THE CONTINUOUS BERNOULLI-GENERATED FAMILY OF DISTRIBUTIONS: THEORY AND APPLICATIONS

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Abstract

The continuous Bernoulli distribution is a one-parameter probability distribution which is useful in analysis on machine learning. A handful of studies has been done to generalize the continuous Bernoulli distribution. In this paper, we introduced a wider extension of the continuous Bernoulli distribution by considering its distribution function as a generator. We referred to the proposed family as the continuous Bernoulli-generated family of distributions. Basic statistical treatments of the proposed family such as the density and cumulative distribution functions, survival and hazard rate functions, quantile, moments, moment generating function, and Renyi entropy are derived. The method of maximum likelihood is employed to estimate the unknown parameters of the family and the asymptotic behaviour of the parameter estimates is investigated via Monte Carlo simulation study. The waiting time (in minutes) of 100 Bank customers and the tensile strength measured in GPa, of 69 carbon fibers data sets formed the basis for real-life data fittings. Results obtained from the fitting of the two data sets when compared with some existing non-nested models revealed that the fittings were in favor of the continuous-Bernoulli Weibull distribution over the rest competing distributions.

Keywords: Continuous Bernoulli Distribution; Moments; Quantile; Monte Carlo Simulation Study

1. INTRODUCTION

The cumulative distribution function (cdf) of the one-parameter continuous Bernoulli distribution has been defined by [13] as

$$F(x,\lambda)=\begin{cases}\dfrac{\lambda^{x}\left(1-\lambda\right)^{1-x}+\lambda-1}{2\lambda-1}, & \lambda\neq 1/2,\ 0<x<1,\\[2ex] x, & \lambda=1/2\end{cases} \tag{1}$$

with the probability density function (pdf) associated to (1) obtained as

$$f(x,\lambda)=\begin{cases} C_\lambda \lambda^x (1-\lambda)^{1-x}, & \lambda \neq \tfrac{1}{2},\quad 0<x<1, \\ \\ 1, & \lambda = \tfrac{1}{2} \end{cases} \tag{2}$$

where the normalizing constant C_λ is defined as

$$C_\lambda = \begin{cases} \dfrac{2\tanh^{-1}(1-2\lambda)}{1-2\lambda}, & \lambda \neq \tfrac{1}{2}, \\ \\ 2, & \lambda = \tfrac{1}{2} \end{cases} \tag{3}$$

and $2\tanh^{-1}(1-2\lambda)=\ln(1-\lambda)-\ln(\lambda)$, using the relation $\tanh^{-1}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

We denote a random variable X following the continuous Bernoulli distribution as $X \sim CB(\lambda)$. The continuous Bernoulli distribution has special application in machine learning. Particularly, in simulating the pixel intensities of natural images in deep learning and computer vision, mostly in the development of variational autoencoders. Similar to the one-parameter Topp-Leone and power distributions, the $CB(\lambda)$ distribution is also a one-parameter distribution with support on a unit-interval.

In the theory of statistical analysis of lifetime data, bounded distributions have found a wide variety of applications ranging from the field of engineering, actuarial sciences, economics, biological sciences, etc. Particularly, when the data are recorded in rates, percentages and proportions. Over the years, the beta and Kumaraswamy distributions are the topmost bounded distributions to be reckon with in regards to fitting [0,1]-valued data sets, until the advent of several methodologies in developing unit-interval distributions. Notable among these distributions are the log-Lindley distribution proposed by [10], unit-logistic distribution developed by [14], log-Xgamma distribution introduced by [2], Marshall-Olkin Topp-Leone distribution developed by [17], unit-Burr XII distribution studied by [11], Marshall-Olkin extended unit-Gompertz distribution studied by [15], transmuted Marshall-Olkin extended Topp-Leone Distribution introduced by [18], Kumaraswamy unit-Gompertz distribution proposed by [1], etc. It is noteworthy to mention that the power continuous Bernoulli distribution due to [3] and transmuted continuous Bernoulli distribution due to [4], apparently the only extensions of the classical continuous Bernoulli distribution belong to this list. The goal of this paper is to develop a novel family of distributions based on the continuous Bernoulli distribution, which is hoped to birth more tractable and flexible lifetime distributions in analyzing real data sets.

The rest of the paper is organized in the following sections. Section 2 is devoted to model formulation. Section 3 provides some sub-models from the proposed family of distributions. General mathematical treatments for the proposed family of distributions, the parameter estimation as well as the investigation of the asymptotic behaviour of the parameter estimates of the model via a Monte Carlo simulation are discussed in Section 4. Section 5 provides the applicability of the proposed family of distributions in real-life data fitting. Section 6 concludes the paper.

2. MODEL FORMULATION

Suppose a random variable T follows a known probability distribution with pdf $f(t)$, [20] adopted the beta-generated technique developed by [6] to introduce the Topp-Leone-generated family of distributions with cdf defined by

$$\begin{aligned} F\left(x,\alpha,\xi\right) &= 2\alpha\int_0^{G(x,\xi)}\left(1-t\right)\left(t\left(2-t\right)\right)^{\alpha-1}dt, \quad 0<t<1,\ \alpha>0, \\ &= G\left(x,\xi\right)^{\alpha}\left(2-G\left(x,\xi\right)\right)^{\alpha}, \end{aligned} \tag{4}$$

and the associated pdf obtained as

$$f\left(x,\alpha,\xi\right) = 2\alpha g\left(x,\xi\right)G\left(x,\xi\right)^{\alpha-1}\left(1-G\left(x,\xi\right)\right)\left(2-G\left(x,\xi\right)\right)^{\alpha-1}. \tag{5}$$

As an alternative to the technique in (4), [5] introduced the so-called type II Topp-Leone generated (TIITL-G) family of distributions based on the methodology of [19] who introduced an alternative gamma-generator reported in [22]. The cdf and pdf of TIITL-G family are, respectively, defined by

$$\begin{aligned} F\left(x,\alpha,\xi\right) &= 1-2a\int_0^{1-G(x,\xi)}t^{a-1}\left(1-t\right)\left(2-t\right)^{\alpha-1}dt, \quad 0<t<1,\ \alpha>0, \\ &= 1-\left(1-G^2\left(x,\xi\right)\right)^{\alpha}, \end{aligned} \tag{6}$$

and

$$f\left(x,\alpha,\xi\right) = 2\alpha g\left(x,\xi\right)G\left(x,\xi\right)\left(1-G^2\left(x,\xi\right)\right)^{\alpha-1}. \tag{7}$$

Motivated by the simplicity of the technique in (6) and using the $CB\left(\lambda\right)$ distribution defined in (3) as the generator, we develop a novel class of distributions with the cdf defined by

$$F(t,\lambda,\xi) = \begin{cases} \dfrac{\lambda^{1-G(t,\xi)}\left(1-\lambda\right)^{G(t,\xi)}-\lambda}{1-2\lambda}, & \lambda\neq 1/2,\quad 0<t<1, \\ G(t,\xi), & \lambda = 1/2 \end{cases} \tag{8}$$

The pdf corresponding to (8) is obtained as

$$f(t,\lambda,\xi) = \begin{cases} C_\lambda g(t,\xi)\lambda^{1-G(t,\xi)}\left(1-\lambda\right)^{G(t,\xi)}, & \lambda\neq 1/2,\quad 0<t<1, \\ g(t,\xi), & \lambda = 1/2 \end{cases} \tag{9}$$

A random variable *T* having the cdf and pdf defined in (8) and (9), respectively, is said to follow the continuous Bernoulli-generated ($CB\left(\lambda,\xi\right)-G$) family of distributions.

The survival and hazard rate functions of $CB\left(\lambda,\xi\right)-G$ family of distributions are defined in (10) and (11), respectively, as

$$S(t,\lambda,\xi) = \begin{cases} \dfrac{\lambda^{1-G(t,\xi)}\left(1-\lambda\right)^{G(t,\xi)}+\lambda-1}{2\lambda-1}, & \lambda\neq 1/2,\quad 0<t<1, \\ 1-G(t,\xi), & \lambda = 1/2 \end{cases} \tag{10}$$

and

$$h(t,\lambda,\xi) = \begin{cases} \dfrac{C_\lambda^* g(t,\xi)\lambda^{1-G(t,\xi)}\left(1-\lambda\right)^{G(t,\xi)}}{\lambda^{1-G(t,\xi)}\left(1-\lambda\right)^{G(t,\xi)}+\lambda-1}, & \lambda\neq 1/2,\quad C_\lambda^* = \left(2\lambda-1\right)C_\lambda, \\ \dfrac{g(t,\xi)}{1-G(t,\xi)}, & \lambda = 1/2 \end{cases} \tag{11}$$

Furthermore, the quantile function of the $CB(\lambda,\xi)-G$ family of distributions is obtained as

$$Q_T(u) = G^{-1}\left[\frac{\ln\left[(1-2\lambda)u+\lambda\right]-\ln[\lambda]}{2\tanh^{-1}(1-2\lambda)}\right], \quad 0<u<1. \tag{12}$$

Whereas substituting $u=0.5$ in (12), the median of the $CB(\lambda,\xi)-G$ family of distributions is obtained as

$$Q_T(0.5) = G^{-1}\left[-\frac{\ln[2]+\ln[\lambda]}{2\tanh^{-1}(1-2\lambda)}\right]. \tag{13}$$

The utility of (12) is in generating random numbers from the $CB(\lambda,\xi)-G$ family of distributions, where u is generated from the uniform distribution satisfying $0<u<1$.

3. SUB-MODELS OF THE $CB(\lambda,\xi)-G$ FAMILY OF DISTRIBUTIONS

This section is concerned with the formulation of tractable models from the $CB(\lambda,\xi)-G$ family of distributions based on the Weibull, Topp-Leone, Kumaraswamy and Burr XII distributions as the baseline distribution in (8).

3.1 The continuous Bernoulli Weibull $CBW(\lambda,\alpha,\beta)$ distribution

Let T be a random variable following the Weibull distribution with cdf, $G(t,\alpha,\beta)=1-e^{-\beta t^{\alpha}}$ and pdf, $g(t,\alpha,\beta)=\alpha\beta t^{\alpha-1}e^{-\beta t^{\alpha}}$, $t>0,\ \alpha,\beta>0$. We defined the cdf and pdf of the $CBW(\lambda,\alpha,\beta)$ distribution, respectively, as follows

$$F(t,\lambda,\alpha,\beta)=\begin{cases}\dfrac{\lambda^{e^{-\beta t^{\alpha}}}(1-\lambda)^{1-e^{-\beta t^{\alpha}}}-\lambda}{1-2\lambda}, & \lambda\neq 1/2,\ \alpha,\beta>0,\ t>0.\\ 1-e^{-\beta t^{\alpha}}, & \lambda=1/2,\ \alpha,\beta>0.\end{cases} \tag{14}$$

and

$$f(t,\lambda,\alpha,\beta)=\begin{cases}C_{\lambda}^{w}t^{\alpha-1}e^{-\beta t^{\alpha}}\lambda^{e^{-\beta t^{\alpha}}}(1-\lambda)^{1-e^{-\beta t^{\alpha}}}, & \lambda\neq 1/2,\ \alpha,\beta>0,\ t>0,\ C_{\lambda}^{w}=\alpha\beta C_{\lambda}.\\ \alpha\beta t^{\alpha-1}e^{-\beta t^{\alpha}}, & \lambda=1/2,\ \alpha,\beta>0\end{cases} \tag{15}$$

3.2 The continuous Bernoulli Topp-Leone $CBTL(\lambda,\alpha)$ distribution

The one-parameter Topp-Leone distribution is defined by the density function

$$g(t,\alpha)=2\alpha(1-t)\left[t(2-t)\right]^{\alpha-1}, \quad \alpha\neq 1,\ \alpha>0,\ \ 0<t<1, \tag{16}$$

and the associated cdf is given by

$$G(t,\alpha)=\left[t(2-t)\right]^{\alpha}, \quad \alpha\neq 1,\ \alpha>0,\ \ 0<t<1, \tag{17}$$

By inserting the pdf and cdf in (16) and (17) into (8) and (9), we defined the cdf and pdf of the $CBTL(\lambda,\alpha)$ distribution, respectively, as

$$F(t,\lambda,\alpha)=\begin{cases}\dfrac{\lambda^{1-[t(2-t)]^{\alpha}}(1-\lambda)^{[t(2-t)]^{\alpha}}-\lambda}{1-2\lambda}, & \lambda\neq 1/2,\ \alpha>0,\ 0<t<1,\\ \\ [t(2-t)]^{\alpha}, & \lambda=1/2,\ \alpha>0.\end{cases} \tag{18}$$

and

$$f(t,\lambda,\alpha)=\begin{cases}C_{\lambda}^{TL}(1-t)[t(2-t)]^{\alpha-1}\lambda^{1-[t(2-t)]^{\alpha}}(1-\lambda)^{[t(2-t)]^{\alpha}}, & \lambda\neq 1/2,\ \alpha>0,\ 0<t<1,\ C_{\lambda}^{TL}=2\alpha C_{\lambda}.\\ \\ 2\alpha(1-t)[t(2-t)]^{\alpha-1}, & \lambda=1/2,\quad \alpha>0\end{cases} \tag{19}$$

3.3 The continuous Bernoulli Kumaraswamy $CBK(\lambda,\alpha,\beta)$ distribution

The Kumaraswamy distribution developed by [12] is a bounded distribution with 2 shape parameters having the cdf, $G(t)=1-(1-t^{\alpha})^{\beta}$ and pdf, $g(t)=\alpha\beta t^{\alpha-1}(1-t^{\alpha})^{\beta-1}$, $\alpha,\beta>0$.
By this information, the cdf and pdf of the $CBK(\lambda,\alpha,\beta)$ distribution is defined, respectively, as

$$F(t,\lambda,\alpha,\beta)=\begin{cases}\dfrac{\lambda^{(1-t^{\alpha})^{\beta}}(1-\lambda)^{1-(1-t^{\alpha})^{\beta}}-\lambda}{1-2\lambda}, & \lambda\neq 1/2,\ \alpha,\beta>0,\ 0<t<1,\\ \\ 1-(1-t^{\alpha})^{\beta}, & \lambda=1/2,\ \alpha,\beta>0.\end{cases} \tag{20}$$

and

$$f(t,\lambda,\alpha,\beta)=\begin{cases}C_{\lambda}^{k}t^{\alpha-1}(1-t^{\alpha})^{\beta-1}\lambda^{(1-t^{\alpha})^{\beta}}(1-\lambda)^{1-(1-t^{\alpha})^{\beta}}, & \lambda\neq 1/2,\ \alpha,\beta>0,\ 0<t<1,\ C_{\lambda}^{k}=\alpha\beta C_{\lambda}.\\ \\ \alpha\beta t^{\alpha-1}(1-t^{\alpha})^{\beta-1}, & \lambda=1/2,\quad \alpha,\beta>0\end{cases} \tag{21}$$

3.4 The continuous Bernoulli Burr XII $CBBXII(\lambda,\alpha,\beta)$ distribution

A random variable T is said to follow the two-parameter Burr XII distribution, if the density function of T is defined by

$$g(t,\alpha,\beta)=\alpha\beta t^{\alpha-1}(1+t^{\alpha})^{-(\beta+1)},\quad \alpha,\beta>0,\quad t>0, \tag{22}$$

and the corresponding cdf is given by

$$G(t,\alpha,\beta)=1-(1+t^{\alpha})^{-\beta},\quad \alpha,\beta>0,\quad t>0, \tag{23}$$

By inserting (22) and (23) into (8) and (9), we defined the cdf and pdf of the $CBBXII(\lambda,\alpha,\beta)$ distribution, respectively, as follows

$$F(t,\lambda,\alpha,\beta)=\begin{cases}\dfrac{\lambda^{\left(1+t^{\alpha}\right)^{-\beta}}\left(1-\lambda\right)^{1-\left(1+t^{\alpha}\right)^{-\beta}}-\lambda}{1-2\lambda}, & \lambda\neq \tfrac{1}{2},\ \ \alpha,\beta>0,\ t>0,\\[2ex] 1-\left(1+t^{\alpha}\right)^{-\beta}, & \lambda=\tfrac{1}{2},\ \ \alpha,\beta>0.\end{cases} \tag{24}$$

and

$$f(t,\lambda,\alpha,\beta)=\begin{cases}C_{\lambda}^{k}t^{\alpha-1}\left(1+t^{\alpha}\right)^{-(\beta+1)}\lambda^{\left(1+t^{\alpha}\right)^{-\beta}}\left(1-\lambda\right)^{1-\left(1+t^{\alpha}\right)^{-\beta}}, & \lambda\neq \tfrac{1}{2},\ \ \alpha,\beta>0,\ t>0,\ C_{\lambda}^{B}=\alpha\beta C_{\lambda}.\\[2ex] \alpha\beta t^{\alpha-1}\left(1+t^{\alpha}\right)^{-(\beta+1)}, & \lambda=\tfrac{1}{2},\ \ \alpha,\beta>0\end{cases} \tag{25}$$

4. MATHEMATICAL PROPERTIES OF THE $CB(\lambda,\xi)-G$ FAMILY OF DISTRIBUTIONS

In this section, the mathematical properties of the $CB(\lambda,\xi)-G$ family of distributions such as the r^{th} non-central moments, moment generating function (mgf) and Renyi entropy are discussed. The method of maximum likelihood estimation is employed to estimate the model parameters and the asymptotic behaviour of the parameter estimates are investigated through a Monte Carlo simulation study.

4.1 The r^{th} non-central moments

Let T be a random variable having the density function of the $CB(\lambda,\xi)-G$ family of distributions, then the r^{th} non-central moments of T is defined by

$$\begin{aligned}E\left[T^{r}\right]=\upsilon_{r}^{'}&=\int_{-\infty}^{\infty}t^{r}f\left(t,\lambda,\xi\right)dt, \qquad r=1,2,3,4,...\\ &=C_{\lambda}\int_{-\infty}^{\infty}t^{r}g(t,\xi)\,\lambda^{1-G(t,\xi)}\left(1-\lambda\right)^{G(t,\xi)}\,dt.\end{aligned} \tag{26}$$

Evaluating (26) yields the following results

$$\begin{aligned}E\left[T^{r}\right]&=C_{\lambda}\int_{-\infty}^{\infty}t^{r}g(t,\xi)\exp\left(\left(G(t,\xi)\right)\ln\left(\lambda\right)+\left(1-G(t,\xi)\right)\ln\left(1-\lambda\right)\right)dt,\\ &=\lambda C_{\lambda}\int_{-\infty}^{\infty}t^{r}g(t,\xi)\exp\left(G(t,\xi)\left[\ln\left(1-\lambda\right)-\ln\left(\lambda\right)\right]\right)dt,\\ &=\lambda C_{\lambda}\int_{-\infty}^{\infty}t^{r}g(t,\xi)\exp\left(G(t,\xi)\left[2\tanh^{-1}\left(1-2\lambda\right)\right]\right)dt.\end{aligned} \tag{27}$$

Applying the Maclaurin's series expansion of the exponential function,

$$e^{G(t,\xi)\left[2\tanh^{-1}(1-2\lambda)\right]}=\sum_{n=0}^{\infty}\frac{\left[2\tanh^{-1}\left(1-2\lambda\right)\right]^{n}}{n!}\left[G(t,\xi)\right]^{n},$$

so that (27) now becomes,

$$E\left[T^{r}\right]=\lambda C_{\lambda}\sum_{n=0}^{\infty}\frac{\left[2\tanh^{-1}\left(1-2\lambda\right)\right]^{n}}{n!}\int_{-\infty}^{\infty}t^{r}g(t,\xi)\left[G(t,\xi)\right]^{n}dt,$$

$$= \lambda C_{\lambda} \sum_{n=0}^{\infty} \frac{\left[2\tanh^{-1}(1-2\lambda)\right]^{n}}{n!(n+1)} \int_{-\infty}^{\infty} t^{r} h_{n+1}(t,\xi) dt,$$

$$= \lambda C_{\lambda} \sum_{n=0}^{\infty} \frac{\left[2\tanh^{-1}(1-2\lambda)\right]^{n}}{n!(n+1)} E\left[Y_{n+1}^{r}\right]. \quad (28)$$

Where $h_{n+1}(t,\xi) = (n+1)\, g(t,\xi) \left[G(t,\xi)\right]^{n}$ and $E\left[Y_{n+1}^{r}\right]$ are, respectively, the density function and r^{th} non-central moments of the exp-G family of distributions with power parameter $(n+1)$.

Thus, we can express the r^{th} non-central moments of the $CB(\lambda,\xi)-G$ family of distributions as a linear combination of the r^{th} non-central moments of the exp-G family of distributions with power parameter $(n+1)$.

For the purpose of numerical computation, we consider the two-parameter Weibull distribution as the baseline distribution. Hence, we compute the first four raw moments, variance, measures of skewness and kurtosis of the continuous Bernoulli Weibull $CBW(\lambda,\alpha,\beta)$ distribution in Table 1.

Table 1: *The Moments of the $CBW(\lambda,\alpha,\beta)$ distribution for selected values of the Parameters*

λ	β	α	υ_1'	υ_2'	υ_3'	υ_4'	σ^2	S	K
0.4	0.5	3	1.1721	1.5417	2.2068	3.3774	0.1679	0.0905	2.7315
		5	1.0822	1.2283	1.4485	1.7636	0.0571	-0.3259	2.9750
		7	1.0524	1.1367	1.2549	1.4119	0.0292	-0.5465	3.4994
	3.0	3	0.6450	0.4669	0.3678	0.3098	0.0509	0.0889	2.7397
		5	0.7563	0.5999	0.4944	0.4206	0.0279	-0.3265	2.8830
		7	0.8147	0.6812	0.5822	0.5072	0.0175	-0.5310	3.6310
0.8	0.5	3	0.9696	1.0912	1.3758	1.9007	0.1511	0.4223	2.9993
		5	0.9616	0.9824	1.0551	1.1827	0.0577	-0.0428	2.9123
		7	0.9659	0.9640	0.9896	1.0415	0.0310	-0.2721	3.2045
	3.0	3	0.5336	0.3305	0.2293	0.1743	0.0458	0.4181	2.9976
		5	0.6720	0.4798	0.3601	0.2821	0.0282	-0.0523	3.0015
		7	0.7478	0.5778	0.4592	0.3741	0.0186	-0.2719	3.0701

Information from Table 1 shows that the CBW distribution exhibits a left-skewed, right-skewed, platykurtic and leptokurtic properties which are essential in modeling heavy-tailed distributions.

4.2 The moment generating function

The moment generating function (mgf) of a random variable T with density function $f(t)$ is defined by

$$M_T(q) = E\left[e^{qt}\right] = \int_{-\infty}^{\infty} e^{qt} f(t) dt, \quad (29)$$

Using similar approach in (29), we defined the mgf of the $CB(\lambda,\xi)-G$ family of distributions as

$$M_T(q) = \lambda C_\lambda \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\left[2\tanh^{-1}(1-2\lambda)\right]^n q^p}{n!(n+1)\,p!} E\left[Y_{n+1}^p\right]. \tag{30}$$

Since, $e^{qt} = \sum_{p=0}^{\infty} \frac{(qt)^p}{p!}$.

4.3 The Renyi entropy

An entropy of a random variable say *T*, measures the degree of randomness associated with the random variable *T*. The Renyi entropy of *T* is defined by [18] as

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\int_{-\infty}^{\infty} f^\gamma(t)\,dt, \qquad \gamma > 0,\ \ \gamma \neq 1. \tag{31}$$

By substituting (9) into (31), we defined the Renyi entropy of a random variable *T* following the $CB(\lambda,\xi)-G$ family of distributions as follows

$$\begin{aligned} \tau_R(\gamma) &= \frac{1}{1-\gamma}\log\left[(C_\lambda)^\gamma \int_{-\infty}^{\infty} g^\gamma(t,\xi)\,\lambda^{\gamma(1-G(t,\xi))}(1-\lambda)^{\gamma G(t,\xi)}\,dt\right],\\ &= \frac{1}{1-\gamma}\log\left[(C_\lambda)^\gamma \lambda^\gamma \int_{-\infty}^{\infty} g^\gamma(t,\xi)\exp\left(\gamma G(t,\xi)\left[\ln(1-\lambda)-\ln(\lambda)\right]\right)dt\right],\\ &= \frac{1}{1-\gamma}\log\left[(C_\lambda)^\gamma \lambda^\gamma \int_{-\infty}^{\infty} g^\gamma(t,\xi)\exp\left(\gamma G(t,\xi)\left[2\tanh^{-1}(1-2\lambda)\right]\right)dt\right]. \end{aligned} \tag{32}$$

Again, applying the Maclaurin's series expansion of the exponential function,

$$e^{\gamma G(t,\xi)\left[2\tanh^{-1}(1-2\lambda)\right]} = \sum_{n=0}^{\infty} \frac{\gamma^n\left[2\tanh^{-1}(1-2\lambda)\right]^n}{n!}\left[G(t,\xi)\right]^n,$$

so that (32) now becomes,

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left[(C_\lambda)^\gamma \lambda^\gamma \sum_{n=0}^{\infty} \frac{\gamma^n\left[2\tanh^{-1}(1-2\lambda)\right]^n}{n!}\int_{-\infty}^{\infty} g^\gamma(t,\xi)\left[G(t,\xi)\right]^n dt\right]. \tag{33}$$

Two major properties of the Renyi entropy of a random variable *T* were identify by [9]. These include

(i) The Renyi entropy of *T* can assume a negative value;

(ii) For any $\gamma_1 < \gamma_2, R_{\gamma_2} \leq R_{\gamma_1}$ and equality holds if and only if *T* is a uniform random variable.

Again, we compute the Renyi entropy of the $CBW(\lambda,\alpha,\beta)$ distribution for selected values of the parameters as shown in Table 2.

Table 2: *Numerical computation of the Renyi entropy of the $CBW(\lambda,\alpha,\beta)$ distribution ($\lambda = 0.8$)*

i	γ_i	$\alpha=0.9,\ \beta=0.5$	$\alpha=0.9,\ \beta=3.0$	$\alpha=1.5,\ \beta=3.0$	$\alpha=1.5,\ \beta=0.5$
1	0.1	3.5600	1.5691	0.8868	2.0813
2	0.3	2.4724	0.4815	0.3213	1.5158
3	0.5	1.9849	-0.0060	0.0923	1.2869
4	0.7	1.6766	-0.3142	-0.0433	1.1513
5	0.9	1.4573	-0.5336	-0.1356	1.0589
6	2	0.8522	-1.1387	-0.3746	0.8199
7	4	0.4451	-1.5458	-0.5180	0.6765
8	6	0.2343	-1.7565	-0.5793	0.6152
9	8	0.0647	-1.9262	-0.6147	0.5799

The result in Table 2 validates the aforementioned properties of the Renyi entropy as suggested by [9].

4.4 Parameter estimation

4.4.1 Maximum likelihood estimation

The maximum likelihood estimation method is employed to estimate the parameters of the $CB(\lambda,\xi)-G$ family of distributions. Suppose $(t_1,t_2,....t_n)$ are random samples of size n from the $CB(\lambda,\xi)-G$ family of distributions, then the likelihood function is obtained as

$$L(t,\varphi) = \prod_{i=1}^{n}\left[g(t,\xi)\,\lambda^{1-G(t,\xi)}\,(1-\lambda)^{G(t,\xi)}\right], \quad \varphi=(\lambda,\xi)^T. \tag{34}$$

By taking the natural logarithm of both sides of (34), the log-likelihood function is obtained as

$$\ell(t,\varphi) = \sum_{i=1}^{n}\ln\left[g(t_i,\xi)\right]+\ln[\lambda]\sum_{i=1}^{n}\left(1-G(t_i,\xi)\right)+\ln[1-\lambda]\sum_{i=1}^{n}G(t_i,\xi). \tag{35}$$

The maximum likelihood estimate, say $\hat{\varphi}=\left(\hat{\lambda},\hat{\xi}\right)^T$ is obtained by differentiating the log-likelihood function in (35) with respect to the parameters and equating the corresponding function to zero as shown below

$$\frac{\partial\ell(t,\varphi)}{\partial\lambda} = \frac{1}{\lambda}\sum_{i=1}^{n}\left(1-G(t_i,\xi)\right)-\frac{1}{1-\lambda}\sum_{i=1}^{n}G(t_i,\xi)=0$$

Further simplification yields,

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n}\left(1-G(t_i,\xi)\right),$$

$$\frac{\partial\ell(t,\varphi)}{\partial\xi} = \sum_{i=1}^{n}\frac{g^{'}(t_i,\xi)}{g(t_i,\xi)}+\ln(1-\lambda)\sum_{i=1}^{n}g(t_i,\xi)-\ln(\lambda)\sum_{i=1}^{n}g(t_i,\xi)=0.$$

Where $g^{'}(t_i,\xi)=\frac{\partial g(t_i,\xi)}{\partial\xi_i}$ and $\partial\xi_j$ is the j^{th} element of the vector of parameter ξ.

It is clear from these expressions that the parameters $\hat{\lambda}$ can be solved analytically, whereas the parameter(s) $\hat{\xi}_j$ may require the use of software program such as ***R*** program for estimation.

4.4.2 Simulation study

In this subsection, we investigate the asymptotic behaviour of the parameter estimates of the $CBW(\lambda,\alpha,\beta)$ distribution. Random samples of size $n=(15,25,50,75,100)$ are generated from the $CBW(\lambda,\alpha,\beta)$ distribution at randomly fixed values of the parameters. A Monte Carlo simulation is repeated 1000 times and the following quantities are computed:

i) $\text{bias}=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\varphi}_i-\overline{\varphi}\right),$

ii) root mean square error (RMSE) $= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\varphi}_i - \bar{\varphi}\right)^2}$.

iii) Coverage Probability of the 95% confidence interval of the estimates $\hat{\varphi}_i$ given by

$$CP\left(\hat{\varphi}\right) = \frac{1}{N}\sum_{i=1}^{N} I\left(\hat{\varphi}_i - Z_{\delta/2}\sqrt{\mathrm{var}\left(\hat{\varphi}\right)} < \varphi_0 < \hat{\varphi}_i + Z_{\delta/2}\sqrt{\mathrm{var}\left(\hat{\varphi}\right)}\right).$$

Where $I(.)$ is an indicator function and $\left(\hat{\varphi}\right)$ is the standard error of the estimate φ_i.

Table 3: *Simulation results for bias, RMSE and CP of parameter estimates of* $CBW\left(\lambda,\alpha,\beta\right)$ *distribution*

Parameters	n	Bias			RMSE			CP		
		α	β	λ	α	β	λ	α	β	λ
$\alpha=0.3$, $\beta=0.6$, $\lambda=0.8$	15	0.0042	0.3614	-0.2437	0.0752	0.5992	0.3598	0.986	0.988	0.908
	25	-0.0215	0.3395	-0.2522	0.0623	0.5566	0.3558	0.958	0.972	0.888
	50	-0.0578	0.3019	-0.2781	0.0527	0.4956	0.3441	0.948	0.970	0.864
	75	-0.0704	0.2741	-0.2996	0.0477	0.4877	0.3253	0.938	0.940	0.878
	100	-0.0961	0.2210	-0.3323	0.0421	0.4231	0.2926	0.958	0.964	0.910
$\alpha=0.5$, $\beta=0.3$, $\lambda=0.6$	15	0.0324	0.1972	-0.1020	0.1422	0.2625	0.2880	0.978	0.958	0.918
	25	0.0093	0.1887	-0.1074	0.1057	0.2472	0.2808	0.988	0.986	0.890
	50	-0.0154	0.1628	-0.1158	0.0832	0.2404	0.2749	0.964	0.978	0.876
	75	-0.0184	0.1017	-0.1356	0.0828	0.2361	0.2741	0.942	0.966	0.872
	100	-0.0209	0.0772	-0.1648	0.0724	0.2227	0.2578	0.944	0.952	0.878
$\alpha=0.9$, $\beta=3.0$, $\lambda=0.4$	15	0.1085	0.3271	0.0496	0.3043	1.2171	0.2746	0.956	0.998	0.914
	25	0.0599	0.1131	0.0401	0.2177	0.9368	0.2645	0.964	0.990	0.904
	50	0.0174	0.1082	0.0192	0.1920	0.8197	0.2632	0.926	0.956	0.852
	75	0.0026	0.0824	0.0186	0.1619	0.7159	0.2586	0.914	0.942	0.824
	100	-0.0043	0.0531	0.0079	0.1615	0.6676	0.2499	0.904	0.940	0.814
$\alpha=0.9$, $\beta=0.6$, $\lambda=0.4$	15	0.0932	0.0618	0.0485	0.2758	0.3681	0.2862	0.978	0.940	0.910
	25	0.0439	0.0527	0.0468	0.2190	0.3658	0.2812	0.966	0.938	0.858
	50	0.0266	0.0523	0.0293	0.1871	0.3382	0.2702	0.938	0.928	0.818
	75	0.0082	0.0470	0.0256	0.1551	0.3023	0.2532	0.950	0.938	0.844
	100	0.0073	0.0452	0.0180	0.1524	0.3007	0.2521	0.922	0.912	0.828

From Table 3, we observe that the bias and root mean square errors of the parameter estimates decrease as the sample size n increases. Moreover, the coverage probability of the parameter estimates approaches the nominal level of 95% confidence interval.

5. REAL-LIFE DATA FITTINGS

The applicability of the proposed family of distributions is investigated in this section. To achieve this, two data sets including the waiting time (in minutes) of 100 Bank customers and the tensile strength measured in GPa, of 69 carbon fibers data sets are employed for data fittings. Some well-known non-nested models such as the Kumaraswamy Weibull ($KW\left(\lambda,\alpha,\beta\right)$), Kumaraswamy inverse Weibull ($KIW\left(\lambda,\alpha,\beta\right)$), Topp-Leone inverse Weibull ($TLIW\left(\lambda,\alpha,\beta\right)$), transmuted Weibull ($TW\left(\lambda,\alpha,\beta\right)$) and the two-parameter Weibull distributions are employed alongside with the proposed continuous-Bernoulli Weibull ($CBW\left(\lambda,\alpha,\beta\right)$) distribution to fit the two data sets. The data sets for the analysis are given below.

Data set 1: The first data set represents the waiting time (in minutes) of 100 Bank customers reported in [16]. The data set was first used by [8] to illustrate the flexibility of the Lindley distribution over the exponential distribution in data fittings. The data are given as follows: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9 ,1.9, 2.1, 2.6, 2.7,2.9, 3.1, 3.2, 3.3 ,3.5, 3.6, 4.0, 4.1, 4.2, 4.2,4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9,5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3,6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0,8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6,9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5,11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9,14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0,19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Data set 2: The second data set comprises of the tensile strength measured in GPa, of 69 carbon fibers tested under tension at gauge length of 20mm reported in [21]. This data set was also employed by [7] to demonstrate the applicability of the power Lindley distribution. The data are represented as follows: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Some popularly used model selection criteria such as the maximized log-likelihood (*LL*), Akaike Information Criteria (AIC), and some goodness of fit test statistics such as the Komolgorov-Smirnov (K-S), Crammer von Mises (W^*) and Anderson Darling (A^*) test statistics with their corresponding *p-value* are considered to access the appropriate model for analyzing the two data sets. Tables 4 and 5 present the summary statistics for the fit of the distributions for the two data sets, respectively.

Table 4: *Summary statistics for the waiting time data set*

Models	*Estimates*	*LL*	*AIC*	*K-S (p-value)*	W^* *(p-value)*	A^* *(p-value)*
CBW	$\alpha=1.7229$	-317.3098	640.6196	0.0423	0.0248	0.1682
	$\beta=0.0071$			(0.994)	(0.9904)	0.9968)
	$\lambda=0.9356$					
KW	$\alpha=1.3727$	-317.6755	641.3510	0.0508	0.0414	0.2578
	$\beta=0.2015$			(0.9587)	(0.9263)	(0.9660)
	$\lambda=1.3379$					
KIW	$\alpha=2.6384$	-332.9531	671.9062	0.1099	0.4051	2.6255
	$\beta=1.1424$			(0.1785)	(0.0698)	(0.0427)
	$\lambda=-1.5224$					
TLIW	$\alpha=0.5235$	-327.1056	641.2112	0.0891	0.2449	1.6727
	$\beta=12.5524$			(0.4044)	(0.1951)	(0.1402)
	$\lambda=0.9569$					
TW	$\alpha=1.5692$	-317.8896	641.7791	0.0481	0.0384	0.2599
	$\beta=0.0157$			(0.9746)	(0.9420)	(0.9648)
	$\lambda=0.6181$					
Weibull	$\alpha=1.4584$	-318.7307	641.4614	0.0577	0.0609	0.4051
	$\beta=0.0305$			(0.8929)	(0.8095)	(0.8433)

Table 5: *Summary statistics for tensile strength data set*

Models	*Estimates*	*LL*	*AIC*	*K-S* (*p-value*)	W^* (*p-value*)	A^* (*p-value*)
CBW	$\alpha = 2.7806$ $\beta = 0.1778$ $\lambda = 0.0026$	-49.0740	104.1481	0.0400 (0.9999)	0.0142 (0.9998)	0.1210 (0.9998)
KW	$\alpha = 3.9464$ $\beta = 0.1690$ $\lambda = -0.1312$	-49.9210	105.8421	0.0675 (0.9112)	0.0581 (0.8276)	0.3901 (0.8580)
KIW	$\alpha = 4.2588$ $\beta = 2.8719$ $\lambda = -3.7556$	-56.2704	118.5408	0.1061 (0.4193)	0.1995 (0.2688)	1.3439 (0.2185)
TLIW	$\alpha = 0.5468$ $\beta = 34.8898$ $\lambda = 3.4115$	-58.0304	122.0608	0.1176 (0.2960)	0.2617 (0.1741)	1.7344 (0.1294)
TW	$\alpha = 5.9303$ $\beta = 0.0021$ $\lambda = 0.6363$	-49.1325	104.2650	0.0433 (0.9995)	0.0191 (0.9979)	0.1714 (0.9963)
Weibull	$\alpha = 5.5045$ $\beta = 0.0046$	-49.5961	104.1923	0.0560 (0.9819)	0.0343 (0.9611)	0.2739 (0.9563)

From Tables 4 and 5, based on the conditions to measure superiority of models, the continuous-Bernoulli $CBW(\lambda, \alpha, \beta)$ distribution having the maximized log-likelihood value, least value in terms of the *AIC*, *K-S*, W^* and A^* test statistics with the corresponding highest *p-value*, outperforms the competitor distributions in analyzing the two data sets, and thus becomes the most appropriate model in fitting the data sets.

6. CONCLUSION

In this paper, we have developed a new class of probability distributions based on the continuous Bernoulli distribution. The proposed family is called the continuous Bernoulli-generated family of distributions. Mathematical derivation of some basic properties of the proposed family such as the density and cumulative distribution functions, survival and hazard rate functions, quantile, moments, moment generating function, and Renyi entropy were obtained. The method of maximum likelihood was employed to estimate the unknown parameters of the family and the asymptotic behaviour of the parameter estimates was investigated via Monte Carlo simulation study. Two real-life data sets including the waiting time (in minutes) of 100 Bank customers and the tensile strength measured in GPa, of 69 carbon fibers data sets were employed to illustrate the applicability of the proposed family. Existing non-nested models such as the Kumaraswamy Weibull, Kumaraswamy inverse Weibull, Topp-Leone inverse Weibull, transmuted Weibull and the two-parameter Weibull distributions were employed alongside the proposed continuous-Bernoulli Weibull distribution to fit the two data sets. Results obtained from the fitting of the two data sets when compared using some model selection criteria and goodness of fit test statistics, revealed that the fittings were in favor of the continuous-Bernoulli Weibull distribution over the rest competing distributions.

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