# THE EFFICIENCY OF ESTIMATING A POPULATION AVERAGE USING INDEX-TYPE ESTIMATORS IN SEQUENTIAL SAMPLING

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**Abstract**

*This paper discusses the difficulty of approximating the population average of a variable y by knowledge about a supplementary variable x in the context of two successive (rotation) sampling occasions. The paper proposes a group of exponential-class estimators that includes the regular balanced estimator, produce-class estimator, and proportion-class estimator and suggests that these estimators are superior to existing estimators. The paragraph also mentions that the paper discusses optimal substitute statements and then the implementation of the recommended estimators, which may be important considerations for practical applications of the proposed methods. Finally, an empirical study is mentioned as supporting evidence for the research.*

**Keywords:** Auxiliary variable, Study variable, Bias, Mean squared error, Successive sampling

## 1. Introduction

It is general towards procedure the supplementary knowledge at the approximation stand to achieve developed estimations of the population average y of the analyze variate x. Auxiliary information can be valuable in improving estimates of population parameters by incorporating additional relevant data. Ratio, Product, and Regression methods are good examples in this context. This statement is also generally true. Ratio, Product, and Regression methods are commonly used techniques for estimating auxiliary information. When the auxiliary variable x is positively (high) correlated with the study variable y, the Ratio estimation method is quite effective. The Ratio estimation method is effective when the auxiliary variable and the study variable have a positive correlation. When there is a negative correlation between the auxiliary variable and the study variable, the Product estimation method is commonly employed. The theory of successive sampling has been developed by various researchers, starting with [1] and followed [2], [4], [5] and several others. The mentioned researchers have indeed contributed to developing the theory of successive sampling.

Auxiliary information was utilized by [6] and [7] multiple times to estimate the current population means in successive sampling. Supplementary information implemented by [6] and [7] within the framework of subsequent selection to estimate the population mean on the current occasion. Recent research conducted by [8] where they utilized auxiliary information on both events and proposed various estimators for estimating the population mean on the current occasion within the framework of two-occasion successive (rotation) sampling. In their work on two-occasion subsequent selection, [8] employed auxiliary information to develop estimators designed explicitly for estimating the population mean on the current occasion. These researchers have successfully used supplementary information and developed various estimation methods to evaluate the population mean on the recent occurrence in the context of successive sampling.

## 2. Introduces the formulation and notation of the proposed estimator

Consider a set $V = \{v_1, v_2, \ldots\ldots v_K\}$ Consisting of K elements. Let $[y, x]$ Represent the study and auxiliary variables, respectively, with $[y_i, x_i]$ denoting the values on the $i^{th}$ unit $V_i\{i = 1,2,3, \ldots\ldots K\}$ of the population. Furthermore, let $[\hat{y}, \hat{x}]$ Represent the population values of $[y, x]$ Respectively. In this scenario, we assume that the population mean$\hat{X}$ Of the auxiliary variable $x$ is already known. Subsequently, to estimate the population mean $\hat{Y}$ Of the study variable $y$, a simple random sample of size $k$ is selected from the population V without replacement. The Classical Ratio Estimator and the Ordinary Product Estimator are two methods used to estimate the population mean $\hat{Y}$. The Classical Ratio Estimator is defined as $l_F = \tilde{F}\hat{X}$, where $\tilde{F} = \frac{\hat{y}}{\hat{x}}$ $\hat{x} \neq 0$ is the estimate of the Ratio F of the population means. The unweighted sample means y and x,, represented as $\hat{y} = \frac{1}{k}\sum_{i=1}^{k} y_i$ and $\hat{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$, respectively, are utilized. The efficiency of the classical ratio estimator relies on a strong positive correlation between the variables y and x. On the other hand, the Ordinary Product Estimator for $\hat{Y}$ is defined as $l_S = \frac{\tilde{S}}{\hat{X}}$ where $\tilde{S} = \hat{y}.\hat{x}$ is the estimate of the Product S of the population means. The Product Estimator is commonly employed when there is an expectation of a strong negative correlation between the two variables.

The research on estimating the population mean $\hat{y}$ has led to the proposal of different estimators by various researchers. The Estimator $l_S$, credited to [9] and revisited by [10] is one such example. Additionally, [11] suggested Ratio and Product-Type Exponential Estimators for estimating the population mean $\hat{Y}$.

The Ratio-Type Exponential Estimator $l_{Fe} is\ defined\ as\ \hat{y}\ e^{\left[\frac{\hat{X}-\hat{x}}{\hat{X}+\hat{x}}\right]}$, while the Product-Type Exponential Estimator $l_{Se}\ is\ defined\ as\ \hat{y}\ e^{\left[\frac{\hat{x}-\hat{X}}{\hat{X}x+\hat{x}}\right]}$ The simple Expansion Estimator $l_0$, which is used for estimating the population mean $\hat{Y}$ When none of the previously mentioned estimators are suitable. The Expansion Estimator is defined as$l_0 = \hat{y} = \frac{1}{k}\sum_{i=1}^{k} y_i$Which is the unweighted sample mean of $y$. Overall, the sentence accurately presents the contributions of different researchers in proposing various estimators for the population mean $\hat{y}$.

It discusses the findings of [12] regarding the variability of sample means and the coefficients of variation. The variability of the sample mean $\hat{x}$ is usually less than that of the sample mean $\hat{y}$ . If then introduces $D[\hat{x}]$ as the coefficient of variation of $\hat{x}$ and $D[\hat{y}]$ as the coefficient of variation of $\hat{y}$ . The sentence further provides the relationships between the squared coefficients of variation and the population coefficients of variation. It states that $D^2[\hat{x}] = \frac{[1-g]}{k} D_x^2$ and $D^2[\hat{y}] = \frac{[1-g]}{k} D_y^2$ where $g = \frac{k}{K}$ is the sampling fraction, $D_x = \frac{T_x}{\hat{X}}$ is the population coefficient of variation for x, and $D_y = \frac{T_y}{\hat{Y}}$ is the population coefficient of variation for y, $T_x^2 = [K-1]^{-1} \sum_{i=1}^{K} [x_i - \hat{X}]^2$ and $T_y^2 = [K-1]^{-1} \sum_{i=1}^{K} [y_i - \hat{Y}]^2$ Represent the sums of squared deviations from the respective population means. If $D_x = bD_y$, where b is a constant between 0 and 1, then $D[\hat{x}] = bD[\hat{y}]\,; 0 < b \le 1$ This implies that the coefficient of variation for $\hat{x}$ is proportional to the coefficient of variation for $\hat{y}$. [10] findings on the variability of sample means and the coefficients of variation explicitly focus on the relationship between $D[\hat{x}]$ and $D[\hat{y}]$ when $D[\hat{x}] = bD[\hat{y}]$.

$$l_F \quad if \ \frac{\gamma}{b} > 0.5,$$
$$\hat{y} \ if - 0.5 \le \frac{\gamma}{b} \le 0.5,$$
$$l_F \quad if \ \frac{\gamma}{b} < -0.5\,,$$

Where, $\gamma = \frac{T_{xy}}{T_x T_y}$ and $T_{xy} = [K-1]^{-1} \sum_{i=1}^{K} [x_i - \hat{X}]\,[y_i - \hat{Y}]$

$$\frac{\gamma}{b} \ge -\frac{1}{b} \ and \ \frac{\gamma}{b} \le \frac{1}{b} \ \ as \ |\gamma| \le 1$$

## 3.The Suggested Estimator

Motivated by [13] based on that motivation, they derived a modified Exponential-Type Estimator for estimating the population mean $\hat{Y}$ as

$$l_{Me} = \hat{y} e^{\left\{\frac{[\hat{x}+\emptyset\hat{X}]-[X+\emptyset\hat{x}]}{[\hat{x}+\emptyset\hat{X}]+[X+\emptyset\hat{x}]}\right\}}$$
$$= \hat{y}\, e^{\left\{\frac{\hat{x}\,[1-\emptyset]+[\emptyset-1]\hat{X}}{\hat{x}\,[1+\emptyset]+[\emptyset+1]\hat{X}}\right\}}$$
$$= \hat{y}\ e^{\left\{\frac{[\emptyset-1]\hat{X}-[\emptyset-1]\hat{x}}{[\emptyset+1]\hat{X}+[\emptyset+1]\hat{x}}\right\}}$$
$$= \hat{y}\, e^{\left\{\frac{[\emptyset-1][\hat{X}-\hat{x}]}{[\emptyset+1][\hat{X}+\hat{x}]}\right\}}$$

Where $\emptyset$ is a scalar used as a design parameter. for $\emptyset = 1$ $l_{Me} = \hat{y}$ $\emptyset = 0, l_{Me} = l_{Pe}$ the value of $\emptyset$ (presumably a variable or parameter) is very large, the proposed estimator

$$\lim_{\emptyset\to\infty} l_{Me} = \lim_{\emptyset\to\infty} \hat{y}\, e^{\left\{\frac{[\emptyset-1][\hat{X}-\hat{x}]}{[\emptyset+1][\hat{X}+\hat{x}]}\right\}}$$
$$\cong l_{Fe} = \hat{y}\, e^{\left\{\frac{[\hat{X}-\hat{x}]}{[\hat{X}+\hat{x}]}\right\}}$$

### 3.1.Sampling bias and mean squared error of estimator

Sampling bias refers to the systematic error when the sample used for estimation does not represent the studied population. It can occur due to non-random sampling, non-response bias, or selection bias. Understanding and addressing sampling bias is crucial for obtaining reliable and accurate estimators.

$$\hat{y} = \hat{Y}[1+e_0],\ \hat{x} = \hat{X}[1+e_1] \ni H[h_0] = H[h_1] = 0\ \&$$
$$H[h_0^2] = \frac{[1-g]}{k} D_y^2,\ H[h_1^2] = \frac{[1-g]}{k} D_x^2,\ H[h_0 h_1] = \frac{[1-g]}{k} \gamma D_y D_x$$
$$l_{Mh} = \hat{Y}[1+h_0] e^{\left\{\frac{-[\emptyset-1]h_1}{[\emptyset+1][2+e_1]}\right\}}$$
$$= \hat{Y}[1+h_0] e^{\left\{\frac{[1-\emptyset]}{[1+\emptyset]}\frac{h_1}{2}\left[1+\frac{h_1}{2}\right]^{-1}\right\}}$$
$$= \hat{Y}[1+h_0] e^{\left\{\frac{Hh_1}{2}\left[1+\frac{h_1}{2}\right]^{-1}\right\}} \quad (1)$$

Where $H = \frac{[1-\emptyset]}{[1+\emptyset]}$

$$l_{Mh} \cong \hat{Y}\left[1 + h_0 + \frac{Rh_1}{2} + \frac{Rh_0h_1}{2} + \frac{G[G-2]}{8} h_1^2\right] \quad (2)$$
$$\left[l_{Mh} - \hat{Y}\right] \cong \hat{Y}\left[h_0 + \frac{Rh_1}{2} + \frac{Rh_0h_1}{2} + \frac{R[R-2]}{8} h_1^2\right] \quad (3)$$
$$A[l_{Mh}] = \hat{Y}\frac{[1-g]}{k}\left[\frac{R\gamma D_y D_x}{2} + \frac{R[R-2]D_x^2}{8}\right] \quad (4)$$
$$= A_0\left[\frac{R}{2}\right]\left[\gamma + \frac{[R-2]}{4b}\right]$$

Where $A_0 = \frac{b(1-g)\hat{Y}C_y^2}{k}$ and $A_0 = \frac{(1-\emptyset)}{(1+\emptyset)}$

$$\left(l_{Me} - \hat{Y}\right)^2 \cong \hat{Y}^2\left[h_0^2 + \frac{R^2h_1^2}{4} + Re_0e_1\right] \quad (5)$$

$$\text{MSE}(l_{Mh}) = \text{MSE}(l_{Mh})_I = \frac{(1-g)}{k}\hat{Y}^2\left[D_y^2 + R\gamma D_y D_x + \frac{R^2D_x^2}{4}\right] \quad (6)$$
$$= U_0\left[1 + \frac{R^2b^2}{4} + bR\gamma\right]$$

Where $U_0 = \frac{(1-g)T_y^2}{k} = Var(\hat{y})$

The MSE $(l_{Mh})$ is minimum when $R_0 = -2\left(\frac{\gamma}{b}\right)$

$$\text{MSE of } (l_{Mh}) = U_0(1-\gamma^2) \quad (7)$$
$$\widehat{y_a} = \hat{y} + \hat{\beta}\left(\hat{X} - \hat{x}\right) \quad (8)$$

**Table 1:** PREs Estimator $l_{Mh}$ is better than $\hat{y}$, $l_F$, $l_S$ $l_{Fh}$ and $l_{Sh}$

| **Estimator** | **PREs** $(*\,\hat{y})$ | | | | |
|---|---|---|---|---|---|
| | Population | | | | |
| | 1 | 2 | 3 | 4 | 5 |
| $\hat{y}$ | 100.0000 | 100.0000 | 100.0000 | 100.0000 | 100.0000 |
| $l_F$ | 66.5810 | 30.5860 | 156.3967 | 31.1061 | 56.2431 |
| $l_S$ | 10.5463 | 7.6514 | 25.8171 | 92.9342 | 167.5887 |
| $l_{Fh}$ | 781.3982 | 292.0779 | 197.7846 | 54.9135 | 74.5067 |
| $l_{Sh}$ | 24.2836 | 19.0754 | 47.1121 | 133.0386 | 133.064 |

## 3.2. Effectiveness Relationship

Scalar R(or∅)" and true optimum value:

$$\text{Var}\quad(\hat{y}) = MSE(\hat{y}) = \left(\frac{1-g}{n}\right)T_y^2 = \left(\frac{1-g}{n}\right)\hat{y}^2C_y^2 = U_0 \quad (9)$$
$$A(l_F) = A_0(A-\gamma)$$
$$A(l_S) = A_0\gamma$$
$$A(l_{Fe}) = \left(\frac{A_0}{8}\right)(3b - 4\gamma) \quad (10)$$
$$A(l_{Fe}) = \left(\frac{A_0}{8}\right)(4\gamma - b)$$
$$\text{MSE}(l_F) = U_0[1 + b^2 - 2\gamma b]$$

$$\text{MSE}= U_0[1 + b^2 + 2\gamma b]$$
$$\text{MSE}(l_S) = U_0\left[1 + \left(\frac{b}{4}\right)(b - 4\gamma)\right]$$
$$\text{MSE}(l_S) = U_0\left[1 + \left(\frac{b}{4}\right)(b + 4\gamma)\right] \quad (11)$$
Where $A_0 = \frac{b(1-g)}{k}\hat{Y}D_y^2$
$$MSE(\hat{y}) - \text{min}.MSE(l_{Me}) = U_0\,\gamma^2 \geq 0 \quad (12)$$
$$MSE(l_R) - \text{min}.MSE(l_{Me}) = U_0\,(a-\gamma)^2 \geq 0 \quad (13)$$
$$MSE(l_S) - \text{min}.MSE(l_{Me}) = U_0\,(a+\gamma)^2 \geq 0 \quad (14)$$
$$MSE(l_{Re}) - \text{min}.MSE(l_{Me}) = U_0\left(\frac{a}{2}-\gamma\right)^2 \geq 0$$
$$MSE(l_{Re}) - \text{min}.MSE(l_{Me}) = U_0\left(\frac{a}{2}+\gamma\right)^2 \geq 0 \quad (15)$$

- The usual Unbiased Estimator $\hat{y}$ does not require the correlation between the study variable y and the auxiliary variable $x$ to be zero. when $\gamma = 0$
- Usual Ratio Estimator $l_R$) and mentioning a condition when $b = \gamma$; $i$n this condition, it states that both the Estimators $l_S$ and $l_{Me}$ are equally efficient.
- Ratio-Type Exponential Estimator [11], denoted as $l_{Re}$, does not hold when $b = 2\gamma$ , which represents the scenario where both the estimators $l_{Fe}$ and $l_{Me}$ exhibit equal efficiency.
- Product-Type Exponential Estimator [11], referred to as $l_{Fe}$, is not applicable when $b = -2\gamma$ , which corresponds to the situation where both the estimators $l_{Se}$ and $l_{Me}$ demonstrate equal efficiency.

**Table 2:** The proposed estimator $l_{Mh}$ is superior to $\hat{y}$, $l_F$ , $l_S$ $l_{Fh}$ *and* $l_{Sh}$ when the value of R falls within a specific range.

| **Population** | **The proposed estimator $l_{Mh}$ Outperforms other estimators when the value of R lies within a specific range.** | | | | | **The standard range of R in which $l_{Mh}$ is better than $\hat{y}$, $l_F$ , $l_S$ $l_{Fh}$ and $l_{Sh}$** |
|---|---|---|---|---|---|---|
| | $\hat{y}$ | $l_F$ | $l_S$ | $l_{Fh}$ | $l_{Sh}$ | |
| 1 | (-1.7765,0) | (-2,0.2235) | (-3.7765,2) | (-1,-0.7765) | (-2.7765,1) | (-1,-0.7765) |
| 2 | (-1.3669,0) | (-2,0.6331) | (-3.3669,2) | (-1,-0.3669) | (-2.3669,1) | (-1,-0.3669) |
| 3 | (-2.5742,0) | (-2,-0.5742) | (-4.5742,2) | (-1.5742, -1) | (-3.5742,1) | (-1.5742,- 0.5740) |
| 4 | (0,1.8673) | (-2,3.8673) | (-0.1328,2) | (-1,2.8673) | (0.8673,1) | (0.8673,1) |
| 5 | (0,6.3059) | (-2,8.3059) | (2,4.3054) | (-1,7.3059) | (1,5.3059) | (-1,4.3054) |

## 3.3.When the scalar R (or ∅) does not align with its actual optimum value:

$$MSE(l_{Me}) - Variance\ (\hat{y}) = U_0 b^2\left[\frac{R^2}{4} + \frac{R\gamma}{b}\right]$$
$$= U_0 b^2\left[\frac{R^2}{4} - \frac{RR_0}{b}\right]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 + R_0^2 - R_0^2]$$
$$= \frac{U_0 b^2}{4}[(R - R_0)^2 - R_0^2] \quad (16)$$
$$(R - R_0)^2 < R_0^2 \quad \text{i.e } |R - R_0| < |R_0^2|$$
$$min\left(0, -\frac{4\gamma}{b}\right) < R < max.\left(0, -\frac{4\gamma}{b}\right)$$
From eq. (15) and eq. (16), we have,

$$MSE(l_{Me}) - MSE\ (l_F) = U_0 b^2\left[\frac{R^2}{4} + \frac{R\gamma}{b} - 1 + \frac{2\gamma}{b}\right]$$
$$= U_0 b^2\left[\frac{R^2}{4} + \frac{R\gamma}{b} - 1 - R_0\right]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 - 4 - 4R_0]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 + R_0^2 - 4 - 4R_0 - R_0^2]$$
$$= \frac{U_0 b^2}{4}[(R-R_0)^2 - (2+R_0)^2\ ] \quad (17)$$
$(R-R_0)^2 < (2+R_0)^2$ i.e., $|R-R_0| < |2+R_0|$

min, $\left\{-2, 2\left(1-\frac{2\gamma}{b}\right)\right\} < R < max\left\{-2, 2\left(1-\frac{2\gamma}{b}\right)\right\}$

From eq. (16) and eq. (17), we have,

$$MSE(l_{Me}) - MSE\ (l_S) = U_0 b^2\left[\frac{R^2}{4} + \frac{R\gamma}{b} - 1 - \frac{2\gamma}{b}\right]$$
$$= U_0 b^2\left[\frac{R^2}{4} - \frac{RR_0}{2} - 1 + R_0\right]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 - 4 + 4R_0]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 + R_0^2 - 4 + 4R_0 - R_0^2]$$
$$= \frac{U_0 b^2}{4}[(R-R_0)^2 - (2-R_0)^2\ ] \quad (18)$$
$(R-R_0)^2 < (2-R_0)^2$ i.e., $|R-R_0| < |2-R_0|$

min, $\left\{-2, 2\left(1+\frac{2\gamma}{b}\right)\right\} < R < max\left\{-2, 2\left(1+\frac{2\gamma}{b}\right)\right\}$

From eq. (17) and eq. (18), we have,

$$MSE(l_{Me}) - MSE\ (l_{Fe}) = U_0 b^2\left[\frac{R^2}{4} + \frac{R\gamma}{b} - \frac{1}{4} + \frac{\gamma}{b}\right]$$
$$= U_0 b^2\left[\frac{R^2}{4} - \frac{RR_0}{2} - \frac{1}{4} - \frac{R_0}{2}\right]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 - 1 - 2R_0]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 + R_0^2 - 1 - 2R_0 - R_0^2]$$
$$= \frac{U_0 b^2}{4}[(R-R_0)^2 - (1+R_0)^2\ ] \quad (19)$$
$(R-R_0)^2 < (1+R_0)^2$ i.e., $|R-R_0| < |1+R_0|$

min, $\left\{-1, \left(1-\frac{4\gamma}{b}\right)\right\} < R < max\left\{-1, \left(1-\frac{4\gamma}{b}\right)\right\}$

From eq. (18) and eq. (19), we have

$$MSE(l_{Me}) - MSE\ (l_{Fe}) = U_0 b^2\left[\frac{R^2}{4} + \frac{R\gamma}{b} - \frac{1}{4} - \frac{\gamma}{b}\right]$$
$$= U_0 b^2\left[\frac{R^2}{4} - \frac{RR_0}{2} - \frac{1}{4} + \frac{R_0}{2}\right]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 - 1 + 2R_0]$$
$$= \frac{U_0 b^2}{4}[R^2 - 2RR_0 + R_0^2 - 1 + 2R_0 - R_0^2]$$
$$= \frac{U_0 b^2}{4}[(R-R_0)^2 - (1-R_0)^2\ ] \quad (20)$$
$(R-R_0)^2 < (1-R_0)^2$ i.e., $|R-R_0| < |1-R_0|$

min, $\left\{1-\left(1+\frac{4\gamma}{b}\right)\right\} < R < max\left\{1-\left(1+\frac{4\gamma}{b}\right)\right\}$

- The usual Unbiased Estimator $\hat{x}$ is applicable when $|R-R_0| < |R_0|$ Which can be equivalently expressed as min $\left\{\left(0, -\frac{4\gamma}{b}\right)\right\} < R < max\left\{\left(0, -\frac{4\gamma}{b}\right)\right\}$
- The usual Ratio Estimator $l_F$ is valid when $|R-R_0| < |2+R_0|$ , which can be alternately expressed as min $\left\{2, -2\left(1-\frac{2\gamma}{b}\right)\right\} < R < max\left\{-2,2\ \left(1-\frac{2\gamma}{b}\right)\right\}$
- The usual Product Estimator $l_S$ is valid when $|R-R_0| < |2-R_0|$, which can be alternately expressed as min $\left\{2, -2\left(1+\frac{2\gamma}{b}\right)\right\} < R < max\left\{-2,2\ \left(1+\frac{2\gamma}{b}\right)\right\}$

- The Ratio-Type Exponential Estimator $l_{Fe}$ introduced by Bah and Tuteja 1991[11] is applicable if $|R - R_0| < |1 + R_0|$, which can be equivalently expressed as $\min\left\{-1, \left(1 - \frac{4\gamma}{b}\right)\right\} < R < max\left\{-1, \left(1 - \frac{4\gamma}{b}\right)\right\}$ .
- The Product-Type Exponential Estimator $l_{Fe}$ introduced Bahl and Tuteja 1991[11] if $|R - R_0| < |1 - R_0|$ which can be equivalently expressed as $\min\left\{1 - \left(1 + \frac{4\gamma}{b}\right)\right\} < R < max\left\{1 - \left(1 + \frac{4\gamma}{b}\right)\right\}$

**Table 3:** PREs Estimator $l_{Mh}$ concerning $\hat{y}$, for different values of R

| R | ∅ | PREs $(*\,\hat{y})$ | | | | |
|---|---|---|---|---|---|---|
| | | Population | | | | |
| | | 1 | 2 | 3 | 4 | 5 |
| **-2.0000** | -3.0000 | 66.5810 | 30.5860 | 156.3967 | 31.1061 | 56.2431 |
| **-1.7500** | -3.6667 | 105.4984 | 45.4209 | 182.7966 | 35.5534 | 60.2317 |
| **-1.5000** | -5.0000 | 187.1938 | 73.6467 | 202.4393 | 40.8775 | 64.5841 |
| **-1.2500** | -9.0000 | 383.2829 | 135.4864 | 208.2654 | 47.2636 | 69.3319 |
| **-1.0000** | 0.0000 | 781.3982 | 292.0779 | 197.7846 | 54.9135 | 74.5067 |
| **-0.7500** | 7.0000 | 738.4366 | 585.7795 | 175.3442 | 64.0167 | 80.1386 |
| **-0.5000** | 3.0000 | 353.0569 | 448.2367 | 148.3074 | 74.6863 | 86.2537 |
| **-0.2500** | 1.6667 | 174.9985 | 200.1900 | 122.3232 | 86.8380 | 92.8714 |
| **0.0000** | 1.0000 | 100.0000 | 100.0000 | 100.0000 | 100.0000 | 100.0000 |
| **0.2500** | 0.6000 | 63.7373 | 57.9872 | 81.8494 | 113.0935 | 107.6315 |
| **0.5000** | 0.3333 | 43.8933 | 37.4100 | 67.4411 | 124.3406 | 115.7344 |
| **0.7500** | 0.1429 | 31.9699 | 26.0031 | 56.0835 | 131.5695 | 124.2463 |
| **1.0000** | 0.0000 | 24.2836 | 19.0754 | 47.1121 | 133.0386 | 133.0649 |
| **1.2500** | 0.1111 | 19.0533 | 14.5708 | 39.9776 | 128.3594 | 142.0399 |
| **1.5000** | -0.2000 | 15.3392 | 11.4840 | 34.2528 | 118.7285 | 150.9669 |
| **1.7500** | 0.2727 | 12.6097 | 9.2794 | 29.6138 | 106.2421 | 159.5862 |
| **2.0000** | -0.3333 | 10.5463 | 7.6514 | 25.8171 | 92.9342 | 167.5887 |

## 4. The precision of first-order approximations to mean squared errors (MSEs).

Once we have compared the MSEs of the Proposed Estimator and other estimators using first-order approximations, our attention turns towards evaluating the accuracy of these approximations by deriving second-order approximations for the MSEs. In this analysis, we assume that $C_x = C_x = C$,, taking a value of 1, and that the sample is drawn from a large Bivariate Normal population. However, it is essential to mention that more complex expressions are derived for other scenarios, as discussed in [13]

$$\vartheta_{30} = \vartheta_{03} = \vartheta_{12} = \vartheta_{21} = 0$$
$$\vartheta_{04} = \vartheta_{40} = 3D^4$$
$$\vartheta_{31} = \vartheta_{13} = 3\gamma D^4$$
$$\vartheta_{22} = (1 + 2\gamma^2)D^4$$

Where $E\left(e_0^i e_1^j\right) \cong \frac{\vartheta_{(i,j)}}{k^a}$ $, i, j = 0,1,2,4$ $\quad and\ a = 2\ for i + j = 4$

$$l_{Me} = \hat{Y}(1 + h_0)e^{\left[\frac{Rh_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right]}$$
$$= \hat{Y}(1 + h_0)\left\{1 + \frac{Rh_1}{2} + \frac{R(R-2)h_1^2}{8} + \frac{R(R^2-6R+6)h_1^3}{48} + \frac{R(R^3-12R^2+36R-24)h_1^4}{384}\right\}\text{-} \tag{21}$$
$$b_1 = \frac{R}{2}, \ b_2 = \frac{R(R-2)}{8}$$
$$b_3 = \frac{R(R^2-6R+6)h_1^3}{48}, \qquad b_4 = \frac{R(R^3-12R^2+36R-24)h_1^4}{384}$$

Then $l_{Me} = \hat{Y}(1 + h_0)[1 + bh_1 + b_2h_1^2 + b_3h_1^3 + bh_1^4 + \cdots]$

$$= \hat{Y}[1 + bh_1 + b_1h_0h_1 + bh_1^2 + b_2h_0h_1^2 + bh_1^3 + b_3h_0h_1^3 + b_4h_{1+\cdots\ldots}^4] \tag{22}$$

By disregarding terms of $h$'s with powers higher than four, we obtain the following.

result:

$$l_{Me} \cong \hat{Y}[1 + h_0 + b_1(h_1 + h_0h_1) + b_2(h_1^2 + h_0h_1^2) + b_3(h_1^3 + h_0h_1^3) + b_4h_1^4] \quad (23)$$

By computing the expectation of both sides of equation (23), we derive the bias of $l_{Me}$ as follows:

$$A(l_{Me}) = \frac{\hat{Y}\,D^2}{k}\left\{(\gamma b_1 + b_2) + \frac{3D^2}{k}(\gamma b_3 + b_4)\right\} \quad (24)$$

By squaring both sides of equation (24) and disregarding terms of $h's$ with powers higher than two, we obtain the following result:

$$\left(l_{Me} - \hat{Y}\right)^2 = \hat{Y}^2\{h_0^2 + 2b_1h_0h_1 + b_1^2h_1^2 + 2(b_1^2 + b_2)h_0h_1^2 + 2b_1h_0^2h_1 + 2b_1b_2h_1^3 + (b_1^2 + 2b_2)h_0^2h_1^2 + 2(b_3 + 2b_1b_2)h_0h_1^3 + (b_2^2 + 2b_1b_3)h_1^4\} \quad (25)$$

$$MSE(l_{Me}) = \left\{MSE(l_{Me}) + \left(\frac{D^4\hat{Y}^2}{k^2}\right)[(b_1^2 + 2b_2)(1 + 2\gamma^2) + 6(h_3 + 2b_1b_2) + 3(a_2^2 + 2b_1b_2)]\right\}$$

$$= MSE(l_{Me}) + \left(\frac{D^4\hat{Y}^2}{64k^2}\right)\{7R^4 + 4R^3(14\gamma - 9) + 4R^2(16\gamma^2 - 36\gamma + 17) - 16R(4\gamma^2 - 3\gamma + 2)\} \quad (26)$$

$R \approx R_0 = -2\gamma$(with a =1). Then putting $R = -2\gamma$ in eq. (25), we have,

If a reliable estimate of $R_0$ is available, denoted as $R \approx R_0 = -2\gamma$ (with a = 1), then substituting $= -2\gamma$ into equation (26), we obtain the following result:

$$MSE(l_{Me}) = MSE(l_{Me}) + \frac{D^4\hat{Y}^2}{4k^2}\gamma(4 + 11\gamma - 10\gamma^2 - 5\gamma^3)$$

$$= MSE(l_{Me})\left\{1 + \frac{D^4}{4k}\gamma\frac{(4+15\gamma+5\gamma^2)}{(1+\gamma)}\right\}$$

$$= \frac{D^2\hat{Y}^2}{k}\left\{(1 - \gamma^2) + \frac{D^2}{4k}\gamma(1-\gamma)(4 + 15\gamma + 5\gamma^2)\right\}$$

$$MSE(l_{Me}) = MSE(\hat{y}_{tf}) = \frac{D^4\hat{Y}^2}{k}(1 - \gamma^2) \quad (27)$$

$$MSE(l_{Re}) = MSE(l_{Re})\left\{1 + \frac{D^2}{16k}\left(\frac{(143-248\gamma+128\gamma^2)}{(5-4\gamma)}\right)\right\}$$

$$= \frac{D^2\hat{Y}^2}{k}\left\{\left(\frac{5}{4} - \gamma\right) + \frac{D^2}{64k}(143 - 248\gamma + 128\gamma^2)\right\}$$

$$MSE(l_{Re}) = MSE(l_{Re})\left\{1 + \frac{D^2}{16k}\left(\frac{7-40\gamma}{(5+4\gamma)}\right)\right\}$$

$$= \frac{D^2\hat{Y}^2}{k}\left\{\left(\frac{5}{4} + \gamma\right) + \frac{D^2}{64k}(7 - 40\gamma)\right\} \quad (28)$$

$$MSE(l_{Re}) = \frac{D^2\hat{Y}^2}{4k}(5 - 4\gamma)$$

$$MSE(l_{Re}) = \frac{D^2\hat{Y}^2}{4k}(5 + 4\gamma) \quad (29)$$

$$MSE(l_{Re}) = MSE(l_{Me}) = \frac{D^2\hat{Y}^2}{k}\left\{\left(\frac{1}{2} - \gamma\right)^2 + \frac{D^2}{64k}(143 - 312\gamma - 48\gamma^2 + 160\gamma^3 + 80\gamma^4)\right\} \quad (30)$$

$$\left\{\left(\frac{1}{2} - \gamma\right)^2 + \frac{D^2}{64k}(143 - 312\gamma - 48\gamma^2 + 160\gamma^3 + 80\gamma^4)\right\} > 0$$

$$MSE(l_{Re}) - MSE(l_{Me}) = \frac{D^2\hat{Y}^2}{k}\left\{\left(\frac{1}{2} + \gamma\right)^2 + \frac{D^2}{64k}(7 - 104\gamma - 176\gamma^2 + 160\gamma^3 + 80\gamma^4)\right\}$$

$$\left\{\left(\frac{1}{2} + \gamma\right)^2 + \frac{D^2}{64k}(7 - 104\gamma - 176\gamma^2 + 160\gamma^3 + 80\gamma^4)\right\} > 0 \quad (31)$$

$$MSE(\hat{y}_{th}) = D^2\hat{Y}^2(1 - \gamma^2)\left(\frac{1}{k} + \frac{1}{k^2}\right)$$

$$MSE(\hat{y}_{th}) - MSE(l_{Me}) = \frac{D^2\hat{Y}^2}{k}\left\{(1 - \gamma^2) - \frac{D^2}{k}\gamma(1-\gamma)(4 + 15\gamma + 5\gamma^2)\right\} > 0 \quad (32)$$

$$\text{If}\left\{(1 - \gamma^2) - \frac{D^2}{k}\gamma(1-\gamma)(4 + 15\gamma + 5\gamma^2)\right\} > 0$$

$D_x = D_x = D$, where b=1 the Mean Squared Error (MSE) of the usual Ratio Estimator $l_{F=\hat{y}}\left(\frac{\hat{X}}{\hat{x}}\right)$ and the Product Estimator $l_{S=\hat{y}}\left(\frac{\hat{X}}{\hat{x}}\right)$

$$MSE(l_F)_2 = MSE(l_F)_1\left\{1 + \frac{D^2}{k}(6 - 3\gamma)\right\}$$

$$= \left\{\frac{D^2\hat{Y}^2}{k}\left(2(1-\gamma) + \frac{6D^2}{k}(2 - 3\gamma + \gamma^2)\right)\right\}$$

$$MSE(l_F)_2 = MSE(l_F)_1\left\{1+\frac{D^2}{2k}\frac{(1+2\gamma^2)}{(1+\gamma)}\right\}$$

$$=\left\{\frac{D^2\hat{Y}^2}{k}\left(2(1+\gamma)+\frac{D^2}{k}(1+2\gamma^2)\right)\right\} \quad (33)$$

From eq. (32) and eq. (33 ), we have

$$MSE(l_F)_2 - MSE(l_{Fe})_2 = \left(\frac{D^2\hat{Y}^2}{k}\right)\left\{\left(\frac{3}{4}-\gamma\right)+\frac{D^2}{k}\left[6(1-\gamma)(2-\gamma)-\frac{1}{64}(143-248\gamma+128\gamma^2)\right]\right\}$$

$$\left\{\left(\frac{3}{4}-\gamma\right)+\frac{D^2}{k}\left[6(1-\gamma)(2-\gamma)-\frac{1}{64}(143-248\gamma+128\gamma^2)\right]\right\} > 0 \quad (34)$$

from eq. (33) and eq. (34), we have,

$$MSE(l_S)_2 - MSE(l_{Sh})_2 = \left(\frac{D^2\hat{Y}^2}{k}\right)\left\{\left(\frac{3}{4}+\gamma\right)+\frac{D^2}{64k}(57+168\gamma)\right\}$$

$$\left\{\left(\frac{3}{4}+\gamma\right)+\frac{D^2}{64k}(57+168\gamma)\right\} > 0 \quad (35)$$

## 5.Conclusion

The research findings highlight the significant impact of introducing new product-type and ratio-type estimators on the efficiency of estimating a population average by index-type estimators in the sequential random sample. The findings demonstrate that incorporating these new types of estimators increases the population average approximations' accuracy, thereby enhancing the overall efficiency of the sampling process. These findings have practical implications for researchers and practitioners in various fields, providing valuable insights to optimize their sampling strategies and obtain more reliable population mean estimates. Future research can explore additional aspects of these new estimators and their potential applications in different sampling scenarios to enhance the efficiency of sequential sampling techniques further.

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