A STUDY ON THE PROPERTIES OF A NEW EXPONENTIATED EXTENDED INVERSE EXPONENTIAL DISTRIBUTION WITH APPLICATIONS

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Abstract

In this paper, a new continuous probability distribution called a new exponentiated extended inverse exponential distribution with four parameters is introduced. The mathematical and statistical properties of the proposed distribution, such as the quantile function, moments, moment generating function, survival function, hazard function, odds function, and reversed hazard function, were studied to understand its nature. The probability density function of the order statistics for this distribution was also obtained. The parameters of the model were estimated using the maximum likelihood method of estimation. The proposed model was applied to two real datasets relating to the relief times of twenty patients receiving an analgesic and the sum of skin folds in 202 athletes collected at the Australian Institute of Sports. The results showed that the new model outperformed its comparators and provides better fit than Topp-Leone exponentiated inverse exponential, Topp-Leone inverse exponential, exponentiated inverse exponential, inverse exponential and exponential distributions.

Keywords: Akaike information criterion, breast cancer, skin fold, inverse exponential, adequacy model

I. Introduction

The creation of novel, all-encompassing statistical models is an important field of study in distribution theory. Such distributions, which are extremely valuable in forecasting and simulating real-world phenomena, are abundant in the literature. The modeling of data in various practical domains, such as bio-medical analysis, reliability engineering, economics, forecasting, astronomy, demography, and insurance, has extensively used a number of classical distributions throughout the past few decades.

The majority of exponential distribution generalizations have constant, non-increasing, non-decreasing, and bathtub hazard rates. However, in real-world situations, it is possible for the data to display a unimodal (first increasing and then decreasing) inverted bathtub hazard rate. In the analysis of breast cancer data, we found that the mortality rises early, reaches a peak after some time, and then drops gradually; the related hazard rate is thus inverted bathtub-shaped or notably unimodal. For this type of data, the one parameter inverted exponential (IEx) distribution, which has the inverted bathtub hazard rate, has been proposed as another extension of the exponential distribution in statistical literature.

In the literature, an inverted exponential distribution which was introduced in [1]. The distribution has an inverted bathtub hazard rate and can be used to simulate real-world events that have inverted bathtub failure rates. [2] have also addressed an example of its use with breast cancer data. According to [3], it has also been described as a model that is helpful in survival analysis.

To increase the modeling flexibility of current probability distributions utilizing various families of distributions, recent research in this field has focused on extending existing probability distributions. Some families of distributions proposed in literature include Kumaraswamy generalized family of distributions by [4], Topp Leone generalized family of distributions by [5], exponentiated extended generalized family of distributions by [6], Power Lindley generalized family of distributions by [7], Topp Leone exponentiated generalized family of distributions by [8], Topp Leone Kumaraswamy generalized family of distributions by [9], Odd Chen generalized family of distributions by [10], Modi generalized family of distributions by [11], A new generalized family of distributions by [12], Type I half-Logistic exponentiated generalized family of distributions by [13], Type II half-Logistic generalized family of distributions by [14], etc.

In line with this, some of the recent developments and extensions of the inverse exponential distribution using generalized families of continuous distribution can be found in [15], [16], [17], [18], [19].

In this context, we developed a generalization of the inverse exponential distribution based on [6], which is derived from the following general construction: if G denotes the baseline of a cumulative distribution function, then a generalized family of distributions can be defined with cumulative distribution function and probability density function give respectively as

$$F(x;\alpha,\lambda,\theta)=\left[1-\left[1-G(x;\omega)\right]^{\alpha\lambda}\right]^{\theta} \tag{1}$$

and

$$f(x;\alpha,\lambda,\theta)=\alpha\lambda\theta g(x;\omega)\left[1-G(x;\omega)\right]^{\alpha\lambda-1}\left[1-\left[1-G(x;\omega)\right]^{\alpha\lambda}\right]^{\theta-1} \tag{2}$$

where ω is the vector of parameters of the baseline distribution.
where $G(x;\omega)$ is the cumulative distribution function (cdf) of the baseline distribution with vector of parameter ω.
for $x\geq 0,\alpha,\lambda,\theta,\omega\geq 0$, where equations (1) and (2) are the cumulative distribution function and probability density function (pdf) of the family of distributions proposed by [6].
The cdf and pdf of the inverse exponential (IEx) distribution are given by

$$G(x;\beta)=e^{-\left(\frac{\beta}{x}\right)} \tag{3}$$

$$g(x;\beta)=\frac{\beta}{x^2}e^{-\left(\frac{\beta}{x}\right)} \tag{4}$$

The major goal of this study is to build a more flexible model by adding three more shape parameters to the inverse exponential distribution to increase its goodness-of-fit to real-world data sets. The main reasons for creating the NEtEIEx distribution in practice are to make the kurtosis more flexible compared to the baseline inverse exponential model, to produce skewness for symmetrical distributions, to build heavy-tailed distributions that are not longer-tailed for modeling real data, to have distributions with symmetric, left-skewed, right-skewed, and inverted bathtub shapes, and to consistently offer better fits than other generated models under certain conditions.

II. Methods

2.1. A New Exponentiated Extended Inverse Exponential (NEtEIEx) Distribution

This section developed a new continuous probability distribution function called new exponentiated extended inverse exponential (NEtEIEx) distribution and provide some plots of its pdf, cdf survival function and hazard rate function (hrf) in order to assess the shape of the new distribution. The cdf of the NEtEIEx distribution is obtained by inserting (3) into (1) given as:

$$F(x;\alpha,\beta,\lambda,\theta)=\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta} \tag{5}$$

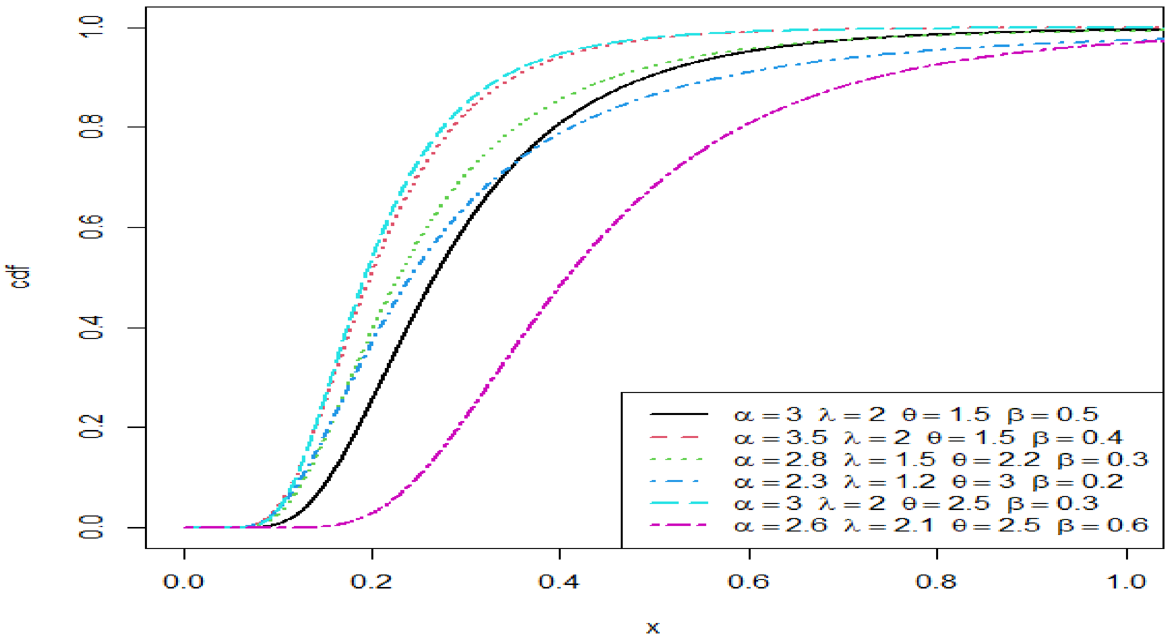

Figure 1: *Plots of CDF of the NEtEIEx distribution for different parameter values*

On differentiating equation (5) with respect to x, we obtained the pdf of a NEtEIEx distribution given in equation (6)

$$f(x;\alpha,\beta,\lambda,\theta)=\alpha\lambda\theta\frac{\beta}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda-1}\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta-1} \tag{6}$$

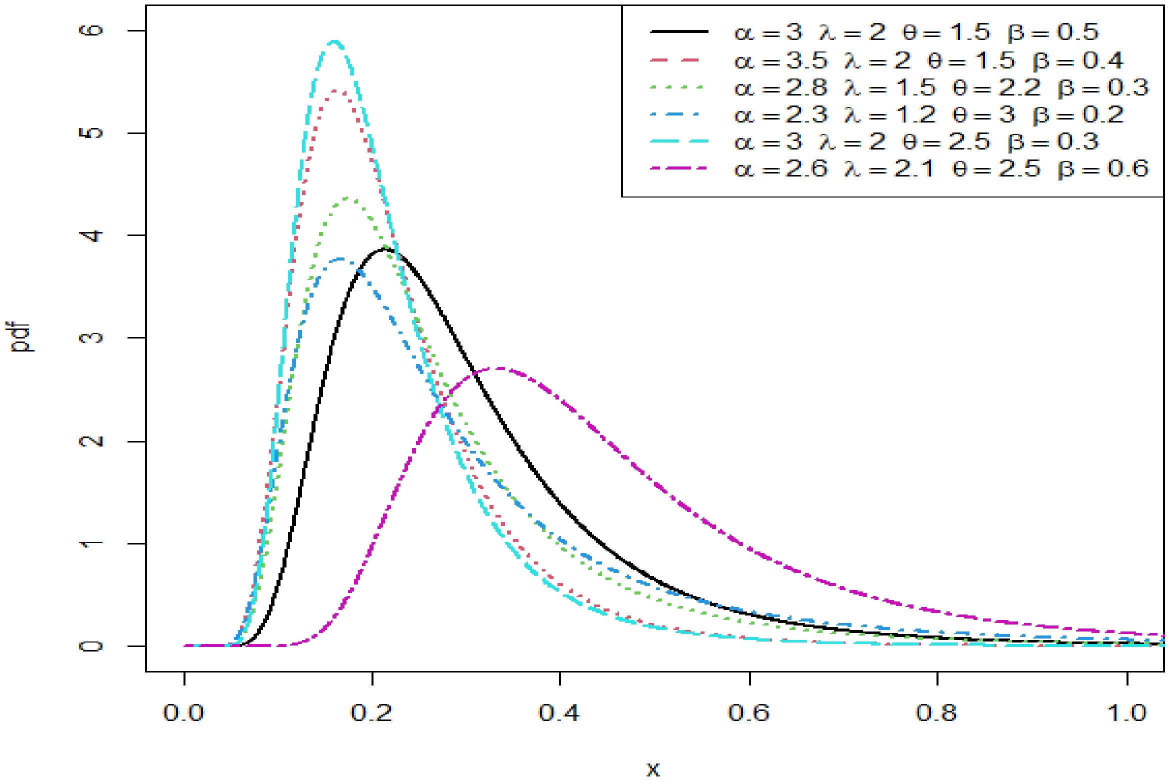

Figure 2: *Plots of PDF of the NEtEIEx distribution for different parameter values*

Where $x \geq 0$, $\beta > 0$ is the scale parameter and $\alpha, \theta, \lambda > 0$ are the shape parameters respectively.

2.1.1. Expansion of Density

In this section the pdf in equation (6) is expanded using binomial expansion. Expanding the last term in equation (6), we have

$$\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta-1} = \sum_{i=0}^{\infty}(-1)^i \binom{\theta-1}{i}\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda i}$$

$$\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda(i+1)-1} = \sum_{j=0}^{\infty}(-1)^j \binom{\alpha\lambda(i+1)-1}{j}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{j}$$

$$f(x;\alpha,\lambda,\theta) = \alpha\lambda\theta\frac{\beta}{x^2}\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\theta-1}{i}\binom{\alpha\lambda(i+1)-1}{j}\left[e^{-\frac{\beta}{x}}\right]^{j+1} \quad (7)$$

Equation (7) is the expansion of equation (6) which will be used to derive some of the properties of the distribution.

2.1.2. Properties of the New Exponentiated Extended Inverse Exponential (NEtEIEx) Distribution

In this section, some of the mathematical and statistical properties of NEtEIEx distribution such as the quantile function, moments, moment generating function, reliability measure, odds function, reversed hazard function and order statistics are derived.

2.1.2.1. Moments

$$E(X^r)=\int_0^\infty x^r f(x)dx \tag{8}$$

$$E\left(X^r\right)=\alpha\lambda\theta\frac{\beta}{x^2}\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\begin{pmatrix}\theta-1\\ i\end{pmatrix}\begin{pmatrix}\alpha\lambda\left(i+1\right)-1\\ j\end{pmatrix}\int_0^\infty x^r\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{1+j} \tag{9}$$

Consider the integral part of equation (9), we have

$$\int_0^\infty x^r\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{1+j}$$

Let $y=(1+j)\left(\frac{\beta}{x}\right)$; $x=(1+j)\left(\frac{\beta}{y}\right)$ and dx$=\frac{dyx^2}{\beta(1+j)}$

$$E\left(X^r\right)=\alpha\lambda\theta\frac{\beta}{x^2}\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\begin{pmatrix}\theta-1\\ i\end{pmatrix}\begin{pmatrix}\alpha\lambda\left(i+j\right)-1\\ j\end{pmatrix}\int_0^\infty\left[\left(1+j\right)\left(\frac{\beta}{y}\right)\right]^r e^{-y}\frac{dyx^2}{\beta\left(1+j\right)}$$

$$E\left(X^r\right)=\alpha\lambda\theta\beta^r\left(1+j\right)^{r-1}\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\begin{pmatrix}\theta-1\\ i\end{pmatrix}\begin{pmatrix}\alpha\lambda\left(i+j\right)-1\\ j\end{pmatrix}\int_0^\infty y^{-r}e^{-y}dy$$

Where $\int_0^\infty y^{-r}e^{-y}dy=\Gamma\left(1-r\right)$

Therefore

$$E\left(X^r\right)=\alpha\lambda\theta\beta^r\left(1+j\right)^{r-1}\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\begin{pmatrix}\theta-1\\ i\end{pmatrix}\begin{pmatrix}\alpha\lambda\left(i+j\right)-1\\ j\end{pmatrix}\Gamma\left(1-r\right) \tag{10}$$

Equation (10) is the moments of NEtEIEx distribution. To obtain the mean, we set *r = 1* in equation (10).

2.1.2.2. Moment Generating Function (mgf)

$$M_x(t)=\int_0^\infty e^{tx}f(x)dx \tag{11}$$

$$M_x(t)=\alpha\lambda\theta\beta^m\left(1+j\right)^{m-1}\frac{t^m}{m!}\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\begin{pmatrix}\theta-1\\ i\end{pmatrix}\begin{pmatrix}\alpha\lambda\left(i+j\right)-1\\ j\end{pmatrix}\Gamma\left(1-m\right)$$

where the expansion of $e^{tx}=\sum_{m=0}^{\infty}\frac{t^m x^m}{m!}$

and following the process of moments above, we have

$$M_x(t)=\alpha\lambda\theta\frac{\beta}{x^2}\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\begin{pmatrix}\theta-1\\ i\end{pmatrix}\begin{pmatrix}\alpha\lambda\left(i+1\right)-1\\ j\end{pmatrix}\int_0^{\infty}e^{tx}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{1+j} \tag{12}$$

2.1.2.3. Quantile Function

Quantile function has a significant position in probability theory and it is the inverse of the cdf. The quantile function is obtained using

$$Q(u)=F^{-1}(u) \tag{13}$$

Using the inverse of equation (5), we have the quantile function given as

$$x=\frac{\beta}{-\log\left[1-\left[1-u^{\frac{1}{\theta}}\right]\right]^{\frac{1}{\alpha\lambda}}} \tag{14}$$

The median is obtained by setting *u = 0.5* in equation (14) given as

$$x_{median}=\frac{\beta}{-\log\left[1-\left[1-0.5^{\frac{1}{\theta}}\right]\right]^{\frac{1}{\alpha\lambda}}} \tag{15}$$

2.1.2.4. Hazard Function

Hazard function is given as

$$\tau(x;\alpha,\beta,\lambda,\theta)=\frac{f(x;\alpha,\beta,\lambda,\theta)}{R(x;\alpha,\beta,\lambda,\theta)} \tag{16}$$

The hazard function of the NEtEIEx distribution is given as

$$\tau(x;\alpha,\beta,\lambda,\theta)=\frac{\alpha\lambda\theta\frac{\beta}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda-1}\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta-1}}{1-\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta}} \tag{17}$$

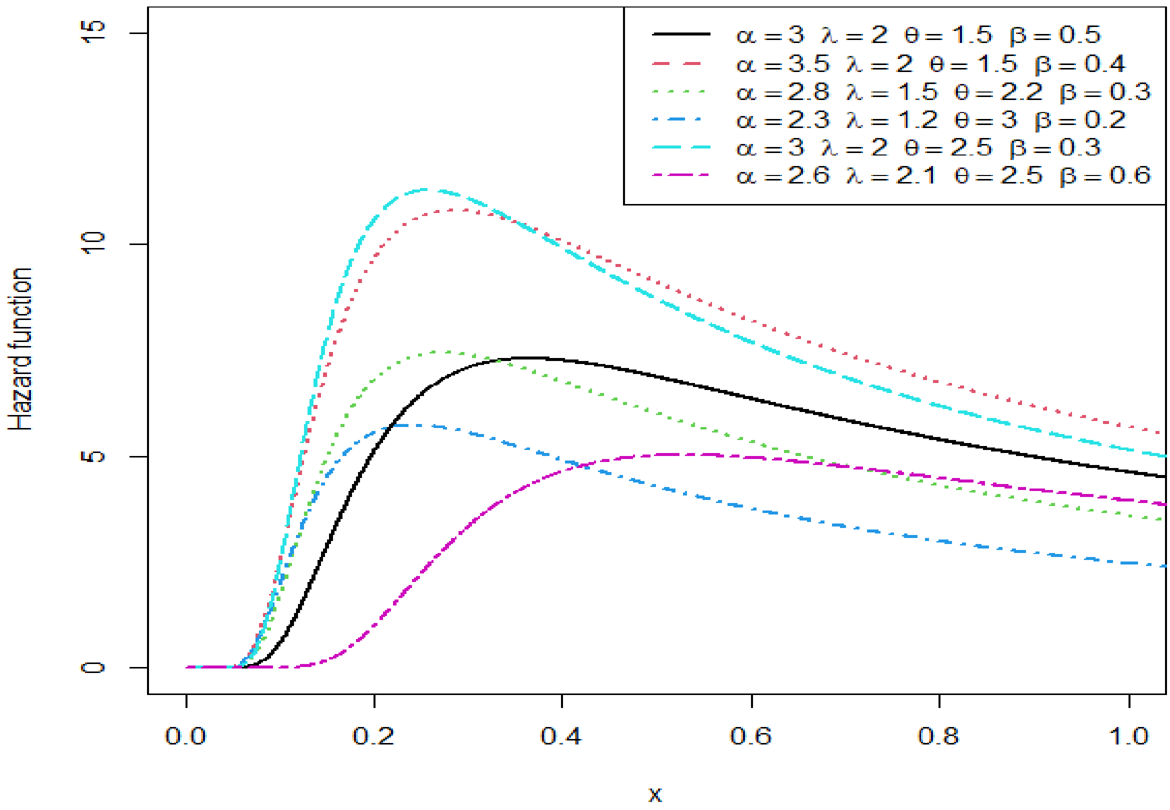

Figure 3: *Plots of hazard function of the NEtEIEx distribution for different parameter values*

2.1.2.5. Survival Function

The reliability function is also known as survival function, which is the probability that a system will survive beyond a specified time [20]. It can be defined as

$$R(x;\alpha,\beta,\lambda,\theta)=1-F(x;\alpha,\beta,\lambda,\theta) \tag{18}$$

The survival function of the NEtEIEx distribution is given as

$$R(x;\alpha,\lambda,\theta,\beta)=1-\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta} \tag{19}$$

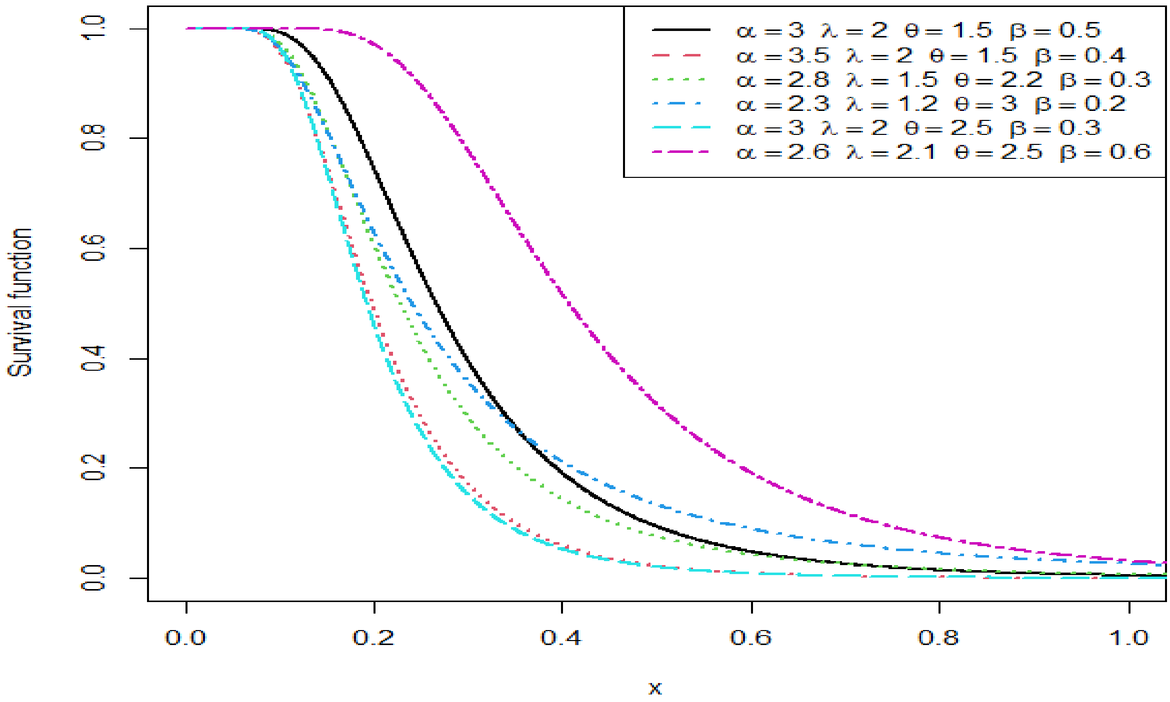

Figure 4: *Plots of survival function of the NEtEIEx distribution for different parameter values*

2.1.2.6. Reversed Hazard Function

Reversed hazard function of a random variable x is given as

$$\Re(x;\alpha,\beta,\lambda,\theta)=\frac{f(x;\alpha,\beta,\lambda,\theta)}{F(x;\alpha,\beta,\lambda,\theta)} \tag{20}$$

The reverse hazard rate function of the NEtEIEx distribution is given as

$$\Re(x;\alpha,\beta,\lambda,\theta)=\frac{\alpha\lambda\theta\frac{\beta}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda-1}\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta-1}}{\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta}} \tag{21}$$

2.1.2.7. Odds Function

The odds function of the NEtEIEx distribution is given as

$$\Pi(x;\alpha,\beta,\lambda,\theta)=\frac{\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta}}{1-\left[1-\left[1-\left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta}} \tag{22}$$

2.2. Order Statistics

Let $X_1, X_2, ..., X_n$ be n independent random variable from the NEtEIEx distributions and let $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1,2,3,...n$ denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. The pdf of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{\infty} \frac{(-1)^i \overline{|(n-r+1)}}{i! \overline{|(n-r+1-i)}} [F(x)]^{r+i-1} f(x) \tag{23}$$

$$f_{r:n}(x) = \frac{\alpha\lambda\theta \frac{\beta}{x^2} e^{-\left(\frac{\beta}{x}\right)}}{B(r, n-r+1)} \sum_{i=0}^{\infty} \frac{(-1)^i \overline{|(n-r+1)}}{i! \overline{|(n-r+1-i)}} \left[1 - \left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda - 1} \left[1 - \left[1 - \left[e^{-\left(\frac{\beta}{x}\right)}\right]\right]^{\alpha\lambda}\right]^{\theta(r+i)-1} \tag{24}$$

Equation (24) is the r^{th} order statistics of the NEtEIEx distribution. To obtain the maximum and minimum order statistics, we set *r* = *1*and *r* = *n* in equation (24) respectively.

2.3. Estimation Method

The method of maximum likelihood estimation (MLE) is used in this section to estimate the parameters of the NEtEIEx distribution. For a random sample, $X_1, X_2, ..., X_n$ of size n from the NEtEIEx$(\alpha, \beta, \theta, \lambda)$, the log-likelihood function L$(\alpha, \beta, \theta, \lambda)$ of (6) is given as

$$\log L = n\log(\alpha) + n\log(\lambda) + n\log(\theta) + n\log(\beta) + \sum_{i=1}^{n} \frac{1}{X_i^2} + (\alpha\lambda - 1) \sum_{i=1}^{n} \log\left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right] + (\theta - 1) \sum_{i=1}^{n} \log\left[1 - \left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha\lambda}\right] \tag{25}$$

Differentiating the log-likelihood with respect to $\lambda, \alpha, \theta, \beta$ and equating the result to zero, we have

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + (\lambda - 1) \sum_{i=1}^{n} \log\left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right] + \lambda(\theta - 1) \sum_{i=1}^{n} \frac{\left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha(\lambda-1)} \left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha} \log\left[e^{-\left(\frac{\beta}{x_i}\right)}\right]}{1 - \left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha\lambda}} = 0 \tag{26}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + (\alpha - 1) \sum_{i=1}^{n} \log\left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right] - (\theta - 1) \sum_{i=1}^{n} \frac{\left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha\lambda} \log\left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha}}{1 - \left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha\lambda}} = 0 \tag{27}$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log\left[1 - \left[1 - \left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha\lambda}\right] = 0 \tag{28}$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}+\left(\alpha\lambda-1\right)\sum_{i=1}^{n}\frac{e^{-\left(\frac{\beta}{x_i}\right)}\frac{1}{x_i}}{1-\left[e^{-\left(\frac{\beta}{x_i}\right)}\right]}-\alpha\lambda\left(\theta-1\right)\sum_{i=1}^{n}\frac{\left[1-\left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha(\lambda-1)}\left[1-\left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha-1}e^{-\left(\frac{\beta}{x_i}\right)}\frac{1}{x_i}}{1-\left[1-\left[e^{-\left(\frac{\beta}{x_i}\right)}\right]\right]^{\alpha\lambda}}=0 \quad (29)$$

Now, equations (26), (27), (28) and (29) do not have a simple analytical form and are therefore not tractable. As a result, we have to resort to non-linear estimation of the parameters using iterative method.

III. Results

3.1. Applications

This section tests the new distribution's flexibility against a few other existing distributions using two actual data sets. AdequacyModel, a package in the R software, is used to produce the analyses' results in this study. Using the Akaike information criterion (AIC) and Bayesian information criterion (BIC), respectively, the performance of the distribution was compared to other existing distributions that were consistent with the baseline distribution in terms of providing good parametric fit to the data sets.

$$AIC=-2ll+2k \quad (30)$$

$$BIC=-2ll+k\log(n) \quad (31)$$

The model selection is carried out using the AIC and the BIC. Where ll denotes the log-likelihood function evaluated at the maxinum likelihood estimates, k is the number of parameters, and n is the sample size from the data. The model with minimum value of AIC and BIC is chosen as the best model to fit the data set. The comparators presented are Topp-Leone exponentiated inverse exponential (TLExIEx), Topp-Leone inverse exponential (TLIEx), exponentiated inverse exponential (ExIEx), inverse exponential (IEx) and exponential (Ex) distributions.

The first data set represents the relief times of twenty patients receiving an analgesic. This data set has been used by [21]. The data set is given as
1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.0.

The second data set represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports, has been used by [22]. The data set is given as
28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2,101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6,52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48, 61.2, 171.1, 113.5, 148.9, 49.9,59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2,70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9.

Table 1: *The ML estimates and goodness of fit measurement for the first data set.*

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC	BIC
NEtEIEx	3.735	1.832	13.235	1.831	15.575	39.150	43.133
TLExIEx	2.772	0.322	1.791	-	46.038	98.077	101.064
ExIEx	1.309	1.317	-	-	32.669	69.337	71.329
TLIEx	-	12.432	0.526	-	22.432	49.984	51.976
IEx	-	1.725	-	-	32.669	67.337	68.333
Ex	-	0.526	-	-	32.837	67.674	68.670

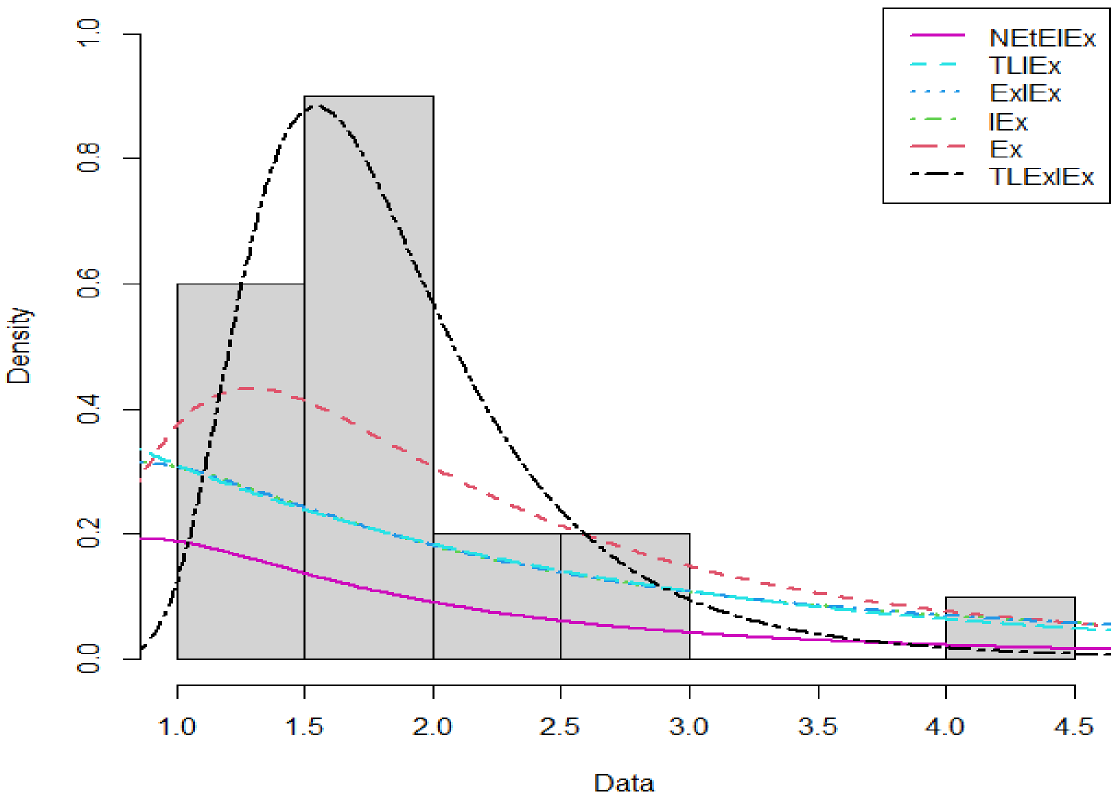

Figure 5: *Histogram and fitted pdfs for the NetEIEx, TLIEx, ExIEx, TLExIEx, IEx and Ex models for the first data set*

Table 2: *The ML estimates and goodness of fit measurement for the second data set*

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-l$	AIC	BIC
NEtEIEx	0.096	13.103	79.636	30.736	955.251	1918.502	1931.735
TLExIEx	3.883	1.789	6.995	-	1521.690	3049.381	3059.305
ExIEx	9.867	5.771	-	-	1055.772	2115.544	2122.160
TLIEx	-	25.015	6.607	-	980.481	1964.962	1971.578
IEx	-	56.953	-	-	1055.772	2113.544	2116.852
Ex	-	0.014	-	-	1057.353	2116.707	2120.015

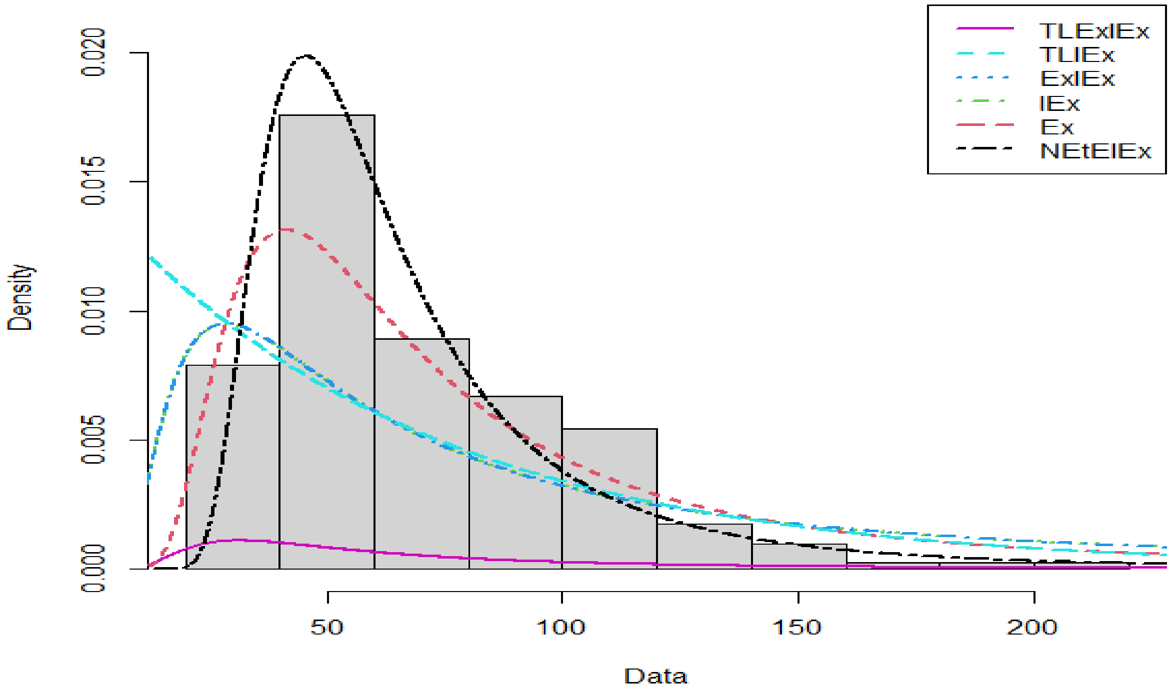

Figure 6: *Histogram and fitted pdfs for the NetEIEx, TLIEx, ExIEx, TLExIEx, IEx and Ex models for the second data set*

IV. Discussion

The estimated values for each parameter and the models' goodness of fits are shown in Tables 1 and 2. AIC and BIC are two metrics for goodness of fits. The model performs better when the AIC and BIC values are lower. Tables 1 and 2 show that the NEtEIEx distribution has the lowest AIC and BIC, respectively. This property makes the new model more adaptable and suitable for handling biomedical data sets.

The new model's forms, fit, and adaptability in connection to the data sets under consideration are shown in Figures 5 and 6. The black line, which represents the new model, more closely matched the data's pattern than the competitors. The histogram and fitted plots make it clear that the black line, which represents the NEtEIEx distribution, matches the two data sets under consideration better.

This study extends the inverse exponential distribution by creating a new continuous distribution known as the new exponentiated extended inverse exponential distribution. It was possible to obtain the survival function, hazard rate function, quantile function, inverted hazard function, odds function, and order statistics from the new distribution. Plotting the pdf and hazard rate function graphs revealed the contours of the suggested distribution. It was discovered that the hazard function is shaped like an upside-down bathtub. Adequacy Model is a package in R that was used to estimate the model parameters using the maximum likelihood method. The proposed distribution was applied to two real life data sets, and the outcomes are shown in Tables 1 and 2. The findings demonstrated that the new extended inverse exponential distribution with exponentiation is much more potent and superior at fitting the two data sets under consideration. The density graphs in figures 5 and 6 for the two data sets further show how adaptable the new model is.

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