COST-REVENUE ANALYSIS AND ANFIS COMPUTING OF HETEROGENEOUS QUEUING MODEL WITH A SECOND OPTIONAL SERVICE WITH FEEDBACK UNDER HYBRID VACATION

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Abstract

This research article examines an $M/M/2$ heterogeneous queueing model that provides two services: a mandatory first essential service (FES) and an optional second optional service (SOS). The model incorporates breakdown, feedback, and a hybrid vacation policy. Matrix expressions are structured to evaluate the stationary probability distribution of the number of customers in the system and system performance measures using the matrix-geometric approach (MGA). Additionally, formulas are being developed to estimate the model's performance indicators. The cost function is being evaluated to determine the best values of the system's decision variables, and an adaptive neural fuzzy inference system (ANFIS) based on soft computing technology is being utilized to validate the obtained results.

Keywords: Markovian queue, Breakdown, Hybrid vacation, Matrix geometric approach, ANFIS.

1. Introduction

Queuing theory is the study of how people behave in service systems like telephone systems and waiting lines. It is a branch of operations research that focuses on analysing the arrival and departure of packets or customers from a service system. Queuing theory is used to analyse a wide range of scenarios, such as traffic control, banking, manufacturing, computer and telecommunications networks, production and transportation systems, and even healthcare systems. In general, queuing theory involves finding an appropriate mathematical model to describe the system and then analysing it to determine how the system performs and how it can be improved. Morse [17] was the first to consider the concept of heterogeneous servers in multi-server queueing models, which is more realistic than assuming all servers provide service at an equal rate. In queuing systems with human servers, this assumption is impossible to implement as different servers can provide services at varying rates. It's obvious that heterogeneous services are essential to the functioning of almost every industrial system. Li and Stanford [9] explored heterogeneous multi-server accumulating priority queues. Krishnamoorthy et al. [14] presented a two heterogeneous servers queueing model. Chang et al. [5] investigate an unreliable-server retrial queue with customer feedback and impatience. They analyse the performance of the system through simulations and derive analytical expressions for its measures of performance and cost. Recently, Wu and Yang [25] examined a two-phase heterogeneous service model. They initially performed a single objective optimization using Canonical PSO then developed a bi-objective cost optimization model for the system and waiting time in tandem.

In queueing theory, vacation refers to a period of time in which a server is unavailable to serve customers. There are several types of vacations, including planned and random, as well as hybrid vacations which combine the two. Planned vacations occur at predetermined intervals, while random vacations occur randomly throughout the queueing system's operation. Hybrid vacations combine both planned and random vacations in order to optimize the system's performance. Servi and Finn [19] work in queuing theory focused on working vacations (WV) for $M/M/1$ queues. In this approach, the server can continue servicing requests at a slower rate when customers are not present. Bouchentouf et al. [3] researched a multi-station unstable machine model with customer impatience and a working vacation schedule. Ziad et al. [29] examined a $M/M/c$ queueing system with waiting servers, balking, reneging, and a K-variant working vacation interrupted by a Bernoulli schedule. Bouchentouf et al. [4] examine the performance and economics of a single server queueing model with feedback, impatient customers, and a vacation policy, finding an optimal control policy for vacation times, and providing their results, which can be used to make decisions regarding the system's parameters. Anshul Kumar et al. [16] investigated the hybrid holiday policy and a two-stage service procedure using matrix geometric techniques. This hybrid holiday is a combo of working vacation and complete vacation (CV), in which the server may begin in WV and continue to give service at a decreased rate when the server is idle. If there are clients present in WV, the server will linger in WV and offer service. If not, it will go to CV. The server will revert to its usual operations after the CV has been completed, and it will begin providing service to any customers who are ready at that time. Dual servers with varying service rates are a logical outgrowth of this concept.

The term "essential service" refers to services that servers must offer in order to satisfy customer needs. Optional services, on the other hand, are those that are provided based on customer demand. Essential services are usually core services that must be provided in order for the server to fulfil its role, while optional services are additional services that customers may choose to opt-in to receive. Laxmi and Jyothsna [22] examined a finite buffer impatient customer queue with WV, where the server offers two phases of service: essential and optional. Yang et al. [26] used SOS to an M/M/R queueing model and offered economic analysis. Several queueing models for optional services have been studied by researchers such as Anitha et al. [2], Chandrika and Kalaiselvi [6], Li and Wang [10], Yang and Chen [27], and others.

Realistically, it's not possible to have a completely reliable server because it could break at any time. Repair setups such as thresholds, backups, or restarts must be applied to restore service. Significant research on breakdown provides a model to investigate how such changes may reduce customer wait times in real-time service systems with a total failure server, such as those found in banks, manufacturing plants, contact centres, and so on.

Ye and Liu [28] applied the MGA to discover the steady-state solutions for a Markovian arrival single server queuing system with an operational breakdown. Vijayalakshmi et al. [23] used a matrix technique to examine the restricted capacity of a Markovian queueing model with working interruptions and two-phase service. The $M/M/1$ model with working breakdowns and recovery policies based on k-threshold recovery time and setup recovery was studied by Ezeagu et al. [8]. A queueing model for a service system with a secondary server was presented by Chakravarthy et al.

The Matrix Geometric Approch was first developed in the 1960s and has since become a fundamental tool in queueing theory. It was widely used in Markov chain models and queuing systems to solve difficult real-time problems. In particular, this method helps to determine the stationary behaviour of a queueing model by calculating the expected number of customers and expected times in the system. M.F. Neuts [18] was a key figure in the development of the Matrix Geometric Approach (MGA). Ke et al. [13] conducted research on an $M/M/R$ queueing system with SOS. Shekhar et al. [21] employed metaheuristics to find effective emergency vacation queueing techniques. Anshul and Madhu Jain [16] conducted an investigation into an unreliable server, they studied the effects of an MGA-based Markovian queueing model for a two-stage service system that utilises a hybrid vacation policy.

ANFIS, or Adaptive Neuro-Fuzzy Inference System, is an intelligent system that combines neural networks and fuzzy logic to model complex non-linear systems. It can be used for many applications, such as classification, prediction, control, and optimization. ANFIS can also be used for transient analysis, where it can generate more accurate insights about system dynamics. Jang [11] proposed the ANFIS as a tool to model complex systems. He demonstrated the application of ANFIS in modelling the nonlinear dynamics of a continuous-time system and provided references on the uses of ANFIS in various research fields. The article was an important milestone in furthering the application of ANFIS in the sciences. The content of Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence was analysed by Jang et al. [12]. ANFIS has been used in many research papers, including Ahuja et al. study of the transient of an unreliable single-server queueing model with multi-stage service and a working vacation. Sethi et al. [20] proposed a mathematical model to analyse the system and demonstrated that the parameters of the system can be optimised using an ANFIS approach. They also discussed the performance of different policies in terms of cost and found that an N-policy was the most efficient for cost optimization. An analysis of an $M^X/G/1$ retrial queue with impatient customers, an unreliable server, a modified vacation policy, delayed repair, and a Bernoulli feedback system was presented by Upadhyaya and Kushwaha [24]. They also used an ANFIS computing approach to compare their numerical results to those obtained from explicit analytical formulas.
The structure of this paper is outlined in the following way: Section 2 outlines the model description and associated mathematical assumptions, as well as provides an explanation for the transition rate matrix. In Section 3, the stability condition is established and the matrix-geometric approach is explained. Performance measures such as the expected number of customers in the system in terms of system state probabilities and the total cost accumulated through different activities and cost elements are derived in Section 4. Section 5 provides numerical results, while the conclusions in Section 6 illustrate noteworthy features and potential areas for future research.

2. Model description

Under the hybrid vacation policy, we propose a heterogeneous Markovian queueing model with a second optional service with feedback and breakdown. This model is detailed as follows:
The arrival pattern: In this model, customers arrive according to a Poisson process with an arrival rate of λ.
The service pattern: The first-in-first-out policy directs the service discipline.We investigate a queueing model with two heterogeneous servers: the first server is constantly available and totally reliable, charges a service rate of μ_0. while the second server has two distinct phases - an first essential phase (FES) and a second optional phase (SOS) - and is only occasionally accessible and unstable. server 2 charges μ_1 for FES and μ_2 for SOS, with the service rates for both following an exponential distribution.
Breakdown and repair rule : The queueing system has two servers, with the first server consisting of a single, reliable phase, and the second server consisting of two phases: FES and SOS. If both phases of the second server experience breakdowns with rates η_1, η_2 and, it is immediately sent for repair with rates θ_1 and θ_2, respectively.
Feedback rule: After receiving a service, an unsatisfied customer can decide to re-enter the system for another service, with a probability $\bar{\kappa}$ referred to as "feedback", or they can choose to permanently exit the system with a probability κ (=1-$\bar{\kappa}$). Feedback service is consider as a new arival λ.

The hybrid vacation strategy: During vacation periods, the server starts WV, which has an exponential distribution with a slower service rate of μ_v. When the server gets empty during WV, it will switch to CV mode. While customers enter the system at CV, the server will return to its usual busy state and start serving customers. CV duration has an exponential distribution with a mean θ_v. The server enters the working period in the CV state with the probability ξ of providing service to the customer. which is always 1.

Let us consider $F(t) = \{(\mathfrak{W}(t), \mathfrak{J}(t)); t \geq 0\}$be the bivariate Markov process (BMP) with a

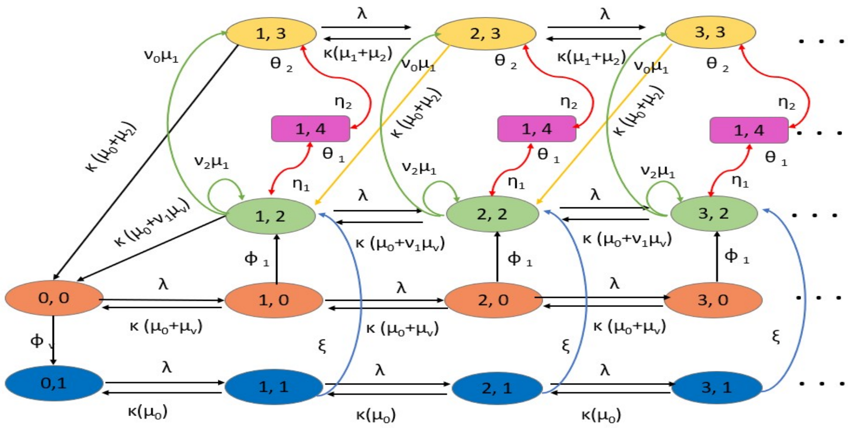


Figure 1: *State transition diagram of the model*

state space at time t. where $\mathfrak{W}(t)$ represents the number of customers in the system, and

$$\mathfrak{J}(t) = \begin{cases} 0, & \text{if server 2 is in } \textit{Working vacation} \\ 1, & \text{if server 2 is in } \textit{Complete vacation} \\ 2, & \text{if server 2 is in } \textit{FES} \\ 3, & \text{if server 2 is in } \textit{SOS} \\ 4, & \text{if server 2 is in } \textit{Breakdown} \end{cases}$$

All stochastic processes in the system are independent of one another. The structure of this model's transition diagram is depicted in the below figure. 1.

2.1. Governing equations

By using birth- death process, Governing equations can be formulated as follows:

$$(\lambda + \phi_v)\iota_{0,0} = (\mu_0 + \mu_v)\iota_{1,0} + \kappa\mu_0\iota_{0,1} + \kappa(\mu_0 + \nu_1\mu_2)\iota_{1,2} + \kappa(\mu_0 + \mu_2)\iota_{1,3}$$
$$[\kappa(\mu_0 + \mu_v) + \lambda + \phi_1]\iota_{1,0} = \lambda\iota_{0,0} + \kappa(\mu_0 + \mu_v)\iota_{2,0}$$
$$[\kappa(\mu_0 + \mu_v) + \lambda + \phi_1]\iota_{m,0} = \lambda\iota_{m-1,0} + \kappa(\mu_0 + \mu_v)\iota_{m+1,0}$$
$$(\kappa\mu_0 + \lambda)\iota_{0,1} = \phi_v\iota_{0,0} + \kappa\mu_0\iota_{1,1}$$
$$(\kappa\mu_0 + \lambda)\iota_{1,1} = \phi_v\iota_{0,1} + \kappa\mu_0\iota_{2,1}$$
$$(\kappa\mu_0 + \lambda)\iota_{m,1} = \phi_v\iota_{m-1,1} + \kappa\mu_0\iota_{m+1,1}$$
$$[\lambda + \kappa(\mu_0 + \nu_1\mu_1) + \nu_0\mu_1 + \eta_1]\iota_{1,2} = \xi\iota_{1,1} + \phi_1\iota_{0,1} + \nu_2\mu_1\iota_{1,2} + \theta_1\iota_{1,4} + \kappa(\mu_0 + \mu_1)\iota_{2,2} + \kappa(\mu_0 + \mu_2)\iota_{2,3}$$
$$[\lambda + \kappa(\mu_0 + \nu_1\mu_1) + \nu_0\mu_1 + \eta_1]\iota_{m,2} = \xi\iota_{m,1} + \phi_1\iota_{m-1,1} + \nu_2\mu_1\iota_{m,2} + \theta_1\iota_{m,4} + \kappa(\mu_0 + \mu_1)\iota_{m+1,2} + \kappa(\mu_0 + \mu_2)\iota_{m+1,3}$$
$$[\kappa(\mu_0 + \mu_2) + \lambda + \eta_2]\iota_{1,3} = \nu_0\mu_1\iota_{1,2} + \theta_2\iota_{1,4}$$
$$[\kappa(\mu_0 + \mu_2) + \lambda + \eta_2]\iota_{m,3} = \nu_0\mu_1\iota_{m,2} + \theta_2\iota_{m,4}$$

$$[\theta_1 + \theta_2]\iota_{1,4} = \eta_1\iota_{1,2} + \eta_2\iota_{1,3}$$
$$[\theta_1 + \theta_2]\iota_{m,4} = \eta_1\iota_{m,2} + \eta_2\iota_{m,3}$$

To make obtaining solutions to our model easier and faster, we are employing the MGA. This method is used to obtain steady-state probabilities when the state-space increases rapidly. This technique helps us to achieve effective and numerically stable solutions that would otherwise be difficult and time-consuming to obtain.

2.2. Matrix Geometric Solution

The system state is symbolized by $\mathfrak{W}(t)$ and $\mathfrak{J}(t)$. Let $\{(\mathfrak{W}(t), \mathfrak{J}(t)); t \geq 0\}$, with the state space organised in lexicographical manner as follows.

$$Y = (0,0) \bigcup (0,1) \bigcup (m,n); n \geq 1, m = 0,1,2,3,4$$

The set of equations in section 2.1 are utilised to create the model's steady-state probability using the matrix-geometric approach. The block tridiagonal pattern is represented by the associated infinitesimal generator matrix $\mathbb{G}$ of this Markov chain, which is expressed as follows:

$$\mathbb{G} = \begin{bmatrix} S_0 & T_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ V_0 & U_1 & U_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & U_2 & U_1 & U_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & U_2 & U_1 & U_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & U_2 & U_1 & U_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{bmatrix}$$

where

$$S_0 = \begin{bmatrix} -(\lambda+\phi_v) & \phi_v \\ \beta & -(\lambda+\beta) \end{bmatrix}; \quad T_0 = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \end{bmatrix}; \quad V_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \\ \gamma & 0 \\ \delta & 0 \\ 0 & 0 \end{bmatrix}$$

$$U_0 = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}; \quad U_2 = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $$U_1 = \begin{bmatrix} -[\lambda+\phi_1+\alpha] & 0 & \phi_1 & 0 & 0 \\ 0 & -[\lambda+\xi+\beta] & \xi & 0 & 0 \\ 0 & 0 & -[\lambda+\eta_1+\nu_1\mu_1+\gamma-\nu_2\mu_1] & \nu_0\mu_1 & \nu_1 \\ 0 & 0 & 0 & -[\lambda+\eta_2+\delta] & \eta_2 \\ 0 & 0 & \theta_1 & \theta_2 & -[\theta_1+\theta_2] \end{bmatrix}$$

Here, $\alpha = \kappa[\mu_0 + \mu_v]$, $\beta = \kappa\mu_0$, $\gamma = \kappa[\mu_0 + \nu_1\mu_1]$, $\delta = \kappa[\mu_0 + \mu_2]$.

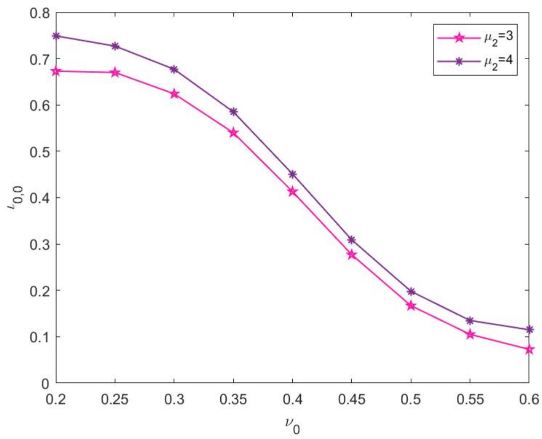


Figure 2: *Influence of ν_0 on the idle state $\iota_{0,0}$ of the servers*

3. Stability Condition

Theorem 1. The system is stable if and only if $\rho = \frac{\lambda}{\beta}\left(\frac{\mathbb{A}\theta_1+\mathbb{B}\theta_2}{\mathbb{A}\theta_1+\mathbb{B}\theta_2+\mathbb{C}}\right) < 1$

Proof. Let us define the matrix $U = U_0 + U_1 + U_2$ given by

$$U = \begin{bmatrix} -\phi_1 & 0 & \phi_1 & 0 & 0 \\ 0 & -\xi & \xi & 0 & 0 \\ 0 & 0 & -[\eta_1+(\nu_0-\nu_2)\mu_1] & \nu_0\mu_1 & \eta_1 \\ 0 & 0 & -\delta & \delta & 0 \\ 0 & 0 & \theta_1 & \theta_2 & -[\theta_1+\theta_2] \end{bmatrix} \tag{1}$$

There exists a stationary probability $\mathcal{I} = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4)$ of U such that

$$\mathcal{I}U = 0; \quad \mathcal{I}\mathfrak{e} = 1 \tag{2}$$

where $\mathfrak{e} = [1,1,1,1,1]^T$. Using theorem 3.1.1 of Netus [18], the necessary and sufficient condition for the stability of the system is as follows:

$$\mathcal{I}U_0\mathfrak{e} < \mathcal{I}U_2\mathfrak{e} \tag{3}$$

Solving 1 and 2, we get

$$\lambda[\mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4] < \alpha\mathcal{I}_0 + \beta\mathcal{I}_1 + \gamma\mathcal{I}_2 + \delta\mathcal{I}_3 \tag{4}$$

$$\frac{\lambda}{\sigma_1}[(\nu_2 + \nu_0\mu_1 + \delta))\theta_1 + (\delta + \eta_1 + \mu_1(\nu_0 + \nu_1 - \nu_2))\theta_2] < \frac{\beta}{\sigma_1}[(\kappa\mu_1\nu_1 + \delta + \nu_0\mu_1)\theta_1 + \kappa\mu_2 + (\delta + \nu_1 + \mu_1(\nu_0 + \nu_1 - \nu_2)\theta_2)] \tag{5}$$

$$\frac{\lambda}{\sigma_1}[\mathbb{A}\theta_1 + \mathbb{B}\theta_2] < \frac{\beta}{\sigma_1}[\mathbb{A}\theta_1 + \mathbb{B}\theta_2 + \mathbb{C}] \tag{6}$$

$$\frac{\lambda}{\beta}\left[\frac{\mathbb{A}\theta_1 + \mathbb{B}\theta_2}{\mathbb{A}\theta_1 + \mathbb{B}\theta_2 + \mathbb{C}}\right] < 1 \tag{7}$$

Here

$$\begin{aligned} \mathbb{A} &= \nu_2 + \nu_0\mu_1 + \delta \\ \mathbb{B} &= \delta + \eta_1 + \mu_1(\nu_0 + \nu_1 - \nu_2) \\ \mathbb{C} &= \kappa(\mu_1\nu_1 + \mu_2) \end{aligned}$$

Table 1: *Impact of arrival rate λ on perfomance measures*

λ	$E[L_s]$	$E[L_q]$	P_{Idle}	$P_{WV}^{S_2}$	$P_{CV}^{S_2}$	$P_{FES}^{S_2}$	$P_{SOS}^{S_2}$	$P_{Bd}^{S_2}$
0.5	0.1862	0.0271	0.8393	0.9450	0.0365	0.0121	0.0032	0.0016
0.6	0.2378	0.0447	0.8069	0.9409	0.0358	0.0160	0.0046	0.0023
0.7	0.2904	0.0647	0.7744	0.9359	0.0349	0.0198	0.0062	0.0029
0.8	0.3480	0.0891	0.7416	0.9297	0.0347	0.0240	0.0081	0.0035
1	0.4780	0.1495	0.6757	0.9149	0.0321	0.0339	0.0134	0.0052
1.1	0.5539	0.1800	0.6423	0.9053	0.0310	0.0396	0.0172	0.0063
1.2	0.6411	0.2488	0.6085	0.8938	0.0299	0.0460	0.0218	0.0077
1.3	0.7384	0.3073	0.5757	0.8936	0.0185	0.0517	0.0266	0.0089
1.4	0.8442	0.3851	0.5398	0.8648	0.0274	0.0614	0.0344	0.0109
1.5	0.9796	0.4725	0.5044	0.8453	0.0261	0.0706	0.0431	0.0130

Table 2: *Impact of service rates* $(\mu_0, \mu_1, \mu_2, \mu_v)$ *on some perfomance measures*

$(\mu_0, \mu_1, \mu_2, \mu_v)$	$E[L_s]$	$E[L_q]$	P_{Idle}	$P^{S_2}_{WV}$	$P^{S_2}_{CV}$	$P^{S_2}_{FES}$	$P^{S_2}_{SOS}$	$P^{S_2}_{Bd}$
(**3**,2,1.5,1)	1.2737	0.6971	0.4440	0.8191	0.0236	0.0783	0.0611	0.0163
(**4**,2,1.5,1)	0.8209	0.3774	0.5652	0.9063	0.0228	0.0424	0.0208	0.0071
(**5**,**2.5**,1.5,1)	0.5589	0.2008	0.6414	0.9409	0.0204	0.0247	0.0098	0.0037
(**6**,**3**,1.5,1)	0.4409	0.1351	0.6941	0.9580	0.0181	0.0161	0.0054	0.0023
(**7**,**3.5**,**2**,1)	0.2693	0.0713	0.7334	0.9007	0.0161	0.0103	0.0028	0.0015
(**7.5**,**4**,**2.5**,1)	0.3323	0.0822	0.7495	0.9716	0.0151	0.0093	0.0024	0.0012
(8,4,3,**1.5**)	0.2884	0.0641	0.7752	0.9746	0.0146	0.0077	0.0017	0.0009

where

$$\mathcal{I}_0 = 0, \quad \mathcal{I}_1 = 0, \quad \mathcal{I}_2 = \frac{[\nu_2 + \delta]\theta_1 + \delta\theta_2}{\sigma_1}, \quad \mathcal{I}_3 = \frac{[\eta_1 + \mu_1(\nu_0 + \nu_1 - \nu_2)\theta_2 + \mu_1\nu_0\theta_1]}{\sigma_1},$$

$$\mathcal{I}_4 = \frac{[\nu_2(\eta_1 - \mu_1\nu_2) + \mu_1\nu_2(\nu_0 + \nu_1)] + [\kappa\nu_1 + \kappa\mu_1(\nu_1 - \nu_2)](\mu_0 + \mu_2)}{\sigma_1} \tag{8}$$

Here,

$$\sigma_1 = \eta_1(\nu_1 + \theta_2) + \nu_2(\theta_1 - \mu_1\nu_2) + \eta_1\delta + \mu_1\nu_2(\nu_0 + \nu_1) + (\theta_1 + \theta_2)(\delta + \nu_0\mu_1) + \mu_1(\theta_2 + \delta)(\nu_1 - \nu_2)$$

Hence, System stability is ensured that ρ is equal to or less than 1. ■

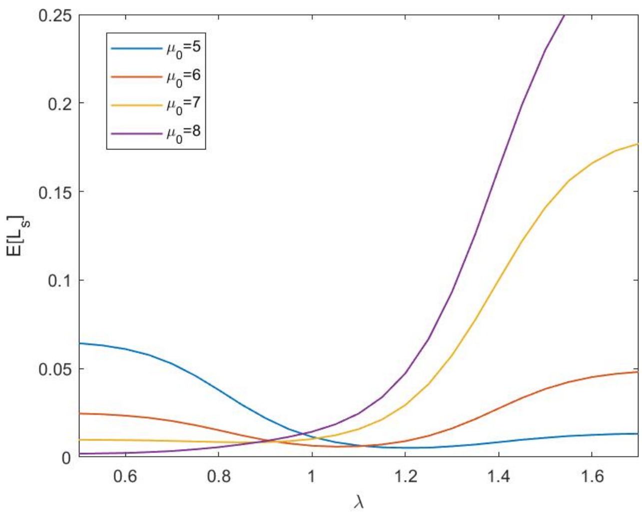


Figure 3: *Influence of* $E[L_s]$ *for different* μ_0 *w.r.t* λ

3.1. Stationary probability distribution

We define $\mathfrak{P}_{mn} = \{(\mathfrak{W}(t) = m, \mathfrak{J}(t) = n)\}$ where m indicates the total number of customers in the queue and n reflects the server state. Under the stability condition $\rho < 1$. The prob. vector is described as follows: $\mathcal{I} = [\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, ...]$, where $\mathcal{I}_0 = [i_{0,0}, i_{0,1}]$, $m = 0$; $\mathcal{I}_i = [i_{m,0}, i_{m,1}, i_{m,2}, i_{m,3}, i_{m,4}]$, $m = 1, 2, 3, 4, ...$. Since the steady-state criterion is achieved, then the

Table 3: *Impact of breakdown and repair rates* $(\eta_1, \theta_1, \eta_2, \theta_2)$ *on some perfomance measures*

$(\eta_1, \theta_1, \eta_2, \theta_2)$	$\mathrm{E}[L_s]$	$\mathrm{E}[L_q]$	P_{Idle}	$P_{WV}^{S_2}$	$P_{CV}^{S_2}$	$P_{FES}^{S_2}$	$P_{SOS}^{S_2}$	$P_{Bd}^{S_2}$
(**0.5**,2,0.5,3)	0.5071	0.8011	0.0227	0.1204	0.0397	0.0159	0.4323	0.0109
(**0.6**,2,**0.55**,3)	0.6796	0.9203	0.0322	0.0297	0.0128	0.0044	0.3679	0.0113
(**0.7**,2,**0.6**,3)	0.6808	0.9219	0.0323	0.0289	0.0124	0.0042	0.3685	0.0113
(**0.8,3,0.7,4**)	0.6744	0.9130	0.0321	0.0391	0.0120	0.0035	0.3665	0.0113
(0.8,**4,0.8,5**)	0.6825	0.9241	0.0323	0.0294	0.0110	0.0027	0.3718	0.0113
(0.8,**5,0.9**,5)	0.6824	0.9240	0.0323	0.0303	0.0103	0.0026	0.3718	0.0113

sub prob.vectors $\mathcal{I}_i$ satisfy the following equations:

$$\mathcal{I}_0 S_0 + \mathcal{I}_1 V_0 + 0 + \ldots = 0 \tag{9}$$
$$\mathcal{I}_0 T_0 + \mathcal{I}_1 U_1 + \mathcal{I}_2 U_2 + 0 + \ldots = 0 \tag{10}$$
$$\mathcal{I}_1 U_0 + \mathcal{I}_2 U_1 + \mathcal{I}_3 U_2 + 0 + \ldots = 0 \tag{11}$$
$$\mathcal{I}_2 U_0 + \mathcal{I}_3 U_1 + \mathcal{I}_4 U_2 + 0 + \ldots = 0 \tag{12}$$
$$\vdots$$
$$\mathcal{I}_i U_0 + \mathcal{I}_{i+1} U_1 + \mathcal{I}_{i+2} U_2 + \ldots = 0 \text{ where } i \geq 2 \tag{13}$$
$$\mathcal{I}_j = \mathcal{I}_1 \mathcal{R}^{j-1}, \text{ where } j \geq 2. \tag{14}$$

Let the matrix $\mathcal{R}$ represents the rate matrix. By substituting equation 12 into the equations 7 to 11, we get

$$\mathcal{I}_0 S_0 + \mathcal{I}_1 V_0 = 0 \tag{15}$$
$$\mathcal{I}_0 T_0 + \mathcal{I}_1 [U_1 + \mathcal{R} U_2] = 0 \tag{16}$$
$$\mathcal{I}_1 [U_0 + \mathcal{R} U_1 + \mathcal{R}^2 U_2] = 0 \tag{17}$$
$$\mathcal{I}_1 \mathcal{R} [U_0 + \mathcal{R} U_1 + R^2 U_2] = 0 \tag{18}$$
$$\mathcal{I}_1 \mathcal{R}^{i-1} [U_0 + \mathcal{R} U_1 + \mathcal{R}^2 U_2] = 0 \quad i \geq 2. \tag{19}$$

The normalizing equation can be expressed as

$$\mathcal{I}_0 \mathfrak{e} + \mathcal{I}_1 [I - \mathcal{R}]^{-1} \mathfrak{e} = 1 \tag{20}$$

Here $\mathfrak{e}$ is a column vector in which all elements are 1's in the corresponding column. By using methodologies from Neuts [18] and Latouche and Ramaswami [15]. we have estimated? the rate matrix $\mathcal{R}$. Thus, $\mathcal{R}$ is a minimal non-negative solution of the matrix quadratic equations.

$$U_0 + \mathcal{R} U_1 + \mathcal{R}^2 U_2 = 0 \tag{21}$$
$$\mathcal{R} = -U_0 U_1^{-1} - \mathcal{R}^2 U_2 U_1^{-1} \tag{22}$$

Where $\mathcal{R} \geq 0$ and it's an irreducible non-negative matrix of spectral radius smaller than one[18]. Matrix $\mathcal{R}$ can be calculated using an iterative approach as shown below.

$$\mathcal{R}_0 = 0 \tag{23}$$
$$\mathcal{R}_{n+1} = -U_0 U_1^{-1} - \mathcal{R}_k^2 U_2 U_1^{-1} \qquad k \geq 1 \tag{24}$$

All values of $\mathcal{R}$ will expand monotonically, and non-negative matrix $\mathcal{R}$ is converging to $-U_1^{-1}$ and $[U_0 + \mathcal{R}^2 U_2]$. The steady state is attained via the MGM.

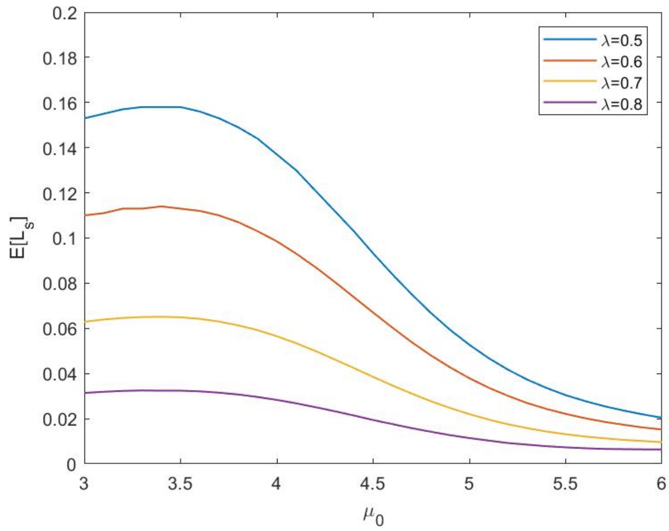


Figure 4: *Influence of* $E[L_s]$ *for different* λ *w.r.t* μ_0

4. Measuring the Performance Characteristics

The following are some performance measurements for the queueing model under consideration in terms of steady-state probability.

4.1. System State Probabilities

- The probability that the servers are idle: $P_{Idle} = \iota_{0,0}$
- The probability of server 2 is in working vacation: $P_{WV}^{S_2} = \sum_{m=1}^{\infty} \iota_{m,0}$
- The probability of server 2 is in complete vacation: $P_{CV}^{S_2} = \sum_{m=1}^{\infty} \iota_{m,1}$
- The probability of server 2 is in FES: $P_{FES}^{S_2} = \sum_{m=1}^{\infty} \iota_{m,2}$
- The probability of server 2 is in SOS: $P_{SOS}^{S_2} = \sum_{m=1}^{\infty} \iota_{m,3}$
- The probability of server 2 is in breakdown: $P_{Bd}^{S_2} = \sum_{m=1}^{\infty} \iota_{m,4}$

4.2. Expected Numbers of Customers in the System and Queue

- The expected no.of customers in the system and queue:

$$E[L_s] = \sum_{m=1}^{\infty} m\iota_{m,0} + \sum_{m=1}^{\infty} m\iota_{m,1} + \sum_{m=1}^{\infty} m\iota_{m,2} + \sum_{m=1}^{\infty} m\iota_{m,3} + \sum_{m=1}^{\infty} m\iota_{m,4}$$

$$E[L_q] = \sum_{m=1}^{\infty} (m-1)\iota_{m,0} + \sum_{m=1}^{\infty} (m-1)\iota_{m,1} + \sum_{m=1}^{\infty} (m-1)\iota_{m,2} + \sum_{m=1}^{\infty} (m-1)\iota_{m,3} + \sum_{m=1}^{\infty} (m-1)\iota_{m,4}$$

- The expected no.of customers served is calculated by:

$$E[SC] = \sum_{m=1}^{\infty} [\mu_0\iota_{m,0} + (\mu_0 + \mu_v)\mu_0\iota_{m,1} + (\mu_0 + \mu_1)\mu_0\iota_{m,2} + (\mu_0 + \mu_2)\mu_0\iota_{m,3}]$$

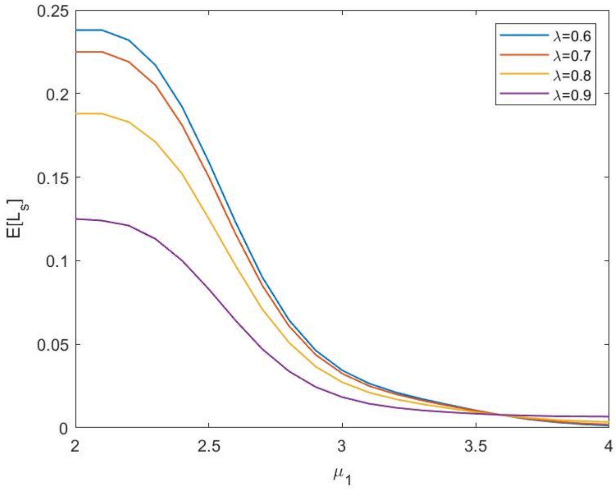


Figure 5: *Influence of* $\mathrm{E}[L_s]$ *for different* μ_0 *w.r.t* λ

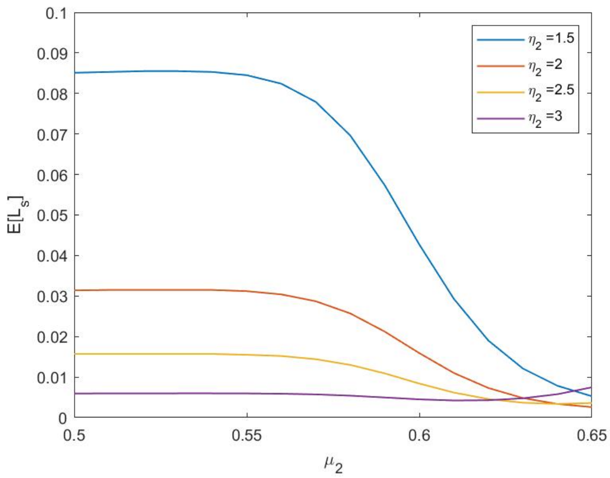


Figure 6: *Influence of* $\mathrm{E}[L_s]$ *for different* μ_0 *w.r.t* λ

4.3. Estimating the Cost and Revenue

Cost and revenue analysis play an important role in queuing systems as they provide an economic interpretation that can be applied to various technical and industrial situations. We define the total expected cost function per unit time and incorporate service rates as selection factors in order to find the optimal service rates that minimize the total cost function. The following factors are incorporated into our prediction:

C_{ls} = Each customer's holding cost per unit time in the system.
C_v = Cost per customer served in the vacation mode of the server 2.
C_F = Cost per customer served in the FES mode of the server 2.
C_S = Cost per customer served in the SOS mode of the server 2.
C_η = Cost per customer incurred when a broken down server is under repair.
C_{μ_0} = Cost per customer served in the busy mode of the server 1.
C_{μ_1} = Cost per customer served in the FES mode of the server 2.
C_{μ_2} = Cost per customer served in the SOS mode of the server 2.

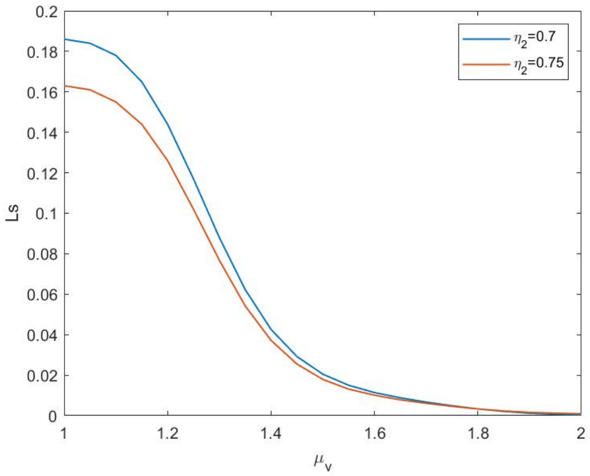
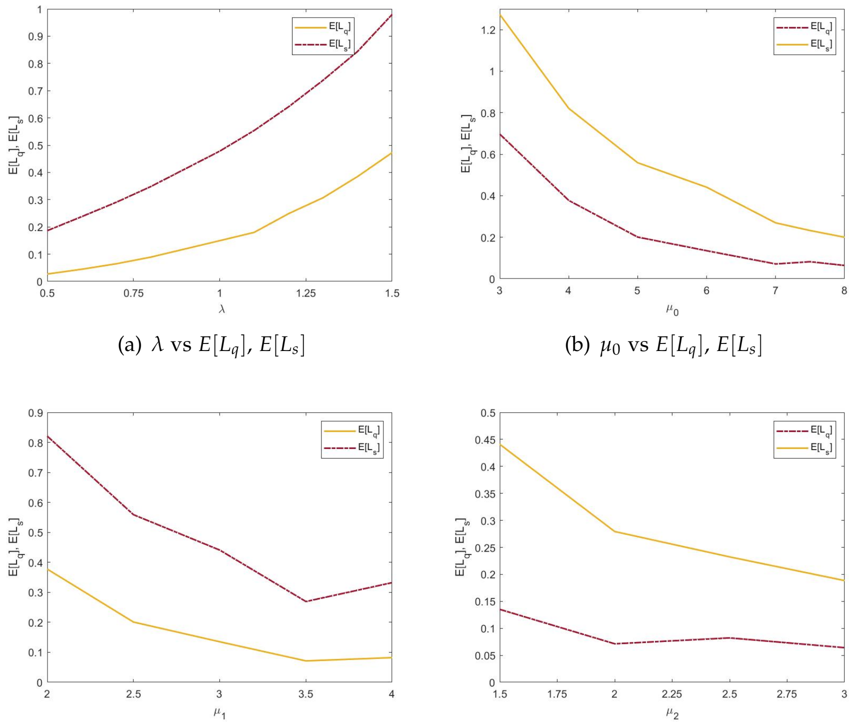


Figure 7: *Influence of* E$[L_s]$ *for different* μ_0 *w.r.t* λ



(a) λ vs $E[L_q]$, $E[L_s]$

(b) μ_0 vs $E[L_q]$, $E[L_s]$

(c) μ_1 vs $E[L_q]$, $E[L_s]$

(d) μ_2 vs $E[L_q]$, $E[L_s]$

Figure 8: *Influence of a few parameters on 2D representation*

Total projected cost (TPC) is defined as:

$$TPC = C_{ls}L_s + C_v(P^{S_2}_{WV} + P^{S_2}_{CV}) + C_F P^{S_2}_{FES} + C_S P^{S_2}_{SOS} + C_\eta P^{S_2}_{Bd} + C_{\mu_0}\mu_0 + C_{\mu_1}\mu_1 + C_{\mu_2}\mu_2$$

If Rev represents customer service revenue, then the system's total anticipated revenue (TAR) is given by:

$$TAR = Rev * E[SC]$$

The total profit is given as:

$$T_{profit} = TAR - TPC$$

The complexity and non-linearity of the cost and revenue functions make it difficult to analyse the behaviour of the cost-revenue model and identify the most suitable values. All computations have been rounded off to two decimal places.

Table 4: *Cost set values for various cost aspects*

Cost set	C_{ls}	C_{wv}	C_{cv}	C_F	C_S	C_{Bd}	$C_{\mu 0}$	$C_{\mu 1}$	$C_{\mu 2}$
I	45	20	10	30	25	15	20	15	10
II	40	20	10	30	25	15	20	15	10
III	45	25	10	30	25	15	20	15	10

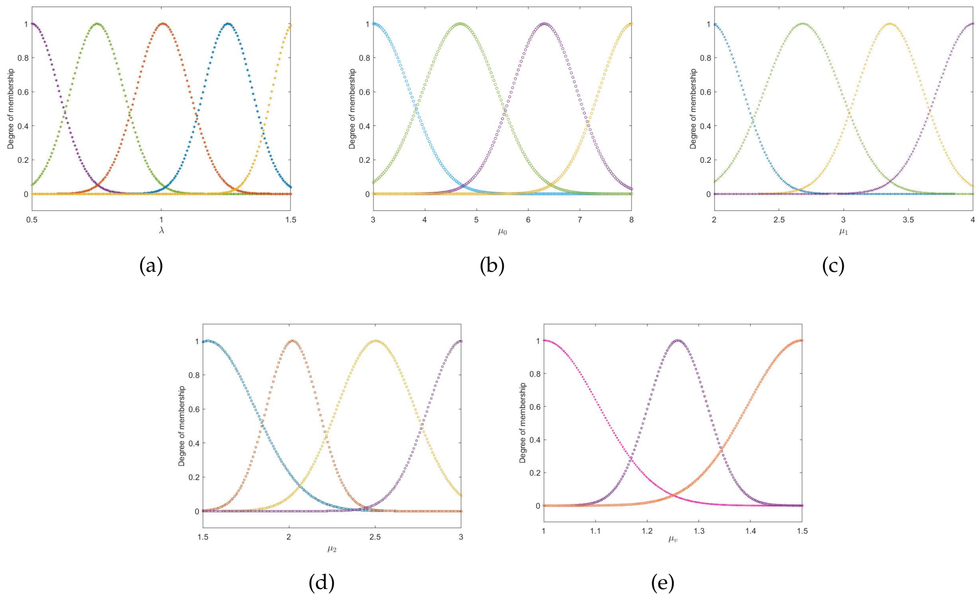


Figure 9: *ANFIS MF for (a) λ, (b) μ_0, (c) μ_1, (d) μ_2 (e) μ_v input variables*

4.4. Adaptive neuro-fuzzy inference system (ANFIS)

Jyh-Shing Roger Jang [11] was the person who first proposed ANFIS in 1992. The system combines both fuzzy logic and neural networks to capture much of the uncertainty and inexactness of real-life systems. In its simplest form, ANFIS consists of a number of inference rules that are used to make decisions or predictions. It can be used for classification, optimization, control, and other tasks where accurate predictions are needed. There are several types of ANFIS, including Type-1, Type-2, and hybrid systems. ANFIS is based on the principles of fuzzy logic, which allows it to consider multiple inputs and outputs simultaneously when making decisions. It also uses neural networks to adjust the strength of the rules. The Tagaki-Sugeno (TS) rule is a type of fuzzy inference system most commonly used for regression and control tasks. It is based on the Takagi-Sugeno-Kang formulation, which is an extension of the Mamdani type of fuzzy logic systems. TS rules combine both fuzzy sets and linear models to allow for more accurate predictions. The architecture of ANFIS can be briefly described with the use of fuzzy parameters. This is done using fuzzy "If-Then" rules, which allow us to perform ANFIS

input-output functions and input-output data pairs. Three components are needed for a fuzzy inference system:
(i) selection of fuzzy rules;
(ii) development of a data structure defining the membership functions (MF) used in the fuzzy rules;
(iii) a reasoning mechanism that performs the inference procedure based on the given fuzzy rules.
The fuzzy rules can be defined as
Rule 1 :"If x is X1 and y is Y1 then $f_1 = p_1x + q_1y + r_1$",
Rule 2 :"If x is X2 nd y is Y2 then $f_2 = p_2x + q_2y + r_2$".
ANFIS networks can be implemented by using the fuzzy toolbox of MATLAB software, where a Gaussian function is used to select fuzzy input parameters, like λ, μ_0, μ_1, μ_2 and μ_v. Moreover, linguistic variables are also defined for input parameters as seen in Table 5.

Table 5: *Values of the membership function for linguistics based on input parameters*

Input parameters	No. of membeship function	Linguistic Values
λ	5	very small, small, medium, large, very large
μ_0, μ_1, μ_2	4	small, medium, large, very large
μ_v	3	small, medium, large

5. Numerical Discussions

5.1. Sensitivity Analysis

In this section, MATLAB is utilized to illustrate how system behavior measurements are influenced by various parameters. The service time, vacation time, breakdown time, and repair time are assumed to be exponentially distributed, and the stability criterion is met by giving the parameters random values. Subsequently, numeric results for the primary performance indicators are obtained.

Variations in λ, (μ_0, μ_1, μ_2, μ_v) , $(\eta_1, \theta_1, \eta_2, \theta_2)$, are provided to reveal average system size (E$[L_s]$), average queue size (E$[L_q]$), and some performance measures in our queueing model.

Table 1 clearly depicts that as arrival rate (λ) rises, E$[L_s]$, E$[L_q]$ also escalates for the value of $\mu_0 = 3, \mu_1 = 2, \mu_2 = 1.5, \mu_v = 1, \eta_1 = 0.5, \eta_2 = 0.7, \theta_1 = 2, \theta_2 = 3, \kappa = 0.8, \xi = 1, \phi_v = 0.2, \phi_1 = 0.2, \nu_0 = 0.2, \nu_1 = 0.5, \nu_2 = 0.3$.

Table 2 clearly depicts that as $(\mu_0, \mu_1, \mu_2, \mu_v)$ escalates, E$[L_s]$, E$[L_q]$ also diminish for the value of $\lambda = 1.7, \eta_1 = 0.5, \eta_2 = 0.7, \theta_1 = 2, \theta_2 = 3, \kappa = 0.8, \xi = 1, \phi_v = 0.2, \phi_1 = 0.2, \nu_0 = 0.2, \nu_1 = 0.5, \nu_2 = 0.3$.

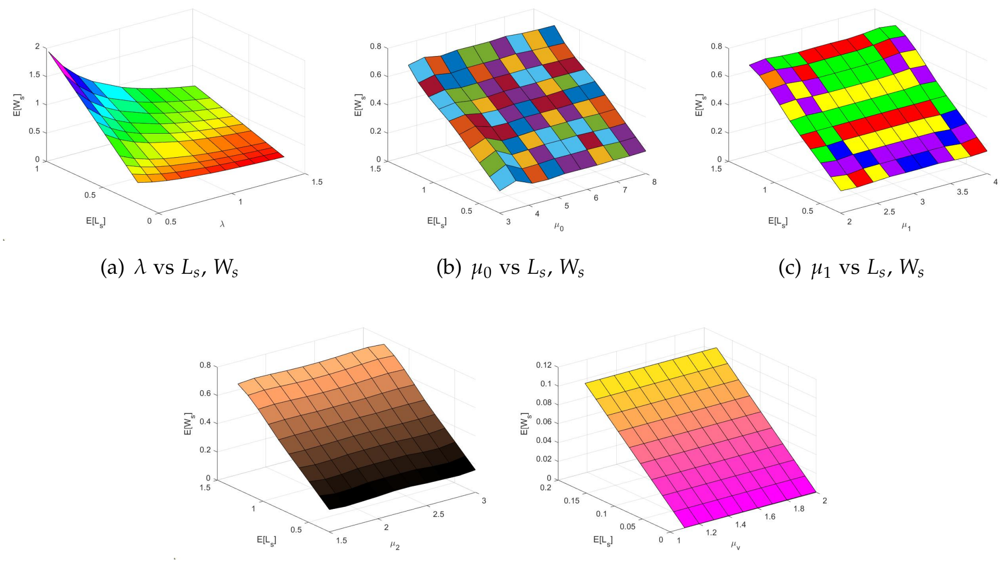
Table 3 clearly depicts that as $(\eta_1, \theta_1, \eta_2, \theta_2)$ varies, E$[L_s]$, E$[L_q]$ are changed for the value of $\lambda = 1.7, \mu_0 = 3, \mu_1 = 2, \mu_2 = 1.5, \mu_v = 1, \kappa = 0.8, \xi = 1, \phi_v = 0.2, \phi_1 = 0.2, \nu_0 = 0.2, \nu_1 = 0.5, \nu_2 = 0.3$.

Figure 2 illustrates the impact of ν_0 for various values of μ_2 on the servers' idle state ($\iota_{0,0}$). The image illustrates that, when the value of μ_2 is held constant, the idle probability, shown by $\iota_{0,0}$, drops as one increases the value of ν_0. This is because when ν_0 rises, the number of customers choosing for SOS increases, and as a consequence, the server's idle probability drops.

Figure 3 shows the increasing nature of the number of customer in the system with varying values of μ_0. Figures. 4, 5, 6 and 7 demonstrate that an increase in the service rate of the second server in busy and vacation modes leads to a decrease in the number of customers (E$[L_s]$) within the system. Moreover, these figures illustrate the effects on E$[L_s]$ when varying values of

λ, η_1 and η_2 are applied.

Figure 8 (a) indicates that as the arrival rate (λ) increases, the expected queue length (E$[L_q]$) and expected system length (E$[L_s]$) also rises. On the other hand, Figures 8 (b–d) show that when the service rate of server 1 (μ_0) or servers 2 (μ_1, μ_2) increases, the expected queue length (E$[L_q]$) and the expected system length (E$[L_s]$) diminish.



(a) λ vs L_s, W_s

(b) μ_0 vs L_s, W_s

(c) μ_1 vs L_s, W_s

(d) μ_2 vs L_s, W_s

(e) μ_v vs L_s, W_s

Figure 10: *Influence of a few parameters on 3D representation*

Figure 10(a – e) shows a three-dimensional graph depicting system performance measures. In Figure 10 (a), the surface illustrates an increase in the arrival rate (λ), expected system length (E$[L_s]$), and expected waiting time in the system (E$[W_s]$) increase. Figures 10 (b) and (c–d) demonstrate that as the service rates of server 1 (μ_0) and server 2 (μ_1, μ_2) increase, the expected system length (E$[L_s]$) and the expected waiting time in the system (E$[W_s]$) decrease. Figure 10(e) further shows that when the vacation rate μ_v increases, expected system length (E$[L_s]$) and expected waiting time in the system (E$[W_s]$) also decrease.

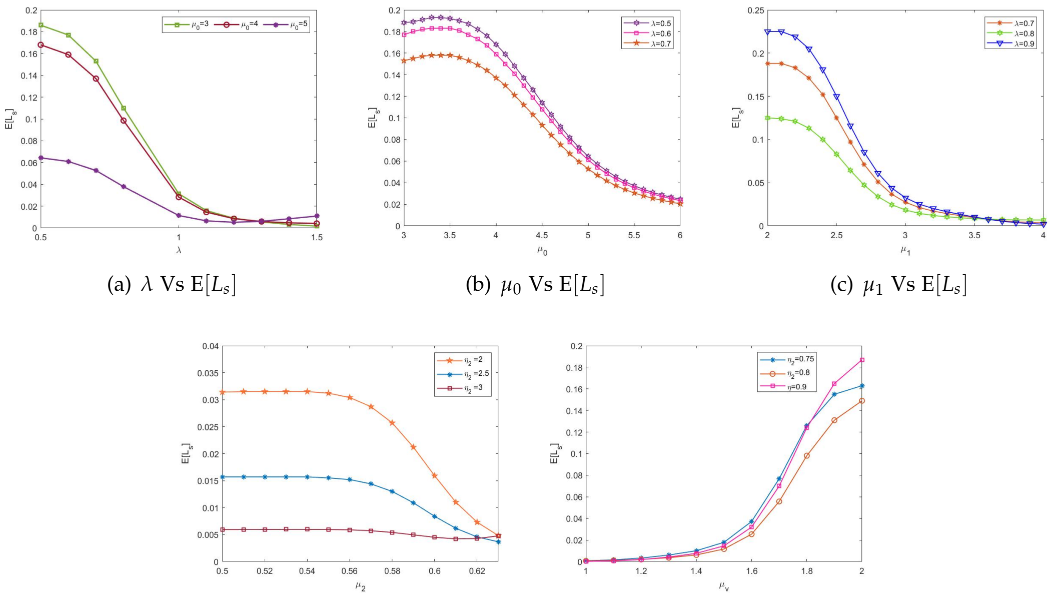
5.2. Anfis Computing and results

Tthe ANFIS results are constructed and verified by the Matlab software by executing the 'neuroFuzzyDesigner' command. The accuracy of the ANFIS outputs for E$[L_s]$ can be examined using the absolute percentage errors. Δ_a is provided by

$$\Delta_a = \frac{\left| \mathrm{E}[L_s] - \mathrm{E}[\overset{*}{L_s}] \right|}{\mathrm{E}[L_s]} \times 100\%$$

where Δ_a is absolute percentage error, E$[L_s]$ exact value of the expected no.of customers in the system by analytical method, E$[\overset{*}{L_s}]$ estimated expected no.of customers in the system by ANFIS technique for varying values of (i) λ (ii) μ_0 (iii) μ_1 (iv) μ_2 (v) μ_v are recorded, and the absolute percentage errors (Δ_a) and accuracy of the estimated value in percentage of E$[L_s]$ are also summarized in Tables 6 - 7.

Lesser The Γ_e value indicates that our ANFIS method is closer to the analytical method's



(a) λ Vs E$[L_s]$ (b) μ_0 Vs E$[L_s]$ (c) μ_1 Vs E$[L_s]$

(d) μ_2 Vs E$[L_s]$ (e) μ_v Vs E$[L_s]$

Figure 11: E$[L_s]$ *Vs (a) λ, (b) μ_0, (c) μ_1, (d) μ_2 (e) μ_v*

results. Figures 7 (a-d) depict the graphs of membership functions for input values (i). *lambda* (ii) μ_0 (iii) μ_1 (iv) μ_v. The tick marks in Fig. 11 (a–d) represent the ANFIS results for the ESL forecast, while the continuous lines represent the analytical results. In these images, there are check marks almost completely covering the curved lines. This suggests that both outcomes are favourable.

Table 6: *Values of the Δ_a, E$[L_s]$, E$[\overset{*}{L_s}]$ by varying input parameters λ, μ_0*

λ	E$[L_s]$	E$[\overset{*}{L_s}]$	Δ_a	μ_0	E$[L_s]$	E$[\overset{*}{L_s}]$	Δ_a
0.5	0.1862	0.186	0.1074	3	1.2737	1.27	0.2905
0.6	0.2378	0.238	0.0841	4	0.8209	0.821	0.0122
0.7	0.2904	0.290	0.1377	5	0.5589	0.559	0.0179
0.8	0.348	0.348	0.0000	6	0.4409	0.441	0.0227
1.0	0.478	0.478	0.0000	7	0.2693	0.269	0.1114
1.1	0.5539	0.554	0.0181	7.5	0.3323	0.332	0.0903
1.2	0.6411	0.641	0.0156	8	0.2884	0.288	0.1387
1.3	0.7384	0.738	0.0542				
1.4	0.8442	0.844	0.0237				
1.5	0.9796	0.98	0.0408				
Average of Δ_a			0.0482	Average of Δ_a			0.0977
Accuracy in predicted value (%)			99.951	Accuracy in predicted value(%)			99.902

5.3. Cost Optimization

The estimated cost per unit of time TPC and total anticipated revenue TAR are found to be **\$132.3475** and **\$243.40** when certain values for the parameters are used such as λ=1, μ_0=6, μ_1=3, μ_2=3.5, μ_v=2, η_1=0.8, θ_1=5 , η_2=0.9, θ_2=5, κ=0.8, ξ=1, ϕ_v= 0.1, ϕ_1=0.2, ν_0=0.2, ν_1=0.5, ν_2=0.3,

Table 7: *Values of the Δ_a, $E[L_s]$, $\overset{*}{E[L_s]}$ by varying input parameters μ_1, μ_2 μ_v*

μ_1	$E[L_s]$	$\overset{*}{E[L_s]}$	Δ_a	μ_2	$E[L_s]$	$\overset{*}{E[L_s]}$	Δ_a	μ_v	$E[L_s]$	$\overset{*}{E[L_s]}$	Δ_a
2	0.8209	0.821	0.0122	1.5	0.4409	0.441	0.0227	1	0.8209	0.821	0.0122
2.5	0.5589	0.559	0.0179	2	0.2693	0.269	0.1114	1.1	0.5589	0.559	0.0179
3	0.4409	0.441	0.0227	2.5	0.3323	0.332	0.0903	1.2	0.4409	0.441	0.0227
3.5	0.2693	0.269	0.1114	3	0.2884	0.29	0.5548	1.3	0.2693	0.269	0.1114
4	0.3323	0.332	0.0903					1.4	0.3323	0.332	0.0903
								1.5	0.2884	0.29	0.5548
Average of Δ_a			0.0509	Average of Δ_a			0.1948	Average of Δ_a			0.1349
Accuracy in predicted value (%)			99.9491	Accuracy in predicted value (%)			99.8052	Accuracy in predicted value (%)			99.8651

Rev=250 are taken into consideration. To gauge the impact of changing the cost parameters, the cost function was examined for three different cost set values, as displayed in Table 4. In the feasible interval, the cost function is convex with regard λ, μ_1, μ_2 and η_1, η_2 as seen in Figs. 13(a-c) - 15(a-c).

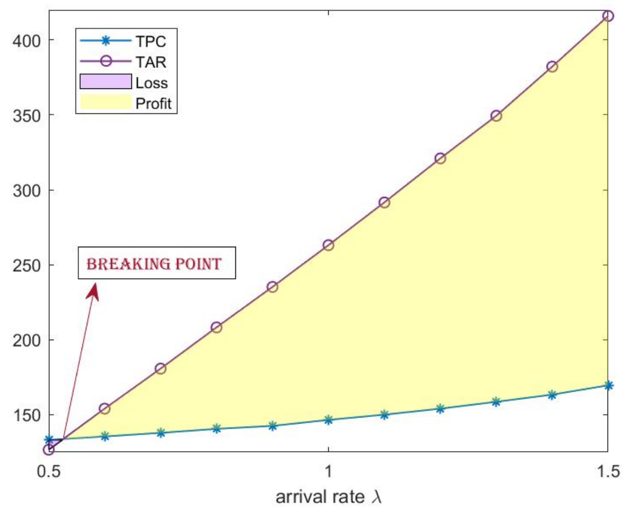
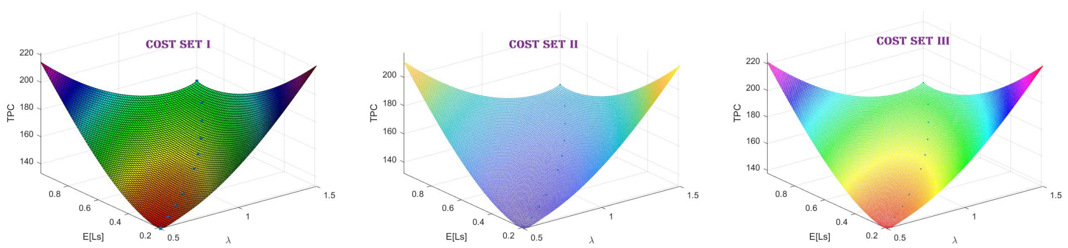


Figure 12: *Impact of λ on TPC, TAR*

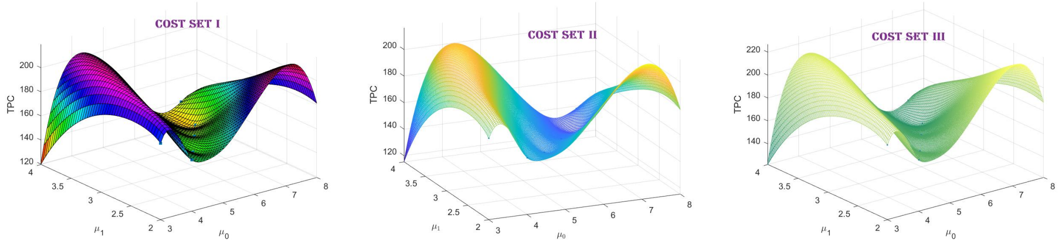


(a) TPC Vs λ, $E[L_s]$

(b) TPC Vs λ, $E[L_s]$

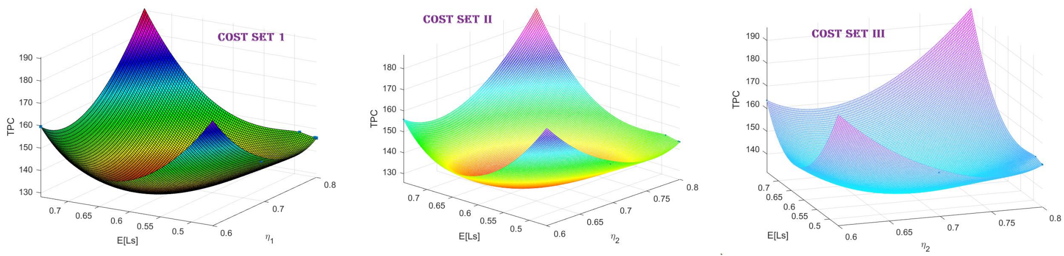
(c) TPC Vs λ, $E[L_s]$

Figure 13: *Total projected cost varying values of λ, $E[L_s]$*



(a) TPC Vs μ_0, μ_1 (b) TPC Vs μ_0, μ_1 (c) TPC Vs μ_0, μ_1

Figure 14: *Total projected cost varying values of* μ_0, μ_1



(a) TPC Vs η_1, E$[L_s]$ (b) TPC Vs η_1, E$[L_s]$ (c) TPC Vs η_1, E$[L_s]$

Figure 15: *Total projected cost varying values of* η_1, E$[L_s]$

The effect of λ on TPC and TAR in the model is depicted in Figure 12. As λ increases, the total elapsed cost and total expected response also increase. The point at which there is neither gain nor loss is seen around $\lambda = 0.53$, based on the set of cost values given for the model. If λ is lower than 0.53, there will be a loss, and when λ is greater than 0.53, the system will see a profit. Thus, with knowledge of the customer arrival rate, appropriate actions can be taken to reduce the TPC and maximize the TAR. Our model's TPC and TAR are shown with the relevant values in Table 8.

6. Conclusion

This research explores a heterogeneous and unreliable server queueing system with the addition of a second optional service, feedback, and breakdown in a hybrid vacation scheme. Through numerical examples, The efficiency of the matrix-geometric technique in determining steady-state probabilities and other performance metrics has been discovered. Utilizing the matrix-geometric approach and Adaptive Network-Based Fuzzy Interference System (ANFIS) for calculating performance indices and the cost function, it has been shown that the model can be used in a real-time system. By comparing the results from ANFIS and the numerical results, we can demonstrate the usefulness of a neural-fuzzy tool to assess the performance of queueing systems based on commercial and technological standards. Lastly, this strategy can be further applied to more realistic scenarios, such as those with the concepts of a phase-type arrival and impatient customer, standby servers and Markovian models like $MAP/PH/1$ and $GI^X/Geo/1$ queues.

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Table 8: *Arrival and service rates and corresponding TPC, TAR*

λ	**1**		**1.1**		**1.2**		**1.3**		**1.4**		**1.5**	
	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR
I	146.55	263.05	150.05	291.57	154.06	320.95	158.63	349.36	163.42	382.17	169.63	415.98
II	144.16	263.05	147.28	291.57	150.85	320.95	154.94	349.36	159.20	382.17	164.74	415.98
III	151.13	263.05	154.58	291.57	158.53	320.95	163.10	349.36	167.74	382.17	173.86	415.98
$(\mu_0, \mu_1, \mu_2, \mu_v)$	(**4**, 2, 1.5, 1)		(**5, 2.5**, 1.5, 1)		(**6, 3,** 1.5, 1)		(**7, 3.5, 2**, 1)		(**7.5, 4, 2.5**, 1)		(8,4,3,**1.5**)	
	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR
I	162.19	458.12	150.21	463.68	144.83	470.60	135.69	355.26	139.89	478.33	137.90	456.97
II	158.08	458.12	147.41	463.68	142.62	470.60	134.34	355.26	138.23	478.33	136.46	456.97
III	166.72	458.12	154.91	463.68	149.62	470.60	140.19	355.26	144.75	478.33	142.77	456.97
$(\eta_1, \theta_1, \eta_2, \theta_2)$	(**0.6**,2,**0.55**,3)		(**0.7**,2,**0.6**,3)		(**0.8,3,0.7,4**)		(0.8,**4,0.8,5**)		(0.8,**5,0.9**,5)		(0.8,5,0.9,**5**)	
	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR	TPC	TAR
I	146.17	258.32	146.02	257.25	147.25	267.52	145.90	256.67	145.94	257.01	133.28	243.40
II	143.82	258.32	143.69	257.25	144.79	267.52	143.58	256.67	143.61	257.01	132.34	243.40
III	150.77	258.32	150.63	257.25	151.81	267.52	150.53	256.67	150.56	257.01	138.14	243.40