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BULK ARRIVAL QUEUEING MODEL WITH SETUP AND m OPTIONAL SERVICE UNDER BERNOULLI VACATION SCHEDULE AND SERVER FAILURE

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Abstract

In the present investigation, we consider a bulk queue model with the assumption that the server may stop working due to random failure during any stage of the service. As soon as the server fails, it is immediately sent for repair. The server offers all incoming units the first mandatory service and any one of the optional services as per the unit's requirements. For computation purposes, we assume that the server offers m+1 services, of which the first one is essential and the remaining are optional. The server may take a vacation in accordance with the Bernoulli vacation schedule with probability p as soon as both service phases of a unit are completed. As the system empties, the server idles and needs some time to set up before initiating the next service. In order to analyse the model and derive various steady-state queue length distributions, we incorporated the supplementary variables corresponding to service time, vacation time, and repair time and applied the probability generating function technique to determine the various system state distributions. Using these probability distributions, we derive the explicit form of various performance indices. To discuss the validity of the present model, we obtained some well-known results from the queueing literature as a special case of the present model by setting appropriate parameters. Finally, to analyse the sensitivity of several performance indices, a numerical demonstration is provided.

Keywords: queue, bulk, essential service, optional service, supplementary variable, queue length

I. Introduction

Most queueing literature makes the assumption that the server in the service station is always available and that the service station never fails. These presumptions, meanwhile, are notably irrational. In real-world systems, it frequently happens that service stations break down and need to be fixed. We frequently experience situations where the entire system pauses owing to a random failure of a unit in computer communication networks, flexible manufacturing systems, production systems, and other areas.

Due to the potential impact on system performance, these types of systems with a repairable service facility are highly worth investigating from both an operational and queueing theory perspective. For detailed related work on queueing models with unreliable servers, we may refer to the work done by Avi-Itzhak and Naor [2], Li et al. [13], Wang and Yang [22], etc. Chaudhury and Tadj [10] discussed the linear cost procedure to obtain the optimal stationary policy of an unreliable queueing model with a Bernaulli vacation schedule. Rajadurai et al. [17] investigated an unreliable queueing model with a modified vacation schedule and applied the supplementary variable technique to obtain the study state queue size distribution. Yang and Wu [23] discussed the M/M/1 queueing model with the assumption that there is a state-dependent breakdown rate under N policy. They assumed that as the system became empty, the server would take a working vacation. Further, Chakravarthy et al. [5] generalised the model of the working-repair-vacation queue by assuming the concept of backup servers, which work at a relatively slow rate during the absence of the main server. Recently, Meena et al. [16] applied the supplementary variable technique to analyse the unreliable non-Markovian machine system, which comprised both operating and standby machines under N policy.

In some queueing situations, servers are unavailable for services for occasional intervals of time; such queueing models are termed vacation models. During vacation, the server may perform other types of service or may perform scheduled maintenance. Due to their variety of applications in computer systems, communication networks, and production and inventory systems, queuing systems with vacations have been extensively investigated. A comprehensive and detailed review of the vacation models can be found by Doshi [11], Choudhury [6] and Tian and Zhang [20] and Takagi [19]. Yang et al. [24] investigated a retrial queueing model with a constant retrial rate under the assumption that as orbit becomes empty, the server takes its first essential vacation. Further, the server may take additional option vacations after availing of the first essential vacation. Ayyapan and Karpagam [3] discussed an unreliable non Markovian queue model with a standby server under Bernoulli's schedule vacation policy. It is assumed that when the main server stops working due to random failure, a standby server starts serving the arriving unit. Ahuga et al. [1] applied the Runge-Kutta method to investigate a Markovian queueing system with multiple stages of service and vacation, where it is assumed that the server may breakdown during the busy period and vacation period. Recently, Rani et al. [18] applied recursive approach to find the steady-state queue size distribution of a finite population Markovian queueing model with vacation and discouragement factors. They apply the particle swarm optimisation technique to determine the optimal total cost.

It happens frequently in various queueing circumstances that when units use the first essential service, they subsequently need further services, or more than one service. For a better understanding, we will use the example of a car's service centre. Here, units arrive for routine maintenance, and if a serious problem is found with any element of the vehicle while it is being serviced, they go for repair or replacement of that component. For some comprehensive work in phase service, we may refer to Madan [14], Wang [21], Choudhury and Paul [8], Ke [12], etc. Choudhury and Deka [9] discussed a queueing model based on the assumption that units arrive one by one and the server is unreliable. But in real life situations where units arrive in groups of random size, units may demand more than one type of optional service apart from the essential one. Further, there may be a need for startup time to start the service again. Such situations motivated us to extend the model of Choudhury and Deka [9] by assuming that

- Units arrive in batches of random size.
- Second-phase services may choose among the available optional services.
- Server need start up time to start the service again.
- Server may go on vacation under Bernoulli's vacation schedule.

The remaining paper is organised as follows: In Section II, we describe the brief model description by making some basic assumptions. In Section III, the governing equations of the present model are described. In Section IV, we derive the steady-state queue size distribution function. In Section V, the performance measures of the present model are carried out. In Section VI, some well-known results are established as special cases of the present model. Finally, in Section VII, numerical illustration and sensitivity analysis of performance measures are done.

II. Medel description

In the present model, we consider a non-Markovian queueing model with the assumption that units arrive in batches of random size, according to poisson arrival fashion. There is a single server that provides the first essential services as well as one of the optional services to each arriving unit. As soon as the system becomes empty, the server gets turned off and needs startup time to start again when at least one or more units arrive. The brief description of notations used for the present model is as follows:

λ : Batch arrival rate of the unit.

$S(x)$: Distribution function of set up time.

$B_0(x)$: Distribution function of essential service time.

$B_i(x)$: Distribution function of i^{th} $(i=1,2,...,m)$ optional service time.

$V(x)$: Distribution function of vacation time.

$G_0(x)$: Distribution function of repair when its fails during essential service of a unit .

$G_i(x)$: Distribution function of repair when its fails during i^{th} $(i=1,2,...,m)$ optional service.

$g_0^{(k)}$:The k^{th} moment of repair time when its fails during essential service of a unit

$g_i^{(k)}$:The k^{th} moment of repair time when its fails during $i^{th}(i=1,2,...,m)$ optional service of a unit

r_i : Probability to opt $i^{th}(i=1,2,...,m)$ optional service after essential service.

p : Probability to opt optional vacation after service completion of a unit.

$N_q(t)$ **:** Denote the queue size in system at time t.

$S^0(t)$: Elapsed set up time at time t..

$B_0^0(t)$: Elapsed service time of essential service at time t.

$B_i^0(t)$: Elapsed service time of $i^{th}(i=1,2,...,m)$ optional service at time t.

$V^0(t)$: Elapsed vacation time at time t..

$G_0^0(t)$: Elapsed repair time at t time when its fails during essential service of a unit .

$G_i^0(t)$: Elapsed repair time at t time when its fails during $i^{th}(i=1,2,...,m)$ optional service.

Let $\gamma(t)$ denote the state of server at time t, where

$$\gamma(t)=\begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is start up at time } t, \\ 2 & \text{if the server is busy with essential service at time } t, \\ 2+i & \text{if the server is busy with ith } (i=1,2,...,m) \text{ optional service at time } t, \\ 3+m & \text{if the server is on vacation at time } t, \\ 4+m & \text{if the server is under repair when it breakdown during essential service at time } t, \\ 4+m+j & \text{if the server is under repair when it breakdown jth } (j=1,2,...,m) \text{ service.} \end{cases}$$

The variables $S^0(t)$, $B_0^0(t)$, $B_i^0(t)$ $(i=1,2,...,m)$, $V^0(t)$, $G_0^0(t)$ and $G_i^0(t)$ $(i=1,2,...,m)$ are added as supplementary variable in order to obtain a bivariate markav process $\{N_q(t), X(t)\}$ where $X(t)$ assumes values,

$0,\ S^0(t),\ B_0^0(t),\ B_i^0(t)$ if $\gamma(t)=0,\ 1,\ 2,\ 2+i$ $(i=1,2..m)$ respectively and values

$V^0(t),\ G_0^0(t),\ G_i^0(t)$ if $\gamma(t)=3+m,\ 4+m,\ 4+m+j$ $(j=1,2..m)$ respectively.

To construct the model, we define the following probabilities

$$L_n(t)=\Pr\{N_q(t)=n, X(t)=0\};\ n\geq 0, \tag{2.1}$$

$$S_n(x,t)=\Pr\{N_q(t)=n,\ X(t)=S^0(t);\ x\leq S^0(t)\leq x+dx\};\ x>0,\ n\geq 1, \tag{2.2}$$

$$P_n^{(0)}(x,t)=\Pr\{N_q(t)=n,\ X(t)=B_0^0(t);\ x\leq B_0^0(t)\leq x+dx\};\ x>0,\ n\geq 1, \tag{2.3}$$

$$P_n^{(i)}(x,t)=\Pr\{N_q(t)=n,\ X(t)=B_i^0(t);\ x\leq B_i^0(t)\leq x+dx\};\ x>0,\ n\geq 1, 1\leq i\leq m, \tag{2.4}$$

$$V_n(y,t)=\Pr\{N_q(t)=n,\ X(t)=V^0(t);\ y\leq V^0(t)\leq y+dy\};\ y>0,\ n\geq 1, \tag{2.5}$$

$$R_n^{(0)}(x,y,t)=\Pr\{N_q(t)=n,\ X(t)=R_0^0(t);\ y\leq R_0^0(t)\leq y+dy/B_0^0(t)=x\}; \quad x>0,\ n\geq 1, \tag{2.6}$$

$$R_n^{(i)}(x,y,t)=\Pr\{N_q(t)=n,\ X(t)=R_i^0(t);\ y\leq R_i^0(t)\leq y+dy/B_i^0(t)=x\};\ x>0, \quad n\geq 1, 1\leq i\leq m. \tag{2.7}$$

Further it is assume that

$$V(0)=0, V(\infty)=1, S(0)=0, S(\infty)=1, B_i(0)=0, B_i(\infty)=1,\ G_i(0)=0, G_i(\infty)=1.$$

Further it is assume that $G\ (y)$, $V(y)$ functions are continuous at $y=0$,while $B_i(x)$, $S(x)$ are continuous at $x=0$.

The hazard rate functions for present system is given by

$$\eta(x)dx=\frac{dS(x)}{1-S(x)},\ \mu_i(x)dx=\frac{dB_i(x)}{1-B_i(x)},\ \nu(y)dy=\frac{dV(y)}{1-V(y)},\ g_i(y)dy=\frac{dG_i(y)}{1-G_i(y)} \text{ for } 0\leq i\leq m.$$

Further we define the following probability generating functions for $i=0,1,2,...,m$ as follows.

$$R^{(i)}(x,y,z)=\sum_{n=1}^{\infty} z^n R_n^{(i)}(x,y)\ ; \qquad R^{(i)}(x,0,z)=\sum_{n=1}^{\infty} z^n R_n^{(i)}(x,0) \quad ; \qquad S(x,z)=\sum_{n=1}^{\infty} z^n S_n(x)$$

$$S(0,z)=\sum_{n=1}^{\infty} z^n S_n(0) \quad ; \quad P^{(i)}(x,z)=\sum_{n=1}^{\infty} z^n P_n^{(i)}(x) \quad ; \quad P^{(i)}(0,z)=\sum_{n=1}^{\infty} z^n P_n^{(i)}(0)$$

$$V(y,z)=\sum_{n=1}^{\infty} z^n V_n(y) \quad ; \quad V(0,z)=\sum_{n=1}^{\infty} z^n V_n(0) \quad ; \quad L(z)=\sum_{n=0}^{\infty} z^n L_n$$

III. Governing Equations

The governing equations of the system are

$$\lambda L_0 = q\left[r_0\int_0^\infty \mu_0(x)P_1^{(0)}(x)dx+\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx+....+\int_0^\infty \mu_m(x)P_1^{(m)}(x)dx\right] + \int_0^\infty \nu(y)V_1(y)dy, \tag{3.1}$$

$$\lambda L_1 + q\left[r_0\int_0^\infty \mu_0(x)P_1^{(0)}(x)dx+\int_0^\infty \mu_1(x)P_1^{(1)}(x)dx+....+\int_0^\infty \mu_m(x)P_1^{(m)}(x)dx\right] + \int_0^\infty \nu(y)V_1(y)dy = \lambda c_1 L_0, \tag{3.2}$$

$$\lambda L_n = \lambda\sum_{k=1}^{n} c_k L_{n-k}\ ,\ n\ge 2, \tag{3.3}$$

$$\frac{d}{dx}S_n(x)+[\lambda+\eta(x)]S_n(x)=\lambda\sum_{j=1}^{n} c_j S_{n-j}(x);\ \ x>0, n\ge 1, \tag{3.4}$$

$$\frac{d}{dx}P_n^{(i)}(x)+[\lambda+\alpha_i+\mu_i(x)]P_n^{(i)}(x)=\lambda\sum_{j=1}^{n} c_j P_{n-j}^{(i)}(x)+\int_0^\infty g_i(y)R_n^{(i)}(x,y)dy; \quad x>0, y>0,\ \ 0\le i\le m, \tag{3.5}$$

$$\frac{d}{dy}V_n(y)+[\lambda+\nu(y)]V_n(x)=\lambda\sum_{j=1}^{n} c_j V_{n-j}(y);\ \ n\ge 1, y>0, \tag{3.6}$$

$$\frac{d}{dy}R_n^{(i)}(x,y)+[\lambda+g_i(y)]R_n^{(i)}(x,y)=\lambda\sum_{j=1}^{n} c_j R_{n-j}^{(i)}(x,y); \quad n\ge 1, x>0, y>0,\ \ 0\le i\le m, \tag{3.7}$$

We will solve the equations (3.1)-(3.7) under the following boundary condition at $x=0$ and $y=0$ given by:

$$S_1(0)=\lambda L_0, \tag{3.8}$$

$$S_n(0)=0; \quad n\ge 2, \tag{3.9}$$

$$P_n^{(0)}(0)=q\left[r_0\int_0^\infty \mu_0(x)P_{n+1}^{(0)}(x)dx+\int_0^\infty \mu_1(x)P_{n+1}^{(1)}(x)dx+...+\int_0^\infty \mu_m(x)P_{n+1}^{(m)}(x)dx\right] + \int_0^\infty \eta(x)S_n(x)dx+\int_0^\infty \nu(y)V_{n+1}(y)dy; \qquad n\ge 1, \tag{3.10}$$

$$P_n^{(i)}(0)=r_i\int_0^\infty \mu_0(x)P_n^{(0)}(x)dx; \quad n\ge 1,\quad 1\le i\le m. \tag{3.11}$$

at $y=0$:

$$V_n(0)=p\left[r_0\int_0^\infty \mu_0(x)P_n^{(0)}(x)dx+\sum_{i=1}^{m}\int_0^\infty \mu_i(x)P_n^{(i)}(x)dx\right]; \quad n\ge 1 \tag{3.12}$$

and at $y=0$ for $i=0,1,2,...,m.$ and fixed value of X.

$$R_n^{(i)}(x;0)=\alpha_i P_n^{(i)}(x); \quad n\ge 1, \qquad i=0,1,2,...,m. \tag{3.13}$$

The normalizing condition for present system is given by

$$\sum_{n=0}^{\infty} L_n + \sum_{n=1}^{\infty}\sum_{i=0}^{m}\left[\int_0^{\infty} P_n^{(i)}(x)dx + \int_0^{\infty}\int_0^{\infty} R_n^{(i)}(x,y)dxdy\right] + \sum_{n=1}^{\infty}\int_0^{\infty} S_n(x)dx + \sum_{n=1}^{\infty}\int_0^{\infty} V_n(y)dy = 1 \tag{3.14}$$

IV. Mathematical Analysis

Apply summation formula after multiplying equation (3.2) and (3.3) by appropriate power of z , we get

$$\lambda L(z) + z\left[q\left\{r_0\int_0^{\infty}\mu_0(x)P_1^{(0)}(x)dx + \sum_{i=1}^{m}\int_0^{\infty}\mu_i(x)P_1^{(i)}(x)dx\right\} + \int_0^{\infty}\nu(y)V_1(y)dy\right]$$
$$= \left[q\left\{r_0\int_0^{\infty}\mu_0(x)P_1^{(0)}(x)dx + \sum_{i=1}^{m}\int_0^{\infty}\mu_i(x)P_1^{(i)}(x)dx\right\} + \int_0^{\infty}\nu(y)V_1(y)dy\right] + \lambda X(z)L(z) \tag{4.1}$$

Substitutes the value of (3.1) into equation (4.1) we get

$$L(z) = \frac{L_0(1-z)}{1-X(z)} \tag{4.2}$$

Solving equation (3.4), (3.6) and (3.7) in usual manner we get

$$S(x,z) = S(0,z)[1-S(x)]\exp\{-a_1(z)x\}; \qquad x>0, \tag{4.3}$$

$$V(y,z) = V(0,z)[1-V(y)]\exp\{-a_1(z)y\}; \qquad y>0, \tag{4.4}$$

$$R^{(i)}(x,y,z) = R^{(i)}(x,0,z)[1-G_i(y)]\exp\{-a_1(z)y\}; \qquad y>0,\ 0\le i\le m. \tag{4.5}$$

On multiplying equations (3.8),(3.9) and (3.13) by appropriate power of z , then after little simplification, we get

$$R^{(i)}(x,0,z) = \alpha_i P^{(i)}(x,z)\ ;\ i = 0,1,2,...,m, \tag{4.6}$$

$$S(0,z) = z\lambda L_0. \tag{4.7}$$

On simplifying equations (4.3) and (4.7) we have

$$S(x,z) = z\lambda L_0[1-S(x)]\exp\{-a_1(z)x\}; \qquad x>0, \tag{4.8}$$

On simplifying equations (3.5) and (4.5), we have

$$\frac{d}{dx}P^{(i)}(x,z) + (a_1(z)+\alpha_i+\mu_i(x))P^{(i)}(x,z) = R^{(i)}(x,0,z)\overline{G}_i(a_1(z)); \qquad 0\le i\le m. \tag{4.9}$$

Solving equations (4.6) and (4.9), we get

$$P^{(i)}(x,z) = P^{(i)}(0,z)[1-B_i(x)]\exp\{-\phi_i(z)x\}; \quad x>0,\ 0\le i\le m, \tag{4.10}$$

where $\phi_i(z) = a_1(z)+\alpha_i(1-\overline{G}_i(a_1(z)))$ and $a_1(z) = \lambda(1-X(z))$.

From equation (4.10), (4.6) and (4.5) we have

$$R^{(i)}(x,y,z) = \alpha_i P^{(i)}(0,z)[1-B_i(x)]\exp\{-\phi_i(z)x\}[1-G_i(y)]\exp\{-a_1(z)y\};$$
$$(x,y)>0,\ 0\le i\le m. \tag{4.11}$$

Further, multiplying equations (3.11), (3.12) by suitable power of z and after simplification we have

$$P^{(i)}(0,z) = r_i P^{(0)}(0,z)\overline{B}_0(\phi_0(z)); \qquad 1\le i\le m. \tag{4.12}$$

$$V(0,z) = pP^{(0)}(0,z)\overline{B}_0(\phi_0(z))\left[r_0 + \sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\right]. \tag{4.13}$$

Similarly, from equation (3.10) we have

$$P^{(0)}(0,z) = \frac{q}{z}\left[r_0\int_0^{\infty}\mu_0(x)P^{(0)}(x,z)dx + \sum_{i=1}^{m}\int_0^{\infty}\mu_i(x)P^{(i)}(x,z)dx\right]$$
$$+\int_0^{\infty}\eta(x)S(x,z)dx + \frac{1}{z}\int_0^{\infty}\nu(y)V(y,z)dy - \lambda L_0 \tag{4.14}$$

Substituting the value of equation (4.4), (4.8), (4.10) in (4.14) and then using the value of equations (4.7), (4.12) - (4.13) we get

$$P^{(0)}(0,z) = \frac{\lambda L_0 z[1-z\overline{S}(a_1(z))]}{\left[\overline{B}_0(\phi_0(z))\{r_0 + \sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a_1(z))\} - z\right]} \tag{4.15}$$

The limiting value of equation (4.15) when $z \to 1$, is given by

$$P^{(0)}(0,1) = \frac{\lambda L_0[1+\lambda E(X)E(S)]}{(1-\rho)} \tag{4.16}$$

where $\rho = \lambda E(X)\{E(B_0)(1+\alpha_0 g_0^{(1)}) + \sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + pE(V)\}.$

Evaluating $z \to 1$ in equation (4.2),(4.4), (4.8)- (4.13) and using the equation (4.16) we have

$$L(1) = \frac{L_0}{E(X)}, \tag{4.17}$$

$$S(x,1) = \lambda L_0[1-S(x)], \tag{4.18}$$

$$P^{(0)}(x,1) = \frac{\lambda L_0[1+\lambda E(X)E(S)][1-B_0(x)]}{(1-\rho)} \; ; \; x>0, \tag{4.19}$$

$$P^{(i)}(x,1) = \frac{r_i \lambda L_0[1+\lambda E(X)E(S)][1-B_i(x)]}{(1-\rho)}; \;\; x>0, \;\; 1\le i\le m, \tag{4.20}$$

$$V(y,1) = \frac{p\lambda L_0[1+\lambda E(X)E(S)][1-V(y)]}{(1-\rho)}; \;\; y>0, \tag{4.21}$$

$$R^{(0)}(x,y,1) = \frac{\alpha_0 \lambda L_0[1+\lambda E(X)E(S)][1-B_0(x)][1-G_0(y)]}{(1-\rho)}; \quad (x,y)>0. \tag{4.22}$$

$$R^{(i)}(x,y,1) = \frac{\alpha_i r_i \lambda L_0[1+\lambda E(X)E(S)][1-B_i(x)][1-G_i(y)]}{(1-\rho)}; \quad (x,y)>0. \tag{4.23}$$

From equations (4.17)-(4.23) and normalizing condition (3.14), we have

$$L_0 = \frac{E(X)(1-\rho)}{1+\lambda E(X)E(S)}. \tag{4.24}$$

Theorem 1: The joint probability distribution functions of system state and queue size, under stability condition, are given by

$$S(x,z) = \frac{z\lambda E(X)(1-\rho)[1-S(x)]\exp\{-a_1(z)x\}}{1+\lambda E(X)E(S)}, \tag{4.25}$$

$$P^{(0)}(x,z) = \frac{\lambda z E(X)(1-\rho)[1-z\bar{S}(a_1(z))][1-B_0(x)]\exp\{-\phi_0(z)x\}}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]}, \tag{4.26}$$

$$P^{(i)}(x,z) = \frac{r_i \lambda z E(X)(1-\rho)\bar{B}_0(\phi_0(z))[1-z\bar{S}(a_1(z))][1-B_i(x)]\exp\{-\phi_i(z)x\}}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]}, \tag{4.27}$$

$1\le i\le m,$

$$V(y,z) = \frac{p\lambda z E(X)(1-\rho)[1-z\bar{S}(a_1(z))]\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}[1-V(y)]\exp\{-a_1(z)y\}}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]}, \tag{4.28}$$

$$R^{(0)}(x,y,z) = \frac{\alpha_0 \lambda z E(X)(1-\rho)[1-z\bar{S}(a_1(z))][1-B_0(x)]\exp\{-\phi_0(z)x\}[1-G_0(y)]\exp\{-a_1(z)y\}}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]}, \tag{4.29}$$

$$R^{(i)}(x,y,z) = \frac{\alpha_i r_i \lambda z E(X)(1-\rho)[1-z\bar{S}(a_1(z))]\bar{B}_0(\phi_0(z))[1-B_i(x)]\exp\{-\phi_i(z)x\}[1-G_i(y)]\exp\{-a_1(z)y\}}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]}, \tag{4.30}$$

$1\le i\le m,$

Theorem 2: The marginal probability distribution function of system state queue size are given by

$$S(z) = \frac{z\lambda E(X)(1-\rho)[1-\bar{S}(a_1(z))]}{(1+\lambda E(X)E(S))a_1(z)}, \tag{4.31}$$

$$P^{(0)}(z) = \frac{\lambda z E(X)(1-\rho)[1-z\bar{S}(a_1(z))][1-\bar{B}_0(\phi_0(z))]}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]\phi_0(z)}, \tag{4.32}$$

$$P^{(i)}(z) = \frac{r_i \lambda z E(X)(1-\rho)\bar{B}_0(\phi_0(z))[1-z\bar{S}(a_1(z))][1-\bar{B}_i(\phi_i(z))]}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i \bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]\phi_i(z)}, \tag{4.33}$$

$1\le i\le m,$

$$V(z)=\frac{p\lambda zE(X)(1-\rho)[1-z\bar{S}(a_1(z))]\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}[1-\bar{V}(a_1(z))]}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]a_1(z)}, \tag{4.34}$$

$$R^{(0)}(z)=\frac{\alpha_0\lambda zE(X)(1-\rho)[1-z\bar{S}(a_1(z))][1-\bar{B}_0(\phi_0(z))][1-\bar{G}_0(a_1(z))]}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]a_1(z)\phi_0(z)}, \tag{4.35}$$

$$R^{(i)}(z)=\frac{\alpha_i r_i\lambda zE(X)(1-\rho)[1-z\bar{S}(a_1(z))]\bar{B}_0(\phi_0(z))[1-\bar{B}_i(\phi_i(z))][1-\bar{G}_i(a_1(z))]}{(1+\lambda E(X)E(S))[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]\phi_i(z)a_1(z)}, \quad 1\le i\le m, \tag{4.36}$$

Proof: See appendix A.

Theorem 3: The stationary queue size distribution at random epoch is given by

$$P(z)=\frac{\lambda E(X)(1-z)(1-\rho)\{1-z\bar{S}(a_1(z))\}[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}]}{a_1(z)[1+\lambda E(X)E(S)][\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]} \tag{4.37}$$

Proof: Adding the equations (4.31)-(4.36) we get required result.
The equation (4.37) can be written as

$$P(z)=\xi(z)\times\omega_{M^X/G/1}^{J\,opt\,with\,Vaca.}(z) \tag{4.38}$$

Where $\xi(z)=\dfrac{\lambda E(X)\{1-z\bar{S}(a_1(z))\}}{a_1(z)[1+\lambda E(X)E(S)]}$

and $\omega_{M^X/G/1}^{J\,opt\,with\,Vaca.}(z)=\dfrac{(1-z)(1-\rho)[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}]}{[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]}$

Equation (4.38) shows that the queue size distribution divides into two independent random variables: the first $\omega_{M^X/G/1}^{J\,opt\,with\,Vaca.}(z)$, the stationary queue size distribution of the unreliable bulk queue with optional service including vacation and repair, and the second $\xi(z)$ is the number of arrivals during idle time including setup time.

Theorem 4: The stationary queue size distribution of system at departure epoch is given by

$$\pi(z)=\frac{(1-\rho)\{1-z\bar{S}(a_1(z))\}[\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}]}{[1+\lambda E(X)E(S)][\bar{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m}r_i\bar{B}_i(\phi_i(z))\}\{q+p\bar{V}(a_1(z))\}-z]} \tag{4.39}$$

Proof: See appendix B.
Equation (4.39) can be written as

$$\pi(z)=\frac{1-X(z)}{E(X)(1-z)}\times P(z) \tag{4.40}$$

$$\pi(z)=\xi(z)\times\frac{1-X(z)}{E(X)(1-z)}\times\omega_{M^X/G/1}^{J\,opt\,with\,Vaca.}(z) \tag{4.41}$$

Thus, the queue size distribution at the departure epoch decomposes into three independent random variables: $\omega_{M^X/G/1}^{J\,opt\,with\,Vaca.}(z)$,the stationary queue size distribution of the unreliable bulk queue with optional service including vacation and repair; $\xi(z)$ the number of arrivals during idle time including setup time; and the third independent random variable $\frac{1-X(z)}{E(X)(1-z)}$, the number of customers placed before a tagged customer.

V. Performance measures

(a) System state probabilities

By considering limit $z \to 1$ in the marginal probability generating function of the server state queue distribution, it is possible to determine the system state probability of the server state.

- The probability that server is under startup is $P_S = \dfrac{\lambda E(X)E(S)(1-\rho)}{\{1+\lambda E(X)E(S)\}}$,
- The probability that server is busy with essential service $P_{B_0} = \lambda E(X)E(B_0)$,
- The probability that server is busy in providing the $i^{th}(1 \le i \le m)$ optional service $P_{B_i} = r_i \lambda E(X)E(B_i)$,
- The probability that server is under optional vacation $P_V = p\lambda E(X)E(V)$,
- The probability that server is under repair when its fail during essential service $P_{R_0} = \alpha_0 \lambda E(X)E(B_0)g_0^{(1)}$
- The probability that server is under repair when its fail during $i^{th}(1 \le i \le m)$ optional service $P_{R_i} = r_i \alpha_i \lambda E(X)E(B_i)g_i^{(1)}$,
- Probability that server is idle is given by $P_L = \dfrac{(1-\rho)}{1+\lambda E(X)E(S)}$.

(b) Average queue length

(i) The mean system size (L_q) at arbitrary epoch can be determined using

$$L_q = \frac{dP(z)}{dz}\Big|_{z=1}$$

$$L_q = \rho + \frac{2\lambda E(X)E(S) + (\lambda E(X))^2 E(S^2) + \lambda E(X^{(2)})E(S)}{2(1+\lambda E(X)E(S))}$$
$$+\frac{(\lambda E(X))^2 \left\{ \begin{aligned} & E(B_0^2)(1+\alpha_0 g_0^{(1)})^2 + \sum_{i=1}^{m} r_i E(B_i^2)(1+\alpha_i g_i^{(1)})^2 + \alpha_0 g_0^{(2)} E(B_0) + \sum_{i=1}^{m} r_i \alpha_i g_i^{(2)} E(B_i) + pE(V^2) \\ & + 2E(B_0)(1+\alpha_0 g_0^{(1)}) \sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + 2pE(V) \sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + 2pE(B_0)(1+\alpha_0 g_0^{(1)})E(V) \end{aligned} \right\}}{2(1-\rho)} \tag{5.1}$$
$$+\frac{\lambda E(X^{(2)}) \left\{ E(B_0)(1+\alpha_0 g_0^{(1)}) + \sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + pE(V) \right\}}{2(1-\rho)}$$

(ii) The mean system size (L_D) at departure epoch can be determined using

$$L_D = \frac{d\pi(z)}{dz}\Big|_{z=1}$$

$$L_D = \rho + \frac{2\lambda E(X)E(S) + (\lambda E(X))^2 E(S^2) + \lambda E(X^{(2)})E(S)}{2(1+\lambda E(X)E(S))}$$

$$+ \frac{(\lambda E(X))^2 \left\{ \begin{array}{l} E(B_0^2)(1+\alpha_0 g_0^{(1)})^2 + \sum_{i=1}^{m} r_i E(B_i^2)(1+\alpha_i g_i^{(1)})^2 + \alpha_0 g_0^{(2)} E(B_0) + \sum_{i=1}^{m} r_i \alpha_i g_i^{(2)} E(B_i) + pE(V^2) \\ + 2E(B_0)(1+\alpha_0 g_0^{(1)}) \sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + 2pE(V)\sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + 2pE(B_0)(1+\alpha_0 g_0^{(1)})E(V) \end{array} \right\}}{2(1-\rho)} \tag{5.2}$$

$$+ \frac{\lambda E(X^{(2)}) \left\{ E(B_0)(1+\alpha_0 g_0^{(1)}) + \sum_{i=1}^{m} r_i E(B_i)(1+\alpha_i g_i^{(1)}) + pE(V) \right\}}{2(1-\rho)} + \frac{E(X^{(2)})}{2E(X)}$$

From (5.1) and (5.2) we can easily observe that $L_D = L_q + \dfrac{E(X^{(2)})}{2E(X)}$

(c) Average waiting time

The average waiting time can be obtained as

$$E(W_q) = \frac{L_q}{\lambda E(X)} \tag{5.3}$$

VI. Special cases

In this section, we evaluate some special case by setting appropriate parameter to validate our result with existing models.

Case (i): By setting $P(S=0)=1,\ P(X=1)=1,\ r_1=1, m=1$; equation (4.39) gives

$$\pi(z) = \frac{(1-\rho)(1-z)[\bar{B}_0(\phi_0(z))\bar{B}_1(\phi_1(z))\{q + p\bar{V}(\lambda(1-z))\}]}{[\bar{B}_0(\phi_0(z))\bar{B}_1(\phi_1(z))\{q + p\bar{V}(a_1(z))\} - z]}$$

where $\phi_i(z) = \lambda(1-z) + \alpha_i(1-\bar{G}_i(\lambda(1-z))),\ i = 0,1.$

The present model reduces to the model studied by Chaudhury and Deka [9].

Case (ii): By setting $P(S=0)=1,\ P(X=1)=1,\ r_1=1, m=1, \alpha_1=\alpha_2=\ldots=\alpha_m=0$; equation (4.39) gives

$$\pi(z) = \frac{(1-\rho)(1-z)[\bar{B}_0(\lambda(1-X(z)))\bar{B}_1(\lambda(1-X(z)))\{q + p\bar{V}(\lambda(1-X(z)))\}]}{[\bar{B}_0(\lambda(1-X(z)))\bar{B}_1(\lambda(1-X(z)))\{q + p\bar{V}(\lambda(1-X(z)))\} - z]}$$

The present model reduces to the model studied by Chaudhury and madan [7].

Case (iii): By setting $\alpha_1=\alpha_2=\ldots=\alpha_m=0,\ p=0$; equation (4.39) gives

$$\pi(z) = \frac{(1-\rho)\{1 - z\bar{S}(a_1(z))\}[\bar{B}_0(\lambda(1-X(z)))\{r_0 + \sum_{i=1}^{m} r_i \bar{B}_i(\lambda(1-X(z)))\}]}{[1+\lambda E(X)E(S)][\bar{B}_0(\lambda(1-X(z)))\{r_0 + \sum_{i=1}^{m} r_i \bar{B}_i(\lambda(1-X(z)))\} - z]}$$

The present model reduces to model investigated by Ke [12].

Case(iv): By setting $\alpha_1=\alpha_2=\ldots=\alpha_m=0, p=0, r_1=r_2=\ldots=r_m=0,$; equation (4.39) gives

$$\pi(z) = \frac{(1-\rho)\{1 - z\bar{S}(a_1(z))\}[\bar{B}_0(\lambda(1-X(z)))]}{[1+\lambda E(X)E(S)][\bar{B}_0(\lambda(1-X(z))) - z]}$$

The present model reduces to the model studied by Choudhury [6]

Case(v): By setting $\alpha_0=\alpha_1=\ldots=\alpha_m=0, p=0, r_2=\ldots=r_m=0, P(X=1)=1$; equation (4.39) gives

$$\pi(z)=\frac{(1-\rho)(1-z)\overline{B}_0(\lambda(1-z))\{r_0+\overline{B}_1(\lambda(1-z))\}}{[\overline{B}_0(\lambda(1-z))\{r_0+\overline{B}_1(\lambda(1-z))\}-z]}$$

The present model reduces to the model studied by Medhi [15].

VII. Numerical illustration

In present section, we will provide the numerical illustration and sensitivity analysis of the various performance measures on different parameters of the model. For this, it assume that the first two moments of the batch size distribution are given by $E(X)=\frac{b}{a},\ E(X^2)=\frac{b(1+b)}{a^2};\ b=1-a.$ It is assumed that the server's start-up time will follow an deterministic distribution with first and second moments $E(S)=\frac{1}{s},E(S^2)=\frac{1}{s^2}.$ The distribution of compulsory and elective service periods is assumed to be exponential, and its first and second moments are therefore derived as $E(B_i)=\frac{1}{\mu_i},\ E(B_i^2)=\frac{2}{\mu_i^2}\ ;\ i=01{,}2.$ where μ_i denote the service rate. Further, the distribution of vacation time is assumed to be Erlangian-2 and has parameter $\gamma_i\,(i=1,2).$ The first and second moments of vacation time distribution are $E(V)=\frac{1}{v},\ E(V^2)=\frac{3}{2v^2}$ .The repair time distribution is further assumed to follow an exponential distribution with a parameter g_i and having the first two moments $g_i^{(1)}=\frac{1}{g_i},\ g_i^{(2)}=\frac{2}{g_i^2};\ i=0{,}1{,}2;$ Coding in MATLAB is used to create computer programmes. We now present the numerical results in tables (1) -(5).

Table 1: $E(X)=2,\mu_1=\mu_2=2\mu_0,\alpha_0=0.01,\alpha_1=\alpha_2=2\alpha_0,r_0=r_1=r_1=1/3,v=15,$
$s=10,g_0=10,g_1=15,g_2=15.$

Table2: $E(X)=2,\mu_1=\mu_2=2\mu_0,\alpha_1=\alpha_2=2\alpha_0,r_0=r_1=r_1=1/3,v=15,$
$s=10,g_0=10,g_1=15,g_2=15,\lambda=0.7,\mu_0=2$

Table 3: $E(X)=2,\mu_1=\mu_2=2\mu_0,\alpha_1=\alpha_2=2\alpha_0,r_0=r_1=r_1=1/3,v=15,p=0.5,$
$s=10,g_0=10,g_1=15,g_2=15,\alpha_0=0.01.$

Table 4: $E(X)=2,\mu_1=\mu_2=2\mu_0,\alpha_1=\alpha_2=2\alpha_0,r_0=r_1=r_1=1/3,v=15,p=0.5,$
$s=10,g_0=10,g_1=15,g_2=15,\lambda=0.7,\mu_0=2.$

Table 5: $E(X)=2,\mu_1=\mu_2=2\mu_0,\alpha_1=\alpha_2=2\alpha_0,r_0=r_1=r_1=1/3,v=15,p=0.5,$
$s=10,g_0=10,g_1=15,g_2=15,\lambda=0.7,\alpha_0=0.01$

Table 1: *Effect of arrival rate and service rate on* L_q (W_q) *for variation in* p

	$\mu=2$				$\mu=2.1$			
	$p=0.3$		$p=0.7$		$p=0.3$		$p=0.7$	
λ	L_q	W_q	L_q	W_q	L_q	W_q	L_q	W_q
0.61	14.174	11.618	18.492	15.157	10.632	8.715	13.275	10.881
0.63	17.922	14.224	24.894	19.758	12.851	10.199	16.634	13.201
0.65	23.607	18.160	36.393	27.994	15.857	12.197	21.611	16.624
0.67	33.253	24.816	63.102	47.091	20.159	15.044	29.755	22.205
0.69	53.215	38.562	194.278	140.781	26.829	19.441	45.498	32.969

Table 2: *Effect of* p *on* L_q (W_q) *for variation in failure rate and* m

		$m=2$		$m=1$		$m=0$	
	p	L_q	W_q	L_q	W_q	L_q	W_q
$\alpha_0=0.01$	0.1	27.349	19.535	17.130	12.236	7.094	5.067
	0.3	33.760	24.114	19.614	14.010	7.623	5.445
	0.5	43.930	31.379	22.869	16.335	8.218	5.870
	0.7	62.558	44.684	27.328	19.520	8.894	6.353
	0.9	107.754	76.967	33.814	24.153	9.668	6.906
$\alpha_0=0.05$	0.1	28.516	20.368	17.578	12.556	7.172	5.123
	0.3	35.519	25.371	20.188	14.420	7.710	5.507
	0.5	46.895	33.496	23.636	16.883	8.317	5.940
	0.7	68.622	49.016	28.406	20.290	9.006	6.433
	0.9	126.698	90.498	35.442	25.316	9.797	6.998

Table 3: *Effect of arrival rate on system state probabilities*

λ	P_L	P_S	P_{B_0}	P_{B_1}	P_{B_2}	P_V	P_{R_0}	P_{R_1}	P_{R_2}
0.61	0.12934	0.01578	0.61000	0.10167	0.10167	0.04067	0.00061	0.00014	0.00014
0.63	0.10399	0.01310	0.63000	0.10500	0.10500	0.04200	0.00063	0.00014	0.00014
0.65	0.07882	0.01025	0.65000	0.10833	0.10833	0.04333	0.00065	0.00014	0.00014
0.67	0.05382	0.00721	0.67000	0.11167	0.11167	0.04467	0.00067	0.00015	0.00015
0.69	0.02900	0.00400	0.69000	0.11500	0.11500	0.04600	0.00069	0.00015	0.00015

Table 4: *Effect of service rate on system state probabilities*

μ_0	P_L	P_S	P_{B_0}	P_{B_1}	P_{B_2}	P_V	P_{R_0}	P_{R_1}	P_{R_2}
2	0.01666	0.00233	0.70000	0.11667	0.11667	0.04667	0.00070	0.00016	0.00016
2.1	0.05569	0.00780	0.66667	0.11111	0.11111	0.04667	0.00067	0.00015	0.00015
2.2	0.09117	0.01276	0.63636	0.10606	0.10606	0.04667	0.00064	0.00014	0.00014
2.3	0.12356	0.01730	0.60870	0.10145	0.10145	0.04667	0.00061	0.00014	0.00014
2.4	0.15326	0.02146	0.58333	0.09722	0.09722	0.04667	0.00058	0.00013	0.00013

Table 5: *Effect of failure rate on system state probabilities*

α_0	P_L	P_S	P_{B_0}	P_{B_1}	P_{B_2}	P_V	P_{R_0}	P_{R_1}	P_{R_2}
0.01	0.01666	0.00233	0.70000	0.11667	0.11667	0.04667	0.00070	0.00016	0.00016
0.02	0.01577	0.00221	0.70000	0.11667	0.11667	0.04667	0.00140	0.00031	0.00031
0.03	0.01488	0.00208	0.70000	0.11667	0.11667	0.04667	0.00210	0.00047	0.00047
0.04	0.01400	0.00196	0.70000	0.11667	0.11667	0.04667	0.00280	0.00062	0.00062
0.05	0.01311	0.00184	0.70000	0.11667	0.11667	0.04667	0.00350	0.00078	0.00078

The impact of arrival rate and service rate on the average queue length (waiting time) $L_q\,(W_q)$ is shown in Table 1. The table clearly shows that the $L_q\,(W_q)$ increases with rising arrivals, however, there is a diminishing trend brought on by a rise in service rate. Additionally, there is an increasing tendency in $L_q\,(W_q)$ with an increase in p for the fixed value of the arrival rate. Table 2 displays the impact of p on the average queue length (waiting time). The table clearly shows that there is an increasing tendency in $L_q\,(W_q)$ as a consequence of the growth in p. Additionally seen is a decline in $L_q\,(W_q)$ as a result of a reduction in the availability of optional services. The variation in system state probability caused by variations in arrival (service) rates is shown in Table 3(4). It is evident from the data that with an increase in arrival (service) rate $P_{B_0}, P_{B_1}, P_{B_2}$ and P_V have growing (declining) trends, whereas P_L and P_S have decreasing (increasing) trends. Table 5 demonstrates that as the failure rate rises, P_L and P_S tend to decline while P_{B_0} , P_{B_1} , P_{B_2} and P_V remain constants. Along with the rise in failure rates, increasing trends can be seen in P_{R_0} , P_{R_1} , and P_{R_2} .

VIII. Conclusion

In the present article, we investigated a queueing model with an unreliable server under the provision of Bernoulli vacation, setup time, and two-phase service, where the first service is essential and the second is optional, and we had to choose among the available options. In the current study, we use the supplementary variable approach to build the model and assess several performance indices expressions. Our model may be useful in more flexible queueing circumstances that occur in many manufacturing and production systems, where some services may be optional based on the customer's desire and where the manufacture of the items must be done in phases, such as assembling, testing, packing, etc. The model studied can be further generalised by incorporating feedback services as well as some more features such as N-Policy, retrial, and extended vacation policies.

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Appendix A

Proof of theorem 1:

Integrating equations (4.25)-(4.27) with respect to x and ussing the result

$$\int_0^\infty e^{-sx}(1-M(x))dx = \frac{1-\overline{M}(s)}{s} \tag{A.1}$$

We get equations (4.31)-(4.33).

Similarly integrating equations (4.28) with respect to y and using (A.1) we get equation (4.34). On repeating the same process for equations (4.29) and (4.30) with variable x, y, and using equation (A.1), we get equations (4.35)-(4.36).

Appendix. B

Proof of theorem 2:

To obtain the queue size distribution at the departure epoch, on the line of Choudhury and Deka [9], we have

$$\pi_j = k_0\left\{r_0\int_0^\infty \mu_0(x)P_{j+1}^{(0)}(x)dx + \int_0^\infty \mu_1(x)P_{j+1}^{(1)}(x)dx + \ldots + \int_0^\infty \mu_m(x)P_{j+1}^{(m)}(x)dx + \int_0^\infty \nu(y)V_{j+1}(y)dy\right\} \tag{B.1}$$

where k_0 is the normalizing constant and $\{\pi_j; j = 0,1,2,\ldots\}$ as the probability that there are j customers in the queue at a departure epoch.

Multiplying equation (B.1) by z^j and using $\pi(z) = \sum_{j=0}^{\infty}\pi_j z^j$ and after simplification,

We get

$$\pi(z) = \frac{k_0\lambda L_0\{1-z\overline{S}(a_1(z))\}[\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a_1(z))\}]}{[\overline{B}_0(\phi_0(z))\{r_0+\sum_{i=1}^{m} r_i\overline{B}_i(\phi_i(z))\}\{q+p\overline{V}(a_1(z))\}-z]} \tag{B.2}$$

Using the condition $\pi(1)=1$, we get

$$k_0 = \frac{(1-\rho)}{\lambda L_0\{1+\lambda E(X)E(S)\}} \tag{B.3}$$

Using the value of equation (B.3) into (B.2), we get required result.