CERTAIN RESULTS OF ALEPH- FUNCTION BASED ON NATURAL TRANSFORM OF FRACTIONAL ORDER

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Abstract

The paper introduces a new type of fractional integral transform called the N-transform of fractional order. This transform is utilized to derive various results for a more generalized function of fractional calculus known as the Aleph-function. The authors present several useful findings and explore the relationship between the N-transform and other existing fractional transforms. Additionally, the paper discusses the relationship between the N-transform of fractional order and other existing fractional transforms. It likely explores how this new transform relates to established transforms in fractional calculus. The authors have also examined special cases or specific examples to further illustrate the applications and properties of the N-transform of fractional order. These cases could involve particular functions or parameter values that offer insight into the behavior of the transform.

Keywords: N-transform of fractional order, **L**-transform of fractional order, S-transform of fractional order Aleph-function

1. Introduction

Our translation of real world problems to mathematical expressions relies on calculus, which in turn relies on the differentiation and integration operations of arbitrary order with a sort of misnomer fractional calculus which is also a natural generalization of calculus and its mathematical history is equally long. It plays a significant role in number of fields such as physics, rheology, quantitative biology, electro-chemistry, scattering theory, diffusion, transport theory, probability, elasticity, control theory, engineering mathematics and many others. Fractional calculus like many other mathematical disciplines and ideas has its origin in the quest of researchers for to expand its applications to new fields. This freedom of order opens new dimensions and many problems of applied sciences can be tackled in more efficient way by means of fractional calculus.

Laplace and Sumudu transformations are closely linked to the natural transform. The Natural transform, also known as the N-transform, was initially introduced by Khan and Khan [6]; Al-Omari [1]; Belgacem and Silambarasan [3] explored its features. Maxwell's equations were solved using the Natural transform in Belgacem and Silambarasan [11] and [2]. Transform methods for solving partial differential equations discussed by Duffy [5]. Sharma and Shekhawat [8] obtained integral transform and the Solution of Fractional Kinetic Equation Involving Some Special Functions

Belgacem and Silambarasan's [4] works on the Natural transform can be found here [11] for more information. If we assume that the function is fractional derivative and continuous, the Natural transform often works with continuous and continuously differentiable functions. The Natural transform, like the Laplace and Sumudu transforms, does not work since the function is not derivative. In a similar vein, we must establish a new term that we will call fractional Natural transform.

2. Definitions and Preliminaries

2.1 Natural transform

In mathematics, the natural transform is an integral transform similar to the Laplace transform and Sumudu transform, introduced by khan and khan [6]. It converges to both Laplace and Sumudu transform just by changing variables. Given the convergence to the Laplace and Sumudu transforms, the N-transform inherits all the applied aspects of the both transforms. Most recently, Belgacem [11] has renamed it the natural transform and has proposed a detail theory and applications. The natural transform of a function f(t), defined for all real numbers t ≥ 0, is the function R(u, s), defined by:

$$R(u,s) = N[f(t)] = \int_0^\infty e^{-st} f(ut)dt \text{ , } Re(S) > 0, u(-\tau_1, \tau_2) \quad (1)$$

Provided the function $f(t) \in R^2$ *is defined in the set*

$$A = \{ f(t) \mid \exists M, \tau_1, \tau_2 > 0. |f(t)| < M \ e^{\frac{|t|}{\tau j}} \} \quad (2)$$

Khan [6] showed that the above integral converges to Laplace transform when $u = 1$, and into Spiegel [7] transform for $s = 1$.

2.2 Fractional Natural transform of order α

$$R_\alpha^+(u,s) = N_\alpha^+[f(x)] = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(ux)(dx)^\alpha \text{ , } 0 < \alpha \le 1 \quad (3)$$

or

$$R_\alpha^+(u,s) = \lim_{M \to \infty} \int_0^M E_\alpha(-s^\alpha x^\alpha) f(ux)(dx)^\alpha \quad (4)$$

where s, u ∈ ℂ, and E_α(x) is the Mittag–Leffler function, $E_\alpha(z) = \sum_{n=0}^\infty \frac{x^n}{\alpha n!}$

2.3 Fractional Laplace transform reported by Estrin and Higgins [10]

From the above definition, when u = 1

$$L_\alpha^+(1, s) = L_\alpha^+[f(x)] = \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha \ , 0 < \alpha \leq 1 \tag{5}$$

or

$$L_\alpha^+(1, s) = \lim_{M\to\infty} \int_0^M E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha \tag{6}$$

Where, s ∈ ℂ, and E_α(x) is the Mittag–Leffler function, $E_\alpha(z) = \sum_{n=0}^\infty \frac{x^n}{\alpha n!}$

2.4 Fractional Sumudu transform

From the above definition, when S = 1

$$S_\alpha^+(u, 1) = S_\alpha^+[f(x)] = \int_0^\infty E_\alpha(-x^\alpha) f(ux)(dx)^\alpha \ , 0 < \alpha \leq 1 \tag{7}$$

or

$$S_\alpha^+(u, 1) = \lim_{M\to\infty} \int_0^M E_\alpha(-x^\alpha) f(ux)(dx)^\alpha \tag{8}$$

where u ∈ ℂ, and E_α(x) is the Mittag–Leffler function , $E_\alpha(z) = \sum_{n=0}^\infty \frac{x^n}{\alpha n!}$

3. Aleph-function

The Aleph-function is defined in terms of the Mellin-Barnes type integral in the following manner is

$$\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\sum_{i=1}^r \tau_i \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + B_{ji}s)\prod_{j=n+1}^{p_i} \Gamma[a_{ji} - A_{ji}s]} z^s \mathrm{ds} \tag{9}$$

Lemma 3.1: For instance the fractional natural transform of the $f(x) = x^{n\alpha}$, $n \in N$ then

$$N_\alpha^+[\,x^{n\alpha}] = \int_0^\infty E_\alpha(-s^\alpha x^\alpha)\,(ux)^{n\alpha}(dx)^\alpha = u^{n\alpha}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)\,(x)^{n\alpha}(dx)^\alpha \tag{10}$$

We put $t = xs$. we get

$$N_\alpha^+[\,x^{n\alpha}] = \frac{u^{n\alpha}}{s^{(n+1)\alpha}}\int_0^\infty E_\alpha(-t^\alpha)\,(t)^{n\alpha}(dt)^\alpha \tag{11}$$

or

$$N_\alpha^+[\,x^{n\alpha}] = \frac{(\alpha!)u^{n\alpha}}{s^{(n+1)\alpha}}\ \Gamma_\alpha(\mathrm{n}+1) \tag{12}$$

Note: $\Gamma_\alpha(\mathrm{n}) = \frac{1}{(\alpha!)}\int_0^\infty E_\alpha(-x^\alpha)\,(x)^{(n-1)\alpha}(dx)^\alpha$

Lemma 3.2**:** For instance the fractional Laplace transform of the $f(x) = x^{n\alpha}$, $n \in N$ then

$$L_\alpha^+[\,x^{n\alpha}] = \int_0^\infty E_\alpha(-s^\alpha x^\alpha)\,(x)^{n\alpha}(dx)^\alpha \tag{13}$$

We put $t = xs$. we get

$$L_\alpha^+[\,x^{n\alpha}] = \frac{1}{s^{(n+1)\alpha}}\int_0^\infty E_\alpha(-t^\alpha)\,(t)^{n\alpha}(dt)^\alpha \tag{14}$$

or

$$L_\alpha^+[\,x^{n\alpha}] = \frac{(\alpha!)}{s^{(n+1)\alpha}}\ \Gamma_\alpha(\mathrm{n}+1) \tag{15}$$

Note: $\Gamma_\alpha(\mathrm{n}) = \frac{1}{(\alpha!)}\int_0^\infty E_\alpha(-x^\alpha)\,(x)^{(n-1)\alpha}(dx)^\alpha$

Lemma 3.3: For instance the fractional Sumudu transform of the $f(x) = x^{n\alpha}$, $n \in N$ then

$$S_\alpha^+[\,x^{n\alpha}] = \int_0^\infty E_\alpha(-x^\alpha)\,(ux)^{n\alpha}(dx)^\alpha = u^{n\alpha}\int_0^\infty E_\alpha(-x^\alpha)\,(x)^{n\alpha}(dx)^\alpha \tag{16}$$

We put $t = x$, we get

$$S_\alpha^+[\,x^{n\alpha}] = u^{n\alpha}\int_0^\infty E_\alpha(-t^\alpha)\,(t)^{n\alpha}(dt)^\alpha \tag{17}$$

or

$$S_\alpha^+[\,x^{n\alpha}] = (\alpha!)u^{n\alpha}\ \Gamma_\alpha(\mathrm{n}+1) \tag{18}$$

Note: $\Gamma_\alpha(\mathrm{n}) = \frac{1}{(\alpha!)}\int_0^\infty E_\alpha(-x^\alpha)\,(x)^{(n-1)\alpha}(dx)^\alpha$

4. Some Main Transformations

4.1 Fractional natural transform of order α

In this section, we derived the fractional natural transform of order α in relationship with the known generalized function of fractional calculus known as Aleph-function.

Theorem 4.1.1: Let $N_\alpha^+[f(x)]$, $0<\alpha\le 1$, be the fractional natural transform of order α associated with Aleph-function . Then there holds the following relationship

$$N_\alpha^+\left\{\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right\}\right\}=$$
$$\frac{1}{s}\aleph_{p_i,q_i;\tau_i;r}^{m,n+1}\left[\frac{u}{s}\left|\begin{matrix}(0,1)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right] \tag{19}$$

Provided the function $f(t)\in R^2$.

Proof: By using the definition of the generalized function of fractional Aleph -function and fractional natural transform of order α we get

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$
$$N_\alpha^+\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}z^k\mathrm{dk}\right\}\quad;\ \mathrm{Re}(\alpha)>0 \tag{20}$$

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$
$$\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\right\}\ N_\alpha^+\{\mathrm{z^k}\} \tag{21}$$

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$
$$\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\right\}\ N_\alpha^+\{\mathrm{z^k}\} \tag{22}$$

By making use of lemma –3.1 in above equation, we get

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$
$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\frac{u^k}{s^{(k+1)}}\ \ \Gamma(\mathrm{k}+1) \tag{23}$$

or

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\left|\begin{matrix}(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$
$$\frac{1}{s}\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)\Gamma(1-0+\mathrm{k})}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\frac{u^k}{s^k} \tag{24}$$

or

$$N_\alpha^+\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right]\right\}=$$
$$\frac{1}{s}\aleph_{p_i,q_i;\tau_i;r}^{m,n+1}\left[\frac{u}{s}\left|\begin{matrix}(0,1)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}\end{matrix}\right.\right] \tag{25}$$

This completes proof of theorem.

4.2 Fractional Laplace transform of order α

In this section, we derived the fractional Laplace transform of order α in relationship with the known function of fractional calculus known as Aleph-function.

Theorem 4.2.1: Let $L_\alpha^+[f(x)]$, $0<\alpha\leq 1$, be the fractional Laplace transform of order α associated with Aleph-function . Then there holds the following relationship

$$L_\alpha^+\left\{\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}= \frac{1}{s}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n+1}\left[s^{-1}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(1,0)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right] \tag{26}$$

Provided the function $f(t)\in R^2$.

Proof: By using the definition of the generalized function of fractional ML -function and fractional Laplace transform of order α we get

$$L_\alpha^+\left\{\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}= L_\alpha^+\left\{\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}z^k\,\mathrm{dk}\right\}\quad ;\ \mathrm{Re}(\alpha)>0 \tag{27}$$

$$L_\alpha^+\left\{\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\ \ L_\alpha^+\{z^k\} \tag{28}$$

$$L_\alpha^+\left\{\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\ L_\alpha^+\{z^k\} \tag{29}$$

By making use of lemma –3.2 in above equation, we get

$$L_\alpha^+\left\{\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{dk}\frac{1}{s^{(k+1)}}\ \ \Gamma(\mathrm{k}+1) \tag{30}$$

or

$$L_\alpha^+\left\{\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-B_jk)\prod_{j=1}^n\Gamma(1-a_j+A_jk)\Gamma(\mathrm{k}+1)\Gamma(1-0+\mathrm{k})}{\sum_{i=1}^r\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\frac{1}{s^{(k+1)}}\mathrm{dk} \tag{31}$$

$$L_\alpha^+\left\{\aleph_{p_i,q_{i;\tau_i;r}}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}= \frac{1}{s}\aleph_{p_i,q_{i;\tau_i;r}}^{m,n+1}\left[s^{-1}\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(1,0)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right] \tag{32}$$

This completes proof of theorem.

4.3 Fractional Sumudu transform of order α

In this section, we derived the fractional Sumudu transform of order α in relationship with the known function of fractional calculus known as ML-function.

Theorem 4.3.1: Let $S_\alpha^+[f(x)]$, $0<\alpha\leq 1$, be the fractional Sumudu transform of order α associated with Aleph-function . Then there holds the following relationship

$$S_{\alpha}^{+}\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}=\frac{1}{s}\,\mathrm{E}_{\alpha}^{1}\left(\frac{u}{s}\right) \tag{33}$$

Provided the function $f(t)\in R^2$.

Proof: By using the definition of the generalized function of fractional Aleph -function and fractional Sumudu transform of order α we get

$$S_{\alpha}^{+}\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}=$$

$$S_{\alpha}^{+}\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-B_jk)\prod_{j=1}^{n}\Gamma(1-a_j+A_jk)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}z^k\,\mathrm{d}k;\quad \mathrm{Re}(\alpha)>0 \tag{34}$$

$$S_{\alpha}^{+}\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-B_jk)\prod_{j=1}^{n}\Gamma(1-a_j+A_jk)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{d}k\;\; S_{\alpha}^{+}\{z^k\} \tag{35}$$

$$S_{\alpha}^{+}\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-B_jk)\prod_{j=1}^{n}\Gamma(1-a_j+A_jk)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{d}k S_{\alpha}^{+}\{\mathrm{z}^{\mathrm{k}}\} \tag{36}$$

By making use of lemma –3.3 in above equation, we get

$$S_{\alpha}^{+}\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-B_jk)\prod_{j=1}^{n}\Gamma(1-a_j+A_jk)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}\mathrm{d}k u^k\;\;\Gamma(\mathrm{k}+1) \tag{37}$$

or

$$S_{\alpha}^{+}\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}=$$

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-B_jk)\prod_{j=1}^{n}\Gamma(1-a_j+A_jk)\Gamma(1-0+\mathrm{k})}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+B_{ji}k)\prod_{j=n+1}^{p_i}\Gamma[a_{ji}-A_{ji}k]}u^k\mathrm{d}k \tag{38}$$

$$S_{\alpha}^{+}\left\{\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[Z\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right]\right\}=$$

$$\aleph_{p_i,q_i;\tau_i;r}^{m,n+1}\left[u\Big|_{(b_j,B_j)_{1,m}[\tau_i(b_j,B_{ji})]_{m+1,q_i}}^{(0,1)(a_j,A_j)_{1,n}[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}}\right] \tag{39}$$

This completes proof of theorem.

5. Special Cases

In this section, we discuss some of the important special cases of the main results established discussed above, If we take $\propto=\tau_i=1$ in the theorems (4.1.1), (4.2.1) and (4.3.1) we get well known results of ordinary calculus like Natural transform of Saxen's i-function, Laplace transform of Saxen's i-function and finally ordinary Sumudu transform of Saxen's i-function as reported in [9].

6. Conclusion

This paper introduces a novel type of fractional integral transform called the N-transform of fractional order. This transform is proposed as a new addition to the theory of fractional order transforms. The paper emphasizes that the contributions made by this new transform are believed to be significant and offer valuable insights to the field. Furthermore, the paper suggests that the N-transform of fractional order has potential applications in solving fractional differential and integral equations. By utilizing this model, it may be possible to find solutions or approaches for various equations involving fractional derivatives or integrals.

Conflict of Interest

The authors declare that they have no conflicts of interest and no personal relationships that could have appeared to influence the work.

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