	0.0395	0.0015			(7,7,8)	0.0122	0.0001
		(10,10,11)	0.0251	0.0010			(10,10,11)	0.0113	0.0001
(1,2,2,2)	**0.3**	(2,2,2)	0.1002	0.0100	**(1,2,2,2)**	**0.746**	(2,2,2)	0.0423	0.0017
		(5,5,5)	0.0984	0.0096			(5,5,5)	0.0296	0.0008
		(7,7,7)	0.0977	0.0095			(7,7,7)	0.0217	0.0004
		(10,10,10)	0.0728	0.0053			(10,10,10)	0.0176	0.0002
		(2,2,3)	0.0717	0.0049			(2,2,3)	0.0228	0.0005
		(5,5,6)	0.0684	0.0046			(5,5,6)	0.0216	0.0004
		(7,7,8)	0.0558	0.0031			(7,7,8)	0.0128	0.0001
		(10,10,11)	0.0301	0.0018			(10,10,11)	0.0121	0.0001
(2,1,2,2)	**0.301**	(2,2,2)	0.0957	0.0091	**(2,1,2,2)**	**0.760**	(2,2,2)	0.0396	0.0015
		(5,5,5)	0.0954	0.0091			(5,5,5)	0.0288	0.0007
		(7,7,7)	0.0947	0.0088			(7,7,7)	0.0215	0.0003
		(10,10,10)	0.0701	0.0049			(10,10,10)	0.0170	0.0001
		(2,2,3)	0.0720	0.0051			(2,2,3)	0.0222	0.0004
		(5,5,6)	0.0636	0.0040			(5,5,6)	0.0211	0.0002
		(7,7,8)	0.0422	0.0017			(7,7,8)	0.0121	0.0001
		(10,10,11)	0.0288	0.0011			(10,10,11)	0.0118	0.0001
(2,2,2,2)	**0.2**	(2,2,2)	0.1028	0.0105	**(2,2,2,2)**	**0.687**	(2,2,2)	0.0430	0.0020
		(5,5,5)	0.0530	0.0098			(5,5,5)	0.0317	0.0009
		(7,7,7)	0.0317	0.0085			(7,7,7)	0.0220	0.0005
		(10,10,10)	0.0301	0.0050			(10,10,10)	0.0178	0.0003
		(2,2,3)	0.0906	0.0082			(2,2,3)	0.0230	0.0006
		(5,5,6)	0.0437	0.0019			(5,5,6)	0.0235	0.0005
		(7,7,8)	0.0323	0.0010			(7,7,8)	0.0130	0.0002
		(10,10,11)	0.0299	0.0010			(10,10,11)	0.0128	0.0002

3. Bayesian Estimation of $\Re_{c_1,c_2,t_1,t_2}$

We will look in this section at the Bayesian estimator of $\Re_{c_1,c_2,t_1,t_2}$ under the assumption that δ_1,δ_2 and δ_3 are random variables.

Following [31], the prior distributions for δ_1,δ_2 and δ_3 are assumed to have the gamma distribution with the following pdfs

$$\pi_1(\delta_1)\propto \delta_1^{a_1-1}e^{-b_1\delta_1},\ \pi_2(\delta_2)\propto \delta_2^{a_2-1}e^{-b_2\delta_2},\ \text{and}\ \pi_3(\delta_3)\propto \delta_3^{a_3-1}e^{-b_3\delta_3},$$

where, the hyper-parameters; a_1, a_2, a_3, b_1, b_2 and b_3 are considered to be known. The joint prior distribution of $\eta=(\delta_1,\delta_2,\delta_3)$, assuming parameters independence is as follows:

$$\pi(\eta)=\delta_1^{a_1-1}\delta_2^{a_2-1}\delta_3^{a_3-1}e^{-(b_1\delta_1+b_2\delta_2+b_3\delta_3)}.$$

Based on the observed samples, the joint density function of $\eta=(\delta_1,\delta_2,\delta_3)$ and the data are:

$$\pi(\eta \mid \underline{r}, \underline{p}, \underline{s}) = \delta_1^{n+a_1-1}\delta_2^{m+a_2-1}\delta_3^{w+a_3-1}e^{-(b_1\delta_1+b_2\delta_2+b_3\delta_3)}$$
$$\times(\gamma)^{n+m+w}(1+r_n)^{-(\gamma+1)}(1+p_m)^{-(\gamma+1)}(1+s_w)^{-(\gamma+1)}[\tau_n]^{\delta_1-1}$$
$$\times[\varphi_m]^{\delta_2-1}[\zeta_w]^{\delta_3-1}\prod_{i=1}^{n-1}\frac{(1+r_i)^{-(\gamma+1)}}{(\tau_i)}\prod_{j=1}^{m-1}\frac{(1+p_j)^{-(\gamma+1)}}{(\varphi_j)}\prod_{u=1}^{w-1}\frac{(1+s_u)^{-(\gamma+1)}}{(\zeta_u)}.$$

As a result, the posterior density function of $\eta = (\delta_1, \delta_2, \delta_3)$ can be expressed as

$$\pi^*\left(\eta \mid \underline{r}, \underline{p}, \underline{s}\right) = \frac{L\left(\underline{r}, \underline{p}, \underline{s} \mid \eta\right)\pi(\eta)}{\int_0^\infty\int_0^\infty\int_0^\infty L\left(\underline{r}, \underline{p}, \underline{s} \mid \eta\right)\pi(\eta)\,d\delta_1 d\delta_2 d\delta_3}.$$

The Bayesian estimator of $\Re_{c_1,c_2,t_1,t_2}$, based on GELF, indicated by $\widehat{\Re}_{c_1,c_2,t_1,t_2}$ is derived as follows:

$$\widehat{\Re}_{c_1,c_2,t_1,t_2} = [E(\Re_{c_1,c_2,t_1,t_2})^{-\xi}]^{\frac{-1}{\xi}} = \left[\int_0^\infty\int_0^\infty\int_0^\infty(\Re_{c_1,c_2,t_1,t_2})^{-\xi}\pi^*\left(\eta \mid \underline{r}, \underline{p}, \underline{s}\right)d\delta_1 d\delta_2 d\delta_3\right]^{\frac{-1}{\xi}}. \quad (10)$$

Additionally, the Bayesian estimator of $\Re_{c_1,c_2,t_1,t_2}$, under LLF indicated by $\breve{\Re}_{c_1,c_2,t_1,t_2}$ is as follows:

$$\breve{\Re}_{c_1,c_2,t_1,t_2} = \exp(E(\log\Re_{c_1,c_2,t_1,t_2})) = \exp\left[\int_0^\infty\int_0^\infty\int_0^\infty \log\Re_{c_1,c_2,t_1,t_2}\,\pi^*\left(\eta \mid \underline{r}, \underline{p}, \underline{s}\right)d\delta_1 d\delta_2 d\delta_3\right]. \quad (11)$$

The Bayesian estimator of $\Re_{c_1,c_2,t_1,t_2}$ for PLF indicated by $\ddot{\Re}_{c_1,c_2,t_1,t_2}$ is as follows:

$$\ddot{\Re}_{c_1,c_2,t_1,t_2} = \sqrt{E(\Re^2_{c_1,c_2,t_1,t_2})} = \left[\int_0^\infty\int_0^\infty\int_0^\infty \Re^2_{c_1,c_2,t_1,t_2}\,\pi^*\left(\eta \mid \underline{r}, \underline{p}, \underline{s}\right)d\delta_1 d\delta_2 d\delta_3\right]^{0.5}. \quad (12)$$

It is difficult to find an explicit formula for (10)–(12) because the posterior density function $\pi^*\left(\eta \mid \underline{r}, \underline{p}, \underline{s}\right)$ has a composite structure. In order to obtain Bayesian estimates, we calculate these integrations using the Metropolis-Hastings (M-H) technique using the MCMCO algorithm.

4. MCMCO Methodology

The MCMCO simulation is used to investigate the behavior $\Re_{c_1,c_2,t_1,t_2}$'s MSS. Bayes estimates (BE) under different loss functions are produced using gamma priors. The $\Re_{c_1,c_2,t_1,t_2}$'s BE accuracy is measured using the ABs, and MSEs. The various LRV options are (n, m, w) = (2, 2, 2), (5, 5, 5), (7, 7, 7), (10, 10, 10), (2, 2, 3), (5, 5, 6), and (7, 7, 8). The possible sets of hyperparameter values are considered to be: Prior I: (2, 1.5, 3, 2, 1.5, 1.1) and Prior II:(1, 1.4, 1, 2, 2.5, 3).

The outcomes are based on 5,000 replications. The M-H process is a popular subgroup of the MCMCO technique in the Bayesian literature for modeling departures from the posterior density and producing accurate anticipated results. The main difficulty with the MCMCO is getting the BEs of $\Re_{c_1,c_2,t_1,t_2}$ from GELF, LLF, and PLF using the M-H approach after simulating samples from the posterior density. It converges to the desired distribution using acceptance/rejection criteria. The M-H algorithm (see [32]) operates as follows:

a) Set the starting parameter value of $\Re^0_{c_1,c_2,t_1,t_2}$ and the sample number N.
b) For i = 2 to N, set $\Re_{c_1,c_2,t_1,t_2} = \Re^{i-1}_{c_1,c_2,t_1,t_2}$.
c) Create u using the uniform (0,1).
d) Choose a candidate parameter $\Re^*_{c_1,c_2,t_1,t_2}$ from the proposal density.

e) If $u \le \dfrac{\pi(\theta^*) g(\theta|\theta^*)}{\pi(\theta) g(\theta|\theta^*)}$, then set $\Re^i_{c_1,c_2,t_1,t_2} = \Re^*_{c_1,c_2,t_1,t_2}$; otherwise, set $\Re^i_{c_1,c_2,t_1,t_2} = \Re_{c_1,c_2,t_1,t_2}$.

f) Return to step (b) and perform the aforementioned actions N times using $i = i+1$.

Using the outputs of the study, which are shown in Tables 2, 3, and are illustrated by Figures 7–12, we come up with the following conclusions:

- The MSEs and ABs of $\Re_{c_1,c_2,t_1,t_2}$ estimates via the GELF, LLF and PLF decrease with increasing the record numbers *n, m, w* rises for all true values of $(c_1,\ c_2,\ t_1,\ t_2)$, (Tables 2, 3).
- The ABs of $\Re_{c_1,c_2,t_1,t_2}$ estimates via the GELF, LLF and PLF have the smallest values at $(c_1,\ c_2,\ t_1,\ t_2)$ = (1, 1, 2, 2), (Tables 2, 3).
- At true value $\Re_{c_1,c_2,t_1,t_2} = 0.748$, the MSE of $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ via PLF take the smallest values in case of prior I except at (7, 7, 8) (see Figure 7).
- At true value $\Re_{c_1,c_2,t_1,t_2} = 0.748$, the AB of $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ at PLF gets the fewest values for a distinct number of records excepting at $(n,m,w) = (7,7,8)$ via prior I (see Figure 8).

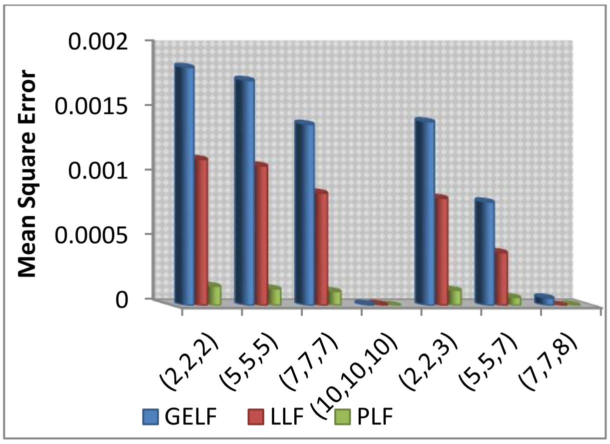


Figure 7: *MSEs of* $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ *at* (c_1,c_2,t_1,t_2) = *(1, 2, 2, 2) for prior I*

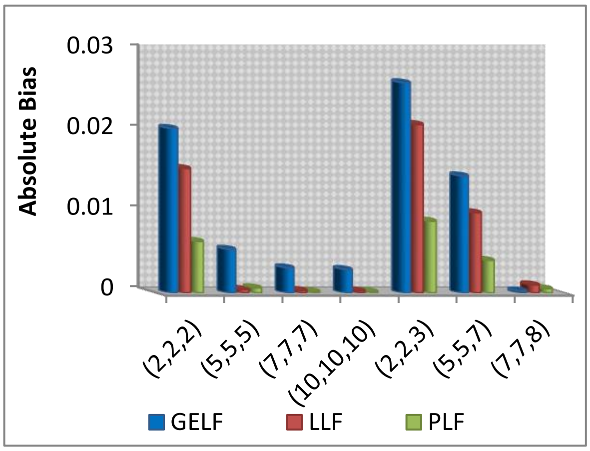


Figure 8: *ABs of* $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ *at* (c_1,c_2,t_1,t_2) = *(1, 1, 2, 2) for prior I*

- The MSEs of $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ under the GELF, LLF and PLF, respectively, decrease as the number of records *n* = *m* = *w* increases via prior II (see Figure 9).
- At true value $\Re_{c_1,c_2,t_1,t_2} = 0.760$, the MSEs of $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ under the GELF, LLF and PLF, respectively, get the least values for similar record values of (*n*, *m*) via prior II (see Figure 10).

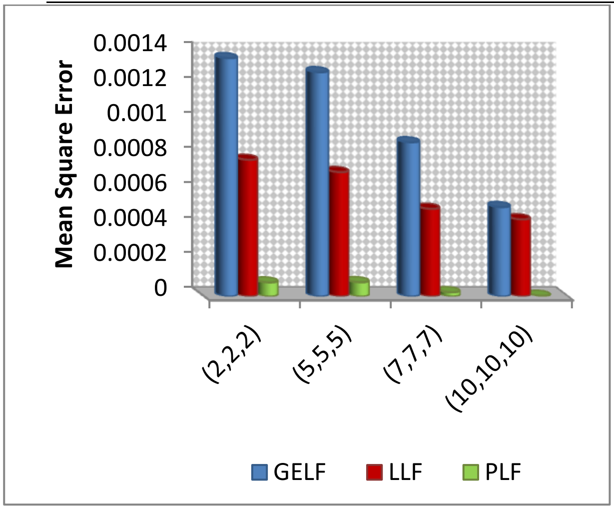


Figure 9: *MSEs of* $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, *and* $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ *at* $(c_1,c_2,t_1,t_2)=(2,1,2,2)$ *for prior II*

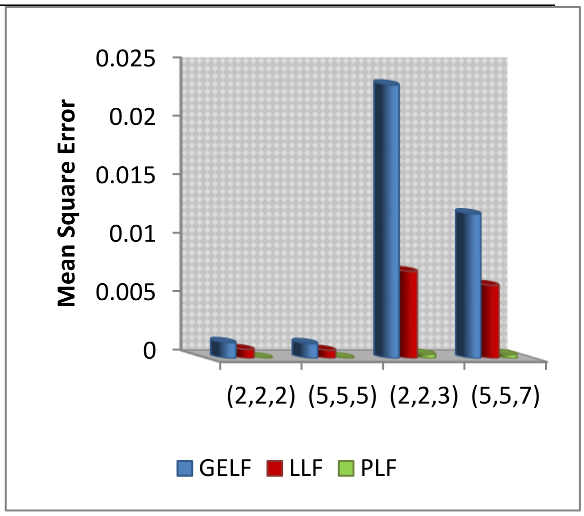


Figure 10: *MSEs of* $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, *and* $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ *at true value* $\Re_{c_1,c_2,t_1,t_2}=0.760$ *for prior II*

- At true value $\Re_{c_1,c_2,t_1,t_2}=0.746$, the MSEs of $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ gets the smallest values compared to $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, and $\breve{\Re}_{c_1,c_2,t_1,t_2}$ for similar record values except at $(n,\ m,\ w)=(10,10,10)$ via prior II (see Figure 11).
- Figure 12 illustrates that the ABs of $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ decrease as true value of $\Re_{c_1,c_2,t_1,t_2}$ increases for $(n,\ m,\ w)=(2,2,2)$.

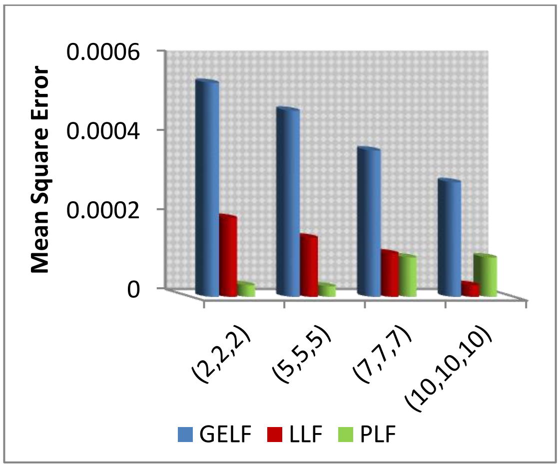


Figure 11: *MSEs of* $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, *and* $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ *at* (c_1,c_2,t_1,t_2) *=(1, 2, 2, 2) for prior II*

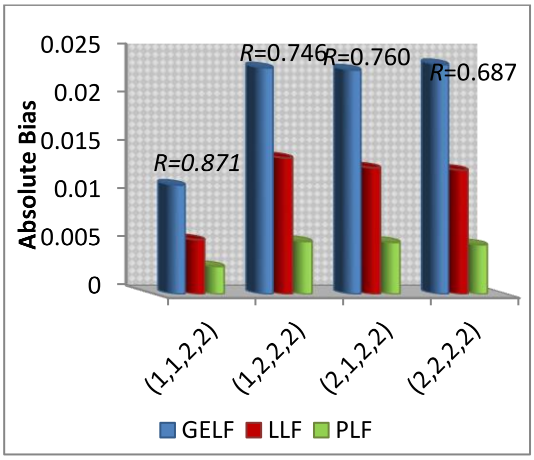


Figure 12: *The ABs for all true values of* $\Re_{c_1,c_2,t_1,t_2}$ *at* $n=m=w=2$ *for prior II*

Table 2: *Numerical results of* $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ *for prior I*

	$(c_1,c_2,t_1,t_2)=(1,2,2,2)$				$(c_1,c_2,t_1,t_2)=(1,2,2,2)$			
Loss function	**Real $\Re_{c_1,c_2,t_1,t_2}$**	(n,m,w)	**AB**	**MSE**	**Real $\Re_{c_1,c_2,t_1,t_2}$**	(n,m,w)	**AB**	**MSE**
GELF	**0.871**		0.02036	0.00041	**0.746**		0.04300	0.00184
LLF		(2, 2, 2)	0.01541	0.00023		(2, 2, 2)	0.03373	0.00113
PLF			0.006413	0.00004			0.01229	0.00015
GELF			0.00542	2.9E-05			0.04175	0.00174
LLF		(5,5,5)	0.00048	2.3E-07		(5,5,5)	0.03295	0.00108
PLF			0.00067	4.5E-07			0.01169	0.00013
GELF		(7,7,7)	0.00314	9.9E-06		(7,7,7)	0.03750	0.00140

Loss function	Real	(n,m,w)	AB	MSE	Real	(n,m,w)	AB	MSE
LLF			0.00036	1.3E-07			0.02955	0.00087
PLF			0.00023	5.4E-08			0.01078	0.00011
GELF			0.00293	8.6E-06			0.00323	1.0E-05
LLF		(10,10,10)	0.00030	1.3E-07		(10,10,10)	0.00289	8.3E-06
PLF			0.00020	4.2E-08			0.00005	3.2E-09
GELF			0.02601	0.00067			0.03778	0.00142
LLF		(2,2,3)	0.02085	0.00043		(2,2,3)	0.02893	0.00083
PLF			0.00891	0.00007			0.01107	0.00012
GELF			0.01449	0.00021			0.02832	0.00080
LLF		(5,5,7)	0.00995	0.00009		(5,5,7)	0.02031	0.00041
PLF			0.00407	1.6E-05			0.00797	6.3E-05
GELF			0.000286	8.1E-06			0.00768	5.9E-05
LLF		(7,7,8)	0.00103	1.0E-06		(7,7,8)	0.00161	2.6E-06
PLF			0.00058	3.3E-07			0.00177	3.1E-06

$(c_1,c_2,t_1,t_2)=(2,1,2,2)$					$(c_1,c_2,t_1,t_2)=(1,2,2,2)$			
Loss function	**Real $\Re_{c_1,c_2,t_1,t_2}$**	(n,m,w)	**AB**	**MSE**	**Real $\Re_{c_1,c_2,t_1,t_2}$**	(n,m,w)	**AB**	**MSE**
GELF	**0.760**		0.04292	0.00184	**0.687**		0.02976	0.00088
LLF		(2,2,2)	0.03398	0.00115		(2,2,2)	0.019006	0.00036
PLF			0.01359	0.00018			0.00830	6.8E-05
GELF			0.03351	0.00112			0.01627	0.00026
LLF		(5,5,5)	0.02530	0.00064		(5,5,5)	0.00513	2.6E-05
PLF			0.00979	9.5E-05			0.00013	1.8E-08
GELF			0.02497	0.00062			0.00552	3.0E-05
LLF		(7,7,7)	0.01697	0.00028		(7,7,7)	0.00244	5.9E-06
PLF			0.00664	4.4E-05			0.00016	2.6E-08
GELF			0.01358	0.00051			0.00491	6.11E-10
LLF		(10,10,10)	0.01511	0.00017		(10,10,10)	0.00235	2.04E-11
PLF			0.00544	3.2E-05			0.00015	2.03E-10
GELF			0.04676	0.00218			0.02680	0.01649
LLF		(2,2,3)	0.04676	0.00144		(2,2,3)	0.01649	0.00027
PLF			0.01530	0.00023			0.00663	4.41E-05
GELF			0.02963	0.00087			0.00534	2.8E-05
LLF		(5,5,7)	0.02181	0.00047		(5,5,7)	0.00193	3.7E-06
PLF			0.01002	0.00010			0.00187	3.5E-06
GELF			0.00357	1.2E-05			0.00017	3.11E-10
LLF		(7,7,8)	0.00270	7.3E-06		(7,7,8)	0.00105	6.55E-11
PLF			0.00031	9.9E-08			0.00135	3.44E-10

Table 3*: Numerical results of* $\widehat{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ *for prior II*

$(c_1,c_2,t_1,t_2)=(1,1,2,2)$					$(c_1,c_2,t_1,t_2)=(1,2,2,2)$			
Loss function	**Real $\Re_{c_1,c_2,t_1,t_2}$**	(n,m,w)	**AB**	**MSE**	**Real $\Re_{c_1,c_2,t_1,t_2}$**	(n,m,w)	**AB**	**MSE**
GELF	**0.871**		0.01134	0.00012	**0.746**		0.02343	0.00054
LLF		(2,2,2)	0.00566	3.2E-05		(2,2,2)	0.01414	0.00020
PLF			0.002903	8.4E-06			0.005499	3.0E-05
GELF			0.00355	1.2E-05			0.02311	0.00047
LLF		(5,5,5)	0.00161	2.6E-06		(5,5,5)	0.01241	0.00015
PLF			0.00072	5.2E-07			0.00421	2.7E-05
GELF			0.00206	4.2E-06			0.02277	0.00037
LLF		(7,7,7)	0.00153	2.3E-06		(7,7,7)	0.01187	0.00011
PLF			0.00103	1.8E-07			0.00365	1.1E-04
GELF		(10,10,10)	0.00201	3.4E-06		(10,10,10)	0.02148	0.00029
LLF			0.00140	1.2E-07			0.01099	3.0E-05

PLF			0.00099	5.2E-08			0.00301	1.0E-04
GELF			0.02524	0.00164			0.03074	0.00426
LLF		(2,2,3)	0.02358	0.00121		(2,2,3)	0.03009	0.00077
PLF			0.00799	0.00080			0.00784	1.7E-05
GELF			0.02470	0.00135			0.02457	0.00333
LLF		(5,5,7)	0.02157	0.00117		(5,5,7)	0.02847	0.00051
PLF			0.00630	0.00060			0.00780	1.1E-05
GELF			0.02110	0.00124			0.01354	0.00251
LLF		(7,7,8)	0.01110	0.00101		(7,7,8)	0.02147	0.00039
PLF			0.00558	0.00038			0.00660	1.8E-06
	$(c_1,c_2,t_1,t_2)=(2,1,2,2)$				$(c_1,c_2,t_1,t_2)=(2,2,2,2)$			
Loss function	**Real** $\mathfrak{R}_{c_1,c_2,t_1,t_2}$	(n,m,w)	**AB**	**MSE**	**Real** $\mathfrak{R}_{c_1,c_2,t_1,t_2}$	(n,m,w)	**AB**	**MSE**
GELF	**0.760**		0.02320	0.00136	**0.687**		0.02377	0.00056
LLF		(2,2,2)	0.01312	0.00078		(2,2,2)	0.01288	0.00016
PLF			0.00445	0.00008			0.00515	2.6E-05
GELF			0.03524	0.00128			0.02228	0.00035
LLF		(5,5,5)	0.02600	0.00071		(5,5,5)	0.01147	0.00012
PLF			0.00931	8.1E-05			0.00478	1.8E-06
GELF			0.03421	0.00088			0.02147	2.4E-05
LLF		(7,7,7)	0.02387	0.00050		(7,7,7)	0.01133	2.9E-06
PLF			0.00900	2.4E-05			0.00330	2.9E-08
GELF			0.03321	0.00051			0.02140	6.1E-08
LLF		(10,10,10)	0.02340	0.00044		(10,10,10)	0.01110	2.0E-07
PLF			0.00875	3.2E-06			0.00250	2.0E-08
GELF			0.03476	0.02336			0.02131	0.01356
LLF		(2,2,3)	0.03554	0.00744		(2,2,3)	0.01109	0.00124
PLF			0.02447	0.00037			0.00190	2.4E-05
GELF			0.03124	0.01235			0.02110	0.01254
LLF		(5,5,7)	0.02490	0.00625		(5,5,7)	0.01148	0.00120
PLF			0.02300	0.00035			0.00166	1.4E-05
GELF			0.03009	0.00147			0.01999	0.00124
LLF		(7,7,8)	0.02370	0.00420		(7,7,8)	0.01122	0.00110
PLF			0.02298	0.00021			0.00150	1.0E-06

Note that: E-0k stands for 10-k, k is integer

5. Actual Data Implementation

In this part, we illustrate our principles using three real datasets. We consider the real data sets reported in [33] where the data represent the time to break down (in minutes) of insulating fluids to electrodes at voltage levels 34 kV, 36 kV and 38 kV. The Kolmogorov-Smirnov (KS) test is used to separately fit each of the three datasets with the EPD along with the corresponding P-value (PV) (see Table 4). The empirical cdf and estimated pdf for these data are explained in Figure 13. At levels 34 kV, 36 kV and 38 kV, the times to break down are reported respectively as follows

Data Group I

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.5	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

Data Group II

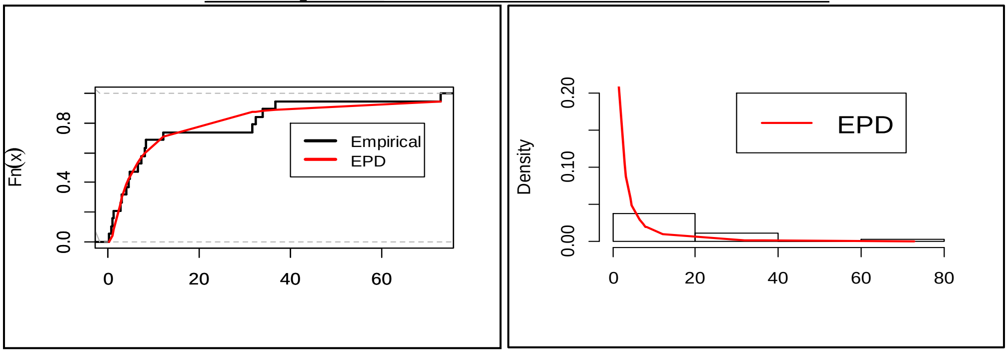
1.97	0.59	2.58	1.69	2.71	25.5	0.35	0.99	3.9	3.67
2.07	0.96	5.35	2.9	13.77					

Data Group III

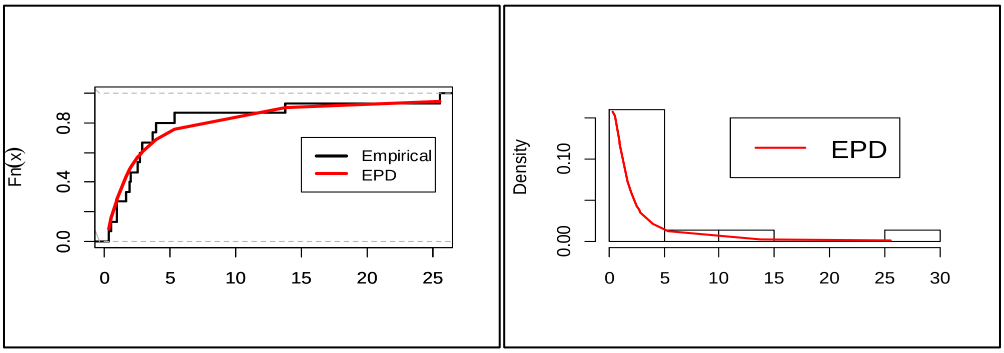
0.47	0.73	1.4	0.74	0.39	1.13	0.09	2.38

Table 4: *The K-S test and corresponding P-values for groups I, II and III*

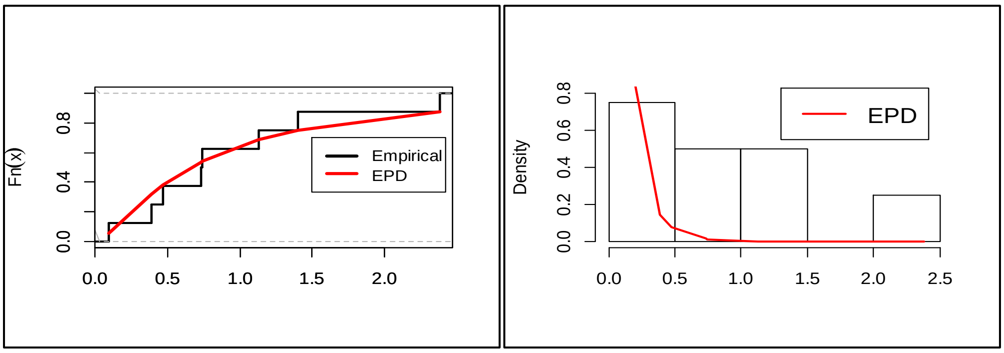
Data	K-S	PV
Group I	0.167	0.6013
Group II	0.185	0.6127
Group III	0.277	0.5013



Data I



Data II



Data III

Figure 13: *Characteristics and limitations of K-S test for the three data groups*

We assume that electrical fluid of specimen considered being good if 1 out of 2 specimens are functioning properly at constant voltage. Form data group I, II and III, three sets of lower record values $\underline{r}$ = (0.96, 0.19), $\underline{p}$ = (1.97, 0.59, 0.35) and $\underline{s}$ = (0.47, 0.39, 0.09) are obtained, respectively. From $\underline{r}$, $\underline{p}$, and $\underline{s}$, we find that $n = 2$, $m = 3$, $w = 3$, then we calculate the estimates of $\Re_{c_1,c_2,t_1,t_2}$ using the ML and Bayesian approaches within GELF, LLF and PLF. Using the above LRVs, the MLE and BE of $\Re_{c_1,c_2,t_1,t_2}$, are calculated in Table 5.

Table 5: *Bayes and ML estimates of* $\Re_{c_1,c_2,t_1,t_2}$ *, for the real data*

MLE of $\Re_{c_1,c_2,t_1,t_2}$	BE of $\Re_{c_1,c_2,t_1,t_2}$		
	GELF	**LLF**	**PLF**
0.5851	0.7673	0.7743	0.7956

6. Concluding Remarks

In the present work, we investigate the stress-strength reliability in a multi-component system with non-identical component strengths where both the stress and strength variables are the EPD. The ML and Bayesian procedures are used to analyse the reliability of MSS. Strength and stress distribution samples are used, and their measurements are presented in LRVs. We use MCMCO techniques in order to evaluate the accuracy of the various Bayesian estimates. The simulation study shows that for four choices of (c_1, c_2, t_1, t_2), the MSEs and ABs decrease with the number of records, supporting the MLE's consistency characteristic of $\Re_{c_1,c_2,t_1,t_2}$. Additionally, as the true value of $\Re_{c_1,c_2,t_1,t_2}$ increases, the MSEs of $\hat{\Re}_{c_1,c_2,t_1,t_2}$ drop. Regarding the MCMCO approach, we deduce that the MSEs and ABs of $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ via PLF generally hold the lowest values in majority of cases. The ABs and MSEs of $\overset{\frown}{\Re}_{c_1,c_2,t_1,t_2}$, $\breve{\Re}_{c_1,c_2,t_1,t_2}$, $\dddot{\Re}_{c_1,c_2,t_1,t_2}$ under different loss functions decrease as the number of records rises. The use of actual data demonstrates that our model's reliability estimates are very near to one, demonstrating its practical usefulness.

Conflicts of Interest: The authors declare no conflict of interest.

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