ANALYSIS OF A FLEXIBLE GROUP SERVICE $MAP\backslash PH\backslash 1$ QUEUEING MODEL WITH, IMMEDIATE FEEDBACK, BALKING AND RENEGING

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Abstract

Queueing models in which the services are provided in groups (or blocks or batches) have found to be very useful in real-world applications and such queues been extensively analysed in the literature. In this paper we see one such group service queueing model with balking, reneging and immediate feedback. The arrival processes is a Markovian arrival , where, the arriving customer may balk the system while the server is idle and the pool is empty. Customers are provided service in groups of varying size from 1 *to the fixed constant, say, N. The service time of a batch follows the phase type distribution corresponding to the each size of the group. A group's service time is taken as the highest of the service times of each customers who make up the group. The group of customers who are dissatisfied with the service then that group will get the service immediately. Here, the feedback of a group is defined as the average of the feedback of each customers who make up the group. During the admission period the customers may renege. We calculated the steady state probabilities by using the matrix geometric method, then, by using it we computed few performance measures. We have studied the busy period and the distribution of waiting time is derived. Results are illustrated with some graphical representations.*

Keywords: Markovian Arrival Process, Flexible group service, Phase type Distributions, Immediate Feedback, Balking, Reneging.

1. Introduction

Queueing models with a group service play a vital role in many real life and engineering systems. And these queues can be generated physically or simulated by computers. Usually in the group service models, the minimum and maximum size of group are presumed. Bailey introduced the bulk service queueing model with fixed group size in [4]. Chaudry and Templeton in [5] studied bulk service queues in detail. A survey paper on bulk service queueing models by Sasikala and Indhira [6] is noteworthy. And the authors Banerjee.A, Gupta.U, Chakravarthy.S in [7] derived the significant results for queues with group service and many models with real-life applications are presented.

Neuts [8] has introduced the general group service rule, according to which the server will start the service only if $''a''$ or more customers are in the queue and the highest service capacity is $''b.''$ At the service completion epoch of a batch, if the number of customers present in the queue is less than $'a'$ then the server has to wait until $'a'$ number of customer arrives. If the number of customers less than or equal to $'b'$ and greater than or equal to $'a'$ then the service commences immediately to existing customers. If the number of customers is greater than $'b'$, then only $'b'$

customers taken in to service. And in the literature very few papers have dealt with group service with non-exponential service times .

The authors Brugno,D'Apice, Dudin, Manzo in [1] have examined a $MAP/PH/1$ queueing model with flexible group service. A predefined integer, let's say N, is typically used in the analysis of group services, and if there are less costumers in the queue than N, service is not initiated. But in this model they predefined the batch size as N, and they assumed that the server's idle time is restricted. Even if there are fewer consumers in the queue than N, service will still start once the idle time runs out. At a service completion moment, if there are N or more than N number of customers, the server provides service for exactly N customers. On the other hand, if the number of customers waiting in the queue is less than N, a admission period starts and its duration follows the PH - distribution. If the number of customers waiting reaches the value N before the admission period expires, the admission period is stopped and the service resumes with N customers. If the admission period expires before the arrival of Nth customer, then the server offers service simultaneously to the group of $'i'$ customers, where $'i'$ ranges between 1 to $N-1$ or if the admission period was over and when there is no one in queue, a new admissions period begins, and the procedure is repeated.

In [1],[7] for a batch service queueing models, a size $'m'$ customer group's service time is assumed to be the highest of $'m'$ identical PH- distributions which in turn a PH- distribution. The batch's service will be completed when the service for the last customer in the batch is completed. This type of batch service models are studied in cloud and grid computing [12].

Ayyappan and Thilagavathy in [9] have studied the $MAP/PH/1$ queueing model with Breakdown, Instantaneous feedback, and server vacation. And Downton in [11] by using random arrivals and a random service time distribution derived the waiting time distribution of bulk service queues.

In this paper we analyse a $MAP/PH/1$ queueing model with flexible group service, balking, reneging and immediate feedback. Here ,the feedback of a group is defined as the average of the feedback of each customers who make up the group.

The article's next sections are organised as follows. The mathematical model is presented with a graphic depiction in section 2. In section 3 we narrated the model and we formulate the QBD matrix. We derive the Ergodicity (stability) condition and the steady state probability vector in section 4. For this model we computed some performance measures in section 5. In section 6 we did busy period analysis. In section 7 waiting time distribution is derived . In section 8 some numerical results with graphical representations are illustrated and the conclusion is given in section 9.

2. The Narration of the Model

In this paper, Markovian arrival process is considered with depiction (D_0, D_1) of order n with the generator matrix $\tilde{D} = D_0 + D_1$. Markovian arrival's fundamental rate is defined as $\lambda = \pi D_1 e_n$, where π is the vector of stationary probability of $\tilde{D}$. Now we assume that the customers will get service in groups of size N, with $N \geq 2$ a fixed integer. If there are $N-1$ customers in the queue and the server is idle an arriving customer will get a immediate service and its PH representation is denoted as $(\beta^{(N)}, S^{(N)})$ of order $M^{(N)}$ with $S_0^{(N)} + S^{(N)}e = 0$ which implies $S_0^{(N)} = -S^{(N)}e$ otherwise based on the sequence of their arrival, the arriving customers are placed in the buffer. And at this moment choosing customers from the buffer at the time a service is complete is defined as follows

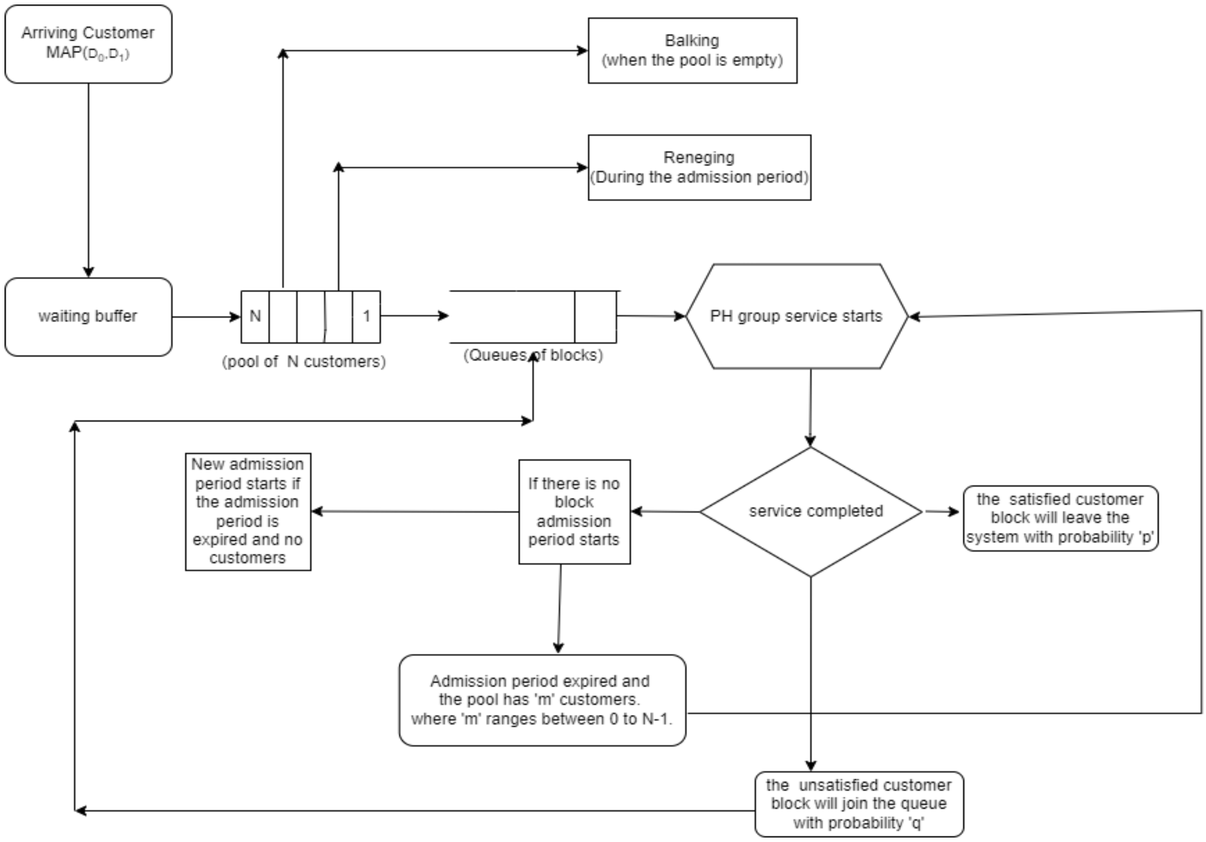


Figure 1: *Diagram illustrating the current model*

- The batch of exactly N consumers (say *Block*) begins service if at this point of time N or more customers are present in the buffer with PH representation $(\beta^{(N)}, S^{(N)})$ of order $M^{(N)}$.
- We refer to a group of consumers as a *pool*, if there are fewer than N customers. Then the so-called admission period begins at this point of time.The admission period follows PH distribution of which the PH representation is denoted by (α, T) of order $M^{(0)}$ with $T_0 + Te = 0$ which implies $T_0 = -Te$. And now
  - When the pool's total customer count equals N, the server starts providing services.
  - or the admission period expires.
  - If the admission period passed and the pool has one or more but fewer than N customers, Then all the customers in the pool strats service immediately and the service of r, customers $1 \leq r \leq N-1$ follows PH distribution of which the PH representation is given by $(\beta^{(r)}, S^{(r)})$ of order $M^{(r)}$ with $S_0^{(r)} + S^{(r)}e = 0$ which implies $S_0^{(r)} = -S^{(r)}e$.
  - A new admission period begins if the admission period passes with the pool empty, and the fundamental rate of admission period is defined as $\eta = [\alpha(-T)^{-1}e]^{-1}$.

Fundamental rate of service to r customers where $1 \leq r \leq N$ is defined as $\gamma_r = [\beta^{(r)}(-S^r)^{-1}e]^{-1}$.

During the server is idle and the pool is empty, with probability b, the customer might quit (balk) the system . During the admission period the customer may renege, which follows exponential distribution with parameter δ. We are dealing the single server queueing model with immediate feedback, this indicates whether a batch of customers is satisfied after receiving service, they leave the system with probability c otherwise the batch of customers will be receiving feedback service right away with probability d such that $c + d = 1$. Here in the group service, we presume that the group's feedback is positive if the average of the individual feedbacks of the group is positive.

2.1. QBD process of system state

Notations for our model

- $\otimes$ - the matrix Kronecker product.
- $\oplus$ - the matrix Kronecker sum .
- I_{m} - an identity matrix of m- dimension.
- e- a column matrix with each entry is 1 of appropriate dimension.
- $diag\{d_k, k \in 1, ..., n\}$ is the diagonal matrix, whose entries are enclosed in brackets.
- F_k is the row matrix of dimension k each of its entries as 0

Let

$$\xi(t) = \{N(t), I(t), R(t), J(t)^{(R(t))}, V(t) : t \geq 0 \ \}$$

is continuous time Markov chain with state level independent Quasi-Birth-and-Death process , where

- $N(t)$ indicates that how many batches are present in the system at time t, which includes the batch in service,
- $I(t)$ indicates that how many customers are there in the pool at time t , $0 \leq I(t) \leq N - 1$,
- $R(t)$ indicates that how many customers are getting the service at time t. Note that $R(t) = 0$ if $N(t) = 0$, as a result the admission period will be ongoing and $1 \ \leq R(t) \leq \ N - 1$ if $N(t) \ \geq \ 1$
- $J(t)^{(R(t))}$ indicates the state of the PH process of customer admission if $R(t) = 0$ with $1 \leq J(t)^{(0)} \leq M^{(0)}$ or it indicates the state of the PH process of customer service process if $1 \leq \ R(t) \ \leq \ N$ with $1 \leq J(t)^{(R(t))} \leq M^{(R(t))}$.
- $V(t)$ shows the state of the Markovian arrival process with $1 \ \leq V(t) \leq \ n$.

$\xi(t)$ has the following state space,

$$B \ = \ l(0) \ \cup \ l(k)$$

where,

$$l(0) = \{(0, i, 0, p, s) : 0 \ \leq i \leq \ N - 1 \ ; \ 1 \ \leq p \ \leq \ M^{(0)} \ ; 1 \ \leq s \ \leq n \ \}$$

and this can be simply written as

$$l(0) = \{(0, i) : 0 \ \leq i \leq \ N - 1 \ \}$$

the PH process of admission period and phases of Markovian arrival are understood.

For $k \ \geq \ 1$,

$$l(k) = \{(k, i, r, p, s) : k \ \geq 1 \ ; \ 0 \ \leq i \leq \ N - 1 \ ; \ 1 \ \leq \ r \ \leq \ N \ ; \ 1 \ \leq p \ \leq \ M^{(r)} \ ; 1 \ \leq s \ \leq n \ \}$$

and this can be simply written as

$$l(k) = \{(k, i, r) : k \geq 1 \,;\, 0 \leq i \leq N-1 \,;\, 1 \leq r \leq N\}$$

the PH process service to r number of customers where $1 \leq r \leq N$ and arrival phases are understood.

The QBD process of infinitesimal matrix generation is given by

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots & \cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}$$

The matrix Q's block matrices are given below

$$B_{00} = \begin{pmatrix} B_{00}{}^{11} & B_{00}{}^{12} & 0 & 0 & 0 & 0 & \cdots & 0 \\ B_{00}{}^{21} & B_{00}{}^{22} & B_{00}{}^{23} & 0 & 0 & 0 & \cdots & 0 \\ 0 & B_{00}{}^{32} & B_{00}{}^{33} & B_{00}{}^{34} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & B_{00}{}^{i\ i-1} & B_{00}{}^{i\ i} & B_{00}{}^{i\ i+1} & & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & B_{00}{}^{N-1\ N-2} & & B_{00}{}^{N-1\ N-1} \end{pmatrix} \text{ where,}$$

$B_{00}{}^{11} = (T + T_0\alpha) \oplus (D_0 + bD_1)$; $\quad B_{00}{}^{12} = (1-b)D_1 \otimes I_{M^{(0)}}$;
$B_{00}{}^{21} = \delta I_n \otimes I_{M^{(0)}}$; $\quad B_{00}{}^{22} = T \oplus (D_0 - \delta I_n)$;
$B_{00}{}^{23} = D_1 \otimes I_{M^{(0)}}$; $\quad B_{00}{}^{i\ i-1} = \delta I_n \otimes I_{M^{(0)}}$;
$B_{00}{}^{i\ i} = T \oplus (D_0 - \delta I_n)$; $\quad B_{00}{}^{i\ i+1} = D_1 \otimes I_{M^{(0)}}$

$$B_{01} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ (B_{01})_{1,0} & & & \\ (B_{01})_{2,0} & \vdots & \ddots & \vdots \\ \vdots & & & \\ (B_{01})_{N-1,0} & 0 & \cdots & 0 \end{pmatrix}, \text{ where}$$

$(B_{01})_{i,0} = (F_{i-1}, T_0 \otimes \beta^{(m)} \otimes I_n, F_{N-i})$ for $1 \leq i \leq N-2$
and $(B_{01})_{N-1,0} = (F_{N-2}, T_0 \otimes \beta^{(N-1)} \otimes I_n, e_{M^{(0)}} \otimes \beta^{(N)} \otimes D_1)$

$$A_1 = \begin{pmatrix} (A_1)_{0,0} & (A_1)_{0,1} & 0 & 0 & \cdots & 0 \\ 0 & (A_1)_{1,1} & (A_1)_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & (A_1)_{2,2} & (A_1)_{2,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & & \ddots & \vdots \\ \vdots & \vdots & & \ddots & & \\ 0 & \vdots & \cdots & & & (A_1)_{N-1,N-1} \end{pmatrix}, \text{ where}$$

$(A_1)_{i,i} = diag\ \{S^{(r)} + dS_0{}^{(r)}\beta^{(r)} \oplus D_0; 1 \leq r \leq N, 0 \leq i \leq N-1\}$

and $(A_1)_{i,i+1} = diag\ \{I_{M^{(r)}} \otimes D_1; 1 \leq r \leq N, 0 \leq i \leq N-2\}$

$$A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ & \vdots & \ddots & \vdots \\ 0 & & & \\ (A_0)_{N-1,0} & 0 & \cdots & 0 \end{pmatrix}, \text{ where } (A_0)_{N-1,0} = diag\ \{I_{M^{(r)}} \otimes D_1; 1 \leq r \leq N\}$$

$$A_2 = \begin{pmatrix} (A_2)_{0,0} & 0 & 0 & \cdots & 0 \\ 0 & (A_2)_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & (A_2)_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & & \cdots & & (A_2)_{N-1,N-1} \end{pmatrix},$$

with $(A_2)_{i,i} = \begin{pmatrix} 0 & \cdots & 0 & cS_0{}^{(1)} \otimes \beta^{(N)} \otimes I_n \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & cS_0{}^{(N)} \otimes \beta^{(N)} \otimes I_n \end{pmatrix}$

$$B_{1,0} = diag\ \{(B_{1,0})_{i,i}\ ;\ 0 \leq i \leq N-1\}, \text{ where } (B_{1,0})_{i,i} = \begin{pmatrix} cS_0{}^{(1)} \otimes \alpha \otimes I_n \\ \vdots \\ cS_0{}^{(N)} \otimes \alpha \otimes I_n \end{pmatrix}.$$

3. Condition for stableness

Let us define the matrix $A = A_0 + A_1 + A_2$, then

$$A = \begin{pmatrix} F & F' & & \\ & \ddots & \ddots & \\ & & F & F' \\ F' & & & F \end{pmatrix}, \ where\ \ F' = \begin{pmatrix} I_{M^{(1)}} \otimes D_1 & & \\ & \ddots & \\ & & I_{M^{(N)}} \otimes D_1 \end{pmatrix},$$

$$and\ \ F = \begin{pmatrix} S^{(1)} + dS_0{}^{(1)}\beta^{(1)} \oplus D_0 & & & cS_0{}^{(1)}\beta^{(N)} \otimes I_n \\ & \ddots & & \vdots \\ & & \ddots & cS_0{}^{(N-1)}\beta^{(N)} \otimes I_n \\ & & & S^{(N)} + dS_0{}^{(N)}\beta^{(N)} \oplus D_0 + cS_0{}^{(N)}\beta^{(N)} \otimes I_n \end{pmatrix}$$

It is clear that A is a square matrix which is an irreducible infinitesimal generator matrix whose order is $N\ M^{(1)}\ n + N\ M^{(2)}\ n + \cdots + N\ M^{(N)}\ n$. The steady-state probability vector of A is indicated by z. And the vector z is denoted as $z = (z_0, z_1, z_2, \cdots, z_{N-1})$, where $z_i = (z_i^1, z_i^2, \cdots, z_i^N)$, $0 \leq i \leq N-1$ which satisfies $zA = 0$ and $ze = 1$. The QBD structure exists for the Markov process. Also there exists Ergodicity (stability) criteria for our model and that it should satisfy $zA_0e < zA_2e$, which is the if and only if condition for stability of a QBD process. By resolving the following equations, the vector z can be determined.

$$z_0^1(S^{(1)} + dS_0{}^{(1)}\beta^{(1)} \oplus D_0) + z_{N-1}^1(I_{M^{(1)}} \otimes D_1) = 0$$
$$z_0^2(S^{(2)} + dS_0{}^{(2)}\beta^{(2)} \oplus D_0) + z_{N-1}^2(I_{M^{(2)}} \otimes D_1) = 0$$
$$\vdots$$

$$\sum_{i=1}^{N-1} z_0^i cS_0{}^{(i)}\beta^{(i)} \otimes I_n + z_0^N cS_0{}^{(N)}\beta^{(N)} \otimes I_n + (S^{(1)} + dS_0{}^{(1)}\beta^{(1)} \oplus D_0) + z_{N-1}^N (I_{M^{(N)}} \otimes D_1) = 0.$$

Similarly, $for\ \ i\ \ ,0\ \leq i \leq N-2$ we have,

$z_{i+1}^1 (S^{(1)} + dS_0{}^{(1)}\beta^{(1)} \oplus D_0) + z_i^1 (I_{M^{(1)}} \otimes D_1) = 0$

$z_{i+1}^2 (S^{(2)} + dS_0{}^{(2)}\beta^{(2)} \oplus D_0) + z_i^2 (I_{M^{(2)}} \otimes D_1) = 0$

$$\vdots$$

$$\sum_{i=1}^{N-1} z_{i+1}^i cS_0{}^{(i)}\beta^{(i)} \otimes I_n + z_{i+1}^N cS_0{}^{(N)}\beta^{(N)} \otimes I_n + (S^{(1)} + dS_0{}^{(1)}\beta^{(1)} \oplus D_0) + z_i^N (I_{M^{(N)}} \otimes D_1) = 0.$$

Following some algebraic calculation, the stability condition $zA_0e \ < \ zA_2e$, which is turns to be

$$\sum_{r=1}^{N} z_{N-1}^r (e_{M^{(r)}} \otimes D_1 e_n) \quad < \quad \sum_{i=0}^{N-1} \sum_{r=1}^{N} z_i^r cS_0{}^{(r)} \otimes e_n.$$

After simplification the Ergodicity condition can be precisely written as $\lambda\gamma_N < N$.

3.1. Study of the Stationary Probability vector

Let x be the Q 's the steady-state probability vector and it is subdivided as $x = (x_0, x_1, x_{2,}, \cdots)$. Note that x_0's dimension is NM^0n and dimension of $x_1, x_2, x_3, \cdots$, are $N(M^{(1)} \ \ + M^{(2)} \ \ + \cdots + M^{(N)})n$. Then x satisfies the condition $xQ = 0$ and $xe = 1$. Once the stability condition is met, the subvectors of x, except for x_0 and x_1 are provided by the following equation, which corresponds to the various level states.

$$x_j = x_1 R^{j-1}, \ \ j \geq 2$$

where R represents the minimum non-negative solution of the matrix quadratic equation as $R^2A_2 + RA_1 + A_0 = 0$, as defined by Neuts [3]. Due to the stability of our system and the fact that the row sums of the sum of square matrices A_0, A_1, and A_2 is zero, R , the rate matrix is a square matrix with order $N(M^{(1)} \ \ + M^{(2)} \ \ + \cdots + M^{(N)})n$. The R matrix is derived from the above quadratic equation and also fulfils $RA_2e = A_0e$.
The following equations were solved to obtain the sub vectors x_0 and x_1.

$$x_0B_{00} + x_1B_{10} = 0$$

$$x_0B_{01} + x_1(A_1 + RA_2) = 0$$

conditioned on the normalising state

$$x_0 e_{NM^{(0)}n} + x_1(1-R)^{-1} e_{N\ \ n(M^{(1)}+M^{(2)}+\cdots+M^{(N)})} = 1.$$

hence, the matrix R could be computed theoriticaly with the reference of Latouche and Ramaswami [2] using necessary steps in the R's Logarithmic reduction algorithm.

4. Performance Measures

- The expected number of customer blocks, including the one receiving service
  $E_{block} = \sum_{k=1}^{\infty} kx_ke = x_1(1-R)^{-2}$.
- The expected number of blocks of customers excluding the one in service
  $\tilde{E}_{block} = \sum_{k=1}^{\infty}(k-1)x_ke = E_{block} - 1 + x_0e_0$.

- The expected number of customers in the pool
$E_{pool} = \sum_{i=0}^{N-1} i x_0 \tilde{e}_{0i} + \sum_{k=1}^{\infty} \sum_{i=0}^{N-1} \sum_{m=1}^{N} i x_{kim} e = \sum_{i=0}^{N-1} i x_0 \tilde{e}_{0i} + \sum_{i=0}^{N-1} i x_1 (1-R)^{-1} \tilde{e}_i.$

  where $\tilde{e}_{0i}$ is the column vector of order $NM^{(0)}n$ with $(i(M^{(0)})n\ +1)$ st to $((i+1)M^{(0)}n)$ th entries are 1 and all other entries are zeros.
  and $\tilde{e}_i$ is the column vector of order $(N(M^{(1)}+\cdots+M^{(N)})n$ with $(i(M^{(1)}+\cdots+M^{(N)})n\ +1)$ st to $((i+1)(M^{(1)}+\cdots+M^{(N)})n)$ th entries are 1 and all other entries are zeros.

- The expected number of customers in the service
$E_{service} = \sum_{k=1}^{\infty} \sum_{i=0}^{N-1} \sum_{m=1}^{N} m x_{kim} e$
$\quad = \sum_{m=1}^{N} m (x_1(1-R)^{-1} e_{0i} + x_1(1-R)^{-1} e_{1m} + \cdots + x_1(1-R)^{-1} e_{N-1m})$
$\quad = \sum_{m=1}^{N} m (\sum_{i=0}^{N-1} x_1(1-R)^{-1} e_{im})$
  where e_{im} are all column vectors of order $(N(M^{(1)}+\cdots+M^{(N)})n$ defined as
  for $m=1$, e_{i1} has $(m(\sum_{k=1}^{N} M^k n)+1)$ st to $(i(\sum_{k=1}^{N} M^k n)+M^{(1)}n)$ th entries are 1 and all other elements are zeros.
  for $2 \leq m \leq N-1$, e_{im} has $(i(\sum_{k=1}^{N} M^k n)+(\sum_{j=1}^{m-1} M^j n)+1)$ st to $(i(\sum_{k=1}^{N} M^k n)+(\sum_{j=1}^{m} M^j n))$ th entries are 1 and all other entries are zeros.
  and for $m=N$, e_{iN} has $(i(\sum_{k=1}^{N} M^k n)+(\sum_{j=1}^{N-1} M^j n)+1)$ st to $((i+1)(\sum_{k=1}^{N} M^k n))$ th entries are 1 and all other entries are zeros.

- The mean size of the system (mean system size or expected system size) at an arbitrary moment including the customers in service
$E_{system} = \sum_{k=1}^{\infty} \sum_{i=0}^{N-1} \sum_{m=1}^{N} (kN+i+m) x_{kim} e + \sum_{i=0}^{N-1} i x_0 \tilde{e}_{0i}$
$\quad = NE_{block} + E_{pool} + E_{service}$

- The mean size of the system at some random time excluding the customers in service
$\tilde{E}_{system} = \sum_{k=1}^{\infty} \sum_{i=0}^{N-1} \sum_{m=1}^{N} ((k-1)N+i+m) x_{kim} e + \sum_{i=0}^{N-1} i x_0 \tilde{e}_{0i}$
$\quad = N\tilde{E}_{block} + E_{pool} + E_{service}$

- The probability of the server is idle at some random time
$P_{idle} = x_0 e_0$

5. Busy Period Analysis

- A busy period is defined as the period of time from when a customer first enters an empty system till the first epoch after that when the system is empty once more. Thus it is the first passing time from level 1 to level 0. A busy cycle is defined as the whole first time at level 0 after visiting a state in any other level at least once.

- We must first define the term "fundamental period" before we can study the busy time. It is the first passing time from level k to level $k-1$ for the QBD process under examination. for $k \geq 2$.

- The case where $k = 0, 1$ correspond to boundary states must be discussed seperately.

- Note that for each level k, for $k \geq 1$, there corresponds $N(M^{(1)}\ + M^{(2)}\ + \cdots + M^{(N)})n$ states. The ordered pair (k, j) represents j th state of level k where the states are ordered in the lexicographic order.

- Let $G_{jj'}(v, x)$ provides the conditional probability of the $Quasi-Birth-Death$ process, this process commences from the state (k,j) at time $t = 0$ accesses the level $k-1$ within the time x. We can alter the v transition move left and get into the the state $(k-1, j')$.

To proceed further, we present the combined transform

$$\tilde{G}_{kk'}(z,s) = \sum_{v=1}^{\infty} z^v \int_0^{\infty} e^{-sx} dG_{kk'}(v,x) \; ; \; |z| \leq 1, Re(s) \geq 0$$

and the matrix is denoted as

$$\tilde{G}(z,s) = \tilde{G}_{kk'}(z,s) \tag{1}$$

then (1) satisfies the equation

$$\tilde{G}(z,s) = z(sI - A_1)^{-1}A_2 + (sI - A_1)^{-1}A_0\tilde{G}^2(z,s)$$

Now $G = G_{kk'} = \tilde{G}(1,0)$ handles the first passage timings, with the exception of the boundary states. Using the result

$$G = -(A_1 + RA_2)^{-1}A_2.$$

G matrix could be determine if R matrix is already known. Otherwise G matrix could be determine using logarithmic reduction algorithm method.
From the above discussions for the boundary levels 1 and 0 we have

$$\tilde{G}^{(1,0)}(z,s) = z(sI - A_1)^{-1}B_{10} + (sI - A_1)^{-1}A_0\tilde{G}(z,s)\tilde{G}^{(1,0)}(z,s),$$
$$\tilde{G}^{(0,0)}(z,s) = (sI - B_{00})^{-1}B_{01}\tilde{G}(z,s)\tilde{G}^{(1,0)}(z,s).$$

Since G, $\tilde{G}^{(1,0)}(1,0)$ and $\tilde{G}^{(0,0)}(1,0)$ are stochastic, using the above matrices we can calculate the below cases.

$$\vec{\mathbb{H}}_1 = -\frac{\partial}{\partial s}\tilde{G}(z,s)|_{z=1,s=0}e = -[A_1 + A_0(I+G)]^{-1}e \tag{2}$$
$$\vec{\mathbb{H}}_2 = -\frac{\partial}{\partial z}\tilde{G}(z,s)|_{z=1,s=0}e = -[A_1 + A_0(I+G)]^{-1}A_2e \tag{3}$$
$$\vec{\mathbb{H}}_1^{(1,0)} = -\frac{\partial}{\partial s}\tilde{G}(z,s)^{(1,0)}|_{z=1,s=0}e = -[A_1 + A_0G]^{-1}(A_0\vec{\mathbb{H}}_1 + e) \tag{4}$$
$$\vec{\mathbb{H}}_2^{(1,0)} = -\frac{\partial}{\partial z}\tilde{G}(z,s)^{(1,0)}|_{z=1,s=0}e = -[A_1 + A_0G]^{-1}(A_0\vec{\mathbb{H}}_2 + B_{10}e) \tag{5}$$
$$\vec{\mathbb{H}}_1^{(0,0)} = -\frac{\partial}{\partial s}\tilde{G}(z,s)^{(0,0)}|_{z=1,s=0}e = -B_{00}^{-1}[B_{01}\vec{\mathbb{H}}_1^{(1,0)} + e] \tag{6}$$
$$\vec{\mathbb{H}}_2^{(0,0)} = -\frac{\partial}{\partial z}\tilde{G}(z,s)^{(0,0)}|_{z=1,s=0}e = -B_{00}^{-1}[B_{01}\vec{\mathbb{H}}_2^{(1,0)}] \tag{7}$$

6. Analysis of Waiting Time Distribution

The first passage time analysis is used in this section to analyse the distribution of a customer's waiting time when they enter the queueing line. Let $W(t)$ be the waiting time distribution function, which takes in to account new customers joining the queue. If there are $N-1$ costumers in line and the server is idle, the arriving customer will receive service right away, otherwise an arrival has to wait. Let $\tilde{\Omega}$ be the state space of the absorption time of a Markov chain,

$$\tilde{\Omega} = (*) \cup \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \cdots\}$$

The absorption state corresponds to the tagged customer will be getting service without waiting. The absorption state is defined as follows

$$(*) = \{(0, N-1)\}.$$

The level state 0 is represented as follows,

$$\bar{0} = \{(0,i); 0 \ \leq i \leq \ N-2\}$$

the level state for p where $p \geq 1$ is given by

$$l(p) = \{(p, l, r, k) : p \geq 1 ; 0 \leq l \leq N-1 ; 1 \leq r \leq N ; 1 \leq k \leq M^{(r)} \}$$

The absorbing Markov chain's transition matrix $\tilde{Q}$ is given by

$$\tilde{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ U_0 & W_0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ U_1 & W_2 & W_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & W_3 & W_1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & W_3 & W_1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}$$

entries of the above matrix are as follows,

$$U_0 = [F_{N-2}]^T \; ; \quad U_1 = [[F_{((N-1)N)}]^T, c{S_0}^{(1)} \otimes \alpha, \cdots, c{S_0}^{(N)} \otimes \alpha]$$

$$W_0 = \begin{pmatrix} T + T_0\alpha & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \delta I_{M^{(0)}} & T - \delta I_{M^{(0)}} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \delta I_{M^{(0)}} & T - \delta I_{M^{(0)}} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & \cdots & \vdots \\ 0 & 0 & \delta I_{M^{(0)}} & T - \delta I_{M^{(0)}} & 0 & & & 0 \\ \vdots & \vdots & \ddots & \ddots & & & & \vdots \\ & & & \ddots & \ddots & & \vdots & \\ 0 & 0 & \cdots & & & & \delta I_{M^{(0)}} & T - \delta I_{M^{(0)}} \end{pmatrix}$$

$$W_2 = \begin{pmatrix} (W_2)_{1,1} & 0 & \cdots & 0 \\ 0 & (W_2)_{2,2} & & \vdots \\ & & \ddots & \\ \vdots & & & (W_2)_{N-2,N-2} \\ 0 & \cdots & & 0 \end{pmatrix}$$

where $(W_2)_{i,i}$ for $0 \leq i \leq N-2$, $(W_2)_{i,i} = \begin{pmatrix} c{S_0}^{(1)} \otimes \alpha \\ \vdots \\ c{S_0}^{(N)} \otimes \alpha \end{pmatrix}$

$$W_1 = \begin{pmatrix} (W_1)_{0,0} & 0 & \cdots & & 0 \\ 0 & (W_1)_{1,1} & \cdots & & 0 \\ 0 & 0 & (W_1)_{2,2} & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & (W_1)_{N-1,N-1} \end{pmatrix}, \text{ where}$$

$(W_1)_{i,i} = diag\ \{S^{(r)} + d{S_0}^{(r)}\beta^{(r)}; 1 \leq r \leq N, 0 \leq i \leq N-1\}$

$$W_3 = \begin{pmatrix} (W_3)_{0,0} & 0 & 0 & \cdots & 0 \\ 0 & (W_3)_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & (W_3)_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & & \cdots & & (W_3)_{N-1,N-1} \end{pmatrix},$$

with $(W_3)_{i,i} = \begin{pmatrix} 0 & \cdots & 0 & c{S_0}^{(1)} \otimes \beta^{(N)} \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & c{S_0}^{(N)} \otimes \beta^{(N)} \end{pmatrix}.$

We begin by calculating the system's state, stationary probability distribution (that is, how many customers were in the system) as observed by the tagged client at the moment of arrival. It is denoted by $y(0) = (y_0(0), y_1(0), y_2(0), \cdots)$ and the system's state conditional probability distribution under tagged customer's arrival can be used to determine it, using $x(0) = (x_0(0), x_1(0), x_2(0), \cdots)$ by the following method

$$y_0(0) = x_0(0)(\frac{D_1 e_n}{\lambda})$$

$$y_j(0) = x_j(0)(I_{N^2(M^{(1)}+M^{(2)}+\cdots+M^{(N)})} \otimes \frac{D_1 e_n}{\lambda})$$

where λ indicates the basic (fundamental) rate of Markovian arrival process. Now define $y(t) = (y_*(t), y_1(t), y_2(t), y_3(t), \cdots)$, where y_0 is of dimension $(1 \times NM^{(0)})$ and $y_i(t)$ for $i \geq 1$ is of dimension $(1 \times (NM^{(1)} + NM^{(2)} + \cdots + NM^{(N)}))$. The elements of $Y_{(t)}$ represents the probability of the CTMC with generator $\tilde{Q}$ is in the respective state of level i at time t. The probability that the tagged customer is in the absorbing state is given by $y_*(t)$. Thus we have $W(t) = y_*(t)$, for all $t \geq 0$. From the differential equation $y'(t) = y(t)\tilde{Q}$ we have,

$$y'_*(t) = \sum_{j=0}^{1} y_i(t) U_j$$

$$y'_0(t) = y_0(t)W_0 + y_1(t)W_2$$
$$y'_i(t) = y_i(t)W_1 + y_{i+1}(t)W_3, \ \ for \ \ i \geq 1.$$

The row vector $\psi(s)$ provides the Laplace-Steeltjes transform (LST) of the first passage through level 1. By Neuts,M.F in [3], we get

$$\psi(s) = \sum_{i=1}^{\infty} y_i(0)[(sI - W_1)^{-1} W_3]^{i-1}$$

We use $\varphi(i,s)$ to represent the LST of the absorbing time to the state $\{*\}$ when the process begins at level $i = 0, 1$. Using $\tilde{Q}$ we have,

$$\varphi(0,s) = (sI - W_0)^{-1} U_0 \tag{8}$$
$$\varphi(1,s) = (sI - W_1)^{-1} W_2 \varphi(0,s) + (sI - W_1)^{-1} U_1.$$

Consequently, the LST for the waiting time distribution is

$$\tilde{W}(s) = y_0(0)\varphi(0,s) + \psi(s)\varphi(1,s). \tag{9}$$

7. Numerical Results

In this section, we use graphical representations of the numerical values to investigate the model's nature. Where the numerical values for arrival process, admission period and service process were referred by Chakravarthy in [21].
Numerical values for Markovian arrival process,

- Exponential Arrival (E-A)

$$D_0 = \begin{pmatrix} -1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 \end{pmatrix}$$

- Erlang Arrival (Er-A)

$$D_0 = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

- Hyper-Exponential Arrival (Hyp-A)

$$D_0 = \begin{pmatrix} -1.90 & 0 \\ 0 & -0.19 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{pmatrix}$$

- MAP-Negative Correlation Arrival (MNC-A)

$$D_0 = \begin{pmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99241 \\ 223.539 & 0 & 2.258 \end{pmatrix}$$

- MAP-Positive Correlation Arrival (MPC-A)

$$D_0 = \begin{pmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.99241 & 0 & 0.01002 \\ 2.258 & 0 & 223.539 \end{pmatrix}$$

Numerical values for Phase type admission period.

- Exponential admission period (E-AP)

$$\alpha = \begin{pmatrix} 1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 \end{pmatrix}$$

- Erlang admission period (Er-AP)

$$\alpha = \begin{pmatrix} -1, & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}$$

- Hyper-Exponential admission period (Hyp-AP)

$$\alpha = \begin{pmatrix} 0.8, & 0.2 \end{pmatrix}, \quad T = \begin{pmatrix} -2.80 & 0 \\ 0 & -0.28 \end{pmatrix}$$

We assume that the numerical values for Phase type distributions for service times to m customers where $1 \leq m \leq N$ are all of exponential distributions. That is all of $(\beta^m, S^(m))$ are exponential distributions irrespective of size. In all the examples we assumed, the arrival rate $\lambda = 1$, the admission period rate $\eta = 3$, the service rate $\gamma = 6$.
The numerical value of the service time is taken as

- Exponential (E)

$$\beta^m = \begin{pmatrix} 1 \end{pmatrix}, \quad S^m = \begin{pmatrix} -1 \end{pmatrix} \ \forall\, 1 \ \leq \ m \ \leq \ N$$

Illustrative Example: 8.1.

We have illustrated the effect of the rate of renege in counter to the mean size of the system in the upcoming figures 2 and 3. We assume $\lambda = 1$, $\eta = 3$, $\gamma = 6$, $b = 0.5$, $c = 0.6$, $d = 0.4$ and we amplify the renege rate such that the values leaves the system to be stable. We execute the example for batch size $N = 2, 3, 4$.

In Figure 2 we fixed the arrival to follow exponential distribution and we assume the admission periods to follow exponential, Erlang and hyper-exponential distribution respectively. We observed that by amplifying the renege rate the system size decreases. We also noticed that the system size decreases slowly in exponential and Erlang whereas quickly in hyper-exponential distribution.

In Figure 3 we fixed the arrival to follow Erlang distribution and we assume the admission periods to follow exponential, Erlang and hyper-exponential distribution respectively.

We observed that by amplifying the renege rate the system size decreases. We also noticed that the system size decreases moderately in all three admission periods.

Illustrative Example: 8.2.

We have analysed the hyper-exponential arrival with exponential admission period case in the following figures 4 and 5. We assume $\lambda = 1$, $b = 0.5$, $c = 0.6$, $d = 0.4$ and increase the renege rate, admission period rate and service rate such that the values leaves the system to be stable. We execute the example for batch size $N = 2, 3, 4$.

In Figure 4 we fixed the service rate as $\gamma = 6$ and we amplify both the admission period rate and the renege rate against the mean system size. We observed that by amplifying the renege rate and admission period rate the mean system size decreases gradually.

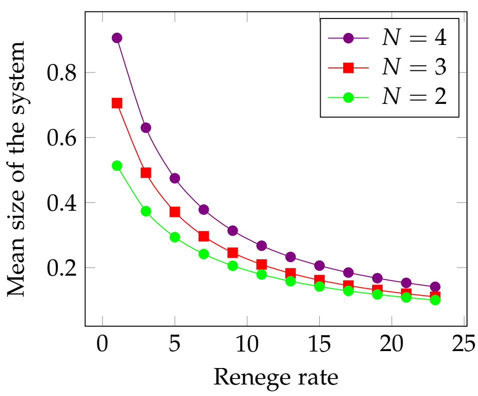
In Figure 5 we fixed the admission period rate as $\eta = 3$ and we amplify both the rate of service and the renege against the mean size of the system. We observed that by amplifying the rate of renege and service the mean size of the system decreases and it falls down rapidly.

Illustrative Example: 8.3.

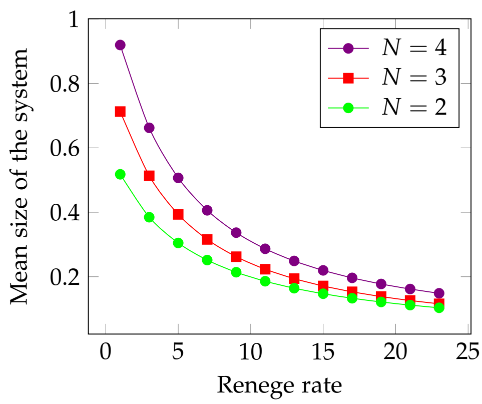
We have analysed the MAP-Positive Correlation Arrival with exponential admission period case in the following figures 6 and 7. We assume $\lambda = 1$, $b = 0.5$, $c = 0.6$, $d = 0.4$ and increase the renege rate, admission period rate and service rate such that the values leaves the system to be stable. We execute the example for batch size $N = 2, 3, 4$.

In Figure 6 we fixed the service rate as $\gamma = 6$ and we amplify both the admission period rate and the renege rate against the mean system size. We observed that by amplifying the renege rate and admission period rate the mean system size decreases slowly.

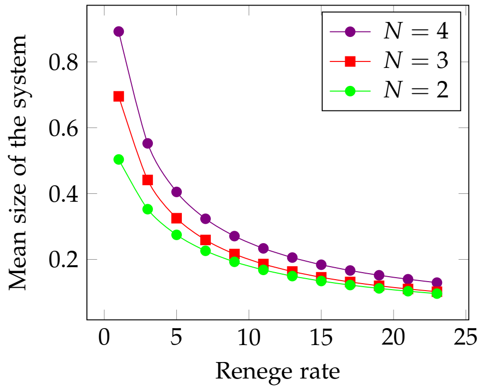
In Figure 7 we fixed the admission period rate as $\eta = 3$ and we amplify both the rate of service and renege against the mean size of the system. We observed that by amplifying the rate of renege and service, the mean size of the system decreases and it falls down moderately.



(a) *E-A and E-AP*

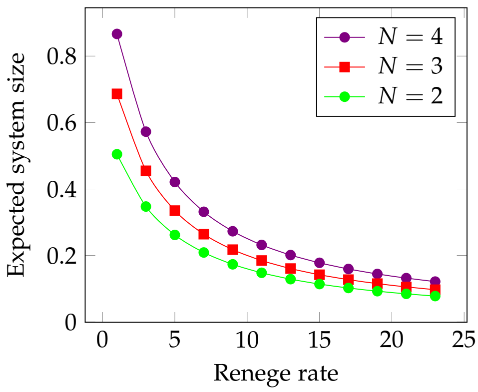


(b) *E-A and Er-AP*

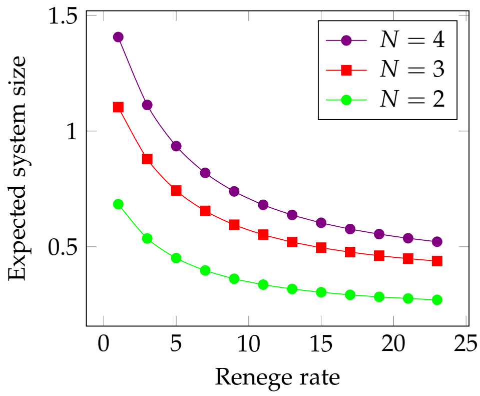


(c) *E-A and Hyp-AP*

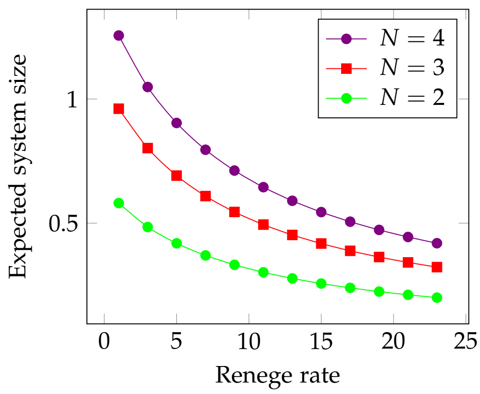
Figure 2: *Renege rate (vs) Expected system size -Exponential Arrival*



(a) *Er-A and E-AP*



(b) *Er-A and Er-AP*



(c) *Er-A and Hyp-AP*

Figure 3: *Renege rate (vs) Expected system size -Erlang Arrival*

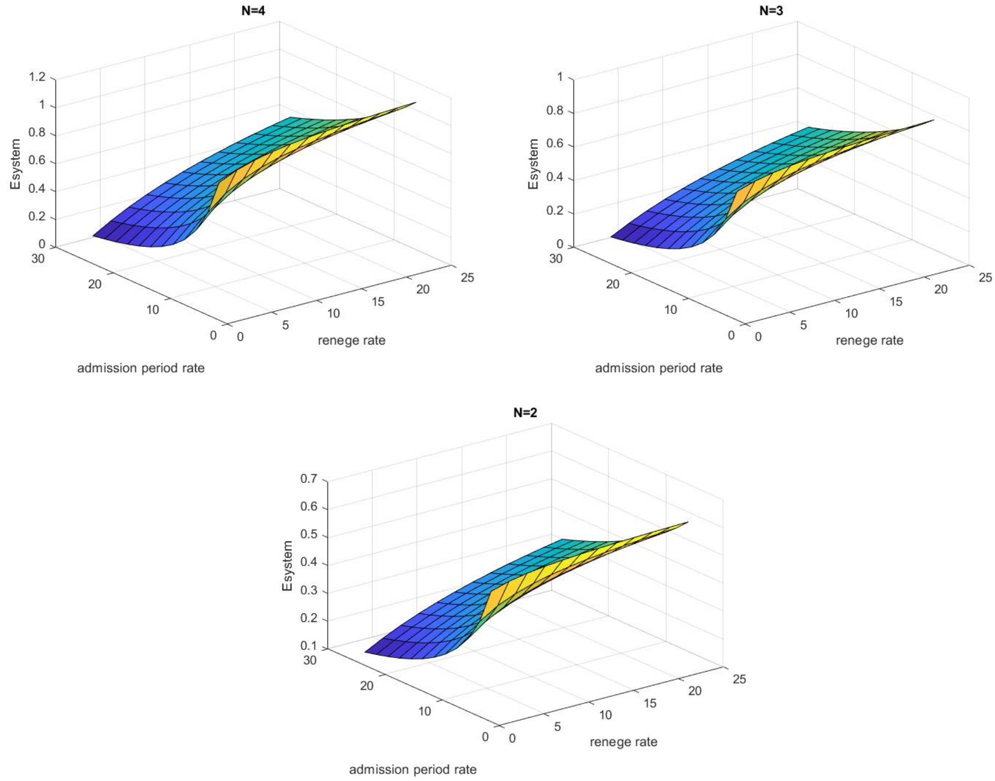


Figure 4: *(Renege rate(δ) and Admission period rate(η) (vs) Esystem)*

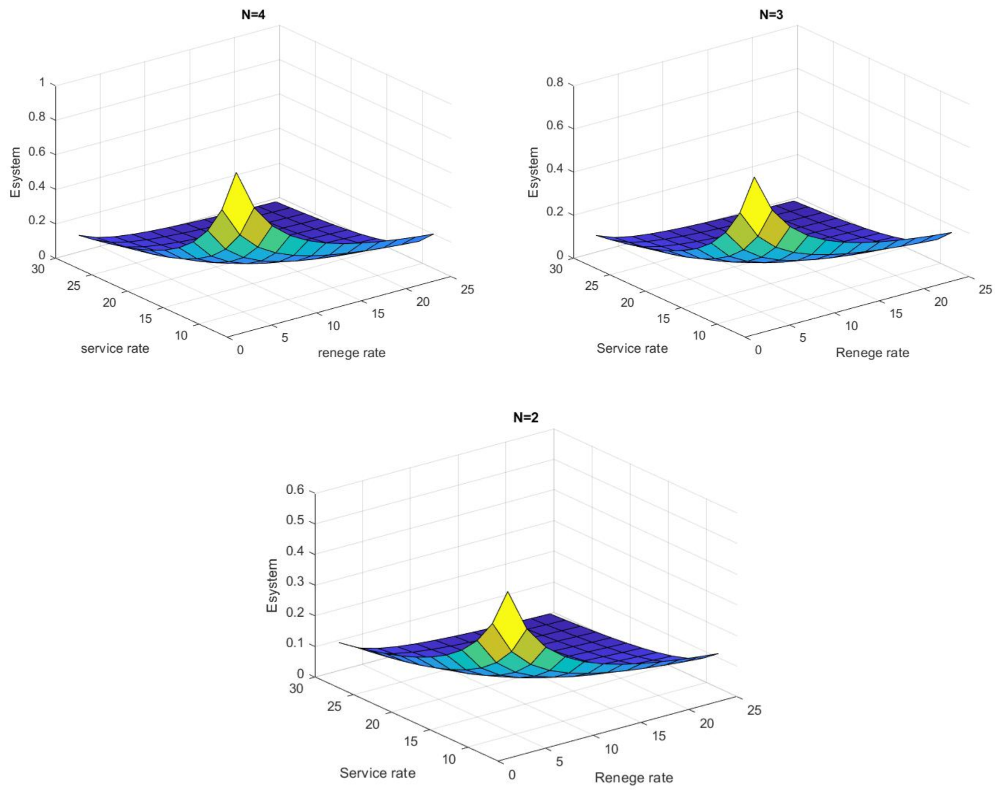


Figure 5: *(Renege rate(δ) and Service rate(γ) (vs) Esystem) [Hyper exponential arrival with Exponential Admission period]*

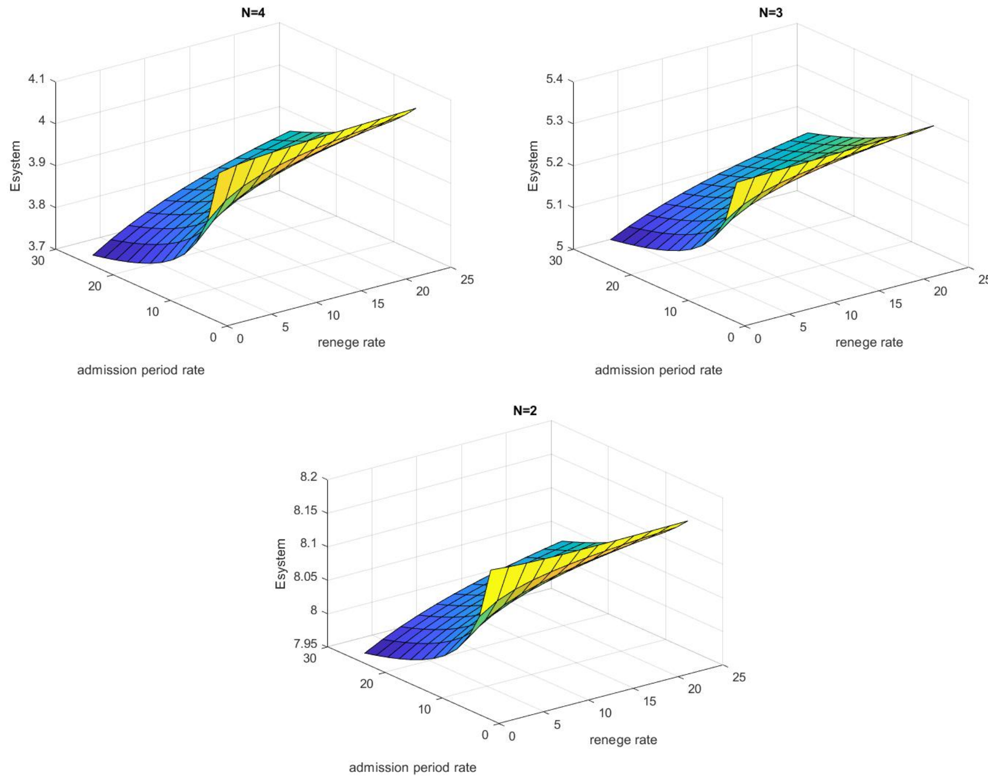


Figure 6: *(Renege rate(δ) and Admission period rate(η) (vs) Esystem)*

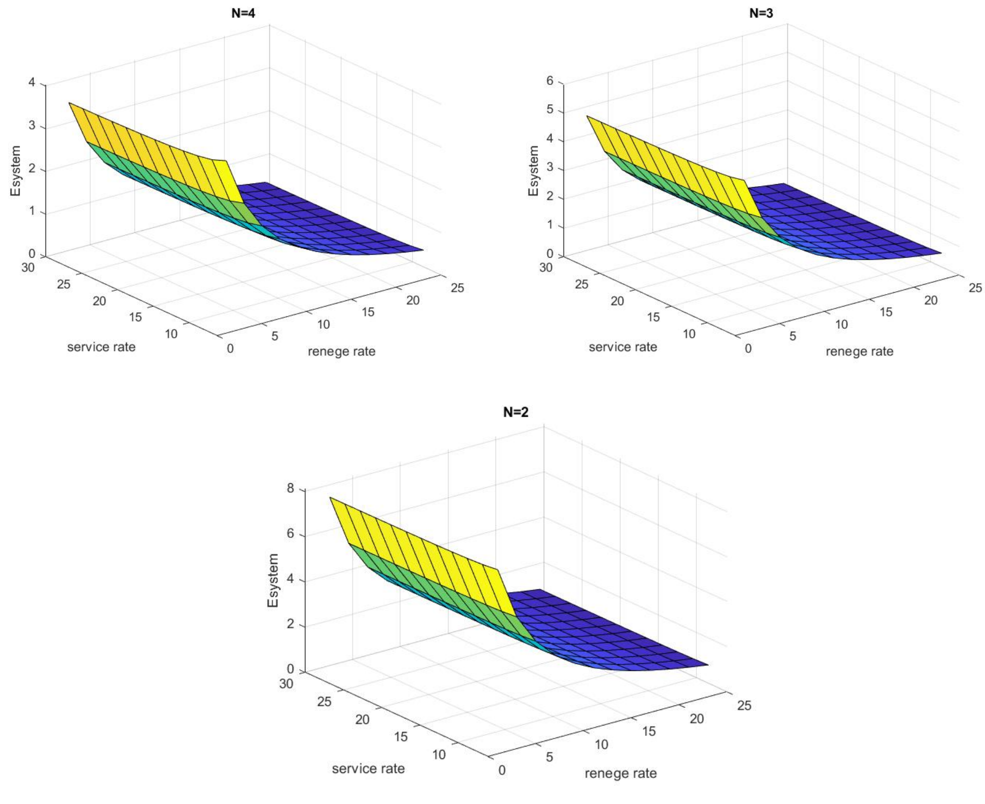


Figure 7: *(Renege rate(δ) and Service rate(γ) (vs) Esystem) [MAP Positive correlation Arrival with Exponential Admission period]*

8. The conclusion

In this paper we studied a group service queueing model with arrivals happen according to a Markovian arrival process in which arrivals may balk or renege the system. The service follows Phase-type distributions in which size of the group may vary and on depending the size of the group, that is, the number of customers getting service, the service time owns different Phase-type distribution representations. If any group of customers would like to receive feedback service, they will receive it immediately. The busy period analysis was done and waiting time distribution was computed. Using the Numerical values of arrival and service times, we compared the mean size of the system counter to renege rate with different batch sizes , which is represented graphically. This model can be extended with various catastrophes on servers, which is currently being probed.

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