STRESS-STRENGTH RELIABILITY MODEL UNDER MULTIVARIATE NORMAL SETUP AND ITS APPLICATIONS

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Abstract

We often see that in a system, the energy is supplied to the system by p_1 sources and its consumed through p_2 sources and the sources are linearly dependent with vector $\boldsymbol{a}'$ and $\boldsymbol{b}'$. The overall representation of the two sets are related to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that they are approximated by $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\boldsymbol{y}$ as in principal component analysis. In this article, a stress strength reliability model R= $Pr(\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y})$, when $\boldsymbol{x}$ and $\boldsymbol{y}$ are distributed dependently multivariate normal distribution is proposed, with $\boldsymbol{a}$ and $\boldsymbol{b}$ are two known vectors. MVUE and MLE of R are obtained. Through simulation studies, their performances are compared using different measures. The two-sided confidence intervals and lower bounds of R are obtained through exact and asymptotic distribution of maximum-likelihood estimators and using bootstrap procedure. Through simulation studies, the performances of these confidence intervals are empirically checked using their coverage and the accuracy. In this study, we proposed to choose the optimal sample size for an experiment assures an adequate power and level. Finally, we applied these interval estimators to a real data set.

Keywords: Stress-strength, Principal component, Maximum Likelihood Estimator (MLE), Minimum Variance Unbiased Estimator (MVUE), Confidence Intervals.

1. Introduction

The stress-strength model consists in estimating R=Pr(X>Y), the lifetime of a component which has a random strength X and it's subjected to random stress Y. In stress-strength model, the system fails if and only if, at any time, the stress is greater than its strength. Birnbaum was first introduced to this model [1] and was developed by Birnbaum and McCarty [2]. There has been a huge number of works in estimation of the reliability R= P(X>Y) in the field of stress-strength models. It has several applications particularly in engineering ideas, like structures, deterioration of rocket motors, static fatigue of ceramic parts, fatigue failure of craft structures, and also in mechanical, civil engineering. The R=Pr(X>Y) has been formulated for the huge majority of the well-known statistical distributions when X and Y are independent random variables belonging to the univariate family and (X,Y) follows bivariate distribution with dependence between X and Y. The R has been established for the bulk of well-known statistical distributions, including Normal, uniform, exponential, gamma, beta, extreme value, Weibull, Laplace, etc [3-7].

This stress-strength reliability model may also be useful in clinical trial. Particularly when comparing two treatment effects, it may be more useful to draw conclusions regarding the unit's free measure, rather than comparing the means [8]. Simonoff, Hochberg and Reiser also used this function to find the effect of the treatment, if Y is the response for a control group, and X refers to a treatment group [9]. A numerical procedure obtained by Birnbaum and McCarty based on the asymptotic distribution to find the sample size needed for setting up an upper confidence bound with the defined width and confidence coefficient [2]. Using this procedure Owen, Craswell and Hanson considered the same problem in case of bivariate normal distribution to obtained the sample size needed for specified confidence bound and the confidence coefficient [10]. Sen obtained the non parametric confidence bounds for P(X<Y) based on independent samples [11]. Govindarazulu obtained two-sided confidence intervals for R when X and Y are independent and also dependent normal variates [12]. Church and Harris obtained confidence intervals for R in case of independent normal varieties [13]. Under the same assumptions, Downton derived the minimum variance unbiased estimator (MVUE) of R [14]. They are suggested that an alternative approximation to obtained the "best" estimate of R and its confidence intervals by Church and Harris. Woodward and Kelley obtained the uniformly minimum variance unbiased estimator (UMVUE) of R based on infinite series [15]. Mukherjee and Sharan obtained the UMVUE for R under the bivariate normal distribution [16] and also obtained their asymptotic variance when parameters other than the means are known and they proposed an estimator R based on maximum likelihood estimators when all the five parameters are unknown. Hor and Seal derived an alternative estimator viz. UMVUE of R under the same case of bivariate normal distribution [17].

All these above works were done under the univariate or bivariate setup. Gupta and Gupta first estimated the reliability under multivariate normal setup [18]. They considered the forms of R when ($\boldsymbol{x}_{p_1\text{x}1}$, $\boldsymbol{y}_{p_2\text{x}1}$) follows multivariate normal distribution with dependence vector between $\boldsymbol{x}_{p_1\text{x}1}$ and $\boldsymbol{y}_{p_2\text{x}1}$. Then, the reliability as R= Pr($\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}$) , where $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are two vectors. This problem arises when a system in the energy is supplied to the system by p_1 sources and is consumed through p_2 sources and the sources of energy supplied and consumed are linearly dependent with known vector $\boldsymbol{a}'$ and $\boldsymbol{b}'$. Under this set up, they obtained and compared the MVUE and MLE estimate of R with some interesting special cases. Enis and Geisser have demonstrated that, how to obtain the exact confidence bounds for R [19]. In this multivariate setup, Reiser and Farragi derived the lower confidence bounds for R=P($\boldsymbol{a'x}$>$\boldsymbol{b'y}$) [20] and solved it iteratively and also derived an approximate lower confidence bounds for R. In a clinical trial require sample size calculations to determine the optimal number of participants (patients) to be included in the trial. Reiser and Guttman introduced the method to obtain the sample size for experiments concerned with inference on R, based on acceptance sampling theory in the univariate normal setup [21].

These two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are to be chosen such that the multivariate behaviours are approximated by $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\boldsymbol{y}$ as in principal component analysis. Thus, the Principal component analysis used to estimate the $\boldsymbol{a}'$ and $\boldsymbol{b}'$where as Gupta and Gupta considered only spatial cases of $\boldsymbol{a}'$ and $\boldsymbol{b}'$ and compare the MVUE and MLE estimates of R using given mean vector and dispersion matrix [18]. The study is carried out on real data set. We do simulation studies to compare the performance of MVUE and MLE in teams of variance (VAR), mean square error (MSE) and mean absolute error (MAE). Then, it is shown that MVUE of R performs better than MLE.

We estimate R=P($\boldsymbol{a'x}$>$\boldsymbol{b'y}$) under the multivariate normal setup, whereas Hor and Seal derived this under the bivariate normal distribution setup [17]. We choose some set of $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{12}$, $\boldsymbol{\Sigma}_{22}$ and to compute L_1 distance between the two distribution functions of MVUE and MLE. We may take different choices of parameters to obtain the L_1 distance, where the parameter is $\sqrt{n}\delta = \frac{-\sqrt{n}(\boldsymbol{b}'\boldsymbol{\mu}_2-\boldsymbol{a}'\boldsymbol{\mu}_1)}{(\boldsymbol{a}'\boldsymbol{\Sigma}_{11}\boldsymbol{a}-2\boldsymbol{a}'\boldsymbol{\Sigma}_{12}\boldsymbol{b}-\boldsymbol{b}'\boldsymbol{\Sigma}_{22}\boldsymbol{b})^{\frac{1}{2}}}$. In this connection, the distributions function of the two estimators MVUE and MLE are derived in section 3. L_1 distance between two functions to compare two estimators is given in section 4. In section 6, we can deal with the problem to obtain two-sided confidence limits and lower bounds for R under the multivariate normal set up. Based on MVUE and MLE, we compare the performance between bootstrap and empirically interval estimator in terms of coverage and accuracy using simulation study. Finally, we applied these interval estimators to a real data set. Finally, in Section 7, we consider the problem of sample size determination.

2. Derivation of the Point Estimation of R

2.1. Maximum Likelihood Estimator of R

Let, $\mathbf{x}_{p1x1}$ and $\mathbf{y}_{p2x1}$ be two random vector such that the distribution of $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \sim N_{p1+p2}(\mu, \Sigma)$

Where, $\mu = \begin{pmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{pmatrix}_{(p_1+p_2)x1}$ and $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}_{(p_1+p_2)x(p_1+p_2)}$

Suppose we have known vectors $\boldsymbol{a}'$ and $\boldsymbol{b}'$. Then, we want to find the reliability in terms of linear combination of $\boldsymbol{a}'\boldsymbol{x}$ and $\boldsymbol{b}'\mathbf{y}$ as R= $\Pr(\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}) = \Pr(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y} > 0)$
Now, the distribution of $u = \boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y}$ follows $N(\mu_u, \sigma_u^2)$,
where, $\mu_u = E(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y}) = \boldsymbol{a}'\boldsymbol{\mu_1} - \boldsymbol{b}'\boldsymbol{\mu_2}$
and $\sigma_u^2 = Var(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y}) = \boldsymbol{a}'\Sigma_{11}\boldsymbol{a} - 2\boldsymbol{a}'\Sigma_{12}\,\boldsymbol{b} + \boldsymbol{b}'\Sigma_{22}\,\boldsymbol{b}$
Now, R= $\Pr(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y} > 0) = \Pr(u > 0)$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left\{-\frac{1}{2}\left(\frac{u-\mu_u}{\sigma_u}\right)^2\right\} du = \int_{-\frac{\mu_u}{\sigma_u}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} dz = \Phi\left(\frac{\mu_u}{\sigma_u}\right)$$

The maximum likelihood estimator of $\mu = \begin{pmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{pmatrix}_{(p_1xp_2)}$ and $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}_{(p_1+p_2)x(p_1+p_2)}$

define as $\begin{pmatrix}\overline{\boldsymbol{x}}\\ \overline{\boldsymbol{y}},\end{pmatrix}$ and $\boldsymbol{S} = \begin{pmatrix}\boldsymbol{S_{11}} & \boldsymbol{S_{12}}\\ \boldsymbol{S_{21}} & \boldsymbol{S_{22}}\end{pmatrix}$ respectively.

We have, $\widehat{\mu_u} = \boldsymbol{a}'\overline{\boldsymbol{x}} - \boldsymbol{b}'\overline{\boldsymbol{y}}$ and $\widehat{\sigma_u^2} = \boldsymbol{a}'\boldsymbol{S_{11}}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{S_{12}}\,\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{S_{22}}\,\boldsymbol{b}$
So, the maximum likelihood estimate of R is define as R*= $\Phi\left(\frac{\widehat{\mu_u}}{\widehat{\sigma_u^2}}\right)$ (1)

2.2. Principal Component Estimation

Let us, compute the estimate of $\boldsymbol{a}'$ *and* $\boldsymbol{b}'$ by Principal component analysis. Principal component analysis explaining the variance-covariance structure Σ_{11} *and* Σ_{22} of a set of variables $\boldsymbol{x}$ *and* $\mathbf{y}$ through a linear combination ($\boldsymbol{a}'$ *and* $\boldsymbol{b}'$) of these variables, i.e, explain maximum variability. It is noted that, the first principal component has the largest possible variance (that is, accounts for as much of the variability in the data as possible), and each succeeding component in turn has the highest variance possible under the constraint that it is orthogonal to the preceding components. We take the maximum likelihood estimate of Σ_{11} as $\boldsymbol{S_{11}}$ and $\boldsymbol{a}'$ as $\boldsymbol{e_1'}$ normalized eigenvectors of $\boldsymbol{S_{11}}$ corresponding to λ_1 eigen value. Similarly, we have estimate of Σ_{22} as $\boldsymbol{S_{22}}$ and $\boldsymbol{b}'$ as $\boldsymbol{l_1'}$ normalized eigenvectors of $\boldsymbol{S_{22}}$ corresponding to λ_1 eigen value. Then from (1) the estimate of R define as R*= $\Phi\left(\frac{\widehat{\mu_u}}{\widehat{\sigma_u^2}}\right)$, where $\widehat{\mu_u} = \boldsymbol{e_1'}\overline{\boldsymbol{x}} - \boldsymbol{l_1'}\overline{\boldsymbol{y}}$ and $\widehat{\sigma_u^2} = \boldsymbol{e_1'}\boldsymbol{S_{11}}\boldsymbol{e_1} - 2\boldsymbol{e_1'}\boldsymbol{S_{12}}\,\boldsymbol{l_1} + \boldsymbol{l_1'}\boldsymbol{S_{22}}\,\boldsymbol{l_1}$.

2.3. Minimum Variance Unbiased Estimator (MVUE) of R

Now, let us find out Minimum Variance Unbiased Estimator (MVUE) of R=$\Pr(\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y} > 0)$. Here, it is assumed that the random sample $\begin{pmatrix}\mathbf{x}_\alpha\\ \mathbf{y}_\alpha\end{pmatrix}$, $\alpha = 1,2,\ldots,n$ are from multivariate normal distribution i.e. $\begin{pmatrix}\mathbf{x}_\alpha\\ \mathbf{y}_\alpha\end{pmatrix} \sim N_{p1+p2}(\mu, \Sigma)$.

Then, $u_{\boldsymbol{\alpha}} = (\boldsymbol{b}'\boldsymbol{y}_{\boldsymbol{\alpha}} - \boldsymbol{a}'\boldsymbol{x}_{\boldsymbol{\alpha}}) \sim N(\mu_u, \sigma_u^2)$, α=1,2,….,n be the random sample of size n. Now, $(\overline{u}, S_u^2)$ is a complete sufficient statistic for (μ_u, σ_u^2), where $\bar{u} = \frac{1}{n}\sum_{\alpha=1}^n u_i$ and $s_u^2 = \frac{1}{n}\sum_{\alpha=1}^n (u_i - \bar{u})^2$, the MVUE of R [22] is

$$\hat{R} = \int_c^1 \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-2}{2})} (1-z^2)^{\frac{n-2}{2}-1} dz, \text{ where } c = \frac{\bar{u}}{(\sqrt{(n-1)}s_u}$$

Then the MVUE of R=Pr($\boldsymbol{a}'\boldsymbol{x} - \boldsymbol{b}'\mathbf{y} > 0$) [18] is

$$\hat{R} = \begin{pmatrix} 0 & \text{if} & c > 1 \\ \frac{1}{2}\left(1 - B\left(c^2; \frac{1}{2}, \frac{n-2}{2}\right)\right) & \text{if} & 0 < c \leq 1 \\ \frac{1}{2}\left(1 + B\left((-c)^2; \frac{1}{2}, \frac{n-2}{2}\right)\right) & \text{if} & -1 < c \leq 0 \\ 1 & \text{if} & c \leq -1 \end{pmatrix} \tag{2}$$

where, $c = \frac{(e_1'\bar{y} - l_1'\bar{x})}{(\sqrt{(n-1)}(e_1'S_{11}e_1 - 2e_1'S_{12}\,l_1 + l_1'S_{22}\,l_1)}$ and $B(k;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^k x^{\alpha-1}(1-x)^{\beta-1}\,dx$

2.4. Simulation Study

The simulation study we performed aim to compare the behaviors of two estimators of R, i.e. $\hat{R}$, the MVUE and R^*= $\Phi\left(\frac{\widehat{\mu_u}}{\widehat{\sigma_u^2}}\right)$, the estimator based on maximum likelihood estimate of $\mu = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}_{(p_1 x p_2)}$ and

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}_{(p_1+p_2)x(p_1+p_2)}.$$

For this purpose we compute the following measures:

(i) Mean of $\hat{R}$ *and* R^*

(ii) Variance of $\hat{R}$ and R^* : $E(\hat{R} - R)^2$ and $E(R^* - R)^2$

(iii) Mean square error of $\hat{R}$ and R^* : Var($\hat{R}$)+Bias($\hat{R}$,R)^2 and Var(R^*)+Bias(R^*,R)^2

(iv) Mean absolute error of $\hat{R}$ and R^* : $E(|\hat{R} - R|)$ and $E(|R^* - R|)$

It is difficult to obtain the analytical form of above expressions for different values of 'R'. So, we figure out these by using simulation study. Hence, we generate the random samples of size n from $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim N_{p1+p2}(\mu, \Sigma)$. For each of sample drown of size n, we compute the above measures by taking 500 replications each time.

For this purpose, here, R programming language is used.

$$\text{Suppose, } \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \sim N_5(\mu, \Sigma), \text{ where , } \mu' = (2,4,2,1,2) \;\; ; \Sigma = \begin{pmatrix} 3.61 & 2.23 & -0.10 & 0.16 & 2.32 \\ 2.23 & 4.74 & 3.32 & -0.69 & 1.76 \\ -0.10 & 3.32 & 5.68 & -2.34 & -1.23 \\ 0.16 & -0.69 & -2.34 & 3.05 & 1.53 \\ 2.32 & 1.76 & -1.23 & 1.53 & 4.45 \end{pmatrix}$$

Therefore, the estimated values of two known vectors based on first principal component are $\boldsymbol{a}'$ = (0.614, 0.789) and $\boldsymbol{b}'$=(0.737, -0.491, -0.465). Now, we want to estimate R= Pr($\boldsymbol{a}'\boldsymbol{x} > \boldsymbol{b}'\boldsymbol{y}$) , then the exact value of stress strength reliability R is 0.886. We take the sample size of n up to 100 in order to achieved exact value for the reliability. Using (1) and (2), calculated mean, variance MSE and MAE of $\hat{R}$ and R^* based on 500 repetitions are reported in Table 1. It can be observed that Variance of $\hat{R}$ is lesser than the Variance of R^* in each sample size. Also, it is noted that the MSE's and MAE's of $\hat{R}$ are less than MSE's and MAE 's of R^*. However, the sample mean of $\hat{R}$ is less than the R^* in each case. But, $\hat{R}$ and R^* are under-estimates the true value of R, when sample size are small. It is also interesting to observe that, the Variance, MSE and MAE of $\hat{R}$ and R^* are reduces as the sample size increases: when n=200 and almost achieved the true value of R.

Table 1: *Sample Mean, Variance, MSE and MAE of $\hat{R}$ and R^**

Sample Size	Sample Mean		Variance		MSE		MAE	
	$\hat{R}$	R^*	$\hat{R}$	R^*	$\hat{R}$	R^*	$\hat{R}$	R^*
10	0.672281	0.675052	0.105394	0.109709	0.150902	0.154031	0.247465	0.250765
20	0.730188	0.732219	0.094903	0.096861	0.119021	0.120346	0.190329	0.192359
30	0.755041	0.756586	0.082294	0.083551	0.099306	0.100158	0.152352	0.163332
40	0.793484	0.795067	0.059074	0.059788	0.067533	0.067956	0.11029	0.11728
50	0.804974	0.806287	0.054406	0.054921	0.060878	0.061181	0.097016	0.104556
60	0.812637	0.813826	0.048111	0.048495	0.053412	0.053621	0.909735	0.092666
70	0.838157	0.839244	0.032573	0.032796	0.034807	0.034926	0.066746	0.068149
80	0.844905	0.84585	0.030538	0.030711	0.032174	0.03227	0.062998	0.063118
90	0.84874	0.849622	0.026384	0.026519	0.027727	0.027797	0.055342	0.055777
100	0.855952	0.856766	0.021074	0.02117	0.021941	0.021988	0.04756	0.04864
200	0.881041	0.881482	0.003745	0.003751	0.003763	0.003765	0.018152	0.018332

3. Distribution Function of $\hat{R}$ and R^*

In this section, we derive the distributions function of $\hat{R}$ and R^*. We have

$$\hat{R} = \begin{pmatrix} 0 & \text{if} \quad c > 1 \\ \frac{1}{2}\left(1 - B\left(c^2; \frac{1}{2}, \frac{n-2}{2}\right)\right) & \text{if} \quad 0 < c \leq 1 \\ \frac{1}{2}\left(1 + B\left((-c)^2; \frac{1}{2}, \frac{n-2}{2}\right)\right) & \text{if} \quad -1 < c \leq 0 \\ 1 & \text{if} \quad c \leq -1 \end{pmatrix},$$

where $B(x; \frac{1}{2}, \frac{n-2}{2})$ is c.d.f $B(\frac{1}{2}, \frac{n-2}{2})$ and it is clear that $0 \leq \hat{R} \leq 1$ for any real number of c.

Let the distribution function of $\hat{R}$ be $F_{\hat{R}}(x)$, then

$$F_{\hat{R}}(x) = 0 \text{ if } x < 0 \text{ and } F_{\hat{R}}(x) = 1 \text{ if } x \geqslant 1$$

If $0 \leqslant x \leqslant \frac{1}{2}$, then the distribution function of $\hat{R}$ is given by

$$F_{\hat{R}}(x) = P(\hat{R} \leqslant x) = P[(\hat{R} \leqslant x) \cap \{(c > 0) \cup (c \leqslant 0)\}] = P[(\hat{R} \leqslant x) \cap (c > 0)] + P[(\hat{R} \leqslant x) \cap (c \leqslant 0)]$$

$$= P[(\hat{R} \leqslant x) | (c > 0)] P[c > 0] + P[(\hat{R} \leqslant x) | (c \leqslant 0)] P[c \leqslant 0]$$

$$= P[\frac{1}{2} - \frac{1}{2} B(c^2; \frac{1}{2}, \frac{n-2}{2}) \leqslant x] P[\frac{(\boldsymbol{b}'\bar{\boldsymbol{y}} - \boldsymbol{a}'\bar{\boldsymbol{x}})}{\sqrt{(n-1)}(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}} > 0]$$

$$= P[B(c^2; \frac{1}{2}, \frac{n-2}{2}) \geqslant 1 - 2x] P[(\boldsymbol{b}'\bar{\boldsymbol{y}} - \boldsymbol{a}'\bar{\boldsymbol{x}}) > 0]$$

$$\text{,where } (\boldsymbol{b}'\bar{\boldsymbol{y}} - \boldsymbol{a}'\bar{\boldsymbol{x}}) \sim N_1((\boldsymbol{b}'\boldsymbol{\mu}_2 - \boldsymbol{a}'\boldsymbol{\mu}_1), \frac{1}{n}(\boldsymbol{a}'\boldsymbol{\Sigma}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{\Sigma}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{\Sigma}_{22}\boldsymbol{b}))$$

$$= P[c^2 \geqslant B^{-1}_{(\frac{1}{2}, \frac{n-2}{2})}(1 - 2x)] \Phi(\frac{\sqrt{n}(\boldsymbol{b}'\boldsymbol{\mu}_2 - \boldsymbol{a}'\boldsymbol{\mu}_1)}{(\boldsymbol{a}'\boldsymbol{\Sigma}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{\Sigma}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{\Sigma}_{22}\boldsymbol{b})^{\frac{1}{2}}})$$

$$= \{P[c \geqslant \left(B^{-1}_{(\frac{1}{2}, \frac{n-2}{2})}(1 - 2x)\right)^{\frac{1}{2}}] + P[c \leqslant -\left(B^{-1}_{(\frac{1}{2}, \frac{n-2}{2})}(1 - 2x)\right)^{\frac{1}{2}}]\} \Phi(-\sqrt{n}\delta)$$

$$\text{,where } \delta = \frac{-(\boldsymbol{b}'\boldsymbol{\mu}_2 - \boldsymbol{a}'\boldsymbol{\mu}_1)}{(\boldsymbol{a}'\boldsymbol{\Sigma}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{\Sigma}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{\Sigma}_{22}\boldsymbol{b})^{\frac{1}{2}}}$$

$$= \{P[\frac{(\boldsymbol{b}'\bar{\boldsymbol{y}} - \boldsymbol{a}'\bar{\boldsymbol{x}})}{\sqrt{(n-1)}(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}} \geqslant (B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}] +$$

$$P[\frac{(\boldsymbol{b}'\bar{\boldsymbol{y}} - \boldsymbol{a}'\bar{\boldsymbol{x}})}{\sqrt{(n-1)}(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}} \leqslant -(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}]\}\, \Phi(-\sqrt{n}\delta)$$

$$= \{P[-\sqrt{n}\hat{\delta} \geqslant (n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}] + P[-\sqrt{n}\hat{\delta} \leqslant -(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}]\}\Phi(-\sqrt{n}\delta)$$

$$,\text{where } \hat{\delta} = \frac{-\sqrt{(n-1)}(\boldsymbol{b}'\bar{\boldsymbol{y}} - \boldsymbol{a}'\bar{\boldsymbol{x}})}{\sqrt{n}(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}}$$

$$= \{P[\sqrt{n}\hat{\delta} \leqslant -(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}] + 1 - P[\sqrt{n}\hat{\delta} \leqslant (n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}]\}\Phi(-\sqrt{n}\delta)$$

$$= \{F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}) +$$

$$[1 - F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}})]\}\Phi(-\sqrt{n}\delta) \quad (3)$$

Using the standard distribution theory [23], if $\boldsymbol{x} \sim N_p(\mu, \Sigma)$ then $\boldsymbol{a}'\boldsymbol{x} \sim N_1(\boldsymbol{a}'\mu, \boldsymbol{a}'\Sigma\boldsymbol{a})$**.** Let, $\bar{\boldsymbol{x}}$ and S be the unbiased estimator of μ and Σ respectively, then $\boldsymbol{a}'\bar{\boldsymbol{x}} \sim N_1(\boldsymbol{a}'\mu, \frac{1}{\boldsymbol{n}}(\boldsymbol{a}'\Sigma\boldsymbol{a}))$ and $\boldsymbol{S} \sim W_p(n-1, \Sigma)$. Thus, we can write, $\frac{\boldsymbol{a}'S\boldsymbol{a}}{\boldsymbol{a}'\Sigma\boldsymbol{a}} \sim \boldsymbol{\chi}^2_{\boldsymbol{n-1}}$, hence $\sqrt{n-1}\hat{\delta} \sim t'_{(n-1),\sqrt{n}\delta}$ where $t'_{(n-1),\sqrt{n}\delta}$ denotes the non - central t - distribution with (n - 1) d.f. We use the unbiased estimator of Σ instead of MLE, then $\sqrt{n}\hat{\delta} \sim t'_{(n-1),\sqrt{n}\delta}$ with non-centrality parameter $\sqrt{n}\delta$ and $F_{t'_{(n-1),\sqrt{n}\delta}}(.)$ be the cdf of non - central t - distribution.

If $\frac{1}{2} < x < 1$, then the distribution function of $\hat{R}$ is given by

$$F_{\hat{R}}(x) = P(\hat{R} \leqslant x) = P[(\hat{R} \leqslant x) \cap \{(c > 0) \cup (c \leqslant 0)\}] = P[(\hat{R} \leqslant x) \cap (c > 0)] + P[(\hat{R} \leqslant x) \cap (c \leqslant 0)]$$

$$= P[(\hat{R} \leqslant x)|(c > 0)]P[c > 0] + P[(\hat{R} \leqslant x)|(c \leqslant 0)]P[c \leqslant 0]$$

$$= P[\frac{1}{2} - \frac{1}{2}B(c^2; \frac{1}{2}, \frac{n-2}{2}) \leqslant x]P[\frac{(\boldsymbol{b}'\bar{\boldsymbol{y}} - \boldsymbol{a}'\bar{\boldsymbol{x}})}{\sqrt{(n-1)}(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a} - 2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}} > 0] +$$

$$P[\frac{1}{2} + \frac{1}{2}B((-c)^2; \frac{1}{2}, \frac{n-2}{2}) \leqslant x]\Phi(\sqrt{n}\delta)$$

$$= \Phi(-\sqrt{n}\delta) + P[B(c^2; \frac{1}{2}, \frac{n-2}{2}) \leqslant 2x - 1]\Phi(\sqrt{n}\delta)$$

$$= \Phi(-\sqrt{n}\delta) + P[c^2 \leqslant B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1)]\Phi(\sqrt{n}\delta)$$

$$= \Phi(-\sqrt{n}\delta) + P[-(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}} \leqslant c \leqslant (B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}}]\Phi(\sqrt{n}\delta)$$

$$= \Phi(-\sqrt{n}\delta) + P[-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}} \leqslant -\sqrt{n}\hat{\delta} \leqslant (n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}}]\Phi(\sqrt{n}\delta)$$

$$= \Phi(-\sqrt{n}\delta) + P[(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}} \geqslant \sqrt{n}\hat{\delta} \geqslant -(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}}]\Phi(\sqrt{n}\delta)$$

$$= \Phi(-\sqrt{n}\delta) + [F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}}) -$$

$$F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}})]\Phi(\sqrt{n}\delta)$$

Thus, the distribution function of $\hat{R}$ is given by

$$F_{\hat{R}}(x) = 0 \text{ if } x < 0 \text{ and } F_{\hat{R}}(x) = 1 \text{ if } x \geqslant 1$$

If $0 \leqslant x \leqslant \frac{1}{2}$,

$$F_{\hat{R}}(x) = \{F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}) +$$

$$[1 - F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}})]\}\Phi(-\sqrt{n}\delta)$$

If $\frac{1}{2} < x < 1$,

$$F_{\hat{R}}(x) = \Phi(-\sqrt{n}\delta) + [F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}}) -$$

$$F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}})]\Phi(\sqrt{n}\delta) \quad (4)$$

The MLE estimate of R, $R^* = \Phi(\frac{-(\boldsymbol{b}'\bar{\boldsymbol{y}}-\boldsymbol{a}'\bar{\boldsymbol{x}})}{(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a}-2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b}-\boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}})$

Let the distribution function of R^* be $F_{R^*}(x)$, then $F_{R^*}(x) = 0$ if $x < 0$ and $F_{R^*}(x) = 1$ if $x \geqslant 1$,

$$F_{R^*}(x) = P(R^* \leqslant x) = P[\Phi(\frac{-(\boldsymbol{b}'\bar{\boldsymbol{y}}-\boldsymbol{a}'\bar{\boldsymbol{x}})}{(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a}-2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b}+\boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}}) \leqslant x]$$

$$= P[\frac{-(\boldsymbol{b}'\bar{\boldsymbol{y}}-\boldsymbol{a}'\bar{\boldsymbol{x}})}{(\boldsymbol{a}'\boldsymbol{S}_{11}\boldsymbol{a}-2\boldsymbol{a}'\boldsymbol{S}_{12}\boldsymbol{b}+\boldsymbol{b}'\boldsymbol{S}_{22}\boldsymbol{b})^{\frac{1}{2}}} \leqslant \Phi^{-1}(x)] = P[\sqrt{n}\hat{\delta} \leqslant \sqrt{(n-1)}\Phi^{-1}(x)]$$

$$= F_{t'_{(n-1),\sqrt{n}\delta}}(\sqrt{(n-1)}\Phi^{-1}(x)) \quad (5)$$

4. Distance Between $F_{\hat{R}}(.)$ and $F_{R^*}(.)$

Let us calculate the distance between two functions (L_1) $F_{\hat{R}}(x)$ and $F_{R^*}(x)$ [17],

$U(n,\delta) = \int_0^1 |F_{R^*}(x) - F_{\hat{R}}(x)|\mathrm{dx}$, for different values of n, μ_1, μ_2, Σ_{11}, Σ_{12}, Σ_{22}.

According to this method, this can be taken as a measure of deviation or equal deviation between $\hat{R}$ and R^*.

Let, $\bar{R}$ be any other estimator of R, then the maximum deviation between distribution of $\bar{R}$ and $\hat{R}$ as

$$M(n,\delta) = \sup_{\bar{R}} \int_0^1 |F_{\bar{R}}(x) - F_{\hat{R}}(x)|\,\mathrm{dx} = \int_0^1 F_{\hat{R}}(x)\mathrm{dx}, \text{if} \int_0^1 F_{\hat{R}}(x)\mathrm{dx} > \frac{1}{2}$$

$$= 1 - \int_0^1 F_{\hat{R}}(x)\mathrm{d}x \text{, if} \int_0^1 F_{\hat{R}}(x)\mathrm{d}x \leqslant \frac{1}{2}$$

The ration $R(n,\delta) = \frac{U(n,\delta)}{M(n,\delta)}$ has to be taken as a relative measure of deviation between $\hat{R}$ and R^* , the maximum deviation between any other estimator of R, i.e. $\bar{R}$ and $\hat{R}$. It' s difficult to get the exact expression of this above measures. So, we compute these measure values numerically using R - programming. Here we take different choices of δ for n = 20. The results are reported in table 2. The results show that the overall output of MVUE and MLE of R are not too distant and the values for these differences are show in columns U(.), M(.) and R(.) of this table. From this table, it is seen that empirical values of the parameters and the performance of MVUE of R is better than MLE and also Figure 1 shows that, MVUE estimator of R is better than the other estimators, i.e. R^*. Also, L_1 distance and graphical impression show this.

5. Derivation of Var($\hat{R}$) and MSE(R^*)

Now, $Var(\hat{R})$ and $MSE(R^*)$ are obtained by using equations (3), (4) and (5) as follows:
Since, 0 <$\hat{R}$ <1, then $E(\hat{R}) = \int_0^1 \{1 - F_{\hat{R}}(x)\}\mathrm{d}x$

If $0 \leqslant x \leqslant \frac{1}{2}$, then we have

$$E_1(\hat{R}) = \int_0^1 [1 - \{F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}) + [1 - F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}})]\}\Phi(-\sqrt{n}\delta)]\,\mathrm{d}x$$

If $\frac{1}{2} < x < 1$, then we have

$$E_2(\hat{R}) = \int_0^1 [1 - \{\Phi(-\sqrt{n}\delta) + [F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}}) - F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}})]\Phi(\sqrt{n}\delta)\}]\,\mathrm{d}x$$

So, $Var(\hat{R}) = \int_0^1 2x\{1 - F_{\hat{R}}(x)\}\mathrm{d}x - \{E(\hat{R})\}^2 = \int_0^1 2x[1 - \{F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}}) +$

$$[1 - F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(1-2x))^{\frac{1}{2}})]\}\Phi(-\sqrt{n}\delta)]\,\mathrm{d}x$$
$$+ \int_0^1 2x[1 - \{\Phi(-\sqrt{n}\delta) + [F_{t'_{(n-1),\sqrt{n}\delta}}((n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}}) -$$
$$F_{t'_{(n-1),\sqrt{n}\delta}}(-(n-1)(B^{-1}_{(\frac{1}{2},\frac{n-2}{2})}(2x-1))^{\frac{1}{2}})]\Phi(\sqrt{n}\delta)\}]\,\mathrm{d}x - \{E_1(\hat{R}) + E_2(\hat{R})\}^2$$

Similarly, we can determine $MSE(R^*) = \int_0^1 2x\left\{1 - F_{t'_{(n-1),\sqrt{n}\delta}}\left(\sqrt{(n-1)}\Phi^{-1}(x)\right)\right\}\mathrm{d}x$ –

$$[\int_0^1 \{1 - F_{t'_{(n-1),\sqrt{n}\delta}}(\sqrt{(n-1)}\Phi^{-1}(x))\}\,\mathrm{d}x]^2$$

From Figure 2 and 3 it is observe that the, values of Var($\hat{R}$) and MSE(R^*) are almost close to zero of δ. Values of Var($\hat{R}$) are less as compared to other values of MSE(R^*). Thus, the performance of MVUE of R is better than MLE.

Table 2: *Performance of point estimators: δ and 100* $\{U(n,\delta), M(n,\delta), R(n,\delta)\}$*

Non-negative values of δ				Negative values of δ			
δ	$U(n,\delta)$	$M(n,\delta)$	$R(n,\delta)$	δ	$U(n,\delta)$	$M(n,\delta)$	$R(n,\delta)$
3	0.087	99.865	0.087	-3	0.087	99.865	0.087
2.898	0.101	99.812	0.101	-2.898	0.101	99.812	0.101
2.797	0.115	99.742	0.115	-2.797	0.115	99.742	0.115
2.695	0.128	99.648	0.129	-2.695	0.128	99.648	0.129
2.593	0.141	99.525	0.141	-2.593	0.141	99.525	0.141
2.492	0.151	99.364	0.152	-2.492	0.151	99.364	0.152
2.39	0.157	99.157	0.158	-2.39	0.157	99.157	0.158
2.288	0.158	98.894	0.16	-2.288	0.158	98.894	0.16
2.186	0.153	98.561	0.155	-2.186	0.153	98.561	0.155
2.085	0.139	98.145	0.142	-2.085	0.139	98.145	0.142
1.983	0.116	97.632	0.119	-1.983	0.116	97.632	0.119
1.881	0.082	97.004	0.084	-1.881	0.082	97.004	0.084
1.78	0.036	96.243	0.038	-1.78	0.036	96.243	0.038
1.678	0.021	95.332	0.022	-1.678	0.021	95.332	0.022
1.576	0.089	94.252	0.094	-1.576	0.089	94.252	0.094
1.475	0.166	92.984	0.179	-1.475	0.166	92.984	0.179
1.373	0.25	91.511	0.273	-1.373	0.25	91.511	0.273
1.271	0.338	89.817	0.376	-1.271	0.338	89.817	0.376
1.169	0.424	87.89	0.482	-1.169	0.424	87.89	0.482
1.068	0.503	85.719	0.587	-1.068	0.503	85.719	0.587
0.966	0.571	83.3	0.685	-0.966	0.571	83.3	0.685
0.864	0.623	80.629	0.772	-0.864	0.623	80.629	0.772
0.763	0.663	77.702	0.853	-0.763	0.663	77.702	0.853
0.661	0.715	74.501	0.959	-0.661	0.715	74.501	0.959
0.559	0.843	70.977	1.188	-0.559	0.843	70.977	1.188
0.458	1.144	67.052	1.706	-0.458	1.144	67.052	1.706
0.356	1.635	62.739	2.606	-0.356	1.635	62.739	2.606
0.254	2.057	58.331	3.527	-0.254	2.057	58.331	3.527
0.153	1.877	54.406	3.45	-0.153	1.877	54.406	3.45
0.051	0.779	51.323	1.518	-0.051	0.779	51.323	1.518

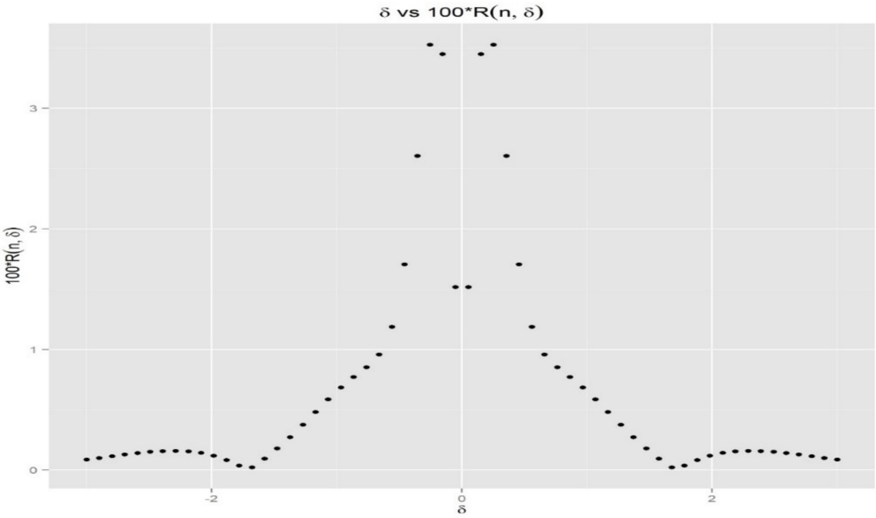

Figure 1: $\delta \ vs \ 100 * R(n, \delta)$

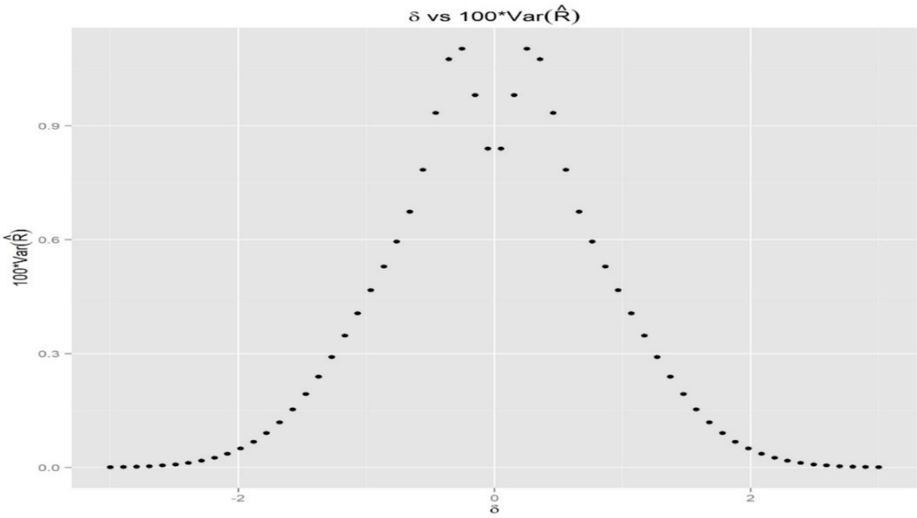

Figure 2: $\delta \ vs \ 100 * Var(\hat{R})$

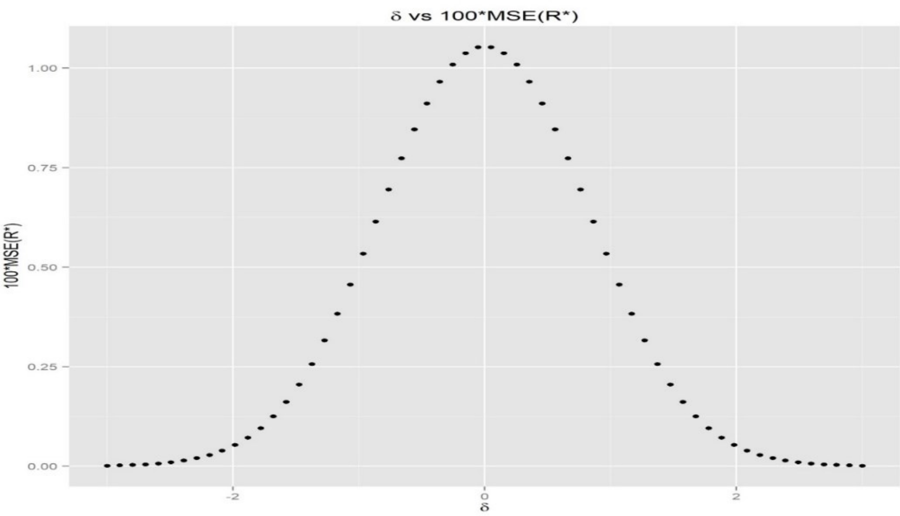

Figure 3: $\delta \ vs \ 100 * MSE(R^*)$

6. Confidence Intervals for R

6.1. Exact Two Sided Confidence Intervals for R

Here, we have $\sqrt{n}\hat{\delta} \sim t'_{(n-1),\sqrt{n}\delta}$, then the tradition approach for finding the lower limit of R , we use the probability p_{δ_L} that $t'_{(n-1),\sqrt{n}\delta}$ exceeds the value of $\sqrt{n}\hat{\delta}$ as

$$p_{\delta_L} = \Pr\left(t'_{(n-1),\sqrt{n}\delta} > \sqrt{n}\hat{\delta}\right) = \alpha/2$$

or,
$$p_{\delta_L} = \Pr\left(t'_{(n-1),\sqrt{n}\delta} < \sqrt{n}\hat{\delta}\right) = 1-\alpha/2 \quad (6)$$

Similarly, we get the upper limit of δ as

$$p_{\delta_U} = \Pr\left(t'_{(n-1),\sqrt{n}\delta} < \sqrt{n}\hat{\delta}\right) = \alpha/2 \quad (7)$$

Equation (6) and (7) can be solved numerically. Finally we get the (1-α) level confidence Intervals for δ as (δ_L, δ_U).

Then, the (1-α) level confidence Intervals for R as $(\Phi(\delta_L), \Phi(\delta_U))$.

6.2. Exact Lower Confidence bound for R

In order to obtain the lower bound of the lower bound of R, we use the probability $p_{\delta_{LB}}$ that $t'_{(n-1),\sqrt{n}\delta}$ exceeds the value of $\sqrt{n}\hat{\delta}$ as

$$p_{\delta_{LB}} = Pr\left(t'_{(n-1),\sqrt{n}\delta} > \sqrt{n}\hat{\delta}\right) = \alpha$$

or,
$$p_{\delta_{LB}} = Pr\left(t'_{(n-1),\sqrt{n}\delta} < \sqrt{n}\hat{\delta}\right) = 1-\alpha \quad (8)$$

Thus, the (1-α) level confidence lower bound for δ can be obtained by solving equation (8). Then, the (1-α) level confidence lower bound for R is $(\Phi(\delta_{LB})$.

6.3. Approximate Two Sided Confidence Intervals for R

From section 3, we have $Pr(\boldsymbol{a'x} > \boldsymbol{b'y}) = \Phi\left[\frac{-(\boldsymbol{b'\mu_2} - \boldsymbol{a'\mu_1})}{(\boldsymbol{a'\Sigma_{11}a} - 2\boldsymbol{a'\Sigma_{12}b} - \boldsymbol{b'\Sigma_{22}b})^{\frac{1}{2}}}\right] = \Phi(\delta)$,

where $\sqrt{n}\hat{\delta} \sim t'_{(n-1),\sqrt{n}\delta}$ with non-centrality parameter $\sqrt{n}\delta$

.

In order to determine the two sided confidence Intervals, we use following well known approximation for large n [24] as

$$Z = \frac{\left[t'_{(n-1),\sqrt{n}\delta} - \sqrt{n}\delta\right]}{\left[1 + \frac{(t'_{(n-1),\sqrt{n}\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}} \sim N(0,1)$$

Using this,

$$Pr[-z_{\alpha/2} \leqslant \frac{\left[\sqrt{n}\hat{\delta} - \sqrt{n}\delta\right]}{\left[1 + \frac{(\sqrt{n}\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}} \leqslant z_{\alpha/2}] = 1\text{-}\alpha$$

or,
$$Pr[-z_{\alpha/2} \leqslant \frac{\left[\hat{\delta} - \delta\right]}{\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}} \leqslant z_{\alpha/2}] = 1\text{-}\alpha$$

or,
$$Pr[\hat{\delta} - z_{\alpha/2}\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}} \leqslant \delta \leqslant \hat{\delta} + z_{\alpha/2}\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}] = 1\text{-}\alpha$$

Thus, an approximate (1-α) level confidence Intervals for δ is given by

$$(\delta_L, \delta_U) = \left\{\hat{\delta} - z_{\alpha/2}\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}, \ \hat{\delta} + \left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}} z_{\alpha/2}\right\}$$

Then, an approximate (1-α) level confidence Intervals for R is represented by

$$(\Phi(\delta_L), \Phi(\delta_U)) = \left\{\Phi\left(\hat{\delta} - z_{\alpha/2}\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}\right), \ \Phi\left(\hat{\delta} + z_{\alpha/2}\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}\right)\right\}$$

Where, $z_{\alpha/2}$ upper critical value for the standard normal distribution

6.4. Approximate Lower Confidence bound for R

The lower bounds based on approximate results is given by

$$Pr(\delta_{LB} \leqslant \delta) = 1 - \alpha$$

or,
$$Pr\left(\frac{\left[\delta_{LB} - \hat{\delta}\right]}{\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}} \leqslant \frac{\left[\delta - \hat{\delta}\right]}{\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}}\right) = 1 - \alpha$$

or,
$$Pr\left(\frac{\left[\hat{\delta} - \delta\right]}{\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}} \leqslant \frac{\left[\hat{\delta} - \delta_{LB}\right]}{\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}}\right) = 1 - \alpha$$

or, $$Pr\left(z \leqslant \frac{\left[\hat{\delta} - \delta_{LB}\right]}{\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}}\right) = 1 - \alpha\,, \qquad \text{or, } \frac{\left[\hat{\delta} - \delta_{LB}\right]}{\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}} = z_{1-\alpha}$$

or, $$\delta_{LB} = \hat{\delta} - z_{1-\alpha}\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}$$

So, an approximate (1-α) confidence lower bound for R as

$$\Phi(\delta_{LB}) = \Phi\left(\hat{\delta} - z_{1-\alpha}\left[\frac{1}{n} + \frac{(\hat{\delta})^2}{2(n-1)}\right]^{\frac{1}{2}}\right)$$

6.5. Bootstrap confidence Intervals for R

In this subsection, we use the confidence intervals based on percentile bootstrap method. Efron suggests the procedure to find out the confidence intervals for a parameter [25] and corrects bias of percentile of bootstrap confidence intervals for R proposed by Efron [26]. It works as follows

(1) Draw random sample $\begin{pmatrix}\mathbf{X}_\alpha \\ \mathbf{Y}_\alpha\end{pmatrix}$, $\alpha = 1,2,\dots,$ n from multivariate normal distribution, where $\begin{pmatrix}\mathbf{X} \\ \mathbf{Y}\end{pmatrix} \sim N_{p1+p2}(\mu, \sum)$
(2) Generate bootstrap samples $\begin{pmatrix}\mathbf{x}^*_\alpha \\ \mathbf{y}^*_\alpha\end{pmatrix}$, $\alpha = 1,2,\dots,$ n , by using random sample of $\begin{pmatrix}\mathbf{X}_\alpha \\ \mathbf{Y}_\alpha\end{pmatrix}$, $\alpha = 1,2,\dots,$ n.
(3) Compute the bootstrap estimates of linear dependent vectors $\boldsymbol{a}'$ and $\boldsymbol{b}'$ using PC1, say $\boldsymbol{e}'^*$ and $\boldsymbol{l}'^*$ respectively. Also, Compute the bootstrap MLE estimates of $\boldsymbol{\mu_1}$, $\boldsymbol{\mu_2}$, $\boldsymbol{\Sigma_{11}}$, $\boldsymbol{\Sigma_{12}}$, $\boldsymbol{\Sigma_{22}}$ by $\overline{\boldsymbol{x}}^*, \overline{\boldsymbol{y}},^* \boldsymbol{S}^*_{11}, \boldsymbol{S}^*_{12}, \boldsymbol{S}^*_{12}$. Using these estimates compute the bootstrap estimate of R, say R^*_B.
(4) Repeat steps 2 and 3, number of boot time B (B sufficiently large, i.e. 1000), thus we obtain the bootstrap distribution of $\{R^*_B\}$.
(5) Estimate $(1-\alpha)$ bootstrap percentile confidence intervals for R from$\{R^*_B\}$ by taking the $\left(\frac{\alpha}{2}\right)$ and $\left(1-\frac{\alpha}{2}\right)$ quantiles as $\left(R^*_{B,\frac{\alpha}{2}},\ R^*_{B,\left(1-\frac{\alpha}{2}\right)}\right)$.
or, $(1-\alpha)$ bootstrap percentile lower bound for R as $R^*_{B,\alpha}$.

6.5. Simulation Study

In this section, we present simulation study to investigate the statistical properties of the interval estimators using the given matrix in section 2.4. The simulation study define as follows

(1) Draw the random samples of size n from $\begin{pmatrix}\mathbf{x} \\ \mathbf{y}\end{pmatrix} \sim N_{p1+p2}(\mu, \sum)$. For each of sample drown of size n, considered different sample sizes (n=50, 100, 150,….etc). We compute the above measures by taking 500 replications each time.
(2) Estimate MLE estimate of R using PC1 for different sample size and a Confidence Intervals (two-sided and lower bound) for R.
(3) Compute exact, approximate and bootstrap confidence intervals using step 2, where number of boot time B=1000.

The results of the simulation study are recorded in Table 3-5. Figure 4-6, represent the exact, approximate and bootstrap confidence belt at 90%, 95% and 99% levels. It has been observed that for a small sample size, the estimate of R is getting high and also confidence intervals. The results get better as the sample sizes increase and the reliability R gets closer to true value. The same phenomenon is observed for the exact, approximate and bootstrap confidence intervals. The overall band of exact and approximate confidence intervals is almost same, whereas bootstrap confidence intervals give the large confidence band for small sample size. But, exact, approximate confidence intervals and Bootstrap confidence intervals all are almost same for large sample size at 90%, 95% and 99% levels. All most the same variation found in confidence belt of exact, approximate Cls, but irregular variation in bootstrap CIs shows in Figure 4-6.

Table 3: *Exact Confidence Intervals*

Sample size	R*	90%			95%			99%		
		L	U	LB	L	U	LB	L	U	LB
50	0.8954	0.8738	0.9349	0.8820	0.8664	0.9394	0.8738	0.8511	0.9473	0.8574
100	0.8953	0.8782	0.9227	0.8838	0.8731	0.9262	0.8782	0.8629	0.9328	0.8671
150	0.8948	0.8785	0.9139	0.8828	0.8746	0.9168	0.8785	0.8669	0.9224	0.8701
200	0.8898	0.8751	0.9128	0.8797	0.8710	0.9159	0.8751	0.8627	0.9218	0.8661
250	0.8888	0.8743	0.9197	0.8800	0.8692	0.9233	0.8743	0.8588	0.9300	0.8631
300	0.8887	0.8732	0.9133	0.8782	0.8688	0.9166	0.8732	0.8599	0.9228	0.8636
350	0.8883	0.8703	0.9070	0.8747	0.8663	0.9101	0.8703	0.8583	0.9159	0.8616
400	0.8876	0.8653	0.9068	0.8704	0.8608	0.9103	0.8653	0.8517	0.9167	0.8554
450	0.8874	0.8677	0.9048	0.8722	0.8637	0.9079	0.8677	0.8556	0.9138	0.8589
500	0.8874	0.8676	0.9047	0.8721	0.8636	0.9079	0.8676	0.8556	0.9138	0.8589
550	0.8868	0.8688	0.9028	0.8729	0.8652	0.9057	0.8688	0.8579	0.9112	0.8609
600	0.8864	0.8640	0.9028	0.8729	0.8595	0.9057	0.8688	0.8503	0.9111	0.8609

Table 4: *Approximate Confidence Intervals*

Sample size	90%			95%			99%		
	L	U	LB	L	U	LB	L	U	LB
50	0.8741	0.9352	0.8823	0.8666	0.9395	0.8741	0.8512	0.9474	0.8576
100	0.8783	0.9228	0.8840	0.8732	0.9263	0.8783	0.8629	0.9328	0.8672
150	0.8786	0.9139	0.8829	0.8747	0.9169	0.8786	0.8669	0.9224	0.8701
200	0.8752	0.9129	0.8798	0.8711	0.9160	0.8752	0.8627	0.9218	0.8661
250	0.8744	0.9198	0.8802	0.8693	0.9234	0.8744	0.8589	0.9300	0.8632
300	0.8733	0.9134	0.8783	0.8689	0.9167	0.8733	0.8600	0.9228	0.8636
350	0.8704	0.9071	0.8748	0.8664	0.9101	0.8704	0.8584	0.9159	0.8616
400	0.8655	0.9069	0.8706	0.8609	0.9103	0.8655	0.8517	0.9167	0.8555
450	0.8678	0.9049	0.8723	0.8637	0.9080	0.8678	0.8556	0.9138	0.8590
500	0.8677	0.9048	0.8722	0.8637	0.9079	0.8677	0.8556	0.9138	0.8589
550	0.8689	0.9029	0.8730	0.8653	0.9057	0.8689	0.8580	0.9112	0.8609
600	0.8689	0.9028	0.8730	0.8652	0.9057	0.8689	0.8579	0.9112	0.8609

Table 5: *Bootstrap Confidence Intervals*

Sample size	90%			95%			99%		
	L	U	LB	L	U	LB	L	U	LB
50	0.8591	0.9592	0.8701	0.8495	0.9688	0.8591	0.8307	0.9876	0.8383
100	0.8804	0.9264	0.8855	0.8760	0.9308	0.8804	0.8674	0.9395	0.8709
150	0.8806	0.9149	0.8844	0.8773	0.9182	0.8806	0.8709	0.9246	0.8735
200	0.8518	0.9419	0.8617	0.8432	0.9505	0.8518	0.8263	0.9674	0.8331
250	0.8787	0.9188	0.8832	0.8749	0.9227	0.8787	0.8674	0.9302	0.8704
300	0.8757	0.9137	0.8799	0.8720	0.9174	0.8757	0.8649	0.9245	0.8678
350	0.8218	0.9629	0.8373	0.8082	0.9764	0.8218	0.7818	0.9896	0.7925
400	0.8676	0.9073	0.8720	0.8638	0.9111	0.8676	0.8564	0.9185	0.8594

450	0.8695	0.9059	0.8735	0.8660	0.9094	0.8695	0.8592	0.9162	0.8619
500	0.8692	0.9045	0.8731	0.8658	0.9079	0.8692	0.8592	0.9145	0.8619
550	0.8703	0.9038	0.8740	0.8671	0.9070	0.8703	0.8608	0.9133	0.8633
600	0.8705	0.9033	0.8741	0.8674	0.9065	0.8705	0.8612	0.9126	0.8637

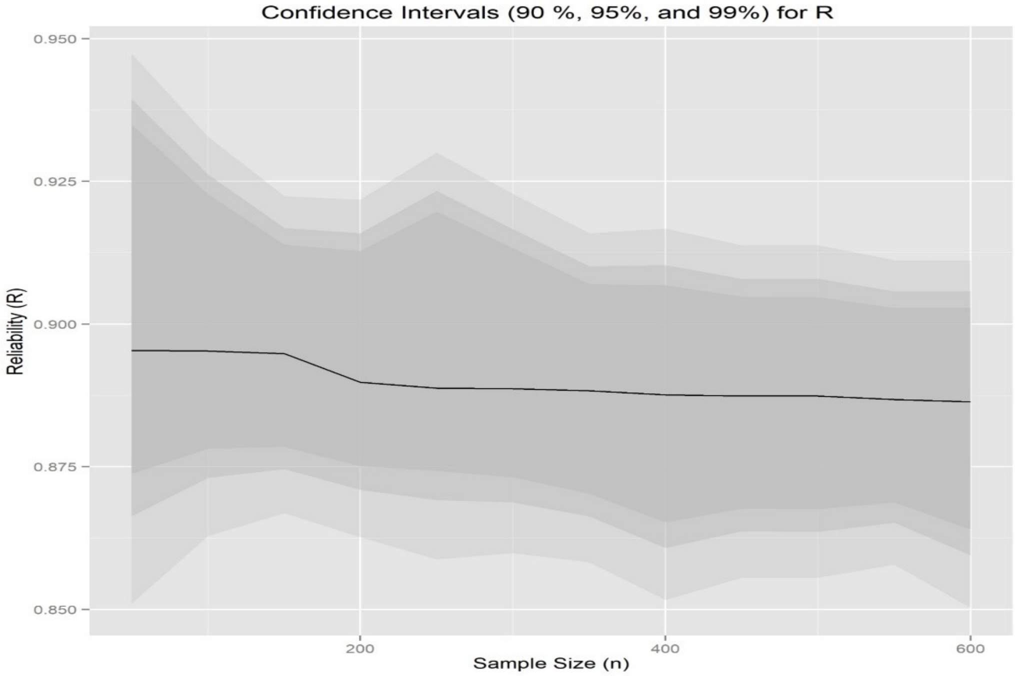

Figure 4: *Exact Confidence Intervals*

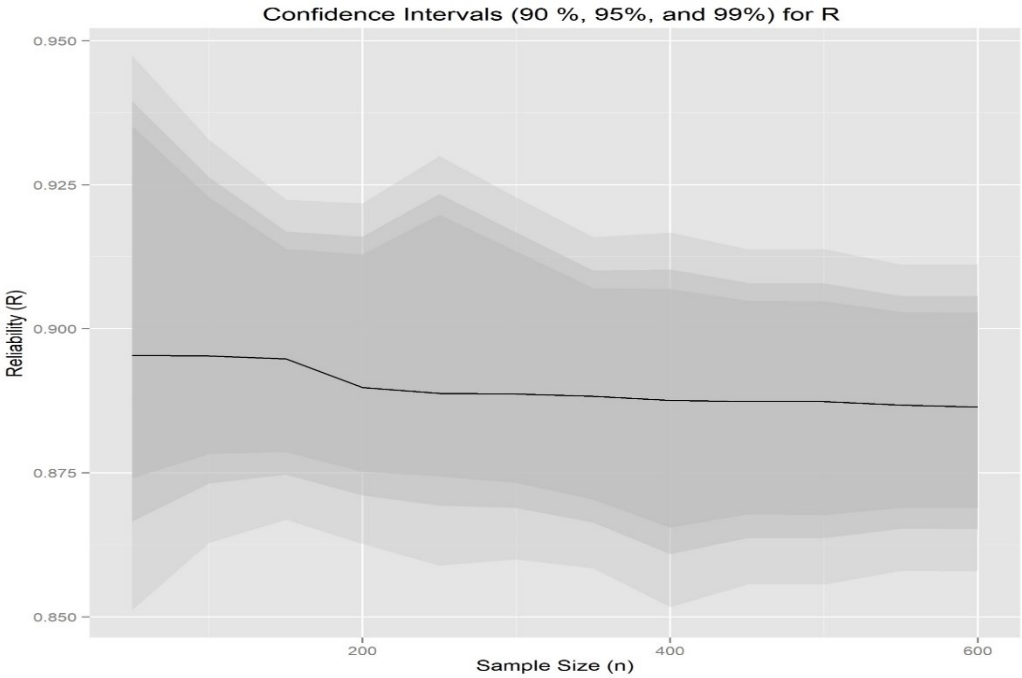

Figure 5: *Approximate Confidence Intervals*

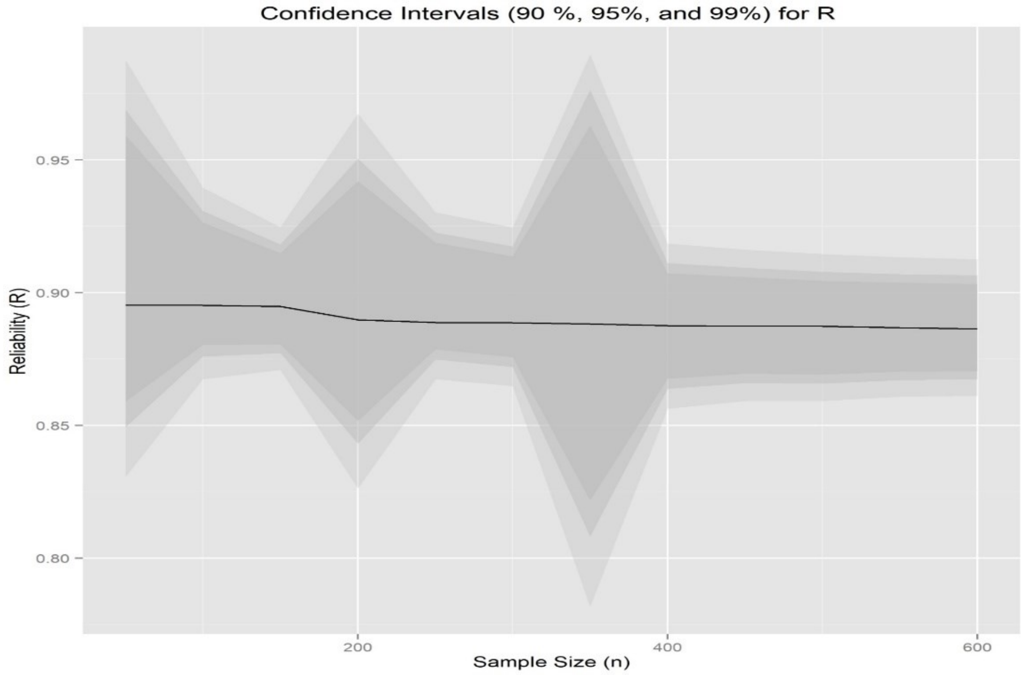

Figure 6: *Bootstrap Confidence Intervals*

6.6. Application on real data set

In this section, the above methods are applied to sample data set taken from Morrison [27]. This data set represent the level of three biochemical compounds found in the brain of twenty mice of the same strain in ten pairs. Both mice in each pair were in the same condition in terms of diet and care and one in each pair was randomly selected and received periodic administrations of the drug. The outcome of tests of the brains of the mice and consists of the amount of the compounds in micrograms per gram of the brain tissue. So, we want to determine the effect of the drug for changes in the level of three bio-chemical compounds found in the brain by estimating the probability.

Here, it is assumed that the $(x_1, x_2, x_3, y_1, y_2, y_3) \sim N_6(\mu, \Sigma)$. For the given data sets, proportion of variance of $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ for PC1: 0.9996 and 0.8662 respectively, then the MVUE of R is $\hat{R} = 0.6219$ and MLE of R is $R^* = 0.6325$. Also, we calculate that the above measure by principal component analysis as $\delta = 0.3386$, $U(n, \delta) = 3.266$, $M(n, \delta) = 60.846$, $R(n, \delta) = 5.367$, $\text{Var}(\hat{R}) = 0.0216$ and $\text{MSE}(R^*) = 0.0226$. The exact, approximate and bootstrap confidence intervals, using the sample data set reported in Table 6 and it is shows that that, the exact and approximate CIs are almost same band, but confidence band of bootstrap CIs is less than these.

Table 6: *Confidence Intervals for the mice dataset*

Confidence	**90%**			**95%**			**99%**		
Intervals	**L**	**U**	**LB**	**L**	**U**	**LB**	**L**	**U**	**LB**
Exact	0.4183	0.8067	0.4649	0.3788	0.8336	0.4183	0.3054	0.8790	0.3344
Approx.	0.4216	0.8092	0.4684	0.3819	0.8359	0.4216	0.3080	0.8807	0.3372
Bootstrap	0.3846	0.7381	0.4236	0.3507	0.7720	0.3846	0.2845	0.8382	0.3113

7. Sample Size Determination for Reliability

In a clinical study, the sample size calculation is to determine the number of subjects needed to have a desired power for detecting a clinically meaningful effect, i.e. the significant changes in clinical parameters. A study conducted with limited budget and/or some medical facilities, to choose the small number of subjects in respect of cost effectiveness and power. Suppose, we are intersected in determining the minimum sample size before the study for effect of the drug of three bio-chemical compounds of mice [27].

In this above context, the hypotheses of interest are

$$H_0 : R = R_0 \ \ against \ H_1 : R = R_1 (> R_0)$$

or, $$H_0 : \delta = \Phi^{-1}(R_0) = \delta_0 \ \ against \ H_1 : \delta = \Phi^{-1}(R_1) = \delta_1 (> \delta_0)$$

Then, the test statistic defined as $t = \sqrt{n}\hat{\delta}$ where, $t \sim t'_{(n-1),\sqrt{n}\delta}$, α = Type I error and 1-β = power of the test. Therefore, there exists an UMP invariant test [28], we rejects H_0 when $t > c$, where 'c' is determined by

$$P_{H_0}(\ t > c) = \ \alpha$$

or, $$P_{H_0}(\ t < c) = 1 - \alpha$$

or, $$c = t'_{(1-\alpha),(n-1),\sqrt{n}\delta_0} \tag{9}$$

Then the power of test,

$$P_{H_1}(\ t > c) = \ 1 - \beta$$

or, $$P_{H_1}\left(\ t'_{(n-1),\sqrt{n}\delta_1} \ < t'_{(1-\alpha),(n-1),\sqrt{n}\delta_0}\right) = \ \beta \tag{10}$$

We can get the sample size (n) by solving the equation (10) numerically for given value of α and β. The values of sample size n are reported in Table 7, in order to calculate sample size for two groups (i.e. treatment or control) are effect or not, we set the null hypothesis as $H_0 : R_0 = 0.5$ against the $R_0 > 0.5$.

Again, we consider the hypotheses,

$$H_0 : R = R_0 \ \ against \ H_1 : R = R_1 (< R_0)$$
or, $$H_0 : \delta = \delta_0 \ \ against \ H_1 : \delta = \delta_1 (< \delta_0)$$

We get,

$$P_{H_0}(\ t < c) = \ \alpha$$
or, $$c = t'_{\alpha,(n-1),\sqrt{n}\delta_0} \tag{11}$$

and, $$P_{H_1}\left(\ t'_{(n-1),\sqrt{n}\delta_1} \ < t'_{\alpha,(n-1),\sqrt{n}\delta_0}\right) = \ 1 - \beta \tag{12}$$

In order to calculate the sample size, Reiser and Guttman used an approximation of a non-central t-distribution by a standard normal distribution [21], valid for large n as

$$n = \frac{z_{\beta}^2(1+\delta_c^2/2)}{(\delta_c - \delta_1)^2} = \frac{z_{(1-\alpha)}^2(1+\delta_c^2/2)}{(\delta_c - \delta_0)^2} \quad \text{, where } \delta_c = \frac{\delta_0 z_\beta + \delta_1 z_\alpha}{z_\beta + z_\alpha} \tag{13}$$

Example 1. Suppose the objective of the study is to compare a test drug with a control for changes in the level of three bio-chemical compounds found in the mice brain.
Suppose the hypotheses of interest are

$$H_0 : R_0 = 0.5 \ \ against \ H_1 : R_1 = 0.7(> R_0)$$

or,
$$H_0 : \delta_0 = 0 \ \ against \ H_1 : \delta_1 = 0.524(> \delta_0)$$

There is no meaningful effect between test drug and control under H_0 and drug has an effect under H_1 . Then, by choosing α= 5%, and β= 20%, we find n≈24 using (9) and (10). Thus, a total number of 24 subjects are required for achieving a 80% power for detection of a clinically meaningful effect at the 5% level of significance.

Example 2. Consider the example of Reiser and Guttman to determine sample size (n) [21] for the case

Prodicer's risk=consumer's risk=0.05

To test,

$$H_0 : R_0 = 0.95 \ \ against \ H_1 : R_1 = 0.90(< R_0)$$

or,
$$H_0 : \delta_0 = 1.645 \ \ against \ H_1 : \delta_1 = 1.282(> \delta_0)$$

Here, Prodicer's risk and consumer's risk are equal, i.e. $z_\alpha = z_\beta = z_{0.05} = 1.645$, then we find n≈170 using (11) and (12). Similarly, we get the same result using (13).

Table 7: *Sample Size Calculation table*

R₀:0.5	$\boldsymbol{Power = 1 - \beta}$								
	70%			**80%**			**90%**		
R₁	$\boldsymbol{Level\ of\ significant = \alpha}$								
	0.01	**0.025**	**0.05**	**0.01**	**0.025**	**0.05**	**0.01**	**0.025**	**0.05**
0.51	12934	9823	7489	15972	12491	9839	20715	16722	13628
0.55	517	393	299	638	499	393	827	667	544
0.6	129	98	75	159	124	98	206	166	135
0.65	57	44	33	70	55	43	90	73	59
0.7	32	24	19	39	31	24	50	40	33
0.75	21	16	12	25	19	15	31	25	20
0.8	14	11	8	17	13	10	21	17	14
0.85	10	8	6	12	9	7	15	12	10
0.9	8	6	5	9	7	5	11	9	7
0.95	6	4	3	7	5	4	8	6	5

7. Conclusions

Under the multivariate normal setup, MVUE of stress-strength model of reliability R is obtained, although the estimator based on MLE of μ_1, μ_2, Σ_{11}, Σ_{12}, Σ_{22}. Simulation studies illustrate that, the Variance and MSE of two estimators reduces as the sample size increases and they almost achieved the true value of R. An application to the given real data set is described and shows that the same result as above. So, that the performance of MVUE of R is better than MLE in this case. In addition, the L_1 distance between distribution functions we see the improvement of such estimators. A difference in terms of MSE is much less as the values are given after multiplying by 100, though detailed calculations are required for other parametric values. Therefore, we may conclude that our recommend estimator performs better.

The exact confidence intervals are preferable for marginally short band of confidence intervals than the approximate confidence intervals. The performance of bootstrap CIs is slightly worse than the exact and approximate CIs in terms of confidence band for small sample size. But the performance of bootstrap confidence intervals and other methods of CIs are almost same for large sample. Thus, the overall performance of the confidence interval is quite good for exact confidence intervals.

The sample size plays the important role using the stress strength reliability model in order to achieve minimum number of observation to evaluate the effectiveness of a new drug. The sample size should be massive enough to adequately answer the analysis question. The determination of the acceptable sample size involves applied statistical criteria additionally as in clinical studies. In order to calculate the sample size, it was necessary to choose the power, the significance level, to produce results that are clinically or experimentally meaningful. Under approximation, with δ_c we can calculate the sample size easily. But, it better to choose the exact method to get sample size.

References

[1] Birnbaum, Z. W. (1956). On a use of the Mann-Whitney statistic. *Proc. of the third Berkeley Symposium on Mathematical Statistics and Probability*, 1:13-17.

[2] Z. W. Birnbaum and R. C. McCarty. (1958). A distribution-free upper confidence bound for P(Y<X) based on independent samples of X and Y. *Annals of Mathematical Statistics*, 29:558–562.

[3] Nadarajah, S. (2002). Reliability for beta models. *Serdica Mathematical Journal*, 28: 267-282.

[4] Nadarajah, S. (2003). Reliability for extreme value distributions. *Mathematical and Computer Modelling,* 37:915–922.

[5] Nadarajah, S. (2003). Reliability for lifetime distributions. *Mathematical and Computer Modelling*, 37:683–688.

[6] Nadarajah, S. (2004). Reliability for Laplace distributions. *Mathematical and Computer Modelling, 2:169–183*

[7] Nadarajah, S. (2004). Reliability for Logistic distributions. *Engineering Simulation, to appear*, 26:81-98.

[8] Weerahandi, S. and Johnson, R. A. (1992). Testing Reliability in a Stress-Strength Model When X and Y Are Normally Distributed. *Technometrics,* 34(1):83-91.

[9] Simonoff, J. S., Hochberg, Y., and Reiser, B. (1986). Alternative estimation procedures for Pr(X<Y) in categorized data. *Biometrics,* 42:895–907.

[10] Owen, D. B., Craswell, K.J. and Hanson, D.L. (1964). Nonparametric Upper Confidence Bounds for Pr{Y<X} and Confidence Limits for Pr{Y<X} When X and Y are Normal. *Journal of the American Statistical Association*, 59 (307):906-924.

[11] Sen, P. K. (1967). A note on asymptotically distribution-free confidence bounds for P(X<Y) based on two independent samples. *Sankhya*, 29:95-102.

[12] Govindarazulu, Zakkula. (1967). Two-Sided Confidence Limits for P(X <Y) Based on Normal Samples of X and Y. *Sankhyā*, 29(1/2):35-40.

[13] Church, J.D., Harris, B. (1970). The estimation of reliability from stress strength relationship.

[14] Downton. F. (1973). The estimation of P(Y<X) in the normal case. *Technometrics*, 15(3): 551–558.

[15] Woodward, W.A. and Kelley, G.D. (1977). Minimum Variance Unbiased Estimation of P[Y < X] in the Normal Case. *Technometrics*, 19(1):95-98.

[16] Mukherjee, S.P. and Sharan, L.K. (1985). Estimation of failure probability from a bivariate normal stress-strength distribution. *Microelectronics Reliability*, 25:699-702.

[17] Hor, P. and Seal, B. (2015). Minimum variance unbiased estimation of stress–strength reliability under bivariate normal and its comparisons. *Communications in Statistics - Simulation and Computation*, 46(3):2447–2456.

[18] Gupta, R. D. and Gupta, R. C. (1990). Estimation on $\Pr(a'x > b'y)$ in the Multivariate Normal Case. *Statistics: A Journal of Theoretical and Applied Statistics,* 21(1):91-97.

[19] Enis, P. and Geisser, S. (1971). Estimation of the Probability that Y < X. J. Amer. *Statist. Assoc.*, 66: 162-168.

[20] Reiser, B. and Farragi, D. (1994). Confidence bounds for P(a'x>b'y). *Statistics*, 25:107–111.

[21] Reiser, B. and Guttman, I. (1989). Sample size choice for reliability verification in strength-stress models. *The Canadian Journal of Statistics / La Revue Canadienne de Statistique*, 17(3): 253-259.

[22] Barton, D.E. (1961). Unbiased estimation of a set of probabilities. *Biometrika*, 48(1-2):227–229.

[23] Mukhopadhyay, P. Multivariate statistical analysis, World Scientific, 2009.

[24] Johnson, N.L. and Welch, B.L. (1939). Applications of the non-central t distribution. *Biometrika*, 31:362-389.

[25] Efron, B. The Jackknife, The Bootstrap, And Other Resampling Plans, CBMS-NSF Regional Conference Series in Applied Mathematics, Monograph 38 (Philadelphia, PA: SIAM), 1982.

[26] Efron, B. (1987). Better bootstrap confidence intervals. *J. Am. Stat. Assoc.*, 82:171–185.

[27] Morrison, D.F. Multivariate statistical methods, McGraw-Hill, New York, 2nd Edt., 1967.

[28] Lehmann, E.L. and Romano, J.P. Testing Statistical Hypotheses, 3rd Edition. Springer, 2008.