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A NEW GENERALIZATION OF TWO PARAMETRIC DIVERGENCE MEASURE AND ITS APPLICATIONS

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Abstract

In this communication, we proposed two parametric generalized divergence measures. The well-known divergence measures available in the literature are a particular case of our new proposed divergence measure. We also looked into its monotonic behaviour and characterization results. We applied the proposed measure to some life-time distributions and observed that the deviation has been reduced. We have shown the mortality rate of two different countries based on COVID-19 data sets.

Keywords: characterization result, Kullback Leibler divergence measure, Havrda and Charvats divergence, monotonic behavior, probability distribution, Renyi's divergence.

1. Introduction

Information measures play an important role in the field of information theory and other applied sciences.[12] pioneered the concept of information measure (uncertainty).He proposed a way to achieve the uncertainty associated with the probability distribution and established that it is an important part of information theory, which today has many applications in various disciplines. Suppose X is a continuous non-negative random variable, then the [12] entropy is defined as

$$H_S(X) = -\int_0^\infty f(x) log f(x) dx \tag{1}$$

where f is defined as the probability density function of X.
Furthermore, it can be written as

$$H_s(X) = E(-log f(x))$$

The $H_S(X)$ is equal to the expected value of (-logf(x)).
The significance of adequate distance measures between probability distributions stems from their function and has extensive application in entropy.The most prominent divergence used in information theory is relative entropy, also known as [6] divergence (KL divergence).It is widely used in contigency tables, ANOVA tables, statistical inference, etc.

If f(x) and g(x) are the two probability distributions for a continuous random variable X and Y, respectively, then the [6] divergence is given by

$$D_{KL}(f||g) = \int_0^\infty f(x) log \frac{f(x)}{g(x)} dx \tag{2}$$

Furthermore, it can be written as

$$D_{KL}(f||g) = E_{KL}\left[log \frac{f(x)}{g(x)}\right]$$

Remarks
1. If g(x)=1, then it just becomes [12].
2. If g(x)= f(x), then [6] divergence is reduced to zero.
In this direction the generalization of [6] divergence of order α was proposed by [11] and is defined as

$$D_R(f||g) = \frac{1}{\alpha - 1} log \int_0^\infty f(x)^\alpha g(x)^{1-\alpha} dx, \quad \alpha \neq 1, \quad \alpha > 0 \tag{3}$$

Remarks
1. If $\alpha \to 1$, then it simply becomes [6] divergence.
Several researchers have developed various generalizations of [6] divergence in different ways, and in this direction, [5] proposed a new generaliztion of the [6] divergence measure of order ′ α′ defined as follows

$$D_{HC}(f||g) = \frac{1}{\alpha - 1}\left[\int_0^\infty f(x)^\alpha g(x)^{1-\alpha} dx - 1\right], \quad \alpha \neq 1, \quad \alpha > 0 \tag{4}$$

Remarks
1. If $\alpha \to 1$, then it just becomes [6]
Our aim is to develop the new two-parametric divergence to reduce the deviations. Applied these proposed measure to life-time distributions having different density functions. Also, we obtained some characterization results.Our proposed new two-parametric measure is defined as

$$D_{\alpha,\beta}(f||g) = \frac{1}{\beta(\alpha - \beta)}\left[\int_0^\infty f(x)^{\alpha-\beta+1} g(x)^{\beta-\alpha} dx - 1\right], \alpha \neq \beta, \quad \beta < \alpha + 1, \alpha, \beta > 0 \tag{5}$$

Remarks
1. When we take f(x) = g(x) then divergence became zero
2. if β = 1 (5) reduced to [5] of order α
3. if β = 1, $\alpha \to$ 1 it converge to simply [6] divergence
4. if g(x) = 1, (5) reduces to simply [12]

1.1. Comparasion between known measure and new proposed measure

Example 1.2. Assume X and Y are two non-negative random variables with probability density functions as $f(x) = 2x; \quad 0 < x < 1$ and $\quad g(x) = 2(1 - x); \quad 0 < x < 1$. The comparison was derived from (1.2.). The table (1) shows the comparison between the known measure and the proposed measure. From, table (1), we conclude that the divergence is reduced in the proposed divergence measure as compared to the known divergence measure. It means that when we introduce a parameter into a known measure, the distance reduces. We say that it is an alternate measure of the known divergence measure. Figure 1 demonstrates this.

Table 1: *Comparasion between known measure and new proposed measure.*

α	β	$D_{HC}(f\|\|g)$	$D_{\alpha,\beta}(f\|\|g)$
0.1	1	0.094	0.076
0.2	1	0.18	0.11
0.3	1	0.26	0.12
0.4	1	0.34	0.12
0.5	1	0.42	0.10
0.6	1	0.51	0.08
0.7	1	0.61	0.05
0.8	1	0.72	0.02
0.9	1	0.85	0.008

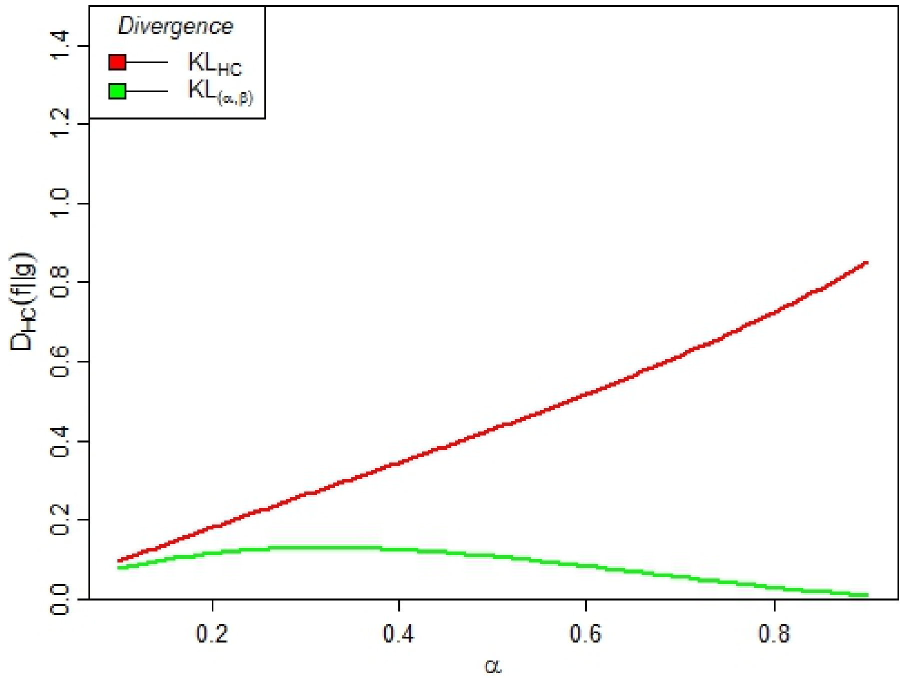

Figure 1: *Divergence between known measure and new proposed measure*

Theorem 1. Assume X and Y are two non-negative random variables with probability density functions f(x) and g(x), then

$$D_{\alpha,\beta}(f||g) \geq 0 \tag{6}$$

with equality if and only if f(x) = g(x).

Proof. By using Gibbs'inequality

$$\int_0^\infty f(x) log \frac{f(x)}{g(x)} dx \geq 0 \tag{7}$$

$$\int_0^\infty \left(f(x)^{\alpha-\beta+1} g(x)^{\beta-\alpha} - 1 \right) log \frac{\left(f(x)^{\alpha-\beta+1} g(x)^{\beta-\alpha} - 1 \right)}{g(x)} dx \geq 0 \tag{8}$$

$$-\int_0^\infty \left(f(x)^{\alpha-\beta+1} g(x)^{\beta-\alpha} - 1 \right) log \frac{g(x)}{\left(f(x)^{\alpha-\beta+1} g(x)^{\beta-\alpha} - 1 \right)} dx \geq 0 \tag{9}$$

then (-log) is a convex function and $\int_0^\infty$ g(x) = 1 we get

$$\int_0^\infty \left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right) log\frac{g(x)}{\left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx \geq -log\int_0^\infty g(x) \tag{10}$$

and

$$\int_0^\infty \left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right) log\frac{g(x)}{\left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx \geq -log(1) \tag{11}$$

After simplification, we obtained the result. ■

Definition 1.1 Let X be an integer of finite measure. If f(x) and g(x) are density functions and x is integrable, then log-sum inequality defined as

$$\int f(x)log\frac{f(x)}{g(x)}dx \geq [f(x)dx]\,log\left[\frac{f(x)dx}{g(x)dx}\right]$$

Theorem 2. Assume X and Y are two non-negative random variables with the probability density functions f(x) and g(x) respectively and $\beta < \alpha+1$, $\alpha,\beta > 0$ then

$$D_{\alpha,\beta}(f||g) \geq \frac{1}{log\beta}D_{KL}(f||g) \tag{12}$$

Proof. By using log-sum inequality we have

$$\int_0^\infty f(x)log\frac{f(x)}{\left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx \geq \int_0^\infty f(x)log\frac{f(x)}{\left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx \tag{13}$$

$$\int_0^\infty f(x)log\frac{f(x)}{\left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx \geq -log\int_0^\infty \left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}dx-1\right) \tag{14}$$

$$\int_0^\infty f(x)log\frac{f(x)}{\left(f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx = \int_0^\infty f(x)logf(x)dx - (\alpha-\beta+1)\int_0^\infty f(x)logf(x)dx - (\beta-\alpha)\int_0^\infty f(x)logg(x)dx + \int_0^\infty f(x)log(1) \tag{15}$$

After simplification, then

$$-log\beta(\alpha-\beta)D_{\alpha,\beta}(f||g) \geq [(\alpha-\beta)H(X)+(\beta-\alpha)I(X,Y)] \tag{16}$$

where, I(X,Y) is a [7]. Hence,we get the desired result. ■

We proposed the weighted generalized divergence measure (WGDM) in Section 2, We studied the montonic properties in Section 3, and in Section 4, we identified divergence for the different life-time distributions.We evaluate this article's conclusion in the final part.

2. Weighted generalized divergence measure(WGDM)

In this section,we propose the weighted generalized divergence measure.In real-life situations,[12] and [6] divergence give equal importance to the random variable, but in practical situations, this may cause problems.To overcome this problem, first [2] introduced a measure known as weighted entropy.The weighted entropy is defined as

$$H_S^w(X) = -\int_0^\infty xf(x)logf(x)dx \tag{17}$$

Remarks 1. If x= 1 then, it becomes simply [12].
The weight function is represented by a factor x that gives more weight to the larger value of the random variable.This measure is known as shift-dependent. Many researchers have proposed various weighted measures [13], [8] and [9] In the recent past, based on the concept of weighted entropy, [14] gave weight to the [6] divergence, defined as

$$D_{KL}^w(f||g) = \int_0^\infty xf(x)log\frac{f(x)}{g(x)}dx \tag{18}$$

Remarks
1. If x=1 then, it becomes simply [6].
Furthermore, it can be written as

$$D_{KL}^w(f||g) = E_{KL}\left[Xlog\frac{f(x)}{g(x)}\right]$$

Definition Similar to (2) and based on (5), the weighted proposed measure is defined as

$$D_{KL}^w(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\int_0^\infty x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}dx - 1\right], \alpha \neq \beta, \quad \beta < \alpha+1, \alpha, \beta > 0 \tag{19}$$

Remarks
1. If $x^\alpha = 1$ then, it became reduced to (5).
To show the importance of random variables in the new two-parametric generalized divergence measure, we consider the following example:

Example 2.1. Suppose X and Y are two non-negative continuous random variables with the density function as follows.
1. $f_1(x) = 1, \quad 0 < x < 1$ and $\quad g_1(x) = nx^{n-1} \quad 0 < x < 1$
2. $f_2(x) = 1, \quad 0 < x < 1$ and $\quad g_2(x) = n(1-x)^{n-1} \quad 0 < x < 1$
Then,the weighted generalized divergence measure characterized the distribution function uniquely.
Using (5) after simplification, we get

$$D_{1(\alpha,\beta)}(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\frac{n^{\beta-\alpha}}{(\beta-\alpha)(n-1)+1} - 1\right] = D_{2(\alpha,\beta)}(f||g) \tag{20}$$

Again using (8) after simplification, we get

$$D_{1(\alpha,\beta)}^w(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\frac{n^{\beta-\alpha}}{\alpha+(\beta-\alpha)(n-1)+1} - 1\right] \tag{21}$$

$$D^{w}_{2(\alpha,\beta)}(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\frac{n^{\beta-\alpha}\Gamma(\alpha+1)\Gamma(t-s)}{\Gamma(\alpha+t-s+1)} - 1\right] \tag{22}$$

Where, t = n(β-α)+1, s = (β-α), and B(u,v) = $\int_0^1 x^{u-1}(1-x)^{v-1}dx =$

$\int_0^1 \frac{x^{u-1}}{(1+x)^{u+v}} = \frac{\Gamma u \Gamma v}{\Gamma(u+v)}$

which is known as the "complete beta function."
We can see from the preceding example that, without weight, our proposed measure has the same value, but given weight, the value is different, so we conclude that the weighted measure uniquely determines the distribution.
From the table (2.1), we conclude that the proposed generalized divergence measure is equal but the weighted generalized divergence measure is different. It can be seen that when different values of alpha, beta, and n are used,
the $D_1(\alpha,\beta)(f||g) = D_2(\alpha,\beta)(f||g)$, but when the proposed divergence measure is weighted, the $Dw_1(\alpha,\beta)(f||g) < D^{w}_{2(\alpha,\beta)}(f||g)$.

Theorem 3. If X and Y are two non-negative continuous random variables with probability density functions f(x) and g(x), then the inequality is as follows

$$D^{w}_{\alpha,\beta}(f||g) \geq \frac{1}{log\beta} D_{KL}(f||g) + \alpha \int_0^\infty f(x)logx, \alpha \neq \beta, \beta < \alpha+1, \alpha,\beta > 0 \tag{23}$$

Proof. By using log-sum inequality, we have

$$\int_0^\infty f(x)log\frac{f(x)}{\left(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx \geq \int_0^\infty f(x)log \frac{f(x)}{\left(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx \tag{24}$$

$$\int_0^\infty f(x)log\frac{f(x)}{\left(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx = -log\int_0^\infty \left(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)dx \tag{25}$$

$$\int_0^\infty f(x)log\frac{f(x)}{\left(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx = -log\left[\beta(\alpha-\beta)D^{w}_{\alpha,\beta}(f||g)\right] \tag{26}$$

Now from L.H.S of (24), we have

$$\int_0^\infty f(x)log\frac{f(x)}{\left(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1\right)}dx = H(x)\left[(\alpha-\beta+1)-1\right] +(\beta-\alpha)I(X,Y) - \alpha\int_0^\infty f(x)logx \tag{27}$$

$$\int_0^\infty f(x)log\frac{f(x)}{(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1)}dx = \frac{1}{log\beta(\alpha-\beta)} \tag{28}$$
$$[H(x)(\alpha-\beta)+(\beta-\alpha)I(X,Y)]-\alpha\int_0^\infty f(x)logx$$

and

$$\int_0^\infty f(x)log\frac{f(x)}{(x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}-1)}dx = -\frac{1}{log\beta}\left[-H(x)-I(X,Y)\right] \tag{29}$$
$$+\alpha\int_0^\infty f(x)logx$$

Using (26) and (30), we obtained the result. ■

Theorem 4. Let X and Y be two random variables with weighted generalized divergence(WGD) $D^w_{\alpha,\beta}(f||g)$ and $\alpha{\neq}\beta$, α, $\beta>0$, then

$$D^w_{\alpha,\beta}(f||g) \leq \frac{1}{\beta(\alpha-\beta)}\int_0^\infty x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}dx - \left[\frac{1}{\beta(\alpha-\beta)}+1\right] \tag{30}$$

Proof. Since we know that for any pasitive number(for any $x>0$) then by using this inequality logx $\leq$ x-1 we get

$$D^w_{\alpha,\beta}(f||g) = log\frac{1}{\beta(\alpha-\beta)}\left[\int_0^\infty x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}dx-1\right] \tag{31}$$

$$D^w_{\alpha,\beta}(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\int_0^\infty x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}dx-1\right]-1 \tag{32}$$

$$D^w_{\alpha,\beta}(f||g) = \frac{1}{\beta(\alpha-\beta)}\int_0^\infty x^\alpha f(x)^{\alpha-\beta+1}g(x)^{\beta-\alpha}dx-\frac{1}{\beta(\alpha-\beta)}-1 \tag{33}$$

After simplification, we obtained the result. ■

3. Montonic properties

Definition 3.1 A function's monotonicity gives insight into how it will behave. If the graph of a function increases only as the equation's values increase, the function is said to be monotonically increasing. Similar to this, a function is said to be monotonically declining if its values exclusively decrease. In this section, we demonstrate the monotonic properties of the proposed divergenec measure. Consider the following numerical examples:
Example 2.2. Assume X and Y are two non-negative random variables with probability density functions as

1. $f_1(x)=2x;\quad 0<x<1$ and $\quad g_1(x)=2(1-x);\quad 0<x<1$
2. $f_2(x)=\frac{x}{2};\quad 0<x<2$ and $\quad g_1(x)=\frac{(2-x)}{2};\quad 0<x<2$

Using (5) after simplification, we get

$$D_{1(\alpha,\beta)}(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\Gamma(\alpha-\beta+2)\Gamma(\beta-\alpha+1)-1\right] \\ = D_{2(\alpha,\beta)}(f||g) \tag{34}$$

Again using (19) after simplification, we get

$$D^{w}_{1(\alpha,\beta)}(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\left(\frac{2\Gamma(2\alpha-\beta+2)}{\Gamma(\alpha+3)}\right)-1\right]$$

$$D^{w}_{2(\alpha,\beta)}(f||g) = \frac{1}{\beta(\alpha-\beta)}\left[\left(\frac{2^{(2\alpha+1)}\Gamma(2\alpha-\beta+2)}{\Gamma(\alpha+3)}\right)-1\right]$$

Where,
B(u,v) = $\int_0^1 x^{u-1}(1-x)^{v-1}dx = \int_0^1 \frac{x^{u-1}}{(1+x)^{u+v}} = \frac{\Gamma u\Gamma v}{\Gamma(u+v)}$ which is known as complete beta function.
Here, from the below graphs (a), (b), and (c), we obtain that for different values of α, β and n then the measure $D_{1(\alpha,\beta)}(f||g), D_{2(\alpha,\beta)}(f||g), D^{w}_{1(\alpha,\beta)}(f||g)$, and $D^{w}_{2(\alpha,\beta)}(f||g)$ respectively, indicate increased behavior.

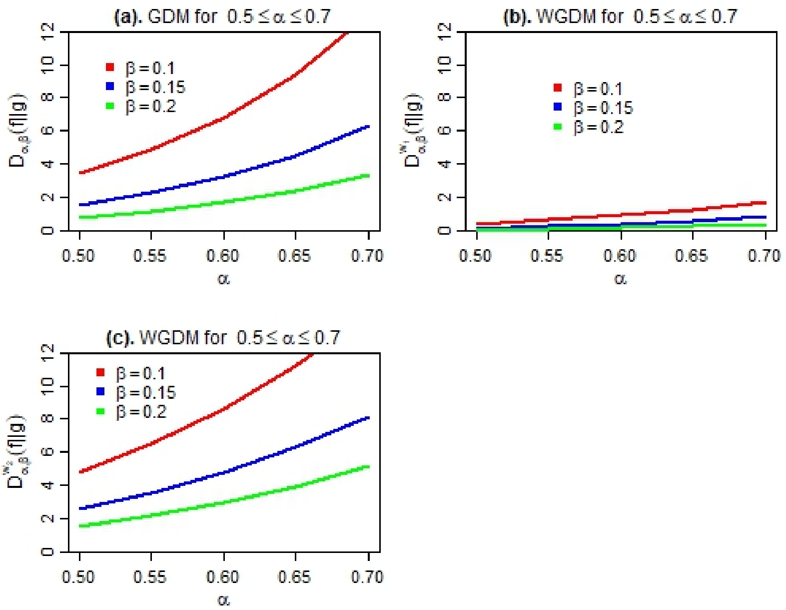

Figure 2: *Monotonic behavior of proposed weighted and non-weighted measure*

4. Divergence measures for some well-known life-time distribution

In this section, we obtained the divergence measures for some life-time distributions using the new proposed divergence measure.
Where,

U = $\frac{1}{\beta(\alpha-\beta)}$, p= $\beta-\alpha$, w= $\theta p+1$, and B(u,v) = $\int_0^1 x^{u-1}(1-x)^{v-1}dx = \int_0^1 \frac{x^{u-1}}{(1+x)^{u+v}} = \frac{\Gamma u\Gamma v}{\Gamma(u+v)}$ beta function.

Table 2: *Proposed divergence measure for some life-time distribution*

Distribution	f(x)	g(x)	x	Proposed Divergence measure
Uniform	$\frac{1}{m}$	$\frac{\theta(m-x)^{(\theta-1)}}{m^{\theta}}$	$0 \leq x \leq m$	$U\left[\frac{\theta^p}{p(\theta-1)+1}-1\right]$
Exponential	ne^{-nx}	$n\theta e^{-n\theta x}$	$x \geq n$, $\theta > 0$	$U\left[\frac{\theta^p}{(\alpha-\beta+w)}-1\right]$
Finite range	$r(1-x)^{r-1}$	$r\theta(1-x)^{(r\theta-1)}$	$0 < x < 1; r, \theta > 0$	$U\left[\frac{\theta^p}{(\alpha-\beta+w)}-1\right]$
Beta	sx^{s-1}	$s\theta x^{s\theta-1}$	$0 < x < 1, \theta, s > 0$	$U\left[\frac{\theta^p}{(\alpha-\beta+w)}-1\right]$
Power	$\frac{bx^{b-1}}{c^b}$	$\frac{b\theta x^{b\theta-1}}{c^{b\theta}}$	$0 < x < c; b, \theta > 0$	$U\left[\frac{\theta^p}{\alpha-\beta+w}-1\right]$

Table 3: *Proposed weighted divergence measure for some life-time distribution*

Distribution	f(x)	g(x)	x	Proposed Divergence measure
Uniform	$\frac{1}{m}$	$\frac{\theta(m-x)^{(\theta-1)}}{m^{\theta}}$	$0 \leq x \leq m$	$U\left[\frac{\theta^p m^{\alpha}\Gamma(\alpha+1)\Gamma(w-p)}{\Gamma(\alpha+w-p+1)}-1\right]$
Exponential	ne^{-nx}	$n\theta e^{-n\theta x}$	$x \geq n$,	$U\left[\frac{\theta^p\Gamma(\alpha+1)}{(\alpha-\beta+w+1)^{\alpha+1}}-1\right]$
Finite	$r(1-x)^{r-1}$	$r\theta(1-x)^{(r\theta-1)}$	$0 < x < 1$	$U\left[\frac{\theta^p r\Gamma(\alpha+1)\Gamma(rw+\alpha-\beta)}{\Gamma(2\alpha+rw-\beta+1)}-1\right]$
Beta	sx^{s-1}	$s\theta x^{s\theta-1}$	$0 < x < 1$	$U\left[\frac{\theta^p}{(\alpha-\beta+w)+\alpha}-1\right]$
Power	$\frac{bx^{b-1}}{c^b}$	$\frac{b\theta x^{b\theta-1}}{c^{b\theta}}$	$0 < x < c$	$U\left[\frac{\theta^p bc^{\alpha}}{\alpha-b\theta(\alpha-\beta)+b(\alpha-\beta+1)}-1\right]$

5. Application

Concerning the applicability of the newly proposed divergence measure, we analyzed two sets of actual data published by Almongy et al.[1] based on COVID-19. The first data set was taken over 108 days from Mexico country. Data was collected from March 4 to July 20, 2020.This data set represents the mortality rate. We consider only 30 observations from 108 observations using a random number table.

Dataset-1

1.041,2.988,5.242,7.903,6.327,7.840,7.267,6.370,2.926,5.985,7.854,3.233,7.151,4.292,2.326,3.298,5.459, 3.440,3.215,4.661,3.499,3.395,2.070,2.506,3.029,3.359,3.778,3.219,4.120,8.551.

The second data set was taken over 30 days from the Netherlands country. Data was collected from March 31 to April 30, 2020.This data set also shows mortality rates.

Dataset-2

1.273,6.027,10.656,12.274,1.974,4.960,5.555,7.584,3.883,4.462,4.235,5.307,7.968,13.211,3.611,3.647,6.940, 7.498,5.928,7.099,2.254,5.431,10.289,10.832,4.097,5.048,1.416,2.857,3.461,14.918. With the parameters θ_1 and θ_2, both sets of data can be fitted as an exponential distribution.Here we used MLE method for unknown parameter estimation.The estimated value of parameter $\hat{\theta}_1$ = 0.220 and $\hat{\theta}_2$=0.1624 with different standard error. The estimated value of weighted proposed divergence measure are
$\hat{D}^{w}_{1\alpha,\beta}(f||g)$ = 1.543
$\hat{D}^{w}_{2\alpha,\beta}(f||g)$ = 0.024. Our analysis demonstrates that Mexico has a higher mortality rate than the Netherlands.

6. Conclusions

In this communication, we proposed a new two parametric weighted generalized divergence measure of orderα and type β. The characterization result is justified by the numerical example that it uniquely determines the distribution functions, and we also studied the mononic behaviour of the proposed divergence measure.Finally, we derived some expressions for some life-time distributions and also showed the mortality rate of two different countries.

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