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ON THE PROPERTIES OF GENERALIZED RAYLEIGH DISTRIBUTION WITH APPLICATIONS

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Abstract

In this study, a new three-parameter lifetime distribution called the generalized Rayleigh distribution was introduced. The new model is an extension of classical Rayleigh distribution. An extension of density of the generalized Rayleigh distribution was derived from which some of the statistical and mathematical properties were derived. Some mathematical properties of the distribution were presented such as moments, moment generating function, quantile function, survival function, hazard function, reversed hazard function and odd function. The distribution of order statistic was obtained in which the maximum and minimum order statistics were derived. Estimation of the parameters by maximum likelihood method was discussed. Two real-life application of the distribution was presented and the analysis showed the fit and flexibility of the generalized Rayleigh distribution over odd Lindley Rayleigh distribution and Rayleigh distribution. The analysis showed that the generalized Rayleigh distribution is more effective and robust in fitting the data sets.

Keywords: flexibility, transect line, myelogeneous leukemia, odd Lindley Rayleigh, quadratic rank transmutation technique

I. Introduction

Over a long period of time, probability distributions have been established through mathematical and statistical study. The assumed probability model or distributions have a significant impact on the effectiveness of the processes used in a statistical study. As a result, numerous common probability distributions and pertinent statistical techniques are described in the literature. However, there are still a number of issues where the real data set does not fit into any of the conventional or classical probability models. In literature numerous generalized distributions have been developed with common feature of having more parameters. Induction of parameters in existing distribution improves the goodness of fit of the distribution under study and tail properties of a distribution increases.

The Rayleigh distribution is one of the most widely applied probability distributions. The Rayleigh distribution is a special case of Weibull distribution, which was first described in [1]. In areas including project effort loading modeling, survival and reliability analysis, communication theory, physical sciences, technology, diagnostic imaging, applied statistics, and clinical research, the Rayleigh distribution is crucial for modeling and interpreting life-time data. Numerous researchers have developed significant expansions to the Rayleigh distribution in light of its significance and the need to provide this distribution more flexibility.

Numerous expansions of Raleigh distribution have been produced as a result of the significance of Raleigh distribution in numerous disciplines. The generalized Raleigh distribution was proposed by [2], and its unknown parameters were determined using several estimation techniques. The novel generalization of Rayleigh distribution was developed by [3] by utilizing the conservability approach. The Bayes estimators for the parameter of the Rayleigh distribution using square error and LINEX loss functions were derived by [4]. The transmuted Rayleigh distribution was developed by [5] using the quadratic rank transmutation technique. The transmuted generalized Rayleigh distribution was proposed by [6]. The Weibull Rayleigh distribution was studied by [7]. The parameters estimation of exponentiated Rayleigh based on type II censored data was deliberated by [8]. A new distribution named as Rayleigh–Rayleigh distribution was derived by [9] and motivated by the transformed transformer technique by [10]. In contrast to the Lindley distribution, the Rayleigh distribution, and other generalizations of the Rayleigh distribution, an extension of the Rayleigh distribution was developed by [11] with two parameters having greater flexibility. An extension of the exponentiated Rayleigh distribution known as the Gompertz-exponentiated Rayleigh distribution was proposed by [12] by utilizing the Transformed-Transformer family of distributions' methodology.

This paper proposes a new distribution that generalizes the Rayleigh distribution using the family of distribution proposed by [13]. This motivation behind this work is to improve the flexibility of the Rayleigh distribution to fit varieties of real life data sets arising from different disciplines including unimodal and bimodal shapes. Also, to make the kurtosis more flexible compared to the baseline Rayleigh model, to produce skewness for symmetrical distributions using type I half-logistic family of distributions derived by [13] and bathtub shapes.

II. Methods

2.1 Generalized Rayleigh (GRa) Distribution

In this section, a new continuous probability distribution function known as generalized Rayleigh distribution is derived. Also, some plots of its pdf, cdf, survival function and hazard rate function (hrf) are shown in order to assess the shape of the new distribution in fitting different kinds of data.

Recently, [13] developed the Type I Half-logistic family of distributions with cdf and pdf given as

$$F(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{1-\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}} \tag{1}$$

$$f(x;\lambda,\alpha,\boldsymbol{\beta})=\frac{2\lambda\alpha h(x;\boldsymbol{\beta})\left[H(x;\boldsymbol{\beta})\right]^{\alpha-1}\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[H(x;\boldsymbol{\beta})\right]^{\alpha}\right]^{\lambda}\right]^{2}} \tag{2}$$

where $\boldsymbol{\beta}$ is the vector of parameters of the baseline distribution.

where $G(x;\boldsymbol{\beta})$ is the cumulative distribution function (cdf) of the baseline distribution with vector of parameter $\boldsymbol{\beta}$.

For $x \geq 0, \alpha, \lambda, \boldsymbol{\beta} \geq 0$, where equations (1) and (2) are the cumulative distribution function and probability density function (pdf) of the family of distributions.

The cdf and pdf of the Rayleigh (Ra) distribution are given respectively as

$$H(x;\theta)=1-e^{-\left(\theta/2\right)x^{2}} \tag{3}$$

$$h(x;\theta)=\theta x e^{-\left(\theta/2\right)x^{2}} \tag{4}$$

To obtain the cdf of the new model, equation (3) is inserted into equation (1) as

$$F(x;\lambda,\alpha,\theta)=\frac{1-\left[1-\left[1-e^{-\left(\theta/2\right)x^{2}}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x^{2}}\right]^{\alpha}\right]^{\lambda}} \tag{5}$$

On differentiating equation (5) with respect to x, the pdf of the GRa distribution is obtained which is given as

$$f(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta x e^{-\left(\theta/2\right)x^{2}}\left[1-e^{-\left(\theta/2\right)x^{2}}\right]^{\alpha-1}\left[1-\left[1-e^{-\left(\theta/2\right)x^{2}}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-e^{-\left(\theta/2\right)x^{2}}\right]^{\alpha}\right]^{\lambda}\right]^{2}} \tag{6}$$

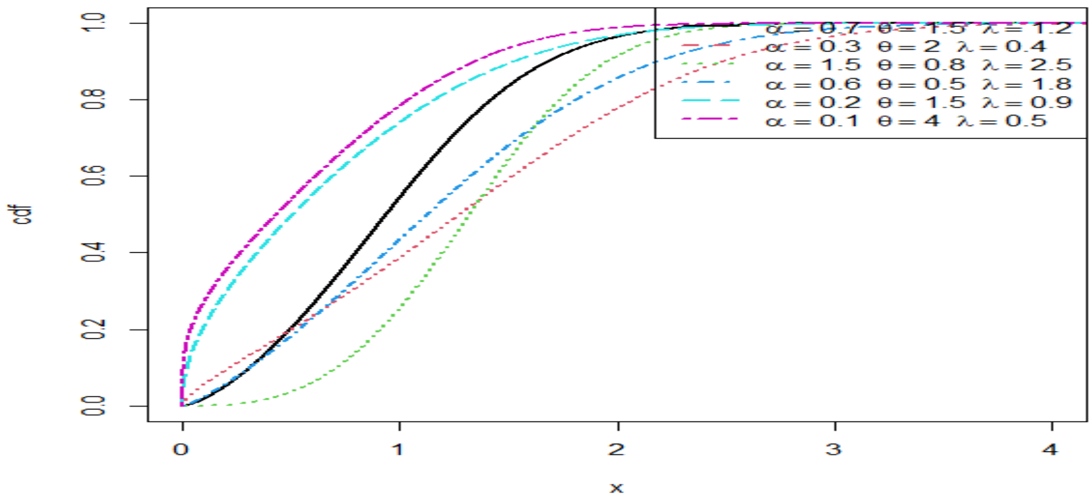

Figure 1: *Plots of cdf of the GRa distribution for different parameter values*

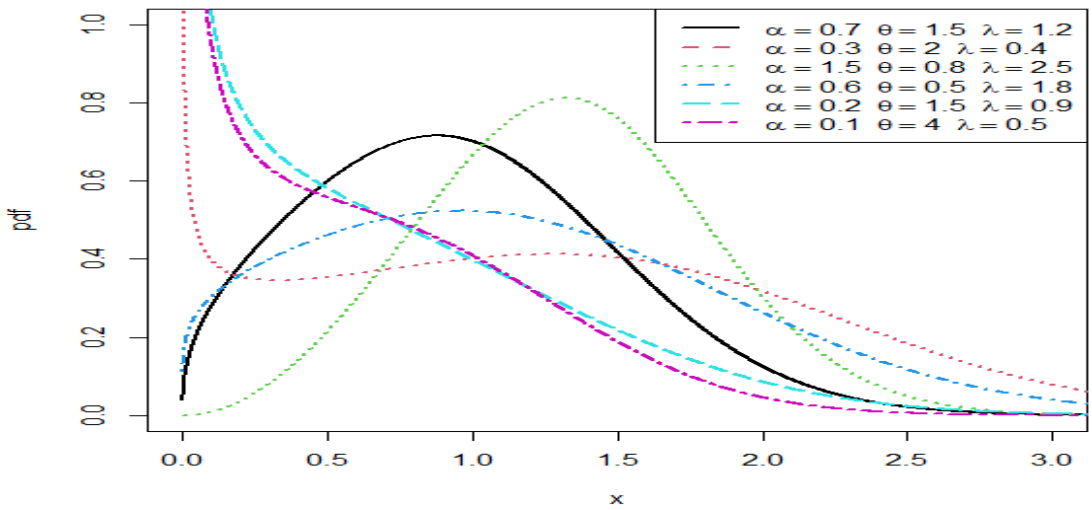

Figure 2: *Plots of pdf of the GRa distribution for different parameter values*

where $x\geq 0$, $\theta>0$ is the scale parameter and $\alpha,\lambda>0$ are the shape parameters respectively.

2.1.1 Expansion of density

In this section the pdf in equation (6) is expanded using binomial expansion.
Expanding the last term in equation (6), we have

$$f(x;\lambda,\alpha,\theta)=2\lambda\alpha\theta xe^{-\left(\theta/2\right)x^2}\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha-1}\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda-1}\left[1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{-2}$$

$$\left[1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{-2}=\sum_{i=1}^{\infty}\left(-1\right)^{i}\binom{1+i}{i}\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda i}$$

$$\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda(i+1)-1}=\sum_{j=1}^{\infty}\left(-1\right)^{j}\binom{\lambda\left(i+1\right)-1}{j}\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha j}$$

$$\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha(j+1)-1}=\sum_{k=0}^{\infty}\left(-1\right)^{k}\binom{\alpha\left(j+1\right)-1}{k}\left[e^{-\left(\theta/2\right)x^2}\right]^{k}$$

$$f(x;\lambda,\alpha,\theta)=2\lambda\alpha\theta x\sum_{i,j,k=0}^{\infty}\left(-1\right)^{i+j+k}\binom{1+i}{i}\binom{\lambda\left(i+1\right)-1}{j}\binom{\alpha\left(j+1\right)-1}{k}\left[e^{-\left(\theta/2\right)x^2}\right]^{k+1} \tag{7}$$

Equation (7) is the expansion of equation (6) which will be used to derive some of the properties of the distribution.
Also, equation (5) is expanded as

$$\left[F(x;\lambda,\alpha,\theta)\right]^{h}=\left[1-\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{h}\left[1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{-h}$$

$$\left[1-\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{h}=\sum_{m=0}^{h}\left(-1\right)^{m}\binom{h}{m}\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda m}$$

$$\left[1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{-h}=\sum_{p=0}^{h}\left(-1\right)^{p}\binom{h+p-1}{p}\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda p}$$

$$\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda(p+m)}=\sum_{z=0}^{\infty}\left(-1\right)^{z}\binom{\lambda\left(p+m\right)}{z}\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha z}$$

$$\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha z}=\sum_{q=0}^{\infty}\left(-1\right)^{q}\binom{\alpha z}{q}\left[e^{-\left(\theta/2\right)x^2}\right]^{q}$$

$$\left[F(x;\lambda,\alpha,\theta)\right]^h = \sum_{m,p=0}^{h}\sum_{z,q=0}^{\infty}(-1)^{m+p+z+q}\binom{h}{m}\binom{h+p-1}{p}\binom{\lambda(p+m)}{z}\binom{\alpha z}{q}\left[e^{-\left(\theta/2\right)x^2}\right]^q \tag{8}$$

Equation (8) is the expansion of equation (5) which will be used to derive some of the properties of the distribution.

2.1.2 Properties of the Generalized Rayleigh distribution

In this section, some of the mathematical and statistical properties of GRa distribution such as the quantile function, moments, moment generating function, reliability measure, odds function, reversed hazard function and order statistics are derived.

2.1.2.1 Moments

$$E\left(X^r\right)=\int_0^\infty x^r f\left(x\right)dx \tag{9}$$

$$=2\lambda\alpha\theta\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\int_0^\infty x^{r+1}\left[e^{-\left(\theta/2\right)x^2}\right]^{k+1}dx$$

$$\int_0^\infty x^{r+1}\left[e^{-\left(\theta/2\right)x^2}\right]^{k+1}dx=\left[\frac{2}{\theta(k+1)}\right]^{\frac{r}{2}}\Gamma\left(1+\frac{r}{2}\right)$$

$$E\left(X^r\right)=2\lambda\alpha(k+1)\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\left[\frac{2}{\theta(k+1)}\right]^{\frac{r}{2}}\Gamma\left(1+\frac{r}{2}\right) \tag{10}$$

Equation (10) is the moments of GRa distribution. To obtain the mean, we set *r = 1* in equation (10).

2.1.2.2 Moment generating function (mgf)

$$M_{(x)}(t)=\int_0^\infty e^{tx}f\left(x\right)dx \tag{11}$$

since the series expansion for e^{tx} is given as

$$e^{tx}=\sum_{w=0}^{\infty}\frac{(tx)^w}{w!}$$

Then, following the method of moments, the mgf is obtained as follows

$$M_{(x)}(t)=2\lambda\alpha(k+1)\sum_{w=0}^{\infty}\frac{t^w}{w!}\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}\left[\frac{2}{\theta(k+1)}\right]^{\frac{w}{2}}\Gamma\left(1+\frac{w}{2}\right) \tag{12}$$

2.1.2.3 Quantile function

Quantile function has a significant position in probability theory and it is the inverse of the cdf. The quantile function is obtained using

$$Q(u) = F^{-1}(u) \tag{13}$$

Using the inverse of equation (5), we have the quantile function of GRa distribution given as

$$x = Q\left(u\right) = \frac{1}{\theta}\left[-\log\left[1-\left[1-\left[\frac{1-u}{u+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]\right]^{\frac{1}{2}} \tag{14}$$

The median is obtained by setting *u = 0.5* in equation (14) given as

$$x_{median} = Q\left(0.5\right) = \frac{1}{\theta}\left[-\log\left[1-\left[1-\left[\frac{0.5}{1.5}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]\right]^{\frac{1}{2}} \tag{15}$$

2.1.2.4 Hazard function

Hazard function is given as

$$\pi(x;\alpha,\lambda,\theta) = \frac{f(x;\alpha,\lambda,\theta)}{S(x;\alpha,\lambda,\theta)} \tag{16}$$

The hazard function of the GRa distribution is given as

$$\pi(x;\alpha,\lambda,\theta) = \frac{\left[\dfrac{2\lambda\alpha\theta x e^{-\left(\theta/2\right)x^2}\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha-1}\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{2}}\right]}{1-\left[\dfrac{1-\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}\right]} \tag{17}$$

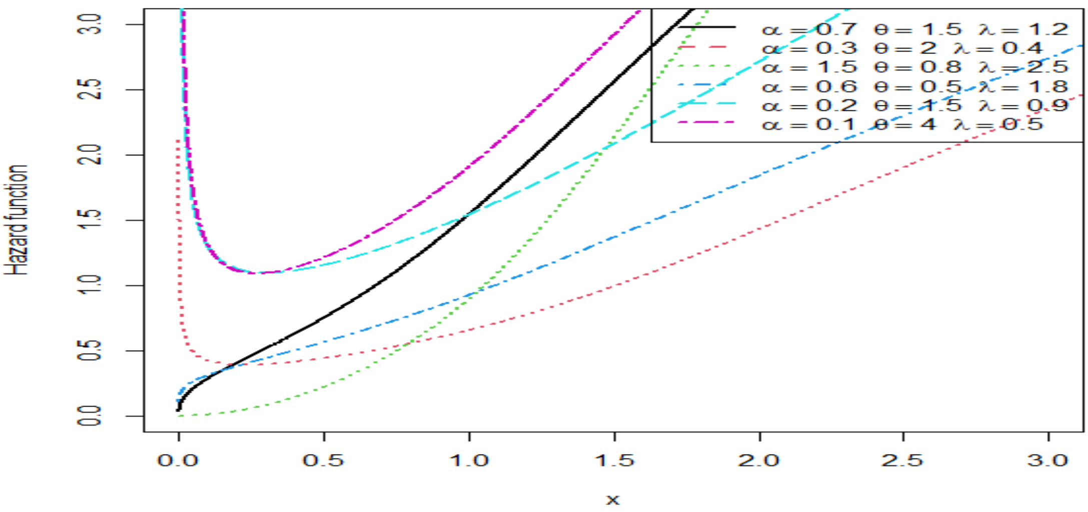

Figure 3: *Plots of hazard function of the GRa distribution for different parameter values*

2.1.2.5 Survival function

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$S(x;\alpha,\lambda,\theta)=1-F(x;\alpha,\lambda,\theta) \tag{18}$$

The survival function of the GRa distribution is given as

$$S(x;\alpha,\lambda,\theta)=1-\left[\frac{1-\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}\right] \quad (19$$

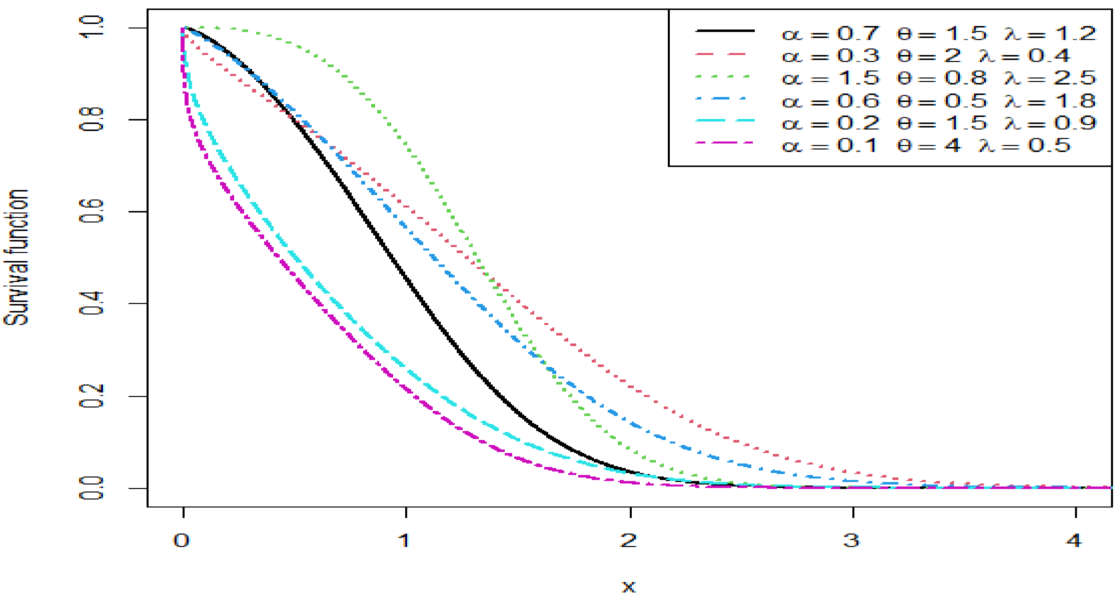

Figure 4: *Plots of survival function of the GRa distribution for different parameter values*

2.1.2.6 Reversed hazard function

Reversed hazard function of a random variable x is given as

$$\Re(x;\alpha,\lambda,\theta)=\frac{f(x;\alpha,\lambda,\theta)}{F(x;\alpha,\lambda,\theta)} \tag{20}$$

The reverse hazard rate function of the GRa distribution is given as

$$\Re(x;\alpha,\lambda,\theta)=\frac{\left[\dfrac{2\lambda\alpha\theta x e^{-\left(\theta/2\right)x^2}\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha-1}\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}\right]^{2}}\right]}{\left[\dfrac{1-\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}\right]} \tag{21}$$

2.1.2.7 Odds function

The odds function of the GRa distribution is given as

$$\Pi(x;\alpha,\lambda,\theta)=\frac{\left[\dfrac{1-\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}\right]}{1-\left[\dfrac{1-\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x^2}\right]^{\alpha}\right]^{\lambda}}\right]} \tag{22}$$

2.2 Order Statistics

Let $X_1, X_2, ..., X_n$ be n independent random variable from the GRa distribution and let $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r=1,2,3,...n$ denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. The pdf of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x)=\frac{f(x)}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}F(x)^{v+r-1} \tag{23}$$

The pdf of r[th] order statistic for the new distribution is obtained by replacing h with v+r-1 in equation (8) as

$$f_{r:n}(x)=2\lambda\alpha\theta x\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i,j,k=0}^{\infty}\sum_{m,p=0}^{v+r-1}\sum_{z,q=0}^{\infty}(-1)^{i+j+k+m+p+z+q+v}\binom{n-r}{v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}$$
$$\binom{v+r-1}{m}\binom{v+r+p-2}{p}\binom{\lambda(p+m)}{z}\binom{\alpha z}{q}\left[e^{-(\theta/2)x^2}\right]^{k+q+1} \tag{24}$$

The pdf of minimum order statistic of the GRa distribution is obtained by setting r=1

$$f_{1:n}(x)=2\lambda\alpha\theta xn\sum_{v=0}^{n-1}\sum_{i,j,k=0}^{\infty}\sum_{m,p=0}^{v}\sum_{z,q=0}^{\infty}(-1)^{i+j+k+m+p+z+q+v}\binom{n-1}{v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}$$
$$\binom{v}{m}\binom{v+p-1}{p}\binom{\lambda(p+m)}{z}\binom{\alpha z}{q}\left[e^{-(\theta/2)x^2}\right]^{k+q+1} \tag{25}$$

Also, the pdf of maximum order statistic of the distribution is obtained by setting r = n

$$f_{n:n}(x)=2\lambda\alpha\theta xn\sum_{m,p=0}^{v+n-1}\sum_{i,j,k=0}^{\infty}\sum_{z,q=0}^{\infty}(-1)^{i+j+k+m+p+z+q+v}\binom{1+i}{i}\binom{\lambda(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}$$
$$\binom{v+n-1}{m}\binom{v+n+p-2}{p}\binom{\lambda(p+m)}{z}\binom{\alpha z}{q}\left[e^{-(\theta/2)x^2}\right]^{k+q+1} \tag{26}$$

2.3 Estimation method

The method of maximum likelihood estimation (MLE) is used in this section to estimate the parameters of the GRa distribution. For a random sample, $X_1, X_2, ..., X_n$ of size n from the GRa distribution(α,θ,λ), the log-likelihood function $L(\alpha,\theta,\lambda)$ of (6) is given as

$$\log(L)=n\log(2)+n\log(\lambda)+n\log(\alpha)+n\log(\theta)-\frac{\theta}{2}\sum_{i=1}^{n}x_i^2+(\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-(\theta/2)x_i^2}\right]$$
$$+(\lambda-1)\sum_{i=1}^{n}\log\left[1-\left[1-e^{-(\theta/2)x_i^2}\right]^{\alpha}\right]-2\sum_{i=1}^{n}\log\left[1+\left[1-\left[1-e^{-(\theta/2)x_i^2}\right]^{\alpha}\right]^{\lambda}\right] \tag{27}$$

Differentiating the log-likelihood with respect to λ,α,θ and equating the result to zero, we have

$$\frac{\partial L}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left[1-\left[1-e^{-(\theta/2)x_i^2}\right]^{\alpha}\right]-2\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-(\theta/2)x_i^2}\right]^{\alpha}\right]^{\lambda}\log\left[1-e^{-(\theta/2)x_i^2}\right]^{\alpha}}{1+\left[1-\left[1-e^{-(\theta/2)x_i^2}\right]^{\alpha}\right]^{\lambda}}=0 \tag{28}$$

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-e^{-\left(\theta/2\right)x_i^2}\right]-(\lambda-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}\log\left[1-e^{-\left(\theta/2\right)x_i^2}\right]}{1-\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}}$$

$$+2\lambda\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}\right]^{\lambda-1}\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}\log\left[1-e^{-\left(\theta/2\right)x_i^2}\right]}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}\right]^{\lambda}}=0 \tag{29}$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}-\frac{1}{2}\sum_{i=1}^{n}x_i^2-(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\left(\theta/2\right)x_i^2}\left(\theta/2\right)x_i^2\left(1/2\right)x_i^2}{1-e^{-\left(\theta/2\right)x_i^2}}+(\lambda-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha-1}e^{-\left(\theta/2\right)x_i^2}\left(\theta/2\right)x_i^2\left(1/2\right)x_i^2}{1-\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}}$$

$$-2\alpha\lambda\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}\right]^{\lambda-1}\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha-1}e^{-\left(\theta/2\right)x_i^2}\left(\theta/2\right)x_i^2\left(1/2\right)x_i^2}{1+\left[1-\left[1-e^{-\left(\theta/2\right)x_i^2}\right]^{\alpha}\right]^{\lambda}}=0 \tag{30}$$

Now, equations (28), (29) and (30) do not have a simple analytical form and are therefore not tractable. As a result, we have to resort to non-linear estimation of the parameters using iterative method.

II. Results

3.1 Applications

In this section, we present two applications of GRa distribution using different data sets from different fields to demonstrate the flexibility of the distribution in modeling real-life data sets. The data are fitted to the GRa distribution and two other distributions as comparators such as Odd Lindley Rayleigh (OLRa) distribution by [11] and Rayleigh (Ra) distribution. This is done to test the new distribution's flexibility against the comparators. *Adequacy Model* which is a package in R-software, is used to produce the results of the analysis. Using the Akaike information criterion (AIC) and Bayesian information criterion (BIC), respectively, the performance of the distribution was compared to other existing distributions that were consistent with the baseline distribution in terms of providing good parametric fit to the data sets.

$$AIC=-2ll+2k \tag{31}$$

$$BIC=-2ll+k\log(n) \tag{32}$$

The model selection is carried out using the AIC and the BIC. Where ll denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters, and n is the sample size from the data.

The model with minimum value of AIC and BIC is chosen as the best model to fit the data set. The comparators presented are odd Lindley Rayleigh (OLRa) distribution and Rayleigh (Ra) distribution.

The first data set, taken from [14] and also reported in [15], shows the locations of the 68 stakes found while walking L = 1000 m and looking w = 20 m on either side of the transect line. The dimensions are:

2.0, 0.5, 10.4, 3.6, 0.9, 1.0, 3.4, 2.9, 8.2, 6.5, 5.7, 3.0, 4.0, 0.1, 11.8, 14.2, 2.4, 1.6, 13.3, 6.5, 8.3, 4.9, 1.5, 18.6, 0.4, 0.4, 0.2, 11.6, 3.2, 7.1, 10.7, 3.9, 6.1, 6.4, 3.8, 15.2, 3.5, 3.1, 7.9, 18.2, 10.1, 4.4, 1.3, 13.7, 6.3, 3.6, 9.0, 7.7, 4.9, 9.1, 3.3, 8.5, 6.1, 0.4, 9.3, 0.5, 1.2, 1.7, 4.5, 3.1, 3.1, 6.6, 4.4, 5.0, 3.2, 7.7, 18.2, 4.1.

The second data set represents the survival times (in weeks) of 33 patients suffering from acute myelogeneous leukemia. These data have been studied by [16]. The data are:

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

Table 1: *The MLEs and Information Criteria of the models based on the first data set*

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$-l$	**AIC**	**BIC**
GRa	0.4774	6.3091	0.0018	185.7156	377.4313	384.0898
OLRa	1.2608	-	0.0114	364.0566	732.1132	736.5522
Ra	-	-	0.0362	202.4675	406.1545	409.1545

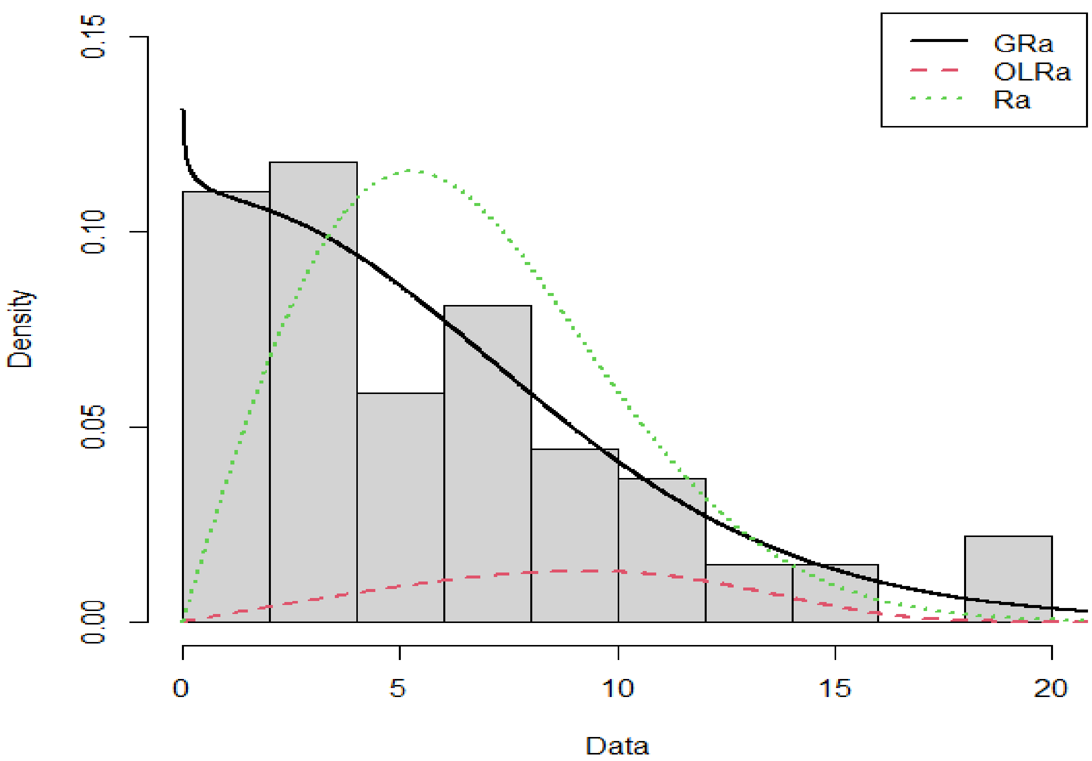

Figure 5: *Histogram and fitted pdfs for the GRa, OLRa and Ra models to the first data set*

Table 2: *The MLEs and Information Criteria of the models based on the second data set*

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$-l$	**AIC**	**BIC**
GRa	0.2293	0.6271	0.0005	158.1467	322.2934	326.7829
OLRa	1.2857	-	0.0001	270.2955	544.5910	547.5840
Ra	-	-	0.0005	188.6356	379.2713	380.7678

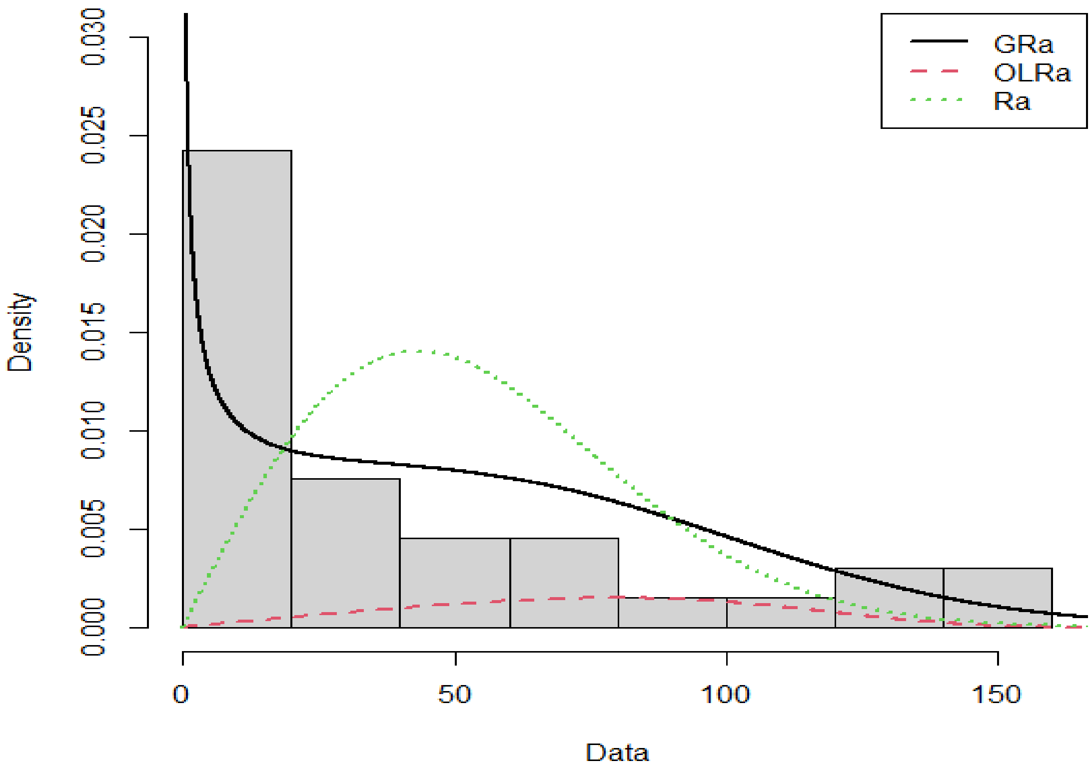

Figure 6: *Histogram and fitted pdfs for the GRa, OLRa and Ra models to the second data set*

IV. Discussion

The estimated values for each parameter and the models' goodness of fits are shown in Tables 1 and 2. AIC and BIC are two metrics for goodness of fits. The model performs better when the AIC and BIC values are lower. Tables 1 and 2 show that the GRa distribution has the lowest AIC and BIC, respectively and this makes the GRa model more adaptable and suitable for handling the data sets.

The new model's forms, fit, and adaptability in connection to the data sets under consideration are shown in Figures 5 and Figure 6. The black line, which represents the GRa model, more closely matched the data's pattern than the competitors. The histogram and fitted plots make it clear that the black line, which represents the GRa distribution, matches the two data sets under consideration better.

This study generalized the Rayleigh distribution by deriving a new continuous distribution known as the generalized Rayleigh distribution. Some of the statistical and mathematical properties of the GRa are obtained such as the survival function, hazard rate function, quantile function, inverted hazard function, odds function, and order statistics from the new distribution. Plotting the pdf and hazard rate function graphs revealed the contours of the suggested distribution. It was discovered that the hazard function is shaped like bathtub. *AdequacyModel* package in R was used to estimate the model parameters using the maximum likelihood method. The generalized Rayleigh distribution and its comparators considered were applied to two real life data sets, and the outcomes are shown in Tables 1 and 2. The findings demonstrated that the GRa distribution is much more potent, robust and superior at fitting the two data sets under consideration. The density graphs in figures 5 and 6 for the two data sets further show how adaptable and robust the new model is.

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