GENERALIZATION OF LENGTH BIASED WEIGHTED GENERALIZED UNIFORM DISTRIBUTION AND ITS APPLICATIONS

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Abstract

In this article, a generalization of length biased weighted generalized uniform distribution called Marshall Olkin length biased weighted generalized uniform distribution is introduced and studied. Some of the statistical properties of the new distribution such as hazard rate function, compounding, quantile function, moments, Renyi and Shannon entropies are discussed. The maximum likelihood estimation of the model parameters is done and a simulation study is conducted for confirming the validity of the estimates and also introduced a minification process with respect to the model and explored its sample path behaviour for different combinations of parameters. Further, the stress strength analysis is carried out and the estimate of the reliability is obtained based on a simulation study.

Keywords: Entropy, Length biased weighted generalized uniform distribution, Maximum likelihood method, Order statistics, Quantiles, Stress strength analysis.

1. Introduction

The theory of weighted distributions provides a collective access for the problems of model specification and data interpretation. Weighted distributions take into account the method of ascertainment, by adjusting the probabilities of the actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded [14].

The uniform distribution is considered as the simplest probability model and is connected to all the distributions. Many characterizations and modifications of the generalized uniform distribution have been introduced and explored by various researchers [see, 19, 10, 8, 18]. Rather and Subramanian [16] introduced and studied the properties of length biased weighted generalized uniform distribution.

The probability density function and cumulative distribution function of length biased weighted generalized uniform distribution (LBWGU) are respectively, given by

$$g_{LBWGU}(x;\theta,\gamma) = \frac{(\theta+2)x^{\theta+1}}{\gamma^{\theta+2}}; \quad 0 < x < \gamma,\ \theta > -1 \tag{1}$$

$$G_{LBWGU}(x;\theta,\gamma) = \left(\frac{x}{\gamma}\right)^{\theta+2}; \quad 0 < x < \gamma,\ \theta > -1 \tag{2}$$

Marshall Olkin [11] introduced a new family of distributions by inserting a new shape parameter to the existing family of distributions. Let G(x) be the cumulative distribution function (cdf) of a random variable X, then the cdf of the Marshall and Olkin family of distributions is

$$F(x) = \frac{G(x)}{1-(1-\beta)(1-G(x))} \tag{3}$$

The corresponding pdf of (3) is given by

$$f(x) = \frac{\beta g(x)}{[1-(1-\beta)(1-G(x))]^2}, \quad (4)$$

where $\beta > 0$ is a shape parameter. Clearly, for β = 1, we obtain the baseline distribution, i.e., F(x) = G(x).

Many authors have introduced various univariate distributions belonging to the Marshall-Olkin family of distributions such as Marshall-Olkin Weibull [5], Marshall-Olkin semi Burr and Marshall-Olkin Burr [7], Marshall-Olkin Frechet distribution [9], Marshall-Olkin generalized exponential distribution [17] and Marshall-Olkin extended generalized Lindley distribution [2]. Recently, introduced Marshall-Olkin form of additive Weibull distribution [1], reliability test plan for the Marshall-Olkin length biased Lomax distribution [12] and Marshall-Olkin length biased Maxwell distribution and its applications [13].

The rest of this paper is planned as follows. In section 2, the Marshall-Olkin length biased weighted generalized uniform (MOLBWGU) distribution is given, with plots of the pdf and cdf. The statistical properties of the new distribution are studied in section 3, including hazard rate function, moments, quantile function, compounding properties, order statistics and Renyi and Shannon entropies. Estimation of the model parameters are discussed in section 4. In section 5, the application of the distribution in time series analysis is discussed. In section 6, the stress strength analysis is carried out using a simulation study. Concluding remarks are presented in section 7.

2. Marshall-Olkin Length Biased Weighted Generalized Uniform Distribution

Let X follows length biased weighted generalized uniform distribution. A new distribution can be defined by inserting (2) in (3). The cdf obtained is

$$F_{MOLBWGU}(x;\theta,\gamma,\beta) = \frac{(\frac{x}{\gamma})^{\theta+2}}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}, \quad 0 < x < \gamma. \quad (5)$$

Based on (5), the survival function of the MOLBWGU distribution can be expressed as

$$S_{MOLBWGU}(x;\theta,\gamma,\beta) = \frac{\beta(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}, \quad 0 < x < \gamma. \quad (6)$$

where $\theta > -1$ and $\beta > 0$.

By putting (1) and (2) in (4), we obtain the pdf of the MOLBWGU distribution as

$$f_{MOLBWGU}(x;\theta,\gamma,\beta) = \frac{(\theta+2)\beta\gamma^{-\theta-2}x^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)\right)^2}, \quad 0 < x < \gamma. \quad (7)$$

We refer to this new distribution as the generalization of length biased weighted generalized uniform distribution with parameters θ, γ and β.

The shape of the pdf $f(x;\theta,\gamma,\beta)$ depends on parameter β. If $\beta \in (0,1)$ then the pdf is a bell shaped function on $(0,\gamma)$ with $f(0;\theta,\gamma,\beta) = 0$ and $f(\gamma,\theta,\gamma,\beta) = \frac{(\theta+2)\beta}{\gamma}$. In the case of $\beta > 1$ then the pdf is an increasing function on $(0,\gamma)$ with $f(0,\theta,\gamma,\beta) = 0$ and $f(\gamma,\theta,\gamma,\beta) = \frac{(\theta+2)\beta}{\gamma}$.

Remark 1. If β = 1, we obtain length biased weighted generalized uniform distribution introduced by Rather and Subramanian [16].

Remark 2. When θ = -1 and β = 1, MOLBWGU distribution reduces to uniform distribution over $(0,\gamma)$.

Remark 3. When γ = 1 and β = 1, MOLBWGU distribution reduces to standard power function distribution.

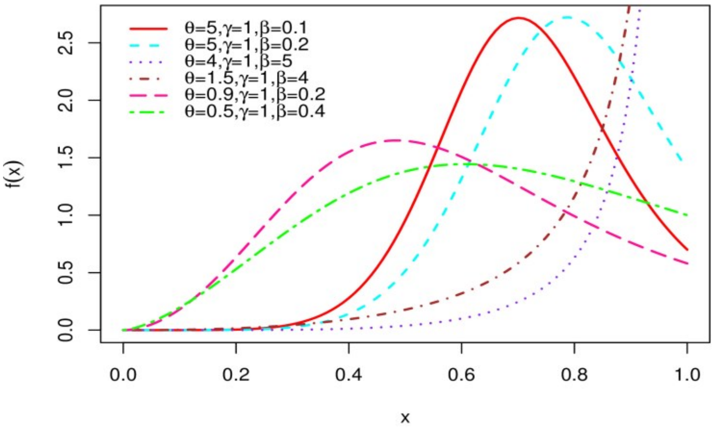

Figure 1 : *Curves of the pdf of the MOLBWGU distribution for different values of the parameters.*

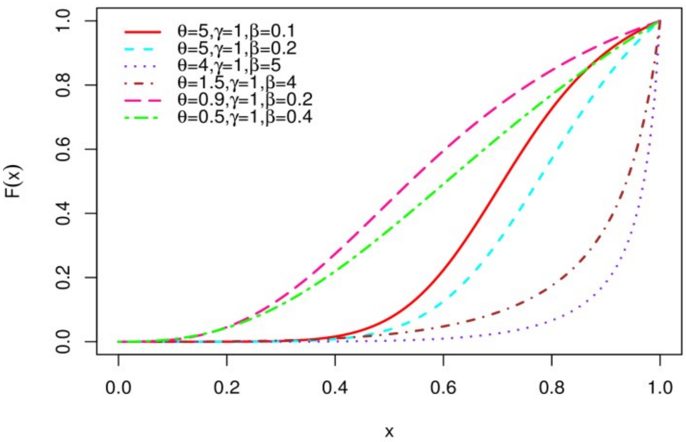

Figure 2 : *Curves of the cdf of the MOLBWGU distribution for different values of the parameters.*

3. Statistical Properties

This section is devoted to some statistical properties of the MOLBWGU distribution.

3.1. Hazard Rate Function

The hrf is given by, $h_{MOLBWGU}(x;\theta,\gamma,\beta) = \frac{(\theta+2)\beta\gamma^{-\theta-2}x^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)\right)^2\left(1-\frac{\left(\frac{x}{\gamma}\right)^{\theta+2}}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}\right)}$,

where, $0 < x < \gamma$.

For $\beta \in (0,1)$ and $\beta > 1$, the hrf is evidently increasing failure rate.

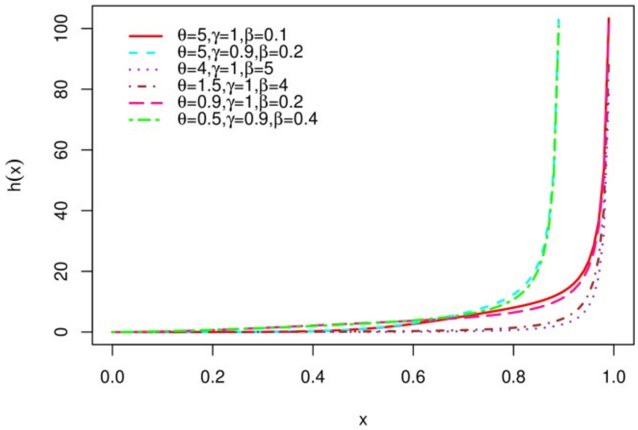

Figure 3: *Curves of the hazard rate function of the MOLBWGU distribution for different values of the parameters.*

The reverse hazard function of $MOLBWGU(\theta,\gamma,\beta)$ is given by,

$$r_{MOLBWGU}(x;\theta,\gamma,\beta)=\frac{(\theta+2)\beta\gamma^{-\theta-2}x^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)\right)^2\left(\frac{\left(\frac{x}{\gamma}\right)^{\theta+2}}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}\right)}.$$

The reverse hazard rate function decreases with $r(0,\theta,\gamma,\beta)=0$ and $r(\gamma,\theta,\gamma,\beta)=\frac{(\theta+2)\beta}{\gamma}$.

3.2. Compounding

The property that Marshall-Olkin family of distributions can be expressed as a compound distribution with exponential distribution as mixing density is useful in obtaining new parameter families of distribution in terms of existing ones, expressed Marshall-Olkin extended forms of Weibull, Lomax, linear exponential and exponential power family of distributions as a compound distribution [see 5, 4, 9].

Theorem 1. Let X be a continuous random variable with conditional survival function on $\Delta=\delta$ expressed as $\bar{F}(x|\delta)=\left(1-(\frac{x}{\gamma})^{\theta+2}\right)e^{-(1-\beta)\delta(\frac{x}{\gamma})^{\theta+2}}, \quad 0<x<\gamma,$

and let Δ follows a distribution function with probability density function

$$m(\delta)=\beta\, e^{-\beta\delta},\ \delta>0.$$

Then the random variable X has the $MOLBWGU(\theta,\gamma,\beta)$ distribution.

Proof: The unconditional survival function of the random variable X is given by,

$$\bar{F}(x)=\int_{-\infty}^{\infty}\bar{F}(x|\delta)m(\delta)d\delta$$

$$=\beta(1-(\frac{x}{\gamma})^{\theta+2})\int_0^{\infty}e^{-[\beta+(1-\beta)(\frac{x}{\gamma})^{\theta+2}]\delta}d\delta$$

$$=\frac{\beta(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}.$$

which is the survival function of the $MOLBWGU(\theta,\gamma,\beta)$ distribution.

Theorem 2. Let $\{X_i,\ i\geq 1\}$ be a sequence of i.i.d. random variables with common survival function $G'(x)$. Let T be a geometric random variable independently distributed of $\{X_i,\ i\geq 1\}$ such that $P(T=n)=\beta(1-\beta)^{n-1}, n=1,2,\ldots,0<\beta<1$. Let $Y_T=min_{1\leq i\leq T}X_i$. Then $\{Y_T\}$ is distributed as MOLBWGU(θ,γ,β) if and only if $\{X_i\}$ follows $LBWGU(\theta,\gamma)$.

Proof: The survival function of the random variable Y_T is

$$H'(x)=P(Y_T>x)$$

$$= \sum_{n=1}^{\infty} P(Y_n > x)P(T = n)$$
$$= \sum_{n=1}^{\infty} [G'(x)]^n \beta(1-\beta)^{n-1}$$
$$= \frac{\beta G'(x)}{1-(1-\beta)G'(x)}$$
$$= \frac{\beta(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}.$$

which is survival function of MOLBWGU(θ,γ,β) distribution.

Theorem 3. Let $\{X_i,\ i \geq 1\}$ be a sequence of i.i.d. random variables with common survival function $G'(x)$. Let T be a geometric random variable independently distributed of $\{X_i,\ i \geq 1\}$ such that $P(T = n) = \beta(1-\beta)^{n-1}, n = 1,2,\dots, 0 < \beta < 1$. Let $Z_T = max_{1\leq i\leq T} X_i$. Then $\{Z_T\}$ is distributed as MOLBWGU$(\theta,\gamma,\frac{1}{\beta})$ if and only if $\{X_i\}$ follows LBWGU(θ,γ) distribution.

Proof: The distribution function of Z_T is

$$K(x) = P(Z_T \leq x)$$
$$= \sum_{n=1}^{\infty} P(Z_n \leq x)P(T = n)$$
$$= \sum_{n=1}^{\infty} [G(x)]^n \beta(1-\beta)^{n-1}$$
$$= \frac{\beta G(x)}{1-(1-\beta)G(x)}$$
$$= \frac{\beta(\frac{x}{\gamma})^{\theta+2}}{1-(1-\beta)\left(\frac{x}{\gamma}\right)^{\theta+2}}.$$

From this it follows that the survival function of the random variable Z_T is

$$K'(x) = \frac{\frac{1}{\beta}(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\frac{1}{\beta})\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)},$$

which implies that Z_T has MOLBWGU$(\theta,\gamma,\frac{1}{\beta})$ distribution.

3.3. Order Statistics

Let $(x_1, x_2, \dots, x_n)$ be a random sample of size n from $MOLBWGU(\theta,\gamma,\beta)$ distribution and let $x_{1:n}, x_{2:n}, \dots, x_{n:n}$ be the corresponding order statistics. Then the pdf of the j^{th} order statistic for the $MOLBWGU$ distribution is given by

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\frac{(\theta+2)\beta\gamma^{-\theta-2}x^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)\right)^2} \times \left[\frac{(\frac{x}{\gamma})^{\theta+2}}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}\right]^{j-1}\left[\frac{\beta(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}\right]^{n-j}.$$

The $MOLBWGU$ distribution has the following pdf for $x_{1:n}$

$$f_{1:n}(x) = \frac{(\theta+2)\beta\gamma^{-\theta-2}x^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)\right)^2}\left[\frac{\beta(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}\right]^{n-1}.$$

and the pdf for $x_{n:n}$ is given by

$$f_{n:n}(x) = n\frac{(\theta+2)\beta\gamma^{-\theta-2}x^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)\right)^2}\left[\frac{(\frac{x}{\gamma})^{\theta+2}}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}\right]^{n-1}, \quad 0 < x < \gamma.$$

3.4. Quantile Function

The q^{th} quantile of $MOLBWGU(\theta,\gamma,\beta)$ distribution is given by

$$x_q = F^{-1}(q) = \gamma\left(\frac{q\beta}{1-q(1-\beta)}\right)^{\frac{1}{\theta+2}}, \qquad 0 \leq q \leq 1.$$

where $F^{-1}(\cdot)$ is the inverse distribution function.

In particular the median of $MOLBWGU(\theta,\gamma,\beta)$ distribution is given by,

$$median(X)=\gamma\left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\theta+2}}.$$

3.5. Moments

If X has the $MOLBWGU(\theta,\gamma,\beta)$ distribution, then the s^{th} order moment is obtained as

$$\mathbb{E}(X^s)=\int_0^\gamma x^s\frac{(\theta+2)\beta\gamma^{-\theta-2}x^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)\right)^2}dx$$
$$=\frac{\beta(\theta+2)}{\gamma^{3(\theta+2)}}\int_0^\gamma\frac{x^{s+\theta+1}}{\left(\beta\gamma^{\theta+2}+(1-\beta)x^{\theta+2}\right)^2}dx.$$

where $\mathbb{E}$ denotes the expectation.
Let $x^{\theta+2}$ = u, above equation reduces to

$$\mathbb{E}(X^s)=\frac{\beta}{\gamma^{3(\theta+2)}}\int_0^{\gamma^{\theta+2}}\frac{u^{\frac{s}{\theta+2}}}{\left(\beta\gamma^{\theta+2}+(1-\beta)u\right)^2}du.$$

From Prudnikov [15],

$$\int_a^b\frac{(x-a)^{\alpha-1}}{(cx+d)^{\alpha+n+1}}dx=\frac{(b-a)^\alpha}{(ac+d)(bc+d)^\alpha}\sum_{k=0}^{n} nC_k\,\frac{B(\alpha+k,n-k+1)}{(bc+d)^k(ac+d)^{n-k}}.$$

where a, b, c and d are real numbers with (ac+d)(bc+d) $>$ 0; Real part of $\alpha>0$ and $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.
Hence if $s/\theta+2$ is a positive integer, we have,

$$\int_0^{\gamma^{\theta+2}}\frac{u^{\frac{s}{\theta+2}}}{\left(\beta\gamma^{\theta+2}+(1-\beta)u\right)^2}du=\frac{1}{\gamma^{\theta+2+s}\beta^{1+s/(\theta+2)}}\sum_{k=0}^{s/\theta+2} s/\theta+2\,\mathrm{C}_k\,\beta^k$$
$$B\left(1+k+\frac{s}{\theta+2},1-k+\frac{s}{\theta+2}\right).$$

Therefore

$$\mathbb{E}(X^s)=\frac{1}{\gamma^{4(\theta+2)+s}\beta^{s/(\theta+2)}}\sum_{k=0}^{s/\theta+2} s/\theta+2\,C_k\,\beta^kB\left(1+k+\frac{s}{\theta+2},1-k+\frac{s}{\theta+2}\right).$$

In particular,

$$\mathbb{E}(X)=\frac{1}{\gamma^{4\theta+9}\beta^{1/(\theta+2)}}\sum_{k=0}^{1/\theta+2} 1/\theta+2\,C_k\,\beta^kB\left(1+k+\frac{1}{\theta+2},1-k+\frac{1}{\theta+2}\right).$$

$$\mathbb{E}(X^2)=\frac{1}{\gamma^{4\theta+10}\beta^{2/(\theta+2)}}\sum_{k=0}^{2/\theta+2} 2/\theta+2\,C_k\,\beta^kB\left(1+k+\frac{2}{\theta+2},1-k+\frac{2}{\theta+2}\right).$$

3.6. Renyi and Shannon Entropies

The Renyi entropy is defined as

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)dx,\gamma>0,\gamma\neq1.$$

Then, $\int_0^\infty f^\gamma(x)dx=\int_0^\gamma\frac{\beta^\gamma(\theta+2)^\gamma}{\gamma^{3(\theta+2)\gamma}}\frac{x^{r(\theta+1)}}{\left(\beta\gamma^{\theta+2}+(1-\beta)x^{\theta+2}\right)^{2\gamma}}dx.$
Let u = $x^{\theta+2}$. Therefore

$$\int_0^\gamma\frac{x^{r(\theta+1)}}{\left(\beta\gamma^{\theta+2}+(1-\beta)x^{\theta+2}\right)^{2\gamma}}dx=\frac{1}{\theta+2}\int_0^{\gamma^{\theta+2}}\frac{u^{\frac{1}{\theta+2}(\gamma-1)(\theta+1)}}{\left(\beta\gamma^{\theta+2}+(1-\beta)u\right)^{2\gamma}}du.$$

Using the equation from Prudnikov [15] and if $\frac{(\gamma-1)(\theta+3)}{(\theta+2)}$ is a positive integer, the above integral becomes

$$\frac{1}{(\theta+2)\gamma^{(\theta+2)+(\gamma-1)(\theta+3)}\beta^{1+\frac{(\gamma-1)(\theta+3)}{(\theta+2)}}}\sum_{k=0}^{\frac{(\gamma-1)(\theta+3)}{(\theta+2)}}\frac{(\gamma-1)(\theta+3)}{(\theta+2)}C_k\,\beta^k$$
$$B\left(1+k+\frac{(\gamma-1)(\theta+3)}{(\theta+2)},1-k+\frac{(\gamma-1)(\theta+3)}{(\theta+2)}\right).$$

Therefore the Renyi entropy is

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left[\frac{\beta^{\frac{(1-\gamma)}{(\theta+2)}}(\theta+2)^{(\gamma-1)}}{\gamma^{(4\theta+9)\gamma-1}}\sum_{k=0}^{\frac{(\gamma-1)(\theta+3)}{(\theta+2)}} {}^{\frac{(\gamma-1)(\theta+3)}{(\theta+2)}}C_k\ \beta^k \right.$$
$$\left. B(1+k+\frac{(\gamma-1)(\theta+3)}{(\theta+2)}, 1-k+\frac{(\gamma-1)(\theta+3)}{(\theta+2)})\right].$$

Thus, the Shannon entropy is

$$E[-\log f(x)] = -\log[\beta(\theta+2)\gamma^{-3(\theta+2)}] - (\theta+1)E[\log(X)] + 2E[\log(\beta\gamma^{\theta+2} + (1-\beta)x^{\theta+2})].$$

4. Maximum Likelihood Estimation

The MLE method is used for the parameter estimation of MOLBWGU distribution. Let $(x_1, x_2, \dots, x_n)$ be a random sample of size n from the MOLBWGU distribution. The likelihood function for the MOLBWGU distribution is,

$$L = \prod_{i=1}^{n} \frac{(\theta+2)\beta\gamma^{-\theta-2}x_i^{\theta+1}}{\left(1-(1-\beta)\left(1-\left(\frac{x_i}{\gamma}\right)^{\theta+2}\right)\right)^2}.$$

from which the log-likelihood function is obtained as

$$\log L = -2\sum_{i=1}^{n} \log\left(1-(1-\beta)\left(1-\gamma^{-\theta-2}x_i^{\theta+2}\right)\right) + (\theta+1)\sum_{i=1}^{n}\log(x_i)$$
$$+n(-(\theta+2)\log(\gamma) + \log(\theta+2) + \log(\beta)).$$

The partial derivatives of this log-likelihood function is given by

$$\frac{\partial \log L}{\partial\theta} = -2\sum_{i=1}^{n} - \frac{(1-\beta)\left(\gamma^{-\theta-2}\log(\gamma)x_i^{\theta+2} - \gamma^{-\theta-2}x_i^{\theta+2}\log(x_i)\right)}{1-(1-\beta)\left(1-\gamma^{-\theta-2}x_i^{\theta+2}\right)}$$
$$+\sum_{i=1}^{n}\log(x_i) + n\left(\frac{1}{\theta+2} - \log(\gamma)\right).$$
$$\frac{\partial logL}{\partial\gamma} = -2\sum_{i=1}^{n}\frac{(-\theta-2)(1-\beta)\gamma^{-\theta-3}x_i^{\theta+2}}{1-(1-\beta)\left(1-\gamma^{-\theta-2}x_i^{\theta+2}\right)} - \frac{(\theta+2)n}{\gamma}.$$
$$\frac{\partial logL}{\partial\beta} = \frac{n}{\beta} - 2\sum_{i=1}^{n}\frac{1-\gamma^{-\theta-2}x_i^{\theta+2}}{1-(1-\beta)\left(1-\gamma^{-\theta-2}x_i^{\theta+2}\right)}.$$

The maximum likelihood estimator $(\hat{\theta}, \hat{\gamma}, \hat{\beta})$ of the parameters (θ, γ, β) can be obtained by solving the equations $\frac{\partial logL}{\partial\theta} = 0$, $\frac{\partial logL}{\partial\gamma} = 0$ and $\frac{\partial logL}{\partial\beta} = 0$.

4.1. Simulation Study

In this section, some simulation results are provided to study the behaviour of the MLEs in terms of the sample size. For this purpose, a Monte Carlo simulation study is conducted for $MOLBWGU(\theta, \gamma, \beta)$ distribution. The results are obtained from 1000 Monte Carlo replications and the simulations are carried out using the statistical software R. In each replication, a random sample of size 25, 50, 100, 150, 200 is generated for different combinations of θ, γ and β . The initial values of parameters are θ = 1.2, γ = 0.3, β = 1.5; θ = 2 , γ = 1.5, β = 2.5; θ = 0.5, γ = 0.5, β = 1.5 and θ = 2, γ = 1, β = 2. Then computed mean of the MLEs of the parameters, biases and mean square errors (MSEs) of the parameter estimates. Tables 1, 2, 3 and 4 gives the values of the estimates, biases and MSEs of the corresponding parameters. From the tables, it can be seen that, as sample size increases the bias and MSE of the estimates decreases.

Table 1: *Estimates, Biases and MSEs for* $\theta = 1.2$, $\gamma = 0.3$ *and* $\beta = 1.5$

Sample Size(n)	Parameters	Estimates	Biases	MSEs
25	θ	1.2155	0.0155	0.0431
	γ	0.3579	0.0579	0.1046
	β	1.5034	0.0034	0.0009
50	θ	1.2143	0.0143	0.0412
	γ	0.3565	0.0565	0.1051
	β	1.5022	0.0022	0.0003
100	θ	1.2135	0.0135	0.0159
	γ	0.3541	0.0541	0.1015
	β	1.5015	0.0015	0.0003
150	θ	1.2111	0.0111	0.0039
	γ	0.3509	0.0509	0.1012
	β	1.5009	0.0009	0.0002
200	θ	1.2097	0.0097	0.0027
	γ	0.3478	0.0478	0.1004
	β	1.5001	0.0001	0.0002

Table 2: *Estimates, Biases and MSEs for* $\theta = 2$, $\gamma = 1.5$ *and* $\beta = 2.5$

Sample Size(n)	Parameters	Estimates	Biases	MSEs
25	θ	2.6271	0.6271	0.4428
	γ	1.5096	0.0096	0.2635
	β	2.5242	0.0242	0.9715
50	θ	2.4927	0.4927	0.3527
	γ	1.5082	0.0082	0.0792
	β	2.5211	0.0211	0.6226
100	θ	2.3813	0.3813	0.3795
	γ	1.5065	0.0065	0.0489
	β	2.5175	0.0175	0.6535
150	θ	2.3145	0.3145	0.1529
	γ	1.5068	0.0068	0.0035
	β	2.5148	0.0148	0.3614
200	θ	2.2501	0.2501	0.1358
	γ	1.5045	0.0045	0.0012
	β	2.5122	0.0122	0.1428

Table 3: *Estimates, Biases and MSEs for θ = 0.5, γ = 0.5 and β = 1.5*

Sample Size(n)	Parameters	Estimates	Biases	MSEs
25	θ	0.5312	0.0312	0.0803
	γ	0.5505	0.0505	1.5112
	β	1.5091	0.0091	0.1583
50	θ	0.5304	0.0304	0.0713
	γ	0.5447	0.0447	0.8218
	β	1.5082	0.0082	0.0685
100	θ	0.5275	0.0275	0.0752
	γ	0.5418	0.0418	0.7943
	β	1.5079	0.0079	0.0082
150	θ	0.5289	0.0289	0.0239
	γ	0.5345	0.0345	0.5728
	β	1.5074	0.0074	0.0047
200	θ	0.5217	0.0217	0.0249
	γ	0.5293	0.0293	0.5355
	β	1.5066	0.0066	0.0032

Table 4: *Estimates, Biases and MSEs for θ = 2, γ = 1 and β = 2*

Sample Size(n)	Parameters	Estimates	Biases	MSEs
25	θ	2.1827	0.1827	0.3912
	γ	1.1578	0.1578	0.2014
	β	2.1666	0.1666	0.0009
50	θ	2.0915	0.0915	0.3373
	γ	1.1579	0.1579	0.2008
	β	2.1643	0.1643	0.0004
100	θ	2.0912	0.0912	0.1838
	γ	1.1458	0.1458	0.2007
	β	2.1712	0.1712	0.0003
150	θ	2.0774	0.0774	0.1425
	γ	1.1435	0.1435	0.2001
	β	2.1626	0.1626	0.0003
200	θ	2.0751	0.0751	0.0564
	γ	1.1315	0.1315	0.0197
	β	2.0125	0.0125	0.0002

5. Application in Autoregressive Time Series Modeling

In this section, some applications of MOLBWGU distribution in autoregressive time series modelling are provided. Now, we construct a first order autoregressive minification process with structure as follows,

$$X_n = \begin{cases} \varepsilon_n, & w.p. \quad \beta \\ \min(x_{n-1}, \varepsilon_n), & w.p. \quad 1-\beta, \quad 0 \le \beta \le 1, \ n \ge 1, \end{cases} \tag{8}$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables following LBWGU(θ, γ) distribution independent of $\{x_{n-1}, x_{n-2}, \dots\}$. Then the process is stationary and is marginally distributed with $MOLBWGU(\theta, \gamma, \beta)$ distribution. This leads to the following theorem.

Theorem 4. In an AR (1) process with structure (8), $\{X_n, \ n \ge 0\}$ defines a stationary AR (1) minification process with $MOLBWGU(\theta, \gamma, \beta)$ marginal distribution iff $\{\varepsilon_n\}$ is a sequence of independently and identically distributed random variable with LBWGU(θ, γ) distributon.

Proof. Consider (8) in terms of survival function

$$\bar{F}_{X_n}(x) = \beta \bar{F}_{\varepsilon_n}(x) + (1-\beta)\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x).$$

Under stationary equilibrium it reduces to

$$\bar{F}_X(x) = \frac{\beta \bar{F}_{\varepsilon_n}(x)}{1-(1-\beta)\bar{F}_{\varepsilon_n}(x)}. \tag{9}$$

and hence

$$\bar{F}_{\varepsilon_n}(x) = \frac{\beta \bar{F}_X(x)}{1-(1-\beta)\bar{F}_X(x)}. \tag{10}$$

If ε_n follows LBWGU(θ, γ) from(9), we get

$$\bar{F}_X(x) = \frac{\beta(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}.$$

which is the survival function of $MOLBWGU(\theta, \gamma, \beta)$.

Conversely, if we take

$$\bar{F}_{X_n}(x) = \frac{\beta(1-(\frac{x}{\gamma})^{\theta+2})}{1-(1-\beta)\left(1-\left(\frac{x}{\gamma}\right)^{\theta+2}\right)}.$$

from (9) it can show that$\bar{F}_{\varepsilon_n}(x)$ is distributed LBWGU(θ, γ) with survival function $(1-(\frac{x}{\gamma})^{\theta+2})$.

5.1. Sample Path

To study the behavior of the process we simulate the sample path for various values of β, the properties of sample path shows that the MOLBWGU AR(1) minification process can be used for modelling a rich variety of real data. Sample path of MOLBWGU AR(1) process for γ = 0.5, θ = 0.9 and β= 0.4, 0.5, 0.6 and 0.8 is given in Figure 4.

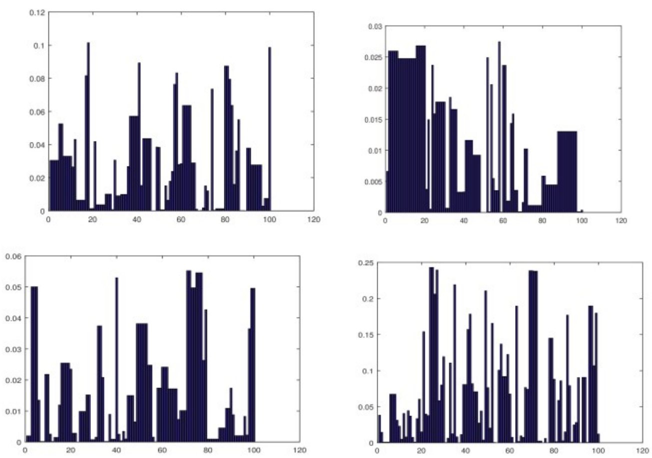

Figure 4: *Sample path of the MOLBWGU AR(1) process for γ = 0.5, θ = 0.9 and β= 0.4, 0.5, 0.6 and 0.8.*

6. Stress Strength Analysis

The stress strength reliability analysis can be regarded as an assessment of reliability of a system in terms of random variables X and Y, where X represents strength and Y represents the stress. If the stress exceeds strength the system would fail and the system will function if strength exceeds stress. The stress strength reliability can be defined as R = P(X >Y). Gupta [6] obtained various results on the MO family in the context of reliability modelling and survival analysis. Then,

$$R = P(X > Y) = \int_{-\infty}^{+\infty} P(X > Y|Y = y) g_Y(y)dy$$
$$= \frac{\frac{\beta_1}{\beta_2}}{(\frac{\beta_1}{\beta_2}-1)^2}\left(-ln\frac{\beta_1}{\beta_2} + \frac{\beta_1}{\beta_2} - 1\right).$$

Let $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ be two independent random samples of sizes m and n from MOLBWGU distribution with tilt parameters β_1 and β_2 respectively, and common unknown parameters γ and θ .
The log likelihood function is given by

$$\log L = -2\sum_{i=1}^{m} \log\left(1 - (1-\beta_1)\left(1-\gamma^{-\theta-2}x_i^{\theta+2}\right)\right) + (\theta+1)\sum_{i=1}^{m} \log(x_i) + m(-(\theta+2)\log(\gamma) + \log(\theta + 2) + \log(\beta_1)) - 2\sum_{i=1}^{n} \log\left(1 - (1-\beta_2)\left(1-\gamma^{-\theta-2}y_i^{\theta+2}\right)\right) + (\theta+1)\sum_{i=1}^{n} \log(y_i) + n(-(\theta+2)\log(\gamma) + \log(\theta+2) + \log(\beta_2))$$

Then MLE of β_1 and β_2 are the solutions of the nonlinear equations $\frac{\partial logL}{\partial \beta_1} = 0$ and $\frac{\partial logL}{\partial \beta_2} = 0$. The elements of Information matrix are given by,

$$I_{11} = -E\left(\frac{\partial^2 L}{\partial \beta_1^2}\right) = \frac{m}{3\beta_1^2}.$$
$$I_{22} = -E\left(\frac{\partial^2 L}{\partial \beta_2^2}\right) = \frac{n}{3\beta_2^2}.$$
$$I_{12} = I_{21} = -E\left(\frac{\partial^2 L}{\partial \beta_1 \partial \beta_2}\right) = 0.$$

By the property of MLE for $m \to \infty$ and $n \to \infty$,

$$(\sqrt{m}(\hat{\beta}_1 - \beta_1), \sqrt{n}(\hat{\beta}_2 - \beta_2)) \to N_2(0, diag\{a_{11}^{-1}, a_{22}^{-1}\}).$$

where $a_{11} = \lim_{m,n\to\infty} \frac{I_{11}}{m} = \frac{1}{3\beta_1^2}$ and $a_{22} = \lim_{m,n\to\infty} \frac{I_{22}}{n} = \frac{1}{3\beta_2^2}$

Now from [6] the 95% confidence interval for R is given by $\hat{R} \pm 1.96\hat{\beta}_1 b_1(\hat{\beta}_1, \hat{\beta}_2)\sqrt{\frac{3}{m} + \frac{3}{n}}$, where $b_1(\beta_1, \lambda_2) = \frac{\partial R}{\partial \beta_1} = \frac{\beta_2}{(\beta_1-\beta_2)^3}\left[-2(\beta_1 - \beta_2) + (\beta_1 + \beta_2)\ln\frac{\beta_1}{\beta_2}\right]$ and $b_2(\lambda_1, \lambda_2) = \frac{\partial R}{\partial \beta_2} = \frac{\beta_1}{(\beta_1-\beta_2)^3}\left[2(\beta_1 - \beta_2) - (\beta_1 + \beta_2)\ln\frac{\beta_1}{\beta_2}\right] = -\frac{\beta_1}{\beta_2} b_1(\beta_1, \beta_2).$

6.1. Simulation Study

For the simulation study, generate N=10,000 sets of X-samples and Y-samples from the MOLBWGU distribution with parameters $(\beta_1, \gamma, \theta)$ and $(\beta_2, \gamma, \theta)$ respectively. The combinations of samples of sizes $m = 20,25,30$ and $n = 20,25,30$ are studied. The validity of the estimate of R is considered by using the following measures, namely average bias of the estimate $(\bar{b})$, average mean square error of the estimate (AMSE), average confidence interval of the estimate and coverage probability defined by,

1. Average bias $(\bar{b})$ of the estimates of R:
$$\frac{1}{N}\sum_{i=1}^{N} (\hat{R}_i - R).$$
2. Average mean square error of the estimates of R:
$$\frac{1}{N}\sum_{i=1}^{N} (\hat{R}_i - R)^2.$$
3. Average length of the asymptotic 95% confidence interval of R:

$$\frac{1}{N}\sum_{i=1}^{N} 2(1.96)\hat{\beta}_{1i}b_{1i}(\hat{\beta}\beta_{1i},\hat{\beta}\beta_{2i})\sqrt{\frac{3}{m}+\frac{3}{n}}.$$

4. The coverage probability of the confidence intervals given by the proportion of such interval that include the parameter R.

The numerical values obtained for the measures are presented in Table 5 and 6. The average bias decreases as the sample size increases. The coverage probability is close to 0.95 as the sample size increases. This simulation results shows that the average bias, average MSE, average confidence interval and coverage probability do not show much variability for various parameter combinations.

Table 5: *Average bias and average MSE of the simulated estimates of R for* $\gamma = 1$ *and* $\theta = 0.9$

(β_1, β_2)								
	Average Bias $(\bar{b})$				Average Mean Square Error (AMSE)			
(m, n)	(0.2,0.4)	(0.4, 0.2)	(0.5, 0.1)	(0.4, 1.2)	(0.2, 0.4)	(0.4, 0.2)	(0.5, 0.1)	(0.4, 1.2)
(20, 20)	-0.0035	-0.0825	-0.0589	0.0286	0.0042	0.0115	0.0040	0.0052
(20, 25)	-0.0066	-0.0832	-0.0605	0.0289	0.0041	0.0117	0.0041	0.0052
(20, 30)	-0.0092	-0.0839	-0.0614	0.0279	0.0039	0.0117	0.0042	0.0050
(25, 20)	-0.0024	-0.0857	-0.0588	0.0233	0.0041	0.0115	0.0040	0.0041
(25, 25)	-0.0052	-0.0877	-0.0606	0.0241	0.0039	0.0118	0.0041	0.0043
(25, 30)	-0.0077	-0.0878	-0.0611	0.0234	0.0038	0.0118	0.0041	0.0042
(30, 20)	-0.0021	-0.0887	-0.0588	0.0199	0.0040	0.0116	0.0040	0.0036
(30, 25)	-0.0048	-0.0910	-0.0602	0.0204	0.0038	0.0118	0.0040	0.0036
(30, 30)	-0.0075	-0.0908	-0.0611	0.0205	0.0037	0.0118	0.0041	0.0037

Table 6: *Average length of the confidence interval and coverage probability of the simulated 95% confidence intervals of R for* $\gamma = 1$ *and* $\theta = 0.9$

(β_1, β_2)								
	Average Confidence Length				Coverage Probability			
(m, n)	(0.2,0.4)	(0.4, 0.2)	(0.5, 0.1)	(0.4, 1.2)	(0.2, 0.4)	(0.4, 0.2)	(0.5, 0.1)	(0.4, 1.2)
(20, 20)	0.3346	0.3505	0.3111	0.3239	0.9592	0.9839	0.9991	0.9805
(20, 25)	0.3167	0.3325	0.2960	0.3074	0.9585	0.9727	0.9898	0.9530
(20, 30)	0.3043	0.3201	0.2853	0.2956	0.9583	0.9611	0.9794	0.9299
(25, 20)	0.3179	0.3334	0.2951	0.3063	0.9582	0.9809	0.9799	0.9793
(25, 25)	0.2991	0.3145	0.2791	0.2889	0.9573	0.9613	0.9999	0.9533
(25, 30)	0.2860	0.3012	0.2675	0.2764	0.9568	0.9434	0.9696	0.9231
(30, 20)	0.3064	0.3215	0.2840	0.2939	0.9574	0.9720	0.9599	0.9783
(30, 25)	0.2867	0.3019	0.2671	0.2759	0.9557	0.9533	0.9599	0.9593
(30, 30)	0.2728	0.2879	0.2551	0.2631	0.9542	0.9264	0.9399	0.9333

7. Conclusion

In this paper, a generalization of LBWGU distribution namely MOLBWGU distribution is developed. Some of the statistical properties of the new distribution such as probability density function, hazard rate function, moments, quantile function, compounding, distribution of order statistics, Renyi and Shannon entropies are derived. We estimated the parameters of the distribution using maximum likelihood estimation method and a simulation study is conducted for proving the validity of the estimates.

Also we developed a minification process using the model and explored its sample path behavior for different combinations of parameters. To check the impact of stress on strength of devices and systems, the stress strength analysis is carried out and the estimate of the reliability is examined based on a simulation study.

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