A TYPE I HALF -LOGISTIC EXPONENTIATED WEIBULL DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

In the area of distribution theory, statisticians have proposed and developed new models for generalizing the existing ones so as to make them more flexible and to aid their application in a variety of fields. In this article, we present a new distribution called the Type I Half-Logistic Exponentiated Weibull (TIHLEtW) Distribution with four positive parameters, which extends the Weibull distribution by two parameters. Some statistical properties of the TIHLEtW distribution, such as explicit expressions for the quantile function, probability weighted moments, moments, generating function, Reliability function, hazard function, and order statistics are discussed. A maximum likelihood estimation technique is employed to estimate the model parameters and the simulation study is presented. The superiority of the new distribution is illustrated with an application to two real data sets. The results showed that the new distribution fits better in the two real data sets amongst the range of distributions considered.

Keywords: Type I Half-Logistic Exponentiated-G, Weibull distribution, Quantile function, Reliability function, Maximum likelihood, Order Statistics.

1. Introduction

All parametric statistical techniques, such as inference, modeling, survival analysis, and reliability, are based on statistical distributions. Fitting the data to a statistical model is a critical step when analyzing lifetime data. For this reason, a number of lifespan distributions have been established in the literature. The majority of lifespan models have a limited set of behaviors. Such distributions are unable to provide a better fit for all real scenarios. As a result, a variety of distribution classes have been created by expanding common continuous distributions. The generated family of continuous distributions is a new enhancement for developing and expanding classic distributions. The newly generated distributions have been extensively researched in a variety of fields, and they provide greater application flexibility.

One of the most well-known lifetime distributions is the Weibull distribution. It adequately represents many distinct forms of failures, both in components and in general. To deal with bathtub-shaped failure rates, various generalizations and extensions of the Weibull distribution have been proposed in the statistical literature. Mudholkar *et al*. [25] pioneered *Exponentiated Weibull* distribution, the modified Weibull extension by Xie *et al*. [32] , flexible Weibull extension (FWEx) by Bebbington *et al.* [10], beta modified Weibull by Silva *et al.* [30], Kumaraswamy Weibull by Cordeiro *et al.* [13], transmuted Weibull by Aryal and Tsokos [8], truncated Weibull distribution by Zhang and Xie [34], Kumaraswamy inverse Weibull by Shahbaz *et al.* [29], exponentiated generalized Weibull by Cordeiro *et al.* [15], McDonald modified Weibull by Merovci and Elbatal [24], beta inverse Weibull by Hanook *et al.* [19], transmuted additive Weibull by Elbatal and Aryal [17], McDonald Weibull by Cordeiro *et al.* [12], Kumaraswamy modified Weibull by Cordeiro *et al.* [14], transmuted complementary Weibull geometric by Afify *et al.* [1], Kumaraswamy transmuted exponentiated additive Weibull by Nofal *et al.* [27], generalized transmuted Weibull by Nofal *et al.* [26], Topp-Leone generated Weibull by Aryal *et al.*,[9], Kumaraswamy complementary Weibull geometric by Afify *et al.* [2], Marshall-Olkin additive Weibull by Afify *et al.* [3], Zubair–Weibull by Ahmad [4], alpha power transformed Weibull by Ahmad *et al.* [5], Topp Leone exponentiated weibull by Ibrahim [20] distributions.

Bello *et al.* [11] proposed a new distribution family called the Type I Half-Logistic Exponentiated-G (TIHLEt-G) with two extra shape parameters. For any arbitrary cumulative distribution function as a baseline (cdf) $H\left(x,\vartheta\right)$, the TIHLEt-G family with two positive shape parameters λ and α has cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{TIHLEt-G}(x;\lambda,\alpha,\vartheta)=\frac{1-[1-H^{\alpha}(x;\vartheta)]^{\lambda}}{1+[1-H^{\alpha}(x;\vartheta)]^{\lambda}},\ x>0,\ \lambda,\alpha>0 \tag{1}$$

and

$$f_{TIHLEt-G}(x;\lambda,\alpha,\vartheta)=\frac{2\lambda\alpha h(x;\vartheta)H^{\alpha-1}_{(x;\vartheta)}[1-H^{\alpha}_{(x;\vartheta)}]^{\lambda-1}}{\left[1+[1-H^{\alpha}_{(x;\vartheta)}]^{\lambda}\right]^{2}},x>0,\ \lambda,\alpha>0 \tag{2}$$

The cdf and pdf of the Weibull distribution are given as

$$H(x;\theta,\beta)=1-e^{-\theta x^{\beta}},\ x>0,\ \theta,\beta>0 \tag{3}$$

$$h(x;\theta,\beta)=\theta\beta x^{\beta-1}e^{-\theta x^{\beta}},\ x>0,\ \theta,\beta>0 \tag{4}$$

The goal of this paper is to develop a more flexible model by extending the two parameter Weibull distribution. The Type II half logistic Weibull (TIHLEtW) distribution is the name given to the new model. We develop the TIHLEtW distribution from Bello *et al.* [11] and provide some essential statistical properties. The layout of this paper is organized as follows: Section 2 defines the TIHLEtW distribution. We obtained very useful and important representations for the TIHLEtW distribution in Section 3. Section 4, some statistical properties such as probability weighted moments, moments, moments generating function, quartile function, reliability function, hazard function and order statistics are derived. The parameters of the new model were estimated using the maximum likelihood estimation (MLEs) approach in Section 5. The simulation study was conducted to show that the estimates are efficient and consistent using MLE in Section 6. The application of the new model to two real data sets was shown in Section 7 to demonstrate the use of the new model. Finally Section 8 concludes the paper.

2. Type I Half-Logistic Exponentiated Weibull (TIHLETW) Distribution

In this section, we define a new model called TIHLEtW model, the random variable *X is* said to have a TIHLEtW model, if its cdf is obtained by inserting equation (3) in equation (1) as follows

$$F_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)=\frac{1-\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}},\ x>0,\lambda,\alpha,\theta,\beta>0 \tag{5}$$

and its corresponding pdf is

$$f_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)=\frac{2\lambda\alpha\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{2}} \tag{6}$$

where β is a scale parameter and λ,α,θ are shape parameters.

3. Important Representation

In this section, we derived a useful representation for the TIHLEtW pdf and cdf. Due to the fact that the generalized binomial series is

$$\left(1+Z\right)^{-\beta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta+i-1}{i}z^{i} \tag{7}$$

For $|z|<1$ and β is a positive real non integer. The density function of the TIHLEtW distribution is then obtained by using the binomial theorem (7) to (6).

$$f_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)=2\lambda\alpha\theta\beta x^{\beta-1}e^{-\theta x^{\beta}}\sum_{i=0}^{\infty}(-1)^{i}\binom{1+i}{i}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda(i+1)-1}$$

Now, using the generalized binomial theorem, we can write

$$\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda(i+1)-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\lambda(i+1)-1}{j}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha j}$$

Also

$$\left[1-e^{-\theta x^{\beta}}\right]^{\alpha(j+1)-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha(j+1)-1}{k}\left(e^{-\theta x^{\beta}}\right)^{k}$$

Then, the pdf can be written as:

$$f_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)=\sum_{i,j,k=0}^{\infty}\eta_{p}x^{\beta-1}\left(e^{-\theta x^{\beta}}\right)^{k+1} \tag{8}$$

where $\eta_{p}=2\lambda\alpha\theta\beta(-1)^{i+j+k}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}$

In addition, an expansion for the $\left[F(x,\lambda,\alpha,\theta,\beta)\right]^{h}$ is produced, with h being an integer, and the binomial expansion is worked out once more.

$$\left[F_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)\right]^h = \underbrace{\left[1-\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}\right]^h}_{A}\underbrace{\left[1+\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{-h}}_{B}$$

$$A=\left[1-\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}\right]^h = \sum_{m=0}^{h}(-1)^m\binom{h}{m}\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda m}$$

$$B=\left[1+\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}\right]^{-h} = \sum_{p=0}^{h}(-1)^p\binom{h+p-1}{m}\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda p}$$

Combining A and B, we obtain

$$\left[F_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)\right]^h = \sum_{p,m=0}^{h}(-1)^{p+m}\binom{h}{m}\binom{h+p-1}{p}\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda(p+m)}$$

Consider

$$\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda(p+m)} = \sum_{z=0}^{\infty}(-1)^z\binom{\lambda(p+m)}{z}\left[1-e^{-\theta x^{\beta}}\right]^{\alpha z}$$

$$\left[1-e^{-\theta x^{\beta}}\right]^{\alpha z} = \sum_{q=0}^{\infty}(-1)^q\binom{\alpha z}{q}(e^{-\theta x^{\beta}})^q$$

The cdf can be written as:

$$\left[F_{TIHLEtW}(x;\lambda,\alpha,\theta,\beta)\right]^h = \sum_{p,m=0}^{h}\varphi_t(e^{-\theta x^{\beta}})^q \tag{9}$$

where $\varphi_t = \sum_{z,q=0}^{\infty}(-1)^{p+m+z+q}\binom{h+p-1}{p}\binom{h}{m}\binom{\lambda(m+p)}{z}\binom{\alpha z}{q}$

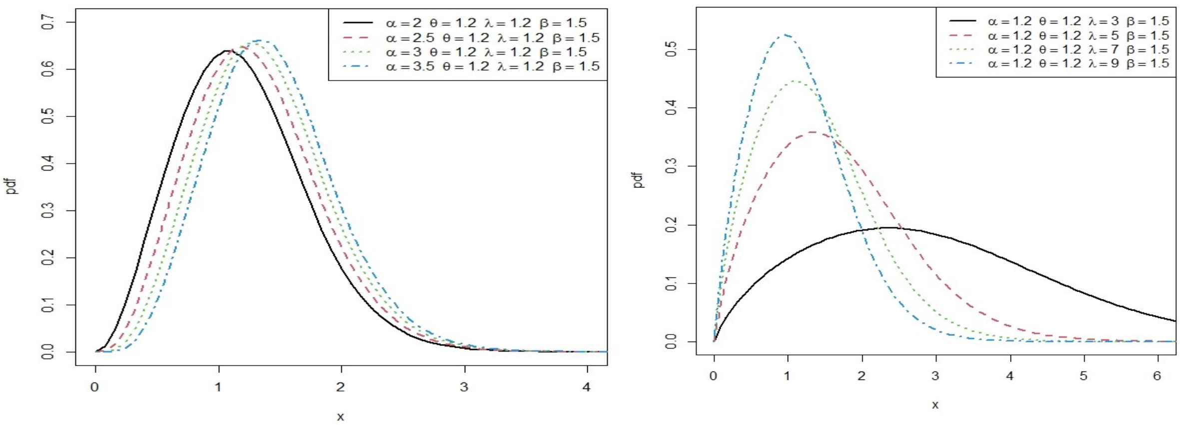

Figure 1: Plots of Pdf of TIHLEtW distribution for different values of parameters.

4. Statistical Properties

In this section, we derived some statistical properties of the new of distribution.

4.1. Probability weighted moments

Greenwood et al. [18] introduced a class of moments known as probability weighted moments (PWMs). This class is used to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, represented by $\tau_{r,s}$, can be derived for a random variable X using the following relationship.

$$\tau_{r,s} = E\left[X^r F(X)^s\right] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx \tag{10}$$

The PWMs of TIHLEtW distribution is derive by substituting (8) and (9) into (10), and replacing h with s, as follows

$$\tau_{r,s} = \sum_{i,j,k=0}^{\infty} \sum_{p,m=0}^{s} \eta_p \varphi_t \int_0^{\infty} x^r (e^{-\theta x^\beta})^{k+1+q} dx \tag{11}$$

Consider the integral

$$\int_0^{\infty} x^r e^{-(k+1+q)\theta x^\beta} dx$$

Let $y = (k+1+q)\theta x^\beta \Rightarrow x = \left[\frac{y}{(k+1+q)\theta}\right]^{\frac{1}{\beta}}; dx = \frac{dy}{(k+1+q)\theta\beta x^{\beta-1}}$

Then

$$\int_0^{\infty} \left[\frac{y}{(k+1+q)\theta}\right]^{\frac{r}{\beta}} e^{-y} \frac{dy}{(k+1+q)\theta\beta x^{\beta-1}} = \frac{1}{\left[(k+1+q)\theta\right]^{\frac{r}{\beta}+1}} \int_0^{\infty} y^{\frac{r}{\beta}} e^{-y} dy$$

$$\int_0^{\infty} y^{\frac{r}{\beta}} e^{-y} dy = \Gamma(\frac{r}{\beta}+1)$$

The PWMs of TIHLEtW can be written as follows

$$\tau_{r,s} = \sum_{i,j,k=0}^{\infty} \sum_{p,m=0}^{s} \frac{\eta_p \varphi_t \Gamma(\frac{r}{\beta}+1)}{(k+1+q)^{\frac{r}{\beta}+1} \theta^{\frac{r}{\beta}}} \tag{12}$$

Now,

$$\varphi_t = \sum_{z,q=0}^{\infty} (-1)^{p+m+z+q} \binom{s+p-1}{p} \binom{s}{m} \binom{\lambda(m+p)}{z} \binom{\alpha z}{q}$$

and

$$\eta_p = 2\lambda\alpha(-1)^{i+j+k} \binom{i+1}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k}$$

4.2. Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the r[th] moment for the new distribution.

$$\mu_r^{'} = E(x^r) = \int_0^\infty x^r f(x)dx \tag{13}$$

By using the important representation of the pdf in equation (8), we have

$$E(X^r) = \sum_{i,j,k=0}^{\infty} \eta_p \int_0^\infty x^r (e^{-\theta x^\beta})^{k+1} dx \tag{14}$$

Consider the integral

$$\int_0^\infty x^r e^{-(k+1)\theta x^\beta} dx$$

Let $w = (k+1)\theta x^\beta \Rightarrow x = \left[\frac{w}{(k+1)\theta}\right]^{\frac{1}{\beta}}; dx = \frac{dw}{(k+1)\theta\beta x^{\beta-1}}$

Then

$$\int_0^\infty \left[\frac{w}{(k+1)\theta}\right]^{\frac{r}{\beta}} e^{-w} \frac{dw}{(k+1)\theta\beta x^{\beta-1}} = \frac{1}{\left[(k+1)\theta\right]^{\frac{r}{\beta}+1}} \int_0^\infty w^{\frac{r}{\beta}} e^{-w} dw$$

$$\int_0^\infty w^{\frac{r}{\beta}} e^{-w} dw = \Gamma(\frac{r}{\beta}+1)$$

The r[th] moment for TIHLEtW distribution can be written as follows

$$E(X^r) = \sum_{i,j,k=0}^{\infty} \frac{\eta_p \Gamma(\frac{r}{\beta}+1)}{(k+1)^{\frac{r}{\beta}+1} \theta^{\frac{r}{\beta}}} \tag{15}$$

Now

$$\eta_p = 2\lambda\alpha(-1)^{i+j+k} \binom{i+1}{i} \binom{\lambda(i+1)-1}{j} \binom{\alpha(j+1)-1}{k}$$

The mean and variance of TIHLEtW distribution are as follows

$$E(X) = \sum_{i,j,k=0}^{\infty} \frac{\eta_p \Gamma(\frac{1}{\beta}+1)}{(k+1)^{\frac{1}{\beta}+1} \theta^{\frac{1}{\beta}}} \tag{16}$$

and

$$\operatorname{var}(X) = \sum_{i,j,k=0}^{\infty} \frac{\eta_p \Gamma(\frac{2}{\beta}+1)}{(k+1)^{\frac{2}{\beta}+1} \theta^{\frac{2}{\beta}}} - \left[\sum_{i,j,k=0}^{\infty} \frac{\eta_p \Gamma(\frac{r}{\beta}+1)}{(k+1)^{\frac{r}{\beta}+1} \theta^{\frac{r}{\beta}}}\right]^2 \tag{17}$$

4.3. Moment generating function (mgf)

The Moment Generating Function of x is given as:

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x)dx \tag{18}$$

where the expansion of $e^{tx} = \sum_{m=0}^{\infty} \frac{t^m x^m}{m!}$

The moment generating function of TIHLEtW distribution is given by

$$M_x(t) = \sum_{i,j,k=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^m \eta_p \Gamma(\frac{m}{\beta}+1)}{(k+1)^{\frac{m}{\beta}+1} \theta^{\frac{m}{\beta}} m!} \tag{19}$$

4.4. Reliability function

The reliability function which is also known as survivor function that gives the probability that a patient will survive longer than specified period of time. It is defined as

$$R(x;\lambda,\alpha,\theta,\beta) = \frac{2\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}} \tag{20}$$

4.5. Hazard function

The hazard function is the probability of an event of interest occurring within a relatively short time frame and is defined as

$$T(x;\lambda,\alpha,\theta,\beta) = \frac{\lambda\alpha\theta\beta x^{\beta-1} e^{-\theta x^{\beta}} \left[1-e^{-\theta x^{\beta}}\right]^{\alpha-1}}{\left[1+\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]^{\lambda}\right]\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\alpha}\right]} \tag{21}$$

4.6. Quantile Function

The quantile function is a vital tool to create random variables from any continuous probability distribution. As a result, it has a significant position in probability theory. For x, the quantile function is F(x) = u, where u is distributed as U(0,1).The TIHLEtW distribution is easily simulated by inverting equation (5) which yields the Quantile function Q(u) defined as:

$$x = Q(u) = \left\{ \frac{-\log\left[1-\left[1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]}{\theta} \right\}^{\frac{1}{\beta}} \tag{22}$$

The first quartile, the median and the third quartile of TIHLEtW distribution are obtained by putting u = 0.25, 0.5 and 0.75, respectively in equation (22).

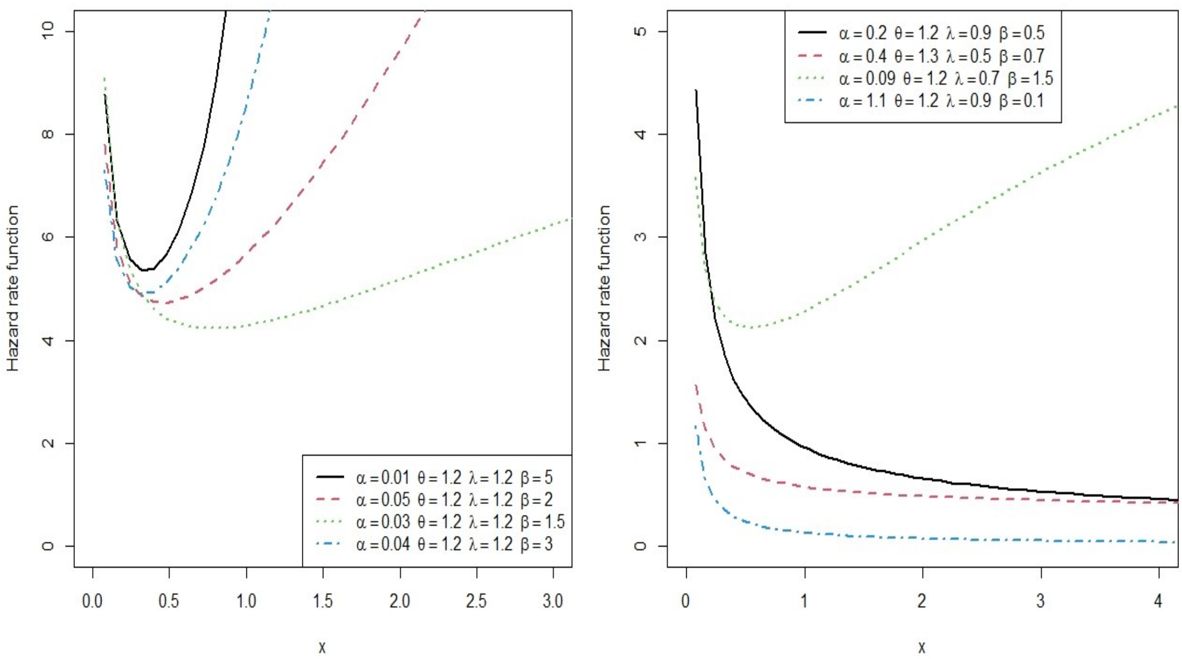

Figure 2: Plots of hazard of the TIHLEtW distribution for different valves of parameters.

4.7. Order Statistics

Many areas of statistics including reliability and life testing have made substantial use of order statistics. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with their corresponding continuous distribution function F (x). Let $X_1, X_2, \ldots, X_n$ be n independently distributed and continuous random variables from the TIHLEtW distribution. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, r = 1, 2, 3, ..., n denote the cdf and pdf of the rth order statistics $X_{r:n}$ respectively. David [16] gave the probability density function of $X_{r:n}$ as:

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \tag{23}$$

By substituting equation (8) and equation (9) into equation (23), also replacing h with v+r-1 in equation (9). We have

$$f_{r:n}(x;\lambda,\alpha,\theta,\beta)=\frac{1}{B(r,n-r+1)}2\lambda\alpha\theta\beta x^{\beta-1}\sum_{v=0}^{n-r}\sum_{i,j,k=0}^{\infty}\sum_{p,m=0}^{v+r-1}(-1)^{i+j+k}(-1)^{p+m+z+q}(-1)^{v}$$
$$\binom{n-r}{v}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k} \tag{24}$$
$$\binom{r+v+p-2}{p}\binom{r+v-1}{m}\binom{\lambda(m+p)}{z}\binom{\alpha z}{q}\varphi_t(e^{-\theta x^{\beta}})^{k+1+q}$$

The equation above is called the rth order statistics for the TIHLEtW distribution.

Let r = n, then the probability density function of the maximum order statistics of TIHLEtW distribution is

$$f_{n:n}(x;\lambda,\alpha,\theta,\beta)=2n\lambda\alpha\theta\beta x^{\beta-1}\sum_{i,j,k=0}^{\infty}\sum_{p,m=0}^{v+n-1}(-1)^{i+j+k}(-1)^{p+m+z+q}(-1)^{v}\binom{i+1}{i}$$
$$\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\binom{n+v+p-2}{p} \tag{25}$$
$$\binom{n+v-1}{m}\binom{\lambda(m+p)}{z}\binom{\alpha z}{q}(e^{-\theta x^{\beta}})^{k+1+q}$$

Also, let r = 1, then the probability density function of the minimum order statistics of TIHLEtW distribution is

$$f_{1:n}(x;\lambda,\alpha,\theta,\beta)=2n\lambda\alpha\theta\beta x^{\beta-1}\sum_{v=0}^{n-1}\sum_{i,j,k=0}^{\infty}\sum_{p,m=0}^{v}(-1)^{i+j+k}(-1)^{p+m+z+q}(-1)^{v}\binom{n-1}{v}\binom{i+1}{i}$$
$$\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\binom{v+p-1}{p} \tag{26}$$
$$\binom{v}{m}\binom{\lambda(m+p)}{z}\binom{\alpha z}{q}(e^{-\theta x^{\beta}})^{k+1+q}$$

5. Parameter Estimation

In this paper, we explore the maximum likelihood technique to estimate the unknown parameters of the TIHLEtW distribution for complete data. Maximum likelihood estimates (MLEs) have appealing qualities that may be used to generate confidence ranges and provide simple approximations that function well in finite samples. In distribution theory, the resulting approximation for MLEs is easily handled, either analytically or numerically. Let $x_1, x_2, x_3, ..., x_n$ be a random sample of size n from the TIHLEtW distribution. Then, the likelihood function based on observed sample for the vector of parameter $(\lambda, \alpha, \theta, \beta)^T$ is given by

$$\begin{aligned}\log\left(\phi\right) &= n\log(2)+n\log(\lambda)+n\log(\alpha)+n\log(\theta)+n\log(\beta)\\ &+(\beta-1)\sum_{i=1}^{n}\log(x_i)-\theta\sum_{i=1}^{n}(x_i^{\beta})\\ &+(\alpha-1)\sum_{i=1}^{n}\log[1-e^{-\theta x_i^{\beta}}]+(\lambda-1)\sum_{i=1}^{n}\log\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]\\ &-2\sum_{i=1}^{n}\log\left[1+\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda}\right]\end{aligned} \tag{27}$$

The components of score vector $\Delta L(\phi)=\left(\frac{\partial L(\phi)}{\partial\lambda},\frac{\partial L(\phi)}{\partial\alpha},\frac{\partial L(\phi)}{\partial\theta},\frac{\partial L(\phi)}{\partial\beta}\right)^{T}$ are given as

$$\begin{aligned}\frac{\partial L(\phi)}{\partial\lambda} &= \frac{n}{\lambda}+\sum_{i=1}^{n}\log\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]\\ &-2\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda}\log\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]}{1+\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda}}\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial L(\phi)}{\partial\alpha} &= \frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i^{\beta}}\right]\\ &+(\lambda-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\log\left[1-e^{-\theta x_i^{\beta}}\right]}{1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}}\\ &+2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda-1}\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\log\left[1-e^{-\theta x_i^{\beta}}\right]}{1+\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda}}\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial L(\phi)}{\partial\theta} &= \frac{n}{\theta}-\sum_{i=1}^{n}x_i^{\beta}+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\theta x_i^{\beta}}x_i^{\beta}}{1-e^{-\theta x_i^{\beta}}}\\ &-(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha-1}e^{-\theta x_i^{\beta}}x_i^{\beta}}{1-\left[1-e^{-\theta x_i^{\beta}}\right]}\\ &-2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda-1}\alpha\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha-1}e^{-\theta x_i^{\beta}}x_i^{\beta}}{1+\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda}}\end{aligned} \tag{30}$$

$$\frac{\partial L(\phi)}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log(x_i)-\theta\sum_{i=1}^{n}x_i^{\beta}\log(x_i)+(\alpha-1)$$

$$-(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha-1}e^{-\theta x_i^{\beta}}\theta x_i^{\beta}\log(x_i)}{1-\left[1-e^{-\theta x_i^{\beta}}\right]} \tag{31}$$

$$-2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda-1}\alpha\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha-1}e^{-\theta x_i^{\beta}}\theta x_i^{\beta}\log(x_i)}{1+\left[1-\left[1-e^{-\theta x_i^{\beta}}\right]^{\alpha}\right]^{\lambda}}$$

The MLEs are obtained by setting $\frac{\partial L(\phi)}{\partial \lambda}, \frac{\partial L(\phi)}{\partial \alpha}, \frac{\partial L(\phi)}{\partial \theta}$ and $\frac{\partial L(\phi)}{\partial \beta}$ to zero and solving these equations simultaneously. These equations cannot be solved analytically, so we have to appeal to numerical method.

6. Simulation Study

In this section, a numerical analysis will be conducted to evaluate the performance of MLE for TIHLEtW Distribution.

Table 1: MLEs, biases and RMSE for some values of parameters

		(3,2,2.5,2)			(3,2,2.5,3)		
N	Parameters	Estimated Values	Bais	RMSE	Estimated Values	Bais	RMSE
	λ	3.0715	0.0715	0.7353	3.0263	0.0263	0.7096
20	α	2.1100	0.1100	0.8549	2.2121	0.2121	0.9444
	θ	2.8745	0.3745	0.9361	2.9235	0.4235	1.0310
	β	2.8745	0.8745	1.2251	3.9235	0.9235	0.9431
	λ	3.0412	0.0412	0.5405	3.0242	0.0242	0.5472
50	α	2.1008	0.1008	0.6248	2.1454	0.1454	0.7306
	θ	2.6588	0.1588	0.5958	2.7069	0.2069	0.6429
	β	2.6588	0.6588	0.8740	3.7069	0.7069	0.6757
100	λ	3.0094	0.0094	0.3889	3.0149	0.0149	0.3930
	α	2.1013	0.1013	0.4425	2.0947	0.0947	0.5194
	θ	2.5938	0.0938	0.3702	2.6052	0.1052	0.4013
	β	2.5398	0.5398	0.6934	3.6052	0.6052	0.5530
250	λ	3.0552	0.0552	0.2985	3.0133	0.0133	0.2430
	α	2.0518	0.0518	0.2713	2.0286	0.0286	0.2740
	θ	2.5044	0.0044	0.2159	2.5114	0.0114	0.1853
	β	2.4044	0.4044	0.5487	3.5114	0.5114	0.5225
500	λ	3.0261	0.0261	0.1840	3.0029	0.0029	0.1756
	α	2.0325	0.0325	0.1856	2.0116	0.0116	0.1831
	θ	2.5011	0.0011	0.1405	2.5033	0.0033	0.1406
		2.3011	0.3011	0.4205	3.2033	0.2033	0.4162

	β						
1000	λ	3.2149	0.0149	0.1380	3.0014	0.0014	0.1136
	α	2.0201	0.0201	0.1243	2.0006	0.0006	0.1067
	θ	2.5009	0.0009	0.1096	2.5010	0.0010	0.0891
	β	2.2029	0.2029	0.3147	3.1010	0.1010	0.2069

The table above shows the values of biases and RMSEs approach zero and the estimates tend to the initial (true) values as the sample increases, which indicates that the estimates are efficient and consistent.

7. Applications to Real Data

In this section, we fit the TIHLEtW distribution to two real data sets and give a comparative study with the fits to the Type II Exponentiated Half Logistic Weibull (TIIEHLW) distribution by Al-Mofleh *et al.* [7], Half-Logistic Generalized Weibull (HLGW) Distribution by Masood and Amna [23], Exponentiated Weibull (EW) by Pal *et al.* [28], Weibull Distribution by Xie and Lai [33] and Topp-Leone Generated Weibull (TLGW) Distribution by Aryal *et al.* [9] as comparator distributions for illustrative purposes.
The TIIEHLW distribution developed by Al-Mofleh *et al.* [7] has pdf defined as:

$$f(x;\alpha,\lambda,\beta,\theta)=2\alpha\lambda\beta\theta x^{\beta-1}e^{-\theta x^{\beta}}\left[1-e^{-\theta x^{\beta}}\right]^{\lambda-1}\frac{\left[1-\left[1-e^{-\theta x^{\beta}}\right]^{\lambda}\right]^{\alpha-1}}{\left[1+\left[1-e^{-\theta x^{\beta}}\right]^{\lambda}\right]^{\alpha+1}} \tag{32}$$

The HLGW distribution developed by Masood and Amna [23] has pdf defined as:

$$f(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta x^{\alpha-1}\left[1+\theta x^{\alpha}\right]^{\lambda-1}exp\left[1-\left[1+\theta x^{\alpha}\right]^{\lambda}\right]}{\left[1+exp\left[1-\left[1+\theta x^{\alpha}\right]^{\lambda}\right]\right]^{2}} \tag{33}$$

The EW distribution proposed by Pal *et al.*, [28] has pdf given as:

$$f(x;\alpha,\lambda,\beta)=\alpha\lambda^{\beta}\beta x^{\beta-1}\left[1-exp(-\lambda x)^{\beta}\right]^{\alpha-1}exp(-\lambda x)^{\beta} \tag{34}$$

The Weibull Distribution proposed by Xie and Lai [33] has pdf given as:

$$f(x;\theta,\beta)=\theta\beta x^{\beta-1}e^{-\theta x^{\beta}} \tag{35}$$

The TLGW distribution developed by Aryal *et al.*, [9] has pdf defined as:

$$f(x;\alpha,\theta,\beta,\lambda)=2\alpha\theta\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta\alpha-1}$$
$$\left[1-\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta}\right]\left[2-\left[1-e^{-(\lambda x)^{\beta}}\right]^{\theta}\right]^{\alpha-1} \tag{36}$$

The two datasets used as examples in the application demonstrate the new proposed distribution flexibility, applicability, and "best fit" in modeling the datasets empirically when compared to the above comparator distributions. All of the calculations are performed using the R programming language.

Data set 1

The first data set shown below represents the remissions times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients, previously used by Lee and Wang [22]:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

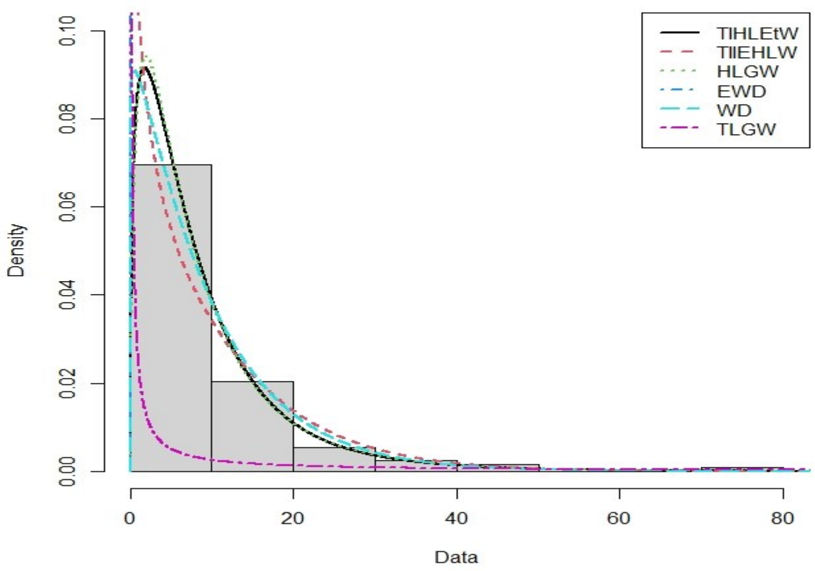

Figure 3: Fitted pdfs for the TIHLEtW, TIIEHLW, HLGW, EWD, WD, and TLGW distributions to the data set 1

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	λ	θ	β	LL	AIC
TIHLEtW	3.4827	1.1097	0.7581	0.5426	-410.6609	829.3218
TIIEHLW	0.2368	0.8929	0.2634	1.1245	-418.4258	844.8516
HLGW	1.0581	0.6613	0.2868		-412.4861	830.9721
EWD	1.1545	0.1188		0.9861	-413.1202	832.2403
WD			0.0939	1.0478	-414.0869	832.1738
TLGW	6.6269	0.0219	4.1785	0.2522	-442.2653	880.5306

Table 2 presents the results of the Maximum Likelihood Estimation of the parameters of the new proposed distribution and the five comparator distributions. Based on the goodness of fit measure, the new proposed distribution reported the minimum AIC value, though followed closely by the HLGW. The visual inspection of the fit presented in Figure 3, also confirms the superiority of the proposed distribution amongst its comparators. Thus the new proposed distribution 'best fit' bladder cancer patients data set amongst the range of distributions considered.

Data set 2

The second data set shown below represents the life times data relating to times (in months from 1st January, 2013 to 31st July, 2018) of 105 patients who were diagnosed with hypertension and received at least one treatment related to hypertension in the hospital where death is the event of interest, previously used by Umeh and Ibenegbu [31]:
45, 37, 14, 64, 67, 58, 67, 55, 64, 62, 9, 65, 65, 43, 13, 8, 31, 30, 66, 9, 10, 31, 31, 31, 46, 37, 46, 44, 45, 30, 26, 28, 45, 40, 47, 53, 47, 41, 39, 33, 38, 26, 22, 31, 46, 47, 66, 61, 54, 28, 9, 63, 56, 9, 49, 52, 58, 49, 53, 63, 16, 67, 61, 67, 28, 17, 31, 46, 52, 50, 30, 33, 13, 63, 54, 63, 56, 32, 33, 37, 7, 56, 1, 67, 38, 33, 22, 25, 30, 34, 53, 53, 41, 45, 59, 59, 60, 62, 14, 57, 56, 57, 40, 44, 63.

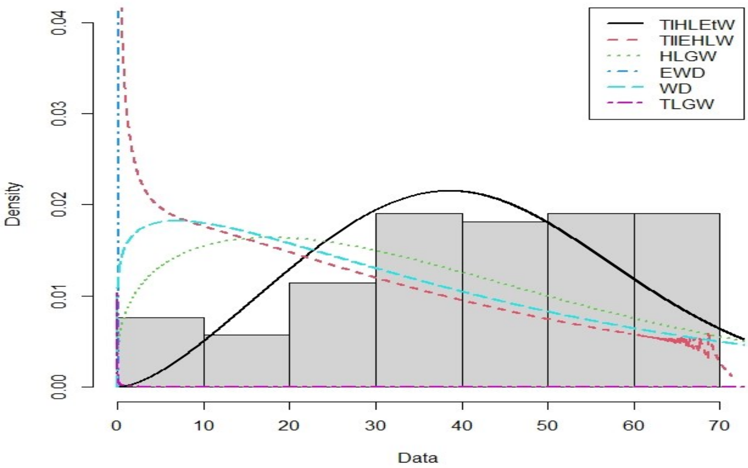

Figure 4: Fitted pdfs for the TIHLEtW, TIIEHLW, HLGW, EWD, WD, and TLGW distributions to the data set 2

Table 3: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	λ	θ	β	LL	AIC
TIHLEtW	0.3906	0.0877	0.0016	2.3771	-446.1673	900.3346
TIIEHLW	0.0487	0.5271	0.3378	1.1028	-495.9288	999.8576
HLGW	1.3181	0.7136	0.0175		-475.625	957.2501
EWD	5.0005	4.4779		0.1826	-464.4788	934.9575
WD			0.0138	1.1406	-487.8239	979.6479
TLGW	10.8957	0.0120	0.0624	8.2172	-471.7036	951.4072

Table 3 shows the results of the Maximum Likelihood Estimation of the parameters of the TIHLEtW distribution and the five comparator distributions. Based on the goodness of fit statistic AIC, the new distribution reported the minimum AIC value suggesting that the distribution is the 'best fit' to the hypertension patients. The visual inspection of the fit presented in Figure 4, also reaffirms the superiority of the new distribution amongst its comparators.

8. CONCLUSION

In this article, we proposed and studied a new distribution called the Type I Half-Logistic Exponentiated Weibull Distribution using the family of distribution proposed by Bello *et al.* (2021). Explicit quantile function, probability weighted moments, moments, generating function, reliability function, hazard function, and order statistics were examined as statistical components of the new proposed distribution. The parameters are estimated using the maximum likelihood technique. We present some simulation results to evaluate the new distribution's performance. In comparison to well-known models, two real data sets are evaluated to highlight the importance and flexibility of the new distribution. The findings reveal that the new distribution appears to be superior to the existing models considered, implying that it can be used to model data in a variety of applications.

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