WEIGHTED GENERALIZED ENTROPY: PROPRTIES AND APPLICATION

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Abstract

Recently, the measurement of uncertainty has attracted the attention of researchers. In this article, we introduce a new weighted uncertainty measure known as weighted generalized entropy. We also study its dynamic (residual) version which is known as weighted generalized residual entropy. These are length-biased shift-dependent uncertainty measures. It is shown that the proposed dynamic uncertainty measure uniquely determines the survival function. The various significant properties and the relationship with other well-known reliability measures of the proposed dynamic uncertainty measure are also studied. Finally, a real life data set is used to illustrate the usefulness of our proposed uncertainty measures.

Keywords: Weighted entropy, weighted residual entropy, hazard rate function and characterization results.

1. Introduction

The notion of entropy that was introduced by Shannon [1] is a very important and well known concept in the area of information theory. For an absolutely continuous non-negative r.v U having p.d.f $g(u)$, the Shannon's entropy (SE) is defined as

$$H_U(g) = -\int_0^\infty g(u)\log g(u)du = -E[\log(U)]. \quad (1)$$

Throughout this article, the notations r.v and p.d.f stands for an absolutely continuous non-negative random variable and the probability density function respectively.

If a lifetime component has survived up to an age t, then the SE is not useful for measuring the uncertainty about its remaining life. To overcome this problem, Ebrahimi [2] has introduced the concept of residual entropy and is defined as

$$H_U(g;t) = -\int_t^\infty \frac{g(u)}{\bar{G}(t)}\log\frac{g(u)}{\bar{G}(t)}du, \quad (2)$$

where, $\bar{G}(t) = 1 - G(t)$ is the survival function (s.f) of the r.v U.

It is clear that the SE is well-known by means of its applications in the area of information theory, but it is a shift-independent uncertainty measure (UM) because it remains unchanged, if for instance U is uniformly distributed in (c, d) or $(c + h, d + h)$ for any $h \in \mathcal{R}$. However, in some applied contexts, such as reliability or mathematical neurobiology, the shift-dependent UM's are desirable. To fulfill this requirement, Belis and Guiasu [3] have introduced the concept of weighted entropy (a shift-independent UM) and is defined as

$$H_{(U,w)}(g) = -\int_0^\infty w(u)g(u)\log g(u)du$$
$$= -ug(u)\log g(u)du\ , \tag{3}$$

where, the coefficient u (i.e the length of the system or component under consideration) represents the weight function of the elementary events.

Similarly, Di Crescenzo and Longobardi [4] have introduced the weighted version of residual entropy (2) and is given by

$$H_{(U,w)}(g;t) = -\int_t^\infty u\frac{g(u)}{\bar{G}(t)}\log\frac{g(u)}{\bar{G}(t)}du\ . \tag{4}$$

In the recent literature, it is seen that the study of weighted UM's have attracted the attention of researchers for introducing the new flexible weighted UM's. For more details see Misagh et al. [5], Misagh and Yari [6], Nourbakhsh and Yari [7], Mirali and Baratpour [8], Kayal [9], Nair et al. [10], Rajesh et al. [11], Khammar and Jahanshahi [12], Bhat and Baig [13] and Bhat et al. [14] etc. Motivated with this research literature, here in this article, our objective is to introduce a new weighted UM and its dynamic (residual) version on the basis of the following new generalization of SE

$$H_U^{(\eta,\mu)}(g) = \frac{1}{2\eta(\mu-\eta)}\log\left(\int_0^\infty g^{2^{\frac{\eta}{\mu}}-1}(u)du\right),\ \frac{\mu}{2} < \eta < \mu,\ \mu \geq 1, \tag{5}$$

where,

$\lim_{\substack{\eta\to 1\\ \mu=1}} H_U^{(\eta,\mu)}(g) = -\int_0^\infty g(u)\log g(u)du$, which is the SE given in (1).

Analogous to (2) and on the basis of (5), the generalized residual entropy can be defined as

$$H_U^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\log\left(\int_t^\infty \left(\frac{g(u)}{\bar{G}(t)}\right)^{2^{\frac{\eta}{\mu}}-1} du\right),\ \frac{\mu}{2} < \eta < \mu, \mu \geq 1. \tag{6}$$

The rest of the article is organized as follows: In section 2, we discuss the weighted generalized entropy (WGE) of order η and type μ in the form of its definition and some properties. The section 3 presents the weighted generalized residual entropy (WGRE) and also some of its significant characterization results. In section 4, we study the various important properties of WGRE and also its relationship with other well-known reliability measures. In section 5, an application of the WGE and WGRE by using a real life data set is presented. Finally, we illustrate some concluding remarks in section 6.

2. Weighted Generalized Entropy (WGE)

In this section, we introduce the weighted version of (5) which is known as weighted generalized entropy (WGE) of order η and type μ.

Definition 2.1 For a r.v U having p.d.f $g(u)$, the WGE of order η and type μ denoted by $H_{(U,w)}^{(\eta,\mu)}(g)$ is defined as

$$H_{(U,w)}^{(\eta,\mu)}(g) = \frac{1}{2\eta(\mu-\eta)}\log\left(\int_0^\infty (ug(u))^{2^{\frac{\eta}{\mu}}-1}du\right), \frac{\mu}{2} < \eta < \mu \geq 1, \tag{7}$$

where, the coefficient u in the integrand denotes the weight function as in (3).

In the following example, we illustrate the importance of WGE.

Exumple 2.1. Let U and V be two r.v's distributed as

$$g_U(u) = \begin{cases} 2u, & 0 < u < 1 \\ 0, & otherwise \end{cases}, \qquad g_V(v) = \begin{cases} 2(1-v), & 0 < v < 1 \\ 0 & otherwise \end{cases}.$$

Here, we can see that

$$H_U^{(\eta,\mu)}(g) = H_V^{(\eta,\mu)}(g) = \frac{1}{2\eta(\mu-\eta)}\log\left(\frac{\mu 2^{2\left(\frac{\eta}{\mu}-1\right)}}{\eta}\right),$$

But, the WGE's of U and V are different with each other as follows

$$H_{(U,w)}^{(\eta,\mu)}(g)=\frac{1}{2\eta(\mu-\eta)}\log\left(\frac{\mu 2^{2\frac{\eta}{\mu}-1}}{4\eta-\mu}\right)$$

and

$$H_{(V,w)}^{(\eta,\mu)}=\frac{1}{2\eta(\mu-\eta)}\log\left(2^{2\frac{\eta}{\mu}-1}\mathrm{B}\left(2\frac{\eta}{\mu},2\frac{\eta}{\mu}\right)\right),$$

where,

$$\mathrm{B}(c,d)=\int_0^1 y^{c-1}(1-y)^{d-1},\ c,d>0=\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}.$$

Thus, even though $H_U^{(\eta,\mu)}(g)=H_V^{(\eta,\mu)}(g)$, but $H_{(U,w)}^{(\eta,\mu)}(g)\neq H_{(V,w)}^{(\eta,\mu)}(g), \forall\ \frac{\mu}{2}<\eta<\mu,\mu\geq 1$.

Example 2.2. Let $g(u)$ be the p.d.f of a r.v U distributed as:

(a) Exponentially with $g(u)=\beta e^{-\beta u}, u>0,\beta>0$, then

$$H_{(U,w)}^{(\eta,\mu)}(g)=\frac{1}{2\eta(\mu-\eta)}\log\left[\frac{\Gamma\left(2\frac{\eta}{\mu}\right)}{\beta\left(2\frac{\eta}{\mu}-1\right)^{2\frac{\eta}{\mu}}}\right].$$

(b) Gamma with $g(u)=\frac{1}{\Gamma(\beta)}e^{-u}u^{\beta-1}, 0<u<\infty,\beta>0$, then

$$H_{(U,w)}^{(\eta,\mu)}(g)=\frac{1}{2\eta(\mu-\eta)}\log\left[\frac{\Gamma\left(\beta\left(2\frac{\eta}{\mu}-1\right)+1\right)}{(\Gamma(\beta))^{2\frac{\eta}{\mu}-1}\left(2\frac{\eta}{\mu}-1\right)^{\beta\left(2\frac{\eta}{\mu}-1\right)+1}}\right].$$

(c) Lomax with $g(u)=\frac{m}{(1+u)^{1+m}}, u>0, m>0$, then

$$H_{(U,w)}^{(\eta,\mu)}(g)=\frac{1}{2\eta(\mu-\eta)}\log\left[\frac{m^{2\frac{\eta}{\mu}-1}\Gamma\left(2\frac{\eta}{\mu}\right)\Gamma\left(m\left(2\frac{\eta}{\mu}-1\right)-1\right)}{\Gamma\left(\left(2\frac{\eta}{\mu}-1\right)(m+1)\right)}\right], m\left(2\frac{\eta}{\mu}-1\right)>1\ .$$

(d) Rayleigh with $g(u)=\beta u e^{-\frac{\beta}{2}u^2}, u\geq o,\beta>0$, then

$$H_{(U,w)}^{(\eta,\mu)}(g)=\frac{1}{2\eta(\mu-\eta)}\log\left[\frac{2^{2\frac{\eta}{\mu}-\frac{3}{2}}\Gamma\left(2\frac{\eta}{\mu}-\frac{1}{2}\right)}{\sqrt{\beta}\left(2\frac{\eta}{\mu}-1\right)^{2\frac{\eta}{\mu}+\frac{3}{2}}}\right].$$

Lemma 2.1. If $Z=mU$, with $m>0$, then

$$H_{(Z,w)}^{(\eta,\mu)}(g)=\frac{1}{2\eta(\mu-\eta)}\log m+H_{(U,w)}^{(\eta,\mu)}(g).$$

Theorem 2.1. For a r.v U having SE $H_U(g)$, we obtain

$$H_{(U,w)}^{(\eta,\mu)}(g)\geq\frac{1}{\eta\mu}\left[H_U(g)-\left(\frac{\mu-2\eta}{2(\mu-\eta)}\right)E(\log U)\right].$$

Proof. By applying the log-sum inequality, we obtain

$$\int_0^\infty g(u)\log\frac{g(u)}{(ug(u))^{2\frac{\eta}{\mu}-1}}du\geq\int_0^\infty g(u)du\log\frac{\int_0^\infty g(u)du}{\int_0^\infty(ug(u))^{2\frac{\eta}{\mu}-1}du}$$

$$=-\log\int_0^\infty\left(ug(u)\right)^{2\frac{\eta}{\mu}-1}du.$$

Due to (7), the desired result is satisfied.

3. Weighted Generalized Residual Entropy (WGRE)

In this section, we introduce the dynamic (residual) version of (7) which is known as weighted generalized residual entropy (WGRE) of order η and type μ. Some important characterization results of this UM are also discussed.

Definition 3.1 Let U be a r.v with p.d.f $g(u)$ and s.f $\bar{G}(t)$, then the WGRE of order η and type μ is defined as

$$H_{(U,w)}^{(\eta,\mu)}(g;t)=\frac{1}{2\eta(\mu-\eta)}\log\left[\int_t^\infty\left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du\right],\ \frac{\mu}{2}<\eta<\mu,\mu\geq 1\ . \tag{8}$$

Here, we evaluate the WGRE of some lifetime distributions.

Example 3.1. Let a r.v U be distributed as:

(a) Exponentially with p.d.f $g(u) = \beta e^{-\beta u}, u\beta > 0, \beta > 0$, then

$$H_{(U,w)}^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\left[R\beta t + \log\left(\frac{\Gamma(R+1,R\beta t)}{\beta R^{R+1}}\right)\right],$$

(b) Gamma with p.d.f $g(u) = \frac{1}{\Gamma(\beta)} e^{-u} u^{\beta-1},\ 0 < u < \infty, \beta > 0$,

$$H_{(U,w)}^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\log\left[\frac{\Gamma(R\beta+1,Rt)}{R^{R\beta+1}(\Gamma(\beta,t))^{R}}\right],$$

(c) Weibull with p.d.f $g(u) = \frac{1}{m} e^{-\left(\frac{u-n}{m}\right)},\ u > n, m > 0, n > 0$, then

$$H_{(U,w)}^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\left[R\frac{t}{m} + \log\left(\frac{m\Gamma\left(R+1,R\frac{t}{m}\right)}{R^{R+1}}\right)\right],$$

(d) Rayleigh with p.d.f $g(u) = \beta u e^{-\frac{\beta}{2}u^2},\ u \geq 0, \beta > 0$, then

$$H_{(U,w)}^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\left[\frac{R\beta t^2}{2} + \log\left\{\frac{2^{R-\frac{1}{2}}\Gamma\left(R+\frac{1}{2},\frac{R\beta t^2}{2}\right)}{\sqrt{\beta}R^{R+\frac{1}{2}}}\right\}\right],$$

where, $\Gamma(n, mz) = m^n \int_z^\infty e^{-mx} x^{n-1} dx,\ m, n > 0$ is an upper incomplete gamma function and $R = 2\frac{\eta}{\mu} - 1$ respectively.

Theorem 3.1 If $H_U^{(\eta,\mu)}(g)$ and $H_{(U,w)}^{(\eta,\mu)}(g;t)$ denotes the GRE and WGRE of a r.v U, then for all $t > 0$, we have

$$H_{(U,w)}^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\log\left[t^{2\frac{\eta}{\mu}-1} exp\left(2\eta(\mu-\eta)H_U^{(\eta,\mu)}(g;t)\right) + \left(2\frac{\eta}{\mu}-1\right)\int_{x=t}^{\infty} x^{2\left(\frac{\eta}{\mu}-1\right)}\left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} exp\left(2\eta(\mu-\eta)H_U^{(\eta,\mu)}(g;x)dx\right)\right].$$

Proof. From (8), we have

$$\begin{aligned}\int_t^\infty \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du &= \int_t^\infty \left[\int_0^u \left(2\frac{\eta}{\mu}-1\right) y^{2\left(\frac{\eta}{\mu}-1\right)} dy\right]\left(\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du \\ &= \left(2\frac{\eta}{\mu}-1\right)\int_t^\infty \left[\int_0^t y^{2\left(\frac{\eta}{\mu}-1\right)} dy + \int_t^u y^{2\left(\frac{\eta}{\mu}-1\right)} dy\right]\left(\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du \\ &= t^{2\frac{\eta}{\mu}-1}\int_t^\infty \left(\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du + \left(2\frac{\eta}{\mu}-1\right)\int_{y=t}^\infty \left[y^{2\left(\frac{\eta}{\mu}-1\right)}\left(\int_{u=y}^\infty \left(\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du\right)\right] dy. \quad (9)\end{aligned}$$

From (6), we have

$$\int_t^\infty \left(\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du = exp\left[2\eta(\mu-\eta)H_U^{(\eta,\mu)}(g;t)\right]. \quad (10)$$

and

$$\int_t^\infty g^{2\frac{\eta}{\mu}-1} du = \bar{G}^{2\frac{\eta}{\mu}-1}(t) exp\left[2\eta(\mu-\eta)H_U^{(\eta,\mu)}(g;t)\right]. \quad (11)$$

Using (9), (10) and (11) in (8), we obtain the required result.

Here, we show that $\bar{G}(t)$ is uniquely determined by $H_{(U,w)}^{(\eta,\mu)}(g;t)$.

Theorem 3.2. Let U be a r.v having p.d.f $g(u)$, s.f $\bar{G}(t)$ and WGRE $H_{(U,w)}^{(\eta,\mu)}(g;t) < \infty, \frac{\mu}{2} < \eta < \mu, \mu \geq 1$ respectively. If $H_{(U,w)}^{(\eta,\mu)}(g;t)$ is increasing in t, then $H_{(U,w)}^{(\eta,\mu)}(g;t)$ uniquely determines $\bar{G}(t)$.

Proof. Rewriting (8) as

$$exp\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right] = \int_t^\infty \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du. \quad (12)$$

Differentiating (12) w.r.t t, we have

$$\frac{\partial}{\partial t} exp\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t) = \left(2\frac{\eta}{\mu}-1\right)\lambda_G(t)\int_t^\infty \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1} du - \left(t\lambda_G(t)\right)^{2\frac{\eta}{\mu}-1}\right], \quad (13)$$

where, $\lambda_G(t) = \frac{g(t)}{\bar{G}(t)}$ represents the hazard rate of U. Using (12), we can rewrite (13) as

$$\left(t\lambda_G(t)\right)^{2\frac{\eta}{\mu}-1} - \left(2\frac{\eta}{\mu}-1\right) exp\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\lambda_G(t)\right]$$

$$2\eta(\mu-\eta)exp\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right]\frac{\partial}{\partial t}H_{(U,w)}^{(\eta,\mu)}(g;t)=0. \tag{14}$$

Hence for fixed $t>0$, $\lambda_G(t)$ is a solution of $\psi(u_t)=0$, where

$$\psi(u_t)=t^{2\frac{\eta}{\mu}-1}-\left(2\frac{\eta}{\mu}-1\right)exp\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right]u_t$$
$$+2\eta(\mu-\eta)exp\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right]\frac{\partial}{\partial t}H_{(U,w)}^{(\eta,\mu)}(g;t).$$

Differentiating both sides w.r.t u_t, we have

$$\frac{\partial}{\partial u_t}\psi(u_t)=\left(2\frac{\eta}{\mu}-1\right)t^{2\frac{\eta}{\mu}-1}u_t^{2\left(\frac{\eta}{\mu}-1\right)}-\left(2\frac{\eta}{\mu}-1\right)exp\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right].$$

Also,

$$\frac{\partial^2}{\partial u_t^2}\psi(u_t)=\left(2\frac{\eta}{\mu}-2\right)\left(2\frac{\eta}{\mu}-1\right)t^{2\frac{\eta}{\mu}-1}u_t^{2\frac{\eta}{\mu}-3}.$$

Now, $\frac{\partial}{\partial u_t}\psi(u_t)=0$ gives

$$u_t=\left[\frac{exp(2\eta(\mu-\eta))H_{(U,w)}^{(\eta,\mu)}(g;t)}{t^{2\frac{\eta}{\mu}-1}}\right]^{2\left(1-\frac{\eta}{\mu}\right)}=u_0 \text{ (say)}.$$

For $\frac{\mu}{2}<\eta<\mu,\ \mu\geq 1,\ \frac{\partial^2}{\partial u_t^2}\psi(u_0)<0$. Thus, $\psi(u_t)$ attains maximum at u_0. Also, $\psi(0)>0$ and $\psi(\infty)=-\infty$. Further it can be easily observed that $\psi(u_t)$ first increases for $0<u_t<u_o$ and then decreases for $u_t>u_o$. So, the unique solution to $\psi(u_t)=0$ is given by $u_t=\lambda_G(t)$. Thus, $H_{(U,w)}^{(\eta,\mu)}(g;t)$ uniquely determines $\lambda_G(t)$ which in turns determines $\bar{G}(t)$.

4. Properties and Inequalities of WGRE

This section presents some interesting properties and inequalities of weighted generalized residual entropy .

Definition 4.1. Let U and V be two r.v's having WGRE's $H_{(U,w)}^{(\eta,\mu)}(g;t)$ and $H_{(V,w)}^{(\eta,\mu)}(g;t)$, then U is said to be smaller than V in WGRE of order η and type μ (denoted by $U\overset{WGRE}{\leq}V$), if$H_{(U,w)}^{(\eta,\mu)}(g;t)\leq H_{(V,w)}^{(\eta,\mu)}(g;t),\ \forall\, t>0$.

Definition 4.2. A r.v U or a s.f $\bar{G}$ will be said to have increasing (decreasing) WGE for residual life of order η and type μ IWGERL (DWGERL), if $H_{(U,w)}^{(\eta,\mu)}(g;t)$ is increasing (decreasing) in t, $t>0$.

Lemma 4.1. If $Y=aU$, with $a>0$ is a constant, then

$$H_{(Y,w)}^{(\eta,\mu)}(g;t)=\frac{1}{2\eta(\mu-\eta)}\log a+H_{(U,w)}^{(\eta,\mu)}\left(g,\frac{t}{a}\right).$$

Proof.

$$H_{(Y,w)}^{(\eta,\mu)}(g;t)=\frac{1}{2\eta(\mu-\eta)}\log\int_t^\infty\left(y\frac{g(y)}{Pr(Y>t)}\right)^{2\frac{\eta}{\mu}-1}dy.$$

Setting $Y=aU$, a strictly increasing function of U, we have

$$H_{(Y,w)}^{(\eta,\mu)}(g;t)=\frac{1}{2\eta(\mu-\eta)}\log\left[a\int_{\frac{t}{a}}^\infty\left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du\right].$$

By using (8), the desired result is obtained.

Theorem 4.1. For two r.v's U and V, let us define $Y_1=a_{1U}$ and $Y_2=a_2V$ with $a_1,a_2>0$. Let $U\overset{WGRE}{\leq}V$ and $a_1\leq a_2$. Then $Y_1\overset{WGRE}{\leq}Y_2$ $Y_1\overset{WGRE}{\leq}Y_2$ if $H_{(U,w)}^{(\eta,\mu)}(g;t)$ or $H_{(V,w)}^{(\eta,\mu)}(g;t)$ is decreasing in $t>0$.

Poof. Suppose $H_{(U,w)}^{(\eta,\mu)}(g;t)$ is decreasing in t.

Now, $U\overset{WGRE}{\leq}V$ implies

$$H_{(U,w)}^{(\eta,\mu)}\left(g;\frac{t}{a_2}\right)\leq H_{(V,w)}^{(\eta,\mu)}\left(g;\frac{t}{a_2}\right). \tag{15}$$

Further, since $\frac{t}{a_1}\geq\frac{t}{a_2}$, we have

$$H_{(U,w)}^{(\eta,\mu)}\left(g;\frac{t}{a_1}\right) \le H_{(U,w)}^{(\eta,\mu)}\left(g;\frac{t}{a_2}\right). \tag{16}$$

Combining (15) and (16), we obtain

$$H_{(U,w)}^{(\eta,\mu)}\left(g;\frac{t}{a_1}\right) \le H_{(V,w)}^{(\eta,\mu)}\left(g;\frac{t}{a_2}\right). \tag{17}$$

Using Lemma 4.1 in (17), we have $Y_1 \overset{WGRE}{\le} Y_2$.

Theorem 4.2. For a r.v U having support $(0,k], k > 0$, p.d.f $g(u)$ and s.f $\bar{G}(t), t > 0$, then for $\frac{\mu}{2} < \eta < \mu,\ \mu \ge 1$, the following upper bound of $H_{(U,w)}^{(\eta,\mu)}(g;t)$ holds

$$H_{(U,w)}^{(\eta,\mu)}(g;t) \le \frac{1}{2\eta(\mu-\eta)}\left[\frac{\int_t^k \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}\log\left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du}{\int_t^k \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du} + \log(k-t)\right].$$

Proof. From log-sum inequality and (8), we have

$$\int_t^k \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}\log\left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du \ge \int_t^k \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du\log\frac{\int_t^k (ug(u))^{2\frac{\eta}{\mu}-1}du}{\int_t^k (\bar{G}(t))^{2\frac{\eta}{\mu}-1}du}$$

$$= \int_t^k \left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du\left[2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t) - \log(k-t)\right].$$

After simplification, we get the desired result.

Theorem 4.3. Let $\bar{G}$ be a IWGRE (DWGRE) and $\mu > \eta$, then

$$\lambda_G(t) \le (\ge)\left[\frac{\left(2\frac{\eta}{\mu}-1\right)exp\left\{2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right\}}{t^{2\frac{\eta}{\mu}-1}}\right]^{\frac{\mu}{2(\eta-\mu)}}.$$

Proof. From (14), we have

$$2\eta(\mu-\eta)\frac{\partial}{\partial t}H_{(U,w)}^{(\eta,\mu)}(g;t) = \left(2\frac{\eta}{\mu}-1\right)\lambda_G(t) - exp\left\{2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right\}\left(t\lambda_G(t)\right)^{2\frac{\eta}{\mu}-1}.$$

Since $\bar{G}$ is IWGERL (DWGERL), therefore, we have

$$\lambda_G(t)\left[\left(2\frac{\eta}{\mu}-1\right) - t^{2\frac{\eta}{\mu}-1}\lambda_G^{2\left(\frac{\eta}{\mu}-1\right)}(t)exp\left\{2\eta(\eta-\mu)H_{(U,w)}^{(\eta,\mu)}(g;t)\right\}\right] \ge (\le) 0.$$

which leads to

$$\lambda_G(t) \le (\ge)\left[\frac{\left(2\frac{\eta}{\mu}-1\right)exp\left\{2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right\}}{t^{2\frac{\eta}{\mu}-1}}\right]^{\frac{\mu}{2(\eta-\mu)}}.$$

Theorem 4.4. If U is IWGERL (DWGERL), then

$$H_{(U,w)}^{(\eta,\mu)}(g;t) \le (\ge)\frac{1}{2\eta(\mu-\eta)}\log\left[\frac{t^{2\frac{\eta}{\mu}-1}}{2\frac{\eta}{\mu}-1}\left(\frac{1+\frac{\partial}{\partial t}m_G(t)}{m_G(t)}\right)^{2\left(\frac{\eta}{\mu}-1\right)}\right],$$

where $m_G(t)$ is the mean residual life function of U.

Proof. From (14), we have

$$\frac{\partial}{\partial t}H_{(U,w)}^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\left[\left(2\frac{\eta}{\mu}-1\right)\lambda_G(t) - \left(t\lambda_G(t)\right)^{2\frac{\eta}{\mu}-1}exp\left\{2\eta(\eta-\mu)H_{(U,w)}^{(\eta,\mu)}(g;t)\right\}\right].$$

Using $\lambda_G(t) = \frac{1+\frac{\partial}{\partial t}m_G(t)}{m_G(t)}$, we have

$$\frac{\partial}{\partial t}H_{(U,w)}^{(\eta,\mu)}(g;t) = \frac{1}{2\eta(\mu-\eta)}\left[\left(2\frac{\eta}{\mu}-1\right)\left(\frac{1+\frac{\partial}{\partial t}m_G(t)}{m_G(t)}\right)\right.$$
$$\left. -t^{2\frac{\eta}{\mu}-1}\left(\frac{1+\frac{\partial}{\partial t}m_G(t)}{m_G(t)}\right)^{2\frac{\eta}{\mu}-1}exp\left\{2\eta(\eta-\mu)H_{(U,w)}^{(\eta,\mu)}(g;t)\right\}\right].$$

Since, $\bar{G}$ is IWGERL (DWGERL), therefore, after simplification, we have

$$H_{(U,w)}^{(\eta,\mu)}(g;t) \ge (\le)\log\left[\frac{t^{2\frac{\eta}{\mu}-1}}{2\frac{\eta}{\mu}-1}\left(\frac{1+\frac{\partial}{\partial t}m_G(t)}{m_G(t)}\right)^{2\left(\frac{\eta}{\mu}-1\right)}\right].$$

Theorem 4.5. Let U be the lifetime of a system with p.d.f $g(u)$ and s.f $\bar{G}(t), t > 0$, then $H_{(U,w)}^{(\eta,\mu)}(g;t)$ attains a lower bound as follows

$$H_{(U,w)}^{(\eta,\mu)}(g;t) \geq \frac{1}{2\eta(\mu-\eta)}\left[\left(2\frac{\eta}{\mu}-1\right)\int_t^\infty \frac{g(u)}{\bar{G}(t)}\log u du + 2\left(1-\frac{\eta}{\mu}\right)H_U(g;t)\right]. \tag{18}$$

Proof. From log-sum inequality, we have

$$\int_t^\infty g(u)\log\frac{g(u)}{\left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}}du \geq \int_t^\infty g(u)du\log\frac{\int_t^\infty g(u)du}{\int_t^\infty\left(u\frac{g(u)}{\bar{G}(t)}\right)^{2\frac{\eta}{\mu}-1}du}$$

$$= \bar{G}(t)\left[\log\bar{G}(t) - \log\left\{2\eta(\mu-\eta)H_{(U,w)}^{(\eta,\mu)}(g;t)\right\}\right]. \tag{19}$$

where (19) is obtained from (8).

The L.H.S of (19) leads to

$$2\left(1-\frac{\eta}{\mu}\right)\int_t^\infty g(u)\log g(u)du - \left(2\frac{\eta}{\mu}-1\right)\int_t^\infty g(u)\log u du + \left(2\frac{\eta}{\mu}-1\right)\bar{G}(t)\log\bar{G}(t). \tag{20}$$

Using (20) in (19), we obtain (18).

5. Application

To illustrate the effectiveness and importance of our proposed UM's, we consider a real life data set. The data set represents the remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang [15] and is given as follows:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Afaq et al. [16] have shown that the length biased Lomax distribution (LD) provides a better fit for this data. Now, in order to compute the entropy of this data set, it is necessary to apply the weighted entropy technique rather than the simple entropy. For the weighted entropy, we need to consider the basic model (i.e LD) of the length biased LD. The MLEs of the parameters of LD from this data set are obtained as: $\theta = 8.431393$(shape parameter) and $\lambda = 70.289624$(scale parameter) respectively. Now, for $\eta = 1.5, \mu = 2.5$ and $t = 10$, we have $H_{(U,w)}^{(\eta,\mu)}(g) = 1.638$ and $H_{(U,w)}^{(\eta,\mu)}(g;t) = 1.694$. Similarly, $\eta = 2.5, \mu = 3$ and $t = 20$, we obtain $H_{(U,w)}^{(\eta,\mu)}(g) = 1.164$ and $H_{(U,w)}^{(\eta,\mu)}(g;t) = 1.481$ respectively.

6. Conclusion

In this article, we have introduced the concepts of weighted generalized entropy and also its dynamic (residual) version which is known as weighted generalized residual entropy. It has been shown that the proposed residual entropy uniquely determines the survival function. The various important properties and the relationship with other well-known reliability measures of the proposed residual entropy are also obtained. Finally, a real data set has been used to investigate the usefulness of the proposed entropy functions.

References

[1] Shannon, C. E. (1948). A mathematical theory of communications, *Bell System Technical Journal*, 27, 379-423.

[2] Ebrahimi, N. (1996). How to measure uncertainty in the residual lifetime distribution. *Sankhya Series A*, 58, 48-56.

[3] Belis, M. and Guiasu, S. (1968). A quantitative-qualitative measure of information in cybernetic systems. *IEEE Transactions on Information Theory*, IT., 4, 593–594.

[4] Di, Crescenzo, A. and Longobardi, M. (2006). On weighted residual and past entropies. *Scientiae Mathematicae Japonicae*, 64, 255–266.

[5] Misagh, F., Panahi, Y., Yari, G. H. and Shahi, R. (2011). Weighted cumulative entropy and its estimation. In Quality and Reliability (ICQR), IEEE International conference, 477-480.

[6] Misagh, F. and Yari, G. H. (2011). On weighted interval entropy. Statistics and Probability Letters, 81, 188–194.

[7] Nourbakhsh, M. and Yari, G. (2016). Weighted Renyi's entropy for lifetime distributions. *Communications in Statistics-Theory and Methods*, doi: 10.1080 /03610926.2016.1148729.

[8] Mirali, M. and Baratpour, S. (2017). Dynamic version of weighted cumulative residual entropy. *Communications in Statistics-Theory and Methods*, *46*(22), 11047-11059.

[9] Kayal, S. (2017). On weighted generalized cumulative residual entropy. *Springer Science+Business Media New York*, 1-17.

[10] Nair, R. S., Sathar, E. I. A. and Rajesh, G. (2017). A study on dynamic weighted failure entropy of order α. *American Journal of Mathematical and Management Sciences*, 36(2), 137-149.

[11] Rajesh, G., Abdul-Sathar, E., and Rohini, S. N. (2017). On dynamic weighted survival entropy of order α. *Communications in Statistics-Theory and Methods, 46*(5), 2139–2150.

[12] Khammar, A., and Jahanshahi, S. (2018). On weighted cumulative residual Tsallis entropy and its dynamic version. *Physica A: Statistical Mechanics and Its Applications,* 491, 678–692.

[13] Bhat, B.A. and Baig, M. A. K. (2019). A New Two Parametric Weighted Generalized Entropy for Lifetime Distributions. *Journal of Modern Applied Statistical Methods*, *18*(2).

[14] Bhat, B. A., Mudasir, S., and Baig. M. A. K. (2019). Some Characterization Results on Length-Biased Generalized Interval Entropy for Lifetime Distributions. *Pakistan Journal of Statistics*, *35*(2), 155-170.

[15] Lee, E. T., and Wang, J. (2003). *Statistical methods for survival data analysis*. *John Wiley & Sons*, Vol. 476.

[16] Ahmad, A., Ahmad, S. P. and Ahmed, A. (2016). Length-biased Weighted Lomax Distribution: Statistical Properties and Application. *Pakistan Journal of Statistics and Operation Research*, 245-255.