A NEW FINITE MIXTURE OF PROBABILITY MODELS WITH APPLICATION

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Abstract

In this research, we present an approach to model lifetime data by a weighted three-parameter probability distribution utilizing the exponential and gamma distributions. We have presented some of the essential characteristics such as the shapes of pdf, cdf, moments, incomplete moments, survival function, hazard function, mean residual life, stochastic ordering, and order statistics of the proposed distribution. Furthermore, we also presented the Bonferroni index and Lorenz curve of the proposed distribution. The maximum likelihood approach is used to estimate the parameters of the distribution. Finally, the proposed probability distribution is compared to goodness of fit with Lindley, Akash, exponential, two-parameter Lindley, cubic transmuted Rayleigh, and Exponential-Gamma distributions for the real-time data set.

Keywords: Lifetime distribution, Hazard function, Mean residual life function, Order statistic, Maximum likelihood estimation.

1. Introduction

A scientific approach to the statistical modeling of a wide variety of random events has been made possible by finite mixture of probability models. Due to its adaptability in representing complicated data, finite mixture models have drawn significant interest recently, both from a theoretical and practical perspective. Karl Pearson [15] conducted one of the earliest significant analyses utilizing mixture models. He modeled a proportional combination of two normal probability density functions with varying means and variances. A variety of probability distributions were subsequently utilized by many authors to fit a combination of probability distributions. Similarly, Lindley [17] also modeled the 'Lindley distribution' which is a combination of an exponential distribution with a scale parameter of θ and a gamma distribution having a shape parameter of 2 and a scale parameter of θ with their corresponding mixing proportions, $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively.

A probability density function (pdf) and cumulative distribution function (cdf) for the Lindley distribution were included below.

$$f(x) = \frac{\theta^2(1+x)e^{-\theta x}}{\theta+1}; x > 0, \theta > 0 \tag{1}$$

$$F(x) = 1 - \left[1 + \frac{\theta x}{\theta+1}\right] e^{-\theta x}; x > 0, \theta > 0 \tag{2}$$

Shanker [22] used the finite mixture model to propose the Akash distribution, which is described by its pdf and cdf.

$$f(x) = \frac{\theta^3(1+x^2)e^{-\theta x}}{\theta^2+2}; x > 0, \theta > 0 \tag{3}$$

$$F(x) = 1 - \left[1 + \frac{\theta x(\theta x+2)}{\theta^2+2}\right] e^{-\theta x}; x > 0, \theta > 0 \tag{4}$$

Furthermore, the finite mixing model is

$$f(x) = w_1 g_1(x) + w_2 g_2(x) \tag{5}$$

Where Shanker [22] uses the mixing proportion for Akash distribution with weights as $w_1 = \frac{\theta^2}{\theta^2+2}$ and $w_2 = \frac{2}{\theta^2+2}$. Here, $g_1(x)$ and $g_2(x)$ denotes pdf of exponential (θ) and gamma (3, θ) distribution respectively.

We make changes to the Akash distribution to make it more inclusive and adaptable. Shanker [22] used the term θ to describe the parameters of an exponential and a gamma distribution. In this study, we presented a new probability distribution, which we called the Exp-Gamma distribution. The proposed distribution is more flexible and it performs like the Generalized version of the Akash distribution. We did this by employing the scale parameter λ for the exponential distribution and shape parameter 3, and the scale parameter β for the gamma distribution with the mixture proportion of $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively.

This paper is also arranged in the following manner. In section 2, we present the Exp-Gamma distribution. Section 3 contains the usual moments and their related measures for the Exp-Gamma distribution. Section 4 deals with reliability analysis. Log-odds rate is calculated in section 5. Section 6 discusses Entropy. Section 7 deals with stochastic ordering. The order statistics for the Exp-Gamma distribution are given in section 8. The Lorenz and Bonferroni curves are presented in Section 9. The section 10 Zenga index is derived. In section 11, it is discussed how to estimate the Exp-Gamma distribution's parameters using the maximum likelihood method. Finally, section 12's proposed distribution as an application makes use of real-time data.

2. Exponential-Gamma Distribution(Exp-Gamma)

The probability distribution of the Exp-Gamma distribution can be described by its probability density function and cumulative distribution function.

$$f(x;\theta,\lambda,\beta) = \frac{1}{\theta^2+2}\left[\theta^2 \lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}\right] \tag{6}$$

$$F(x) = \frac{\theta^2(1-e^{-\lambda x}) + 2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)}{\theta^2+2} \tag{7}$$

for, $x \geq 0, \theta \geq 0, \lambda \geq 0, \beta \geq 0$.

The following images Figure 1 and Figure 2 show a few potential pdf and cdf shapes for an Exp-Gamma distribution for various parameter values. The Akash and Gamma distributions are the special cases of the Exp-Gamma distribution when $\lambda = \beta = \theta$ and $\theta = 0$ respectively. According to Figure 1, the Exp-Gamma distribution presents a variety of pdf patterns, including right-skewed and reversed-J shaped, pdf parameters that have fixed values.

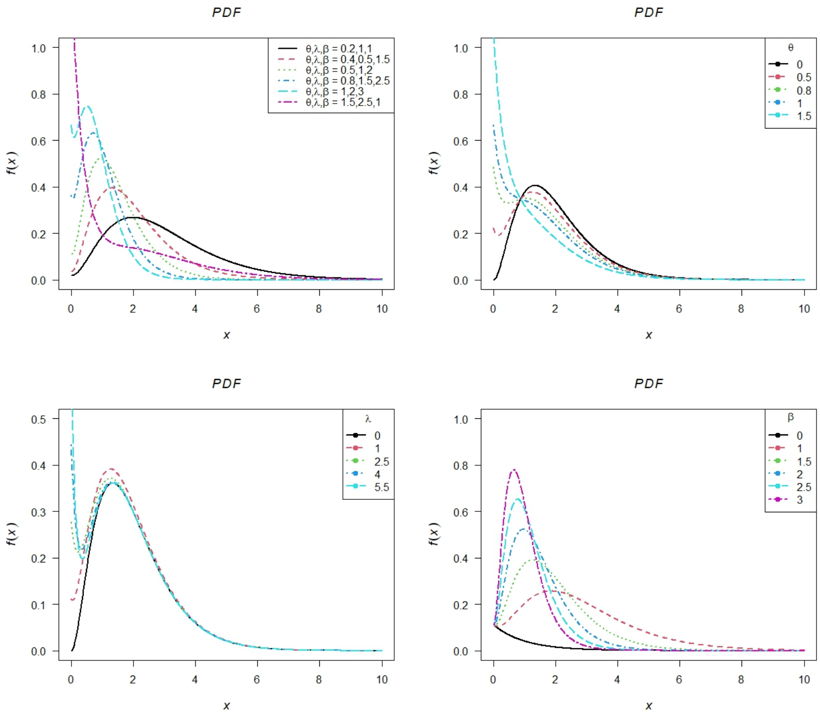

Figure 1: *The shape of the pdf of the Exp-Gamma distribution with varying parameter values.*

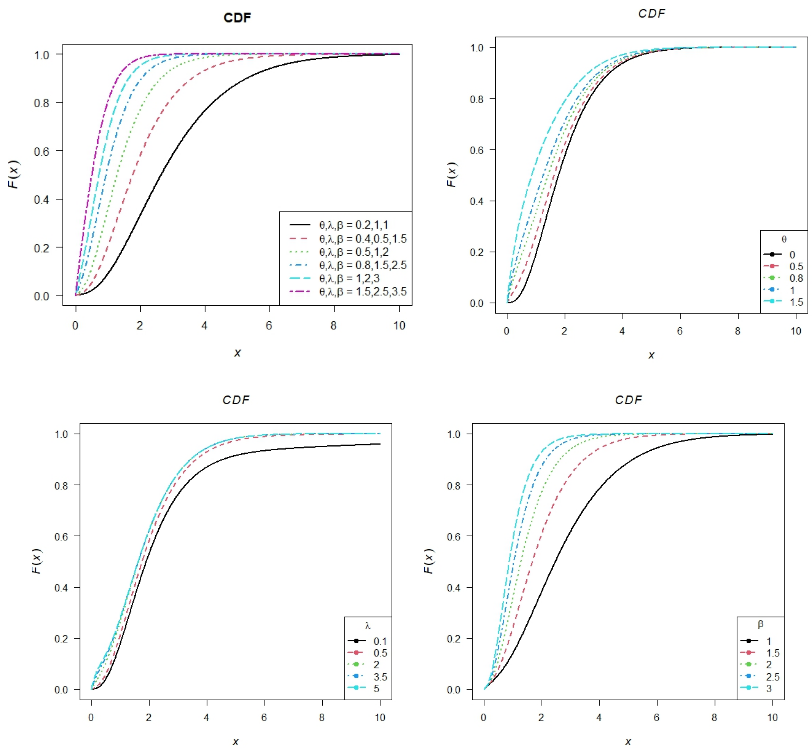

Figure 2: *The form of the Exp-Gamma distribution's cdf changes when the parameter values change.*

3. Moments and related measures

The r^{th} moment (raw moments) has been obtained as

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r f(x)dx \\ &= \int_0^\infty x^r \frac{1}{\theta^2+2}[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}]dx \\ &= \frac{1}{\theta^2+2}\left[\frac{\theta^2\Gamma(r+1)}{\lambda^r} + \frac{\Gamma(r+3)}{\beta^r}\right] \end{aligned} \tag{8}$$

when $r = 1, 2, 3, 4$ then the results follow.

The Exp-Gamma distribution's first four moments are:

$$Mean(\mu) = E(X) = \frac{\beta\theta^2 + 6\lambda}{\beta\lambda(\theta^2+2)}$$

$$E(X^2) = \frac{2\beta^2\theta^2 + 24\lambda^2}{\beta^2\lambda^2(\theta^2+2)}$$

$$E(X^3) = \frac{6\beta^3\theta^2 + 120\lambda^3}{\beta^3\lambda^3(\theta^2+2)}$$

$$E(X^4) = \frac{24\beta^4\theta^2 + 720\lambda^4}{\beta^4\lambda^4(\theta^2+2)}$$

As a result, the Exp-Gamma distribution's central moments are calculated as

$$\mu_2 = Variance = \frac{12\lambda^2 + \theta^2(\beta^2\theta^2 + 24\lambda^2 + 4\beta^2 - 12\lambda\beta)}{\beta^2\lambda^2(\theta^2+2)^2}$$

$$\mu_3 = \frac{2[13\beta^3\theta^2 + 6\beta^3\theta^4 + 60\lambda^3\theta^4 + 24\lambda^3\theta^2 - 36\lambda^2\beta\theta^4 - 36\lambda\beta^2\theta^2 - 54\lambda^2\beta\theta^2 - 84\lambda^3]}{\beta^3\lambda^3(\theta^2+2)^3}$$

$$\begin{aligned} \mu_4 = \Bigg[&24\beta^4\theta^2(0.375\theta^6 + 3\theta^4 + 8\theta^2 + 8) - 48\lambda^4(93 + 114\theta^2 + 51\theta^4 - 1.5\theta^6) + \\ &72\lambda^2\beta^2\theta^2(12 + 2\theta^4 + \theta^2) - 72\beta^3\theta^2(8\lambda + 4\lambda\theta^2 + \theta^4) \\ &-192\lambda^3\beta\theta^2(5.5 + \theta^2 + 2.5\theta^4)\Bigg]\frac{1}{\beta^4\lambda^4(\theta^2+2)^4} \end{aligned}$$

With the use of the aforementioned moments, closed-form formulas for the Exp-Gamma distribution's skewness, kurtosis, variation, and index of dispersion are produced. The variance-to-mean ratio is known as the index of dispersion (DI). The model is appropriate for datasets with low dispersion if the DI value is less than 1. The model works well with overly distributed datasets if the DI value is greater than 1.

$$\begin{aligned} skewness(x) &= \frac{E(X^3) - 3E(X^2)\mu + 2\mu^3}{\sigma^3} \\ &= \frac{2[13\beta^3\theta^2 + 6\beta^3\theta^4 + 60\lambda^3\theta^4 + 24\lambda^3\theta^2 - 36\lambda^2\beta\theta^4 - 36\lambda\beta^2\theta^2 - 54\lambda^2\beta\theta^2 - 84\lambda^3]}{(12\lambda^2 + \theta^2(\beta^2\theta^2 + 24\lambda^2 + 4\beta^2 - 12\lambda\beta))^{\frac{3}{2}}} \\ Kurtosis &= \frac{E(X^4) - 4E(X^3)\mu + 6E(X^2)\mu^2 - 3\mu^4}{\sigma^4} \end{aligned}$$

$$Kurtosis = \Bigg[24\beta^4\theta^2(0.375\theta^6 + 3\theta^4 + 8\theta^2 + 8) - 48\lambda^4(93 + 114\theta^2 + 51\theta^4 - 1.5\theta^6) + 72\lambda^2\beta^2\theta^2(12 + 2\theta^4 + \theta^2) - 72\beta^3\theta^2(8\lambda + 4\lambda\theta^2 + \theta^4) - 192\lambda^3\beta\theta^2(5.5 + \theta^2 + 2.5\theta^4)\Bigg] \frac{1}{(12\lambda^2 + \theta^2(\beta^2\theta^2 + 24\lambda^2 + 4\beta^2 - 12\lambda\beta))^2}$$

$$COV = \frac{\sigma}{\mu} = \frac{(12\lambda^2 + \theta^2(\beta^2\theta^2 + 24\lambda^2 + 4\beta^2 - 12\lambda\beta))^{\frac{1}{2}}}{\beta\theta^2 + 6\lambda}$$

$$DOI(\gamma) = \frac{\sigma^2}{\mu} = \frac{12\lambda^2 + \theta^2(\beta^2\theta^2 + 24\lambda^2 + 4\beta^2 - 12\lambda\beta)}{(\beta\lambda(\theta^2 + 2))(\beta\theta^2 + 6\lambda)}$$

As seen in the table 1 to 5, the mean, variance, skewness, kurtosis, and index of dispersion are all expressed in quantitative terms.

From the tables, we can infer that the proposed distributions have the following features:

* The mean of the proposed function is a declining function of θ, λ, and, β.
* The Exp-Gamma distribution is positively skewed for all the parameter values.
* Every positively skewed set of data can fits the suggested distribution.
* When the parameter values of the Exp-Gamma distribution are less than 1, then the Exp-Gamma distribution belongs to the light-tailed distribution, and when it exceeds the value of 1, then it belongs to the heavy-tailed distribution.
* The Exp-Gamma distribution is appropriate for both over- and under-dispersed datasets, as evidenced by the increasing and diminishing DI behavior.

Table 1: *Mean values of the model*

θ	β	λ					
		0.5	1	1.5	2	2.5	3
1	0.5	4.6667	4.3333	4.2222	4.1667	4.1333	4.1111
	1	2.6667	2.3333	2.2222	2.1667	2.1333	2.1111
	1.5	2.0000	1.6667	1.5556	1.5000	1.4667	1.4444
	2	1.6667	1.3333	1.2222	1.1667	1.1333	1.1111
	2.5	1.4667	1.1333	1.0222	0.9667	0.9333	0.9111
	3	1.3333	1.0000	0.8889	0.8333	0.8000	0.7778
2	0.5	3.3333	2.6667	2.4444	2.3333	2.2667	2.2222
	1	2.3333	1.6667	1.4444	1.3333	1.2667	1.2222
	1.5	2.0000	1.3333	1.1111	1.0000	0.9333	0.8889
	2	1.8333	1.1667	0.9444	0.8333	0.7667	0.7222
	2.5	1.7333	1.0667	0.8444	0.7333	0.6667	0.6222
	3	1.6667	1.0000	0.7778	0.6667	0.6000	0.5556
3	0.5	2.7273	1.9091	1.6364	1.5000	1.4182	1.3636
	1	2.1818	1.3636	1.0909	0.9545	0.8727	0.8182
	1.5	2.0000	1.1818	0.9091	0.7727	0.6909	0.6364
	2	1.9091	1.0909	0.8182	0.6818	0.6000	0.5455
	2.5	1.8545	1.0364	0.7636	0.6273	0.5455	0.4909
	3	1.8182	1.0000	0.7272	0.5909	0.5091	0.4545

Table 2: *The variance of the model*

θ	β	λ					
		0.5	1	1.5	2	2.5	3
1	0.5	12.8889	13.8889	14.4691	14.8056	15.0222	15.1728
	1	3.5556	3.2222	3.3580	3.4722	3.5556	3.6173
	1.5	2.2222	1.4444	1.4321	1.4722	1.5111	1.5432
	2	1.8889	0.8889	0.8025	0.8056	0.8222	0.8395
	2.5	1.7956	0.6622	0.5314	0.5122	0.5156	0.5240
	3	1.7778	0.5556	0.3951	0.3611	0.3556	0.3580
2	0.5	10.2222	10.2222	10.6173	10.8889	11.0756	11.2099
	1	3.8889	2.5556	2.5062	2.5556	2.6089	2.6543
	1.5	3.1111	1.3333	1.1358	1.1111	1.1200	1.1358
	2	2.9722	0.9722	0.7006	0.6389	0.6256	0.6265
	2.5	2.9689	0.8356	0.5195	0.4356	0.4089	0.4010
	3	3.0000	0.7778	0.4321	0.3333	0.2978	0.2840
3	0.5	7.8347	6.7190	6.7769	6.8864	6.9779	7.0496
	1	3.9669	1.9587	1.7190	1.6798	1.6820	1.6942
	1.5	3.5152	1.2094	0.8705	0.7817	0.7542	0.7466
	2	3.4463	0.9917	0.6033	0.4897	0.4473	0.4298
	2.5	3.4552	0.9114	0.4932	0.3647	0.3134	0.2899
	3	3.4821	0.8788	0.4408	0.3023	0.2451	0.2176

Table 3: *Skewness of the model*

θ	β	λ					
		0.5	1	1.5	2	2.5	3
1	0.5	0.0252	0.0224	0.0205	0.0194	0.0187	0.0182
	1	0.1912	0.2015	0.1910	0.1794	0.1705	0.1639
	1.5	0.5940	0.6636	0.6800	0.6587	0.6306	0.6054
	2	1.0277	1.5293	1.5954	1.6119	1.5773	1.5279
	2.5	1.2964	2.9363	3.0324	3.1345	3.1482	3.0980
	3	1.4238	4.7520	5.1614	5.3091	5.4316	5.4400
2	0.5	0.0494	0.0592	0.0574	0.0554	0.0539	0.0528
	1	0.2053	0.3950	0.4645	0.4738	0.4680	0.4593
	1.5	0.3739	0.9375	1.3331	1.5255	1.5893	1.5990
	2	0.4567	1.6424	2.4939	3.1600	3.5401	3.7158
	2.5	0.4828	2.3873	3.9009	5.1726	6.1719	6.8048
	3	0.4856	2.9913	5.5430	7.5000	9.2689	10.6651
3	0.5	0.0877	0.1541	0.1641	0.1639	0.1620	0.1601
	1	0.2215	0.7013	1.0708	1.2328	1.2935	1.3129
	1.5	0.3088	1.2455	2.3669	3.2923	3.8550	4.1606
	2	0.3406	1.7724	3.5959	5.6103	7.3395	8.5668
	2.5	0.3483	2.1934	4.8130	7.8061	10.9577	13.7362
	3	0.3475	2.4707	5.9817	9.9640	14.4031	18.9349

Table 4: *Kurtosis of the model*

θ	β	λ					
		0.5	1	1.5	2	2.5	3
1	0.5	0.0291	0.0239	0.0212	0.0198	0.0189	0.0183
	1	0.4635	0.4655	0.4200	0.3824	0.3567	0.3388
	1.5	2.2113	2.3624	2.3568	2.2050	2.0551	1.9358
	2	4.5271	7.4158	7.5377	7.4486	7.0975	6.7211
	2.5	5.9951	18.3288	18.0734	18.4501	18.1850	17.5152
	3	6.6248	35.3808	37.5423	37.7987	38.2654	37.7084
2	0.5	0.0612	0.0709	0.0657	0.0617	0.0590	0.0571
	1	0.4662	0.9786	1.1455	1.1349	1.0926	1.0512
	1.5	1.0564	3.3125	4.9543	5.6862	5.8242	5.7456
	2	1.3369	7.4593	11.9326	15.6581	17.6348	18.3282
	2.5	1.4021	12.5799	22.7055	31.2223	38.2278	42.3995
	3	1.3871	16.9017	37.7625	53.0000	67.5222	79.2691
3	0.5	0.1318	0.2697	0.2848	0.2798	0.2730	0.2670
	1	0.4966	2.1087	3.6540	4.3159	4.5235	4.5574
	1.5	0.7823	4.7057	10.6753	16.4434	20.0144	21.8491
	2	0.8732	7.9452	18.9774	33.7393	48.0182	58.4636
	2.5	0.8850	10.7268	29.1432	52.9549	82.3714	110.9024
	3	0.8726	12.5162	40.2223	75.2908	119.4368	170.8054

Table 5: *Index of dispersion of the model*

θ	β	λ					
		0.5	1	1.5	2	2.5	3
1	0.5	2.7619	3.2051	3.4269	3.5533	3.6344	3.6907
	1	1.3333	1.3809	1.5111	1.6026	1.6667	1.7135
	1.5	1.1111	0.8667	0.9206	0.9815	1.0303	1.0684
	2	1.1333	0.6667	0.6566	0.6905	0.7255	0.7556
	2.5	1.2224	0.5843	0.5198	0.5299	0.5524	0.5751
	3	1.3333	0.5556	0.4444	0.4333	0.4444	0.4603
2	0.5	3.0667	3.8333	4.3434	4.6667	4.8863	5.0444
	1	1.6667	1.5333	1.7350	1.9167	2.0596	2.1717
	1.5	1.5556	1.0000	1.0222	1.1111	1.2000	1.2778
	2	1.6212	0.8333	0.7418	0.7667	0.8159	0.8675
	2.5	1.7128	0.7833	0.6152	0.5939	0.6133	0.6444
	3	1.8000	0.7778	0.5556	0.5000	0.4963	0.5111
3	0.5	2.8727	3.5195	4.1414	4.5909	4.9203	5.1697
	1	1.8182	1.4364	1.5758	1.7597	1.9273	2.0707
	1.5	1.7576	1.0233	0.9576	1.0116	1.0915	1.1732
	2	1.8052	0.9091	0.7374	0.7182	0.7455	0.7879
	2.5	1.8631	0.8794	0.6459	0.5814	0.5745	0.5906
	3	1.9151	0.8788	0.6061	0.5117	0.4814	0.4788

The r^{th} Incomplete moment for Exp-Gamma distribution has been obtained as

$$\begin{aligned}\phi_r(x) &= \int_0^t x^r f(x)dx \\ &= \int_0^r x^r \frac{1}{\theta^2+2}[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}]dx \\ &= \frac{1}{\theta^2+2}\left[\frac{\theta^2\gamma(r+1,\lambda t)}{\lambda^r} + \frac{\gamma(r+3,\beta t)}{\beta^r}\right]\end{aligned} \tag{9}$$

when $r = 1$ the first incomplete moment of the Exp-Gamma distribution is

$$\phi_1(x) = \frac{1}{\theta^2+2}\left[\frac{\beta\theta^2\gamma(2,\lambda t) + \lambda\gamma(4,\beta t)}{\lambda\beta}\right]$$

The related Exp-Gamma distribution moment-generating function is

$$\begin{aligned}M_X(t) = E(e^{tX}) &= \int_0^\infty e^{tX} f(x)dx \\ &= \sum_{i=0}^{\infty}\frac{t^i}{i!}\left(\frac{1}{\theta^2+2}\left[\frac{\theta^2\Gamma(i+1)}{\lambda^i} + \frac{\Gamma(i+3)}{\beta^i}\right]\right)\end{aligned} \tag{10}$$

The corresponding characteristic function of the Exp-Gamma distribution is

$$\begin{aligned}\phi_X(t) = E(e^{itX}) &= \int_0^\infty e^{itX} f(x)dx \\ &= \sum_{i=0}^{\infty}\frac{it^k}{k!}\left(\frac{1}{\theta^2+2}\left[\frac{\theta^2\Gamma(k+1)}{\lambda^k} + \frac{\Gamma(k+3)}{\beta^k}\right]\right)\end{aligned} \tag{11}$$

The Exp-Gamma distribution's associated cumulant-generating function is

$$\begin{aligned}K_X(t) &= \log_e M_X(t) \\ &= \prod_{i=0}^{\infty}\log_e\left(\frac{t^i}{i!}\left(\frac{1}{\theta^2+2}\left[\frac{\theta^2\Gamma(i+1)}{\lambda^i} + \frac{\Gamma(i+3)}{\beta^i}\right]\right)\right)\end{aligned} \tag{12}$$

Probability-weighted moments are derived using a different method for statistical distributions whose inverse form is difficult to define. The corresponding probability-weighted moment for the Exp-Gamma distribution can be found using the formula below.

$$\begin{aligned}\pi_{r,s} &= E(X^r F(x)^s) \\ &= \int_0^\infty x^r f(x)[F(x)]^s dx \\ &= \frac{1}{(\theta^2+2)^{s+1}}\int_0^\infty x^r[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}][\theta^2 - \theta^2 e^{-\lambda x} + 2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]^s dx\end{aligned} \tag{13}$$

The corresponding n^{th} conditional moment of the Exp-Gamma distribution is defined as

$$E[X^n/X > x] = \frac{1}{S(x)}\int_x^\infty x^n f(x)dx$$

$$E[X^n/X>x] = \frac{\theta^2+2\int_x^\infty x^n\left(\frac{1}{\theta^2+2}[\theta^2\lambda e^{-\lambda x}+\beta^3x^2e^{-\beta x}]\right)dx}{(\theta^2+2)-\theta^2(1-e^{-\lambda x})+[2-e^{-\beta x}(x^2\beta^2+2x\beta+2)]}$$
$$= \frac{-\Gamma(n+1,\lambda x)\theta^2\beta^n-\Gamma(n+3,\beta x)\lambda^n}{\lambda^n\beta^n\left((\theta^2+2)-\theta^2(1-e^{-\lambda x})+[2-e^{-\beta x}(x^2\beta^2+2x\beta+2)]\right)} \quad (14)$$

4. Reliability Analysis

4.1. Survival Function

The odds that an item won't fail before x is specified is the survival function $S(x)$.

$$S(x) = P(X>x) = 1-F(x)$$
$$= 1-\frac{\theta^2(1-e^{-\lambda x})+[2-e^{-\beta x}(x^2\beta^2+2x\beta+2)]}{\theta^2+2} \quad (15)$$
$$= \frac{(\theta^2+2)-\theta^2(1-e^{-\lambda x})+[2-e^{-\beta x}(x^2\beta^2+2x\beta+2)]}{\theta^2+2}$$

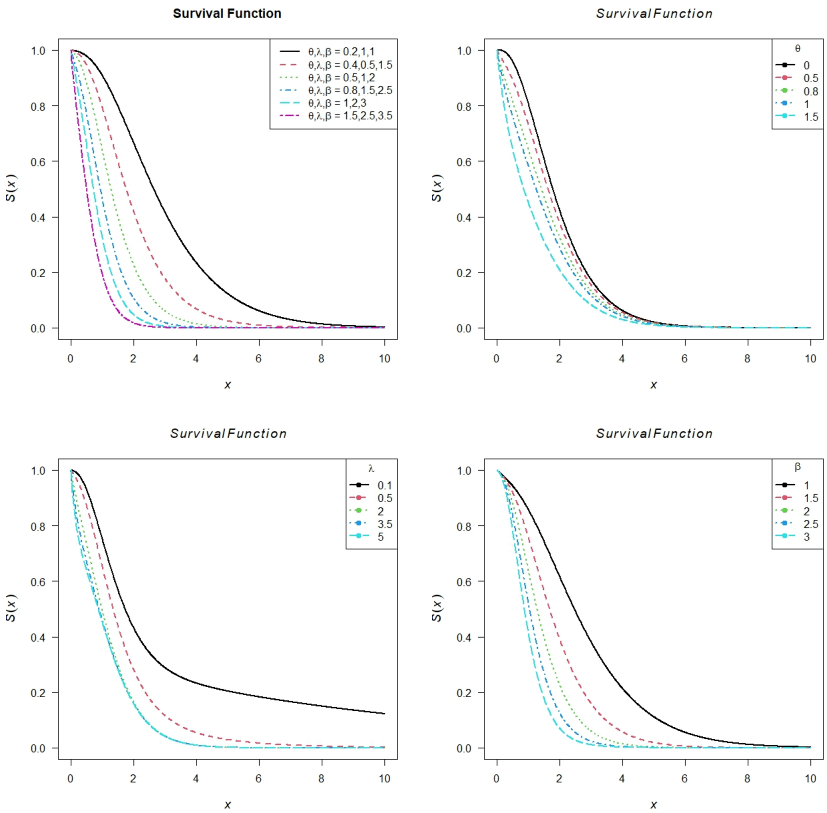

Figure 3: *The different shapes of the sf of an Exp-Gamma distribution for different parameter values.*

4.2. Hazard Rate Function

Assume that X is a continuous random variable with pdf $f(x)$ and cdf $F(x)$. The hazard function of X is

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{[\theta^2 \lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}]}{(\theta^2+2) - \theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]} \tag{16}$$

4.3. Mean Residual Life Function

Assume that X is a continuous random variable with pdf $f(x)$ and cdf $F(x)$. According to X, the mean residual life function is

$$m(x) = E[X - x/X > x] = \frac{1}{1-F(x)} \int_x^\infty [1-F(t)]dt = \frac{\beta\theta^2 e^{-\lambda x} + \lambda(x^2\beta^2 + 4x\beta + 6)e^{-\beta x}}{(\theta^2+2) - \theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]} \tag{17}$$

The Exp-Gamma distribution's hazard function can take three different shapes: decreasing HF, unimodal HF, increasing HF, and decreasing-increasing HF. A declining function is also a property of the mean residual life function.

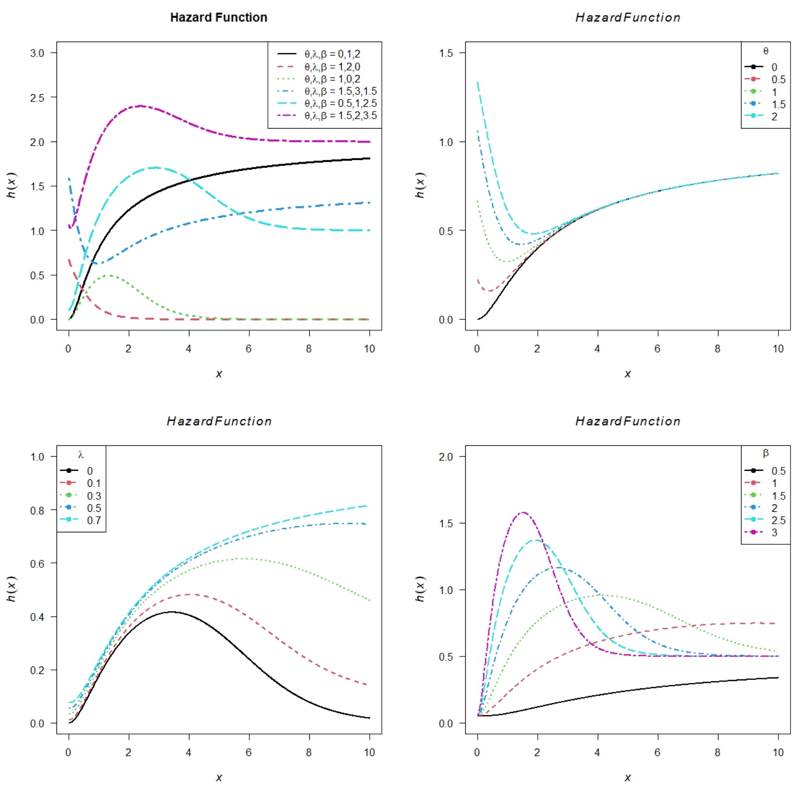

Figure 4: *Hazard function of the Exp-Gamma distribution for different parameter values. The shape of the hazard function changes as the parameter values are varied.*

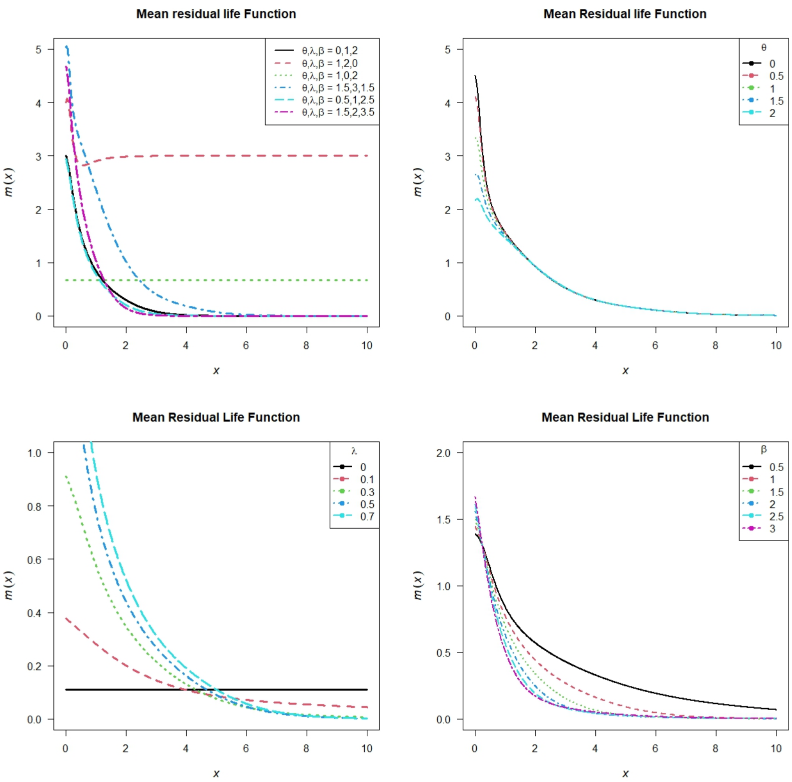

Figure 5: *The various forms of an Exp-Gamma distribution's mean residual life function for various parameter values.*

4.4. Mean Inactivity Time

The mean inactive time is the amount of time that has passed after an item's failure based on the premise that it failed in $(0, t)$.

$$\begin{aligned} \psi_x(t) &= E(X - t / X < t) \\ &= t - \frac{\phi_1(t)}{F(t)} \\ &= t - \frac{\beta\theta^2\gamma(2, \lambda t) + \lambda\gamma(4, \beta t)}{\lambda\beta\Big(\theta^2(1 - e^{-\lambda t}) + [2 - e^{-\beta t}(t^2\beta^2 + 2t\beta + 2)]\Big)} \end{aligned} \tag{18}$$

4.5. Cumulative Hazard

The cumulative hazard function is

$$\begin{aligned} H(x) &= -\log(1 - F(x)) \\ &= \log(\theta^2 + 2) - \log\Big((\theta^2 + 2) - \theta^2(1 - e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]\Big) \end{aligned} \tag{19}$$

4.6. Reversed Hazard Rate

The Reversed Hazard Rate is

$$\begin{aligned}\tau(x) &= \frac{f(x)}{F(x)} \\ &= \frac{[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}]}{\theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]}\end{aligned} \tag{20}$$

5. Log-odds Rate

The log-odds rate was used by Wang et al. (2003) to propose a model for time to failure as well as some definition of failure time distributions. By simulating the failure process in terms of the log odds rate, the model may be used to analyze the distribution of time until failure.

The odds function is given by

$$\begin{aligned}\pi_o(x) &= \frac{F(x)}{S(x)} \\ &= \frac{\theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]}{(\theta^2+2) - \theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]}\end{aligned} \tag{21}$$

The log-odds function is given by

$$\begin{aligned}LO(x) = \log\frac{F(x)}{1-F(x)} = (\log(\theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]) - \log((\theta^2+2) - \\ \theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]))\end{aligned} \tag{22}$$

The log-odds rate is defined as

$$\begin{aligned}LOR(x)(x) &= \frac{h(x)}{\bar{F}(x)} \\ &= \frac{[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}](\theta^2+2)}{(\theta^2+2) - \theta^2(1-e^{-\lambda x}) + [2 - e^{-\beta x}(x^2\beta^2 + 2x\beta + 2)]}\end{aligned} \tag{23}$$

6. Entropy

Entropy is a metric for describing the degree of uncertainty in a random variable (X) for the probability density function obtained from the lifetime distribution.

6.1. Renyi Entropy

Renyi entropy of a random variable $X \sim Exp - Gamma(\theta, \lambda, \beta)$ with pdf is defined as

$$\begin{aligned}I_R(\eta) &= \frac{1}{1-\eta}\log\int_0^\infty f^\eta(x)dx; \eta > 0, \eta \neq 1 \\ &= \frac{1}{1-\eta}\log\int_0^\infty \left(\frac{1}{\theta^2+2}[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}]\right)^\eta dx \\ &= \frac{1}{1-\eta}\log\left(\frac{1}{(\theta^2+2)^\eta}\int_0^\infty [\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}]^\eta\right)dx\end{aligned} \tag{24}$$

6.2. Shannon Entropy

The Shannon Entropy of $X \sim Exp - Gamma(\theta, \lambda, \beta)$ is given by

$$\begin{aligned}
E[-\log f(X)] &= -\int_X f(x)\log f(x)dx \\
&= E\left[-log\left(\frac{1}{\theta^2+2}[\theta^2\lambda e^{-\lambda x}+\beta^3x^2e^{-\beta x}]\right)\right] \\
&= \log(\theta^2+2) - E\left[\log[\theta^2\lambda e^{-\lambda x}+\beta^3x^2e^{-\beta x}]\right] \\
&= -\frac{1}{\theta^2+2}\int_X\left[\theta^2\lambda e^{-\lambda x}+\beta^3x^2e^{-\beta x}\right]\log\left(\frac{1}{\theta^2+2}[\theta^2\lambda e^{-\lambda x}+\beta^3x^2e^{-\beta x}]\right)dx
\end{aligned} \tag{25}$$

6.3. Generalized Entropy

The Generalized Entropy of $X \sim Exp - Gamma(\theta, \lambda, \beta)$ is given by

$$\begin{aligned}
GE(w,\delta) &= \frac{1}{\delta(\delta-1)\mu^\delta}\left[\int_0^\infty x^\delta f(x)dx\right]-1 \\
&= \frac{1}{\delta(\delta-1)\left(\frac{\beta\theta^2+6\lambda}{\beta\lambda(\theta^2+2)}\right)^\delta}\left[\int_0^\infty x^\delta\left(\frac{1}{\theta^2+2}[\theta^2\lambda e^{-\lambda x}+\beta^3x^2e^{-\beta x}]\right)dx\right]-1 \\
&= \frac{\left(\beta^\delta\theta^2\Gamma(\delta+1)+\lambda^\delta\Gamma(\delta+3)\right)\left(\beta\lambda(\theta^2+2)\right)^\delta}{(\theta^2+2)\lambda^\delta\beta^\delta(\delta(\delta-1)(\beta\theta^2+6\lambda)^\delta)}-1
\end{aligned} \tag{26}$$

7. Stochastic ordering

Stochastic ordering can be used to assess the relative performance of positive continuous random variables. The size of random variable X is less than that of random variable Y.

- Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(y)$ for all x.
- Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(y)$ for all x.
- Mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(y)$ for all x.
- Likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(y)}$ decreases in x.

The stochastic ordering of distributions was created by Shaked and Shanthi Kumar (1994) using the results.

The Exp-Gamma distribution is sorted according to the strongest 'likelihood ratio'. Let $X \sim Exp - Gamma(\theta_1, \lambda_1, \beta_1)$ and $Y \sim Exp - Gamma(\theta_2, \lambda_2, \beta_2)$. If, $\beta_1 \geq \beta_2$, then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y, X \leq_{mlr} Y$ and $X \leq_{st} Y$.we have

$$\begin{aligned}
\frac{f_X(x)}{f_Y(x)} &= \frac{(\theta_2^2+2)[\theta_1^2\lambda_1e^{-\lambda_1x}+\beta_1^3x^2e^{-\beta_1x}]}{(\theta_1^2+2)[\theta_2^2\lambda_2e^{-\lambda_2x}+\beta_2^3x^2e^{-\beta_2x}]} \\
\log\frac{f_X(x)}{f_Y(x)} &= \log\left[\frac{(\theta_2{}^2+2)\left[\theta_1{}^2\lambda_1e^{-\lambda_1x}+\beta_1{}^3x^2e^{-\beta_1x}\right]}{(\theta_1{}^2+2)\left[\theta_2{}^2\lambda_2e^{-\lambda_2x}+\beta_2{}^3x^2e^{-\beta_2x}\right]}\right] \\
&= \log\left(\theta_2{}^2+2\right)+\log\left[\theta_1{}^2\lambda_1e^{-\lambda_1x}+\beta_1{}^3x^2e^{-\beta_1x}\right]-\log\left(\theta_1^2+2\right)-\log\left[\theta_2{}^2\lambda_2e^{-\lambda_2x}+\beta_2{}^3x^2e^{-\beta_2x}\right]
\end{aligned}$$

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{{\theta_2}^2{\lambda_2}^2 e^{-\lambda_2 x} - {\beta_2}^3\left(2xe^{-\beta_2 x} - x^2\beta e^{-\beta_2 x}\right)}{\left[{\theta_2}^2\lambda_2 e^{-\lambda_2 x} + {\beta_2}^3 x^2 e^{-\beta_2 x}\right]} - \frac{{\theta_1}^2{\lambda_1}^2 e^{-\lambda_1 x} + \beta_1^3\left(2xe^{-\beta_1 x} - x^2\beta e^{-\beta_1 x}\right)}{\left[{\theta_1}^2\lambda_1 e^{-\lambda_1 x} + {\beta_1}^3 x^2 e^{-\beta_1 x}\right]} \tag{27}$$

Now if $\theta_1 = \theta_2 = \theta, \lambda_1 = \lambda_2 = \lambda, \beta_1 \geq \beta_2$, then it implies $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y, X \leq_{mlr} Y$ and $X \leq_{st} Y$.

8. Order Statistics

If $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ denotes the order statistic of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then the pdf $X_{(r)}$ is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left[F_X(x)\right]^{(r-1)}\left[1 - F_X(x)\right]^{(n-r)}$$

For, $r = 1, 2, \ldots n$. The pdf of the r^{th} order statistic for the Exp-Gamma distribution is calculated, and the pdf of the largest order statistic $X_{(n)}$ and smallest order statistic $X_{(1)}$ are given below.

n^{th} order statistics

$$\begin{aligned} f_{X_{(n)}}(x) &= nf_X(x)\left[F_X(x)\right]^{(n-1)} \\ &= \frac{n}{\theta^2+2}\left[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}\right]\left[\frac{\theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]}{\theta^2+2}\right]^{(n-1)} \end{aligned} \tag{28}$$

1^{st} order statistics

$$\begin{aligned} f_{X_{(1)}}(x) &= nf_X(x)\left[1 - F_X(x)\right]^{(n-1)} \\ &= \frac{n}{\theta^2+2}\left[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}\right]\left[\frac{(\theta^2+2) - \theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]}{\theta^2+2}\right]^{(n-1)} \end{aligned} \tag{29}$$

The pdf of a median of order statistic is given as

$$\begin{aligned} f_{m+1:n}(x) &= \frac{(2m+1)}{m!m!} f_X(x)\left[F_X(x)\right]^m\left[1 - F_X(x)\right]^m \\ &= \frac{(2m+1)}{m!m!}\left(\frac{1}{\theta^2+2}\left[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}\right]\right)\left[\frac{\theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]}{\theta^2+2}\right]^m \\ &\left[\frac{(\theta^2+2) - \theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]}{\theta^2+2}\right]^m \end{aligned} \tag{30}$$

9. Lorenz and Bonferroni Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) are used in a variety of sectors, including economics, demography, insurance, and medicine. An Exp-Gamma distribution's Bonferroni and Lorenz curves are calculated as follows:

$$B_o(x) = \frac{1}{\mu F(x)} \int_0^t x f(x) dx = \frac{L_0(x)}{F(x)}$$
$$= \frac{\beta\theta^2\gamma(2,\lambda t) + \lambda\gamma(4,\beta t)}{\lambda\beta\mu\left(\theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]\right)}$$
$$L_o(x) = \frac{1}{\mu}\int_0^t x f(x) dx = \frac{\phi_1(x)}{E(X)}$$
$$= \frac{\left[\beta\theta^2\gamma(2,\lambda t) + \lambda\gamma(4,\beta t)\right]}{\lambda\beta\mu\left(\theta^2 + 2\right)}$$

10. Zenga index

The Gini index is commonly used to account for the extent of income inequality in a population. The Zenga index (Zenga, 2007) is a relatively new metric and a novel alternative to the Gini index and other current inequality measurements and curves, and the Zenga index is denoted by z.

$$z = 1 - \frac{\mu^-_{(x)}}{\mu^+_{(x)}}$$

where,

$$\mu^-_{(x)} = \frac{1}{F(x)}\int_0^x x f(x) dx = \left[\frac{\beta\theta^2\gamma(2,\lambda x) + \lambda\gamma(4,\beta x)}{\lambda\beta\left(\theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]\right)}\right]$$
$$\mu^+_{(x)} = \frac{1}{1-F(x)}\int_0^\infty x f(x) dx = \frac{\beta\theta^2 + 6\lambda}{\beta\lambda\left(\left(\theta^2+2\right) - \theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]\right)}\Bigg]$$
$$z = 1 - \left[\frac{\beta\theta^2\gamma(2,\lambda x) + \lambda\gamma(4,\beta x)\left(\beta\lambda\left(\left(\theta^2+2\right) - \theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]\right)\right)}{\left(\beta\theta^2 + 6\lambda\right)\lambda\beta\left(\theta^2\left(1-e^{-\lambda x}\right) + \left[2 - e^{-\beta x}\left(x^2\beta^2 + 2x\beta + 2\right)\right]\right)}\right]$$

11. Estimation of Parameters

In this section, the MLE approach is used to estimate the parameters θ, λ, and β. Consider a sample drawn at random from the Exp-Gamma distribution. Then the log-likelihood function is provided by

$$g(x) = \frac{1}{\theta^2+2}\left[\theta^2\lambda e^{-\lambda x} + \beta^3 x^2 e^{-\beta x}\right]$$
$$L\left(x_i, \theta, \lambda, \beta\right) = \prod_{i=1}^{n} g\left(x_i, \theta, \lambda, \beta\right)$$
$$L\left(x_i, \theta, \lambda, \beta\right) = \prod_{i=1}^{n}\left(\frac{1}{\theta^2+2}\left[\lambda\theta^2 e^{-\lambda x_i} + \beta^3 x_i^2 e^{-\beta x_i}\right]\right)$$
$$= \left(\frac{n}{\theta^2+2}\prod_{i=1}^{n}\left[\lambda\theta^2 e^{-\lambda x_i} + \beta^3 x_i^2 e^{-\beta x_i}\right]\right)$$

The respective sample log-likelihood function is

$$\log L(x_i, \theta, \lambda, \beta) = \log n - \log(\theta^2 + 2) + \sum_{i=1}^{n}\log[\lambda\theta^2 e^{-\lambda x_i} + \beta^3 x_i^2 e^{-\beta x_i}]$$

Now that we have differentiating w.r.t. θ, λ, and β, we can write

$$\frac{\partial \log L}{\partial \theta} = \frac{-2\theta}{(\theta^2+2)} \sum_{i=1}^{n} \frac{2\theta\lambda e^{-\lambda x_i}}{\left[\theta^2\lambda e^{-\lambda x_i} + \beta^3 x_i^2 e^{-\beta x_i}\right]} = 0$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\theta^2\left(e^{-\lambda x_i} - \lambda x_i e^{-\lambda x_i}\right)}{\left[\theta^2\lambda e^{-\lambda x_i} + \beta^3 x_i^2 e^{-\beta x_i}\right]} = 0$$

and

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{x_i^2\left(3\beta^2 e^{-\beta x_i} - \beta^3 x_i e^{-\beta x_i}\right)}{\left[\theta^2\lambda e^{-\lambda x_i} + \beta^3 x_i^2 e^{-\beta x_i}\right]} = 0$$

The MLEs are obtained by solving this system of nonlinear equations. The sample likelihood function can be quantitatively improved by using nonlinear optimization techniques, which are frequently more practical. R programming can be used to solve these equations numerically.

12. Application

Biomedical science lifespan data sets have been fitted with Exp-Gamma distribution. This section compares the goodness of fit of the Exp-Gamma model to the one-parameter Akash [22], Lindley [17], Exponential, two-parameter Lindley [26], Cubic transmuted Rayleigh, and Exponential-Gamma [18] distributions on a real-life data set. A density comparison diagram is also included in this section.

The data, according to Gross and Clark (1975, P.105), represents the lifetime data on the minutes of pain alleviation experienced by 20 people who received an analgesic. The details are as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

For a real lifetime dataset, the $-2lnL, AIC, AICC, BIC, K-S, CVM, andAD$ statistics have been calculated and shown in Table 7 to compare the goodness of fit of the Exp-Gamma, Akash, Lindley, Exponential, Cubic transmuted Rayleigh, Two parameter Lindley, Exponential-Gamma distributions.

Table 6: *Estimated parameter values of the distributions for the dataset*

Model	Parameter Estimate	Log-Lik
Exp-Gamma	$\hat{\theta} = 5.3520e^{-05}$, $\hat{\lambda} = 0.2914$	
	$\hat{\beta} = 1.5789$	-22.8873
Akash	$\hat{\theta} = 1.1569$	-29.7613
Lindley	$\hat{\theta} = 0.8161$	-30.2496
Exponential	$\hat{\lambda} = 0.5263$	-32.8371
Cubic transmuted Rayleigh	$\hat{\sigma} = 2.63597$	
	$\hat{\lambda} = 2.5971$	-24.9371
Two parameter Lindley	$\hat{\theta} = 1.48$	-25.8862
	$\hat{\alpha} = -0.2914$	
Exponential-Gamma	$\hat{\lambda} = 0.7361$	-62.2516
	$\hat{\alpha} = 1.7971$	

The variance-covariance matrix of the MLEs is computed as

$$I(\hat{\theta})^{-1} = \begin{pmatrix} 1.1362e^{-01} & -9.5914e^{-06} & 8.3587e^{-07} \\ -9.5914e^{-06} & 8.0969e^{-10} & -8.1641e^{-11} \\ 8.3587e^{-07} & -8.1641e^{-11} & 4.1550e^{-02} \end{pmatrix}$$

The variances of the MLEs of the parameters of Exp-Gamma θ, λ and β are $\text{var}(\bar{\theta}) = 0.1136$, $\text{var}(\hat{\lambda}) = 8.0969e^{-10}$ and $\text{var}(\bar{\beta}) = 0.0415$. And 95% confidence intervals of θ, λ and β are $[-6.60597, 6.60704]$, $[0.29136, 0.29147]$ and $[1.1794, 1.9785]$ respectively.

Table 7: *Criteria for comparison*

Model	**-2lnL**	**AIC**	**AICC**	**BIC**	**AD**	**K-S statistic**	**CVM**
Exp-Gamma	45.7745	51.7745	53.2747	54.7617	1.9324 (0.097)	0.2587 (0.1007)	0.3508 (0.1375)
Akash	59.5226	61.5226	61.7471	62.5206	3.3554 (0.0185)	0.3705 (0.0082)	0.6555 (0.0154)
Lindley	60.4991	62.4991	62.7213	63.4948	3.7504 (0.0118)	0.3911 (0.0044)	0.7550 (0.0086)
Exponential	65.6742	67.6742	67.8964	68.6699	4.6035 (0.0046)	0.4395 (0.0009)	0.9630 (0.0026)
Cubic transmuted Rayleigh	49.8742	53.8742	54.5801	55.8657	2.216 (0.0707)	0.26534 (0.1196)	0.3873 (0.0772)
Two parameter Lindley	51.7724	55.4375	54.7785	55.8564	3.7822 (0.0085)	0.4102 (0.0075)	0.5275 (0.0058)
Exponential -Gamma	124.503	128.5032	130.4946	129.2091	41.855 (0.0000)	1.000 (0.0000)	5.4779 (0.0000)

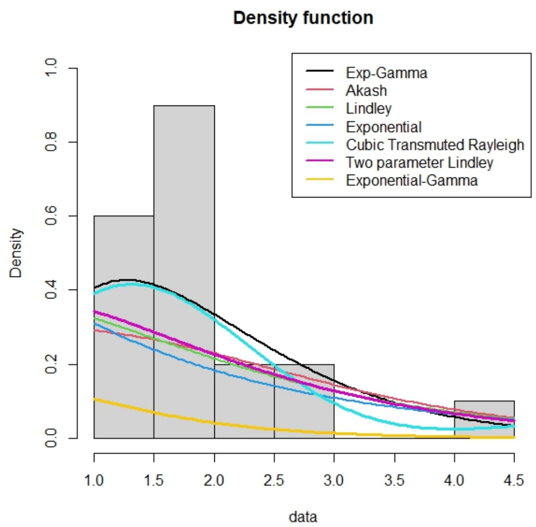

Figure 6: *Comparison of model fit for the distributions.*

The Exp-Gamma distribution fits the dataset better than the Akash, Lindley, exponential, two-parameter Lindley, Cubic transmuted Rayleigh, and Exponential-Gamma distributions as observed from Table 7.

13. Conclusion

A weighted three-parameter probability distribution is developed in this study for modelling skewed lifetime data. We derive expansions of important statistical measures like mean, variance, moments, and moment generating function, etc., as well as maximum likelihood estimation is used to estimate the Exp-Gamma distribution's parameters and hazard and reliability functions are used to examine the distribution's properties. The proposed distribution was fitted using real-time data.

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