A NEW EXPONENTIAL TYPE RATIO ESTIMATOR FOR THE POPULATION MEAN IN SYSTEMATIC SAMPLING

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Abstract

Utilizing auxiliary information effectively in sample surveys can enhance the accuracy of estimations by capitalizing on the relationship between the main variable under study and the auxiliary variable. Estimators such as ratio, product, exponential, and regression estimators are frequently employed either during the estimation process, the design phase, or both. In everyday situations, it is common to incorporate information from one or two auxiliary variables to improve the precision of estimators. Auxiliary information has been in practice in sampling theory since the advent of modern sample surveys. Information on auxiliary variable having high correlation with the variable under study is quite useful in improving the sampling design. Cochran (1940) used the highly positively correlated study and auxiliary variable to propound the ratio estimator. Product estimator requires a high negative correlation between study and auxiliary variable. By reviewing the literature, it is concluded that applying the auxiliary information enhances the efficiencies of the estimators for estimating any parameter under consideration. So it is well established fact that the use of auxiliary variable technique improves the estimation process for target population. It is also noticed that ratio method of estimation is relatively simple and one of the commonly used methods of estimation. Due to limitations in terms of time and cost, sample surveys are often preferred over census surveys for collecting primary data. In these sample surveys, the ratio, product, and regression estimators are frequently employed to estimate the mean or other parameters of interest for the variable under study. To assess their efficiency, these estimators are compared based on their approximate mean squared errors. In this paper we proposed an exponential ratio type estimator for the estimation of finite population mean under systematic sampling. The mean square error of the proposed estimator is computed up to the first order of approximation and we find proposed estimator is efficient as compared to other existing estimators. Furthermore this theoretical result is supported by numerical examples as well.

Keywords: Systematic sampling, exponential ratio type estimator, mean square error, efficiency.

1. Introduction

In the literature of survey sampling, it is well known that the efficiencies of the estimators of the population parameters of the study variable can be increased by the use of auxiliary information related to auxiliary variable x, which is highly correlated with the study variable y. Auxiliary information may be efficiently utilized either at planning stage or at design stage to arrive at an improved estimator compared to those estimators, not utilizing auxiliary information. A simple technique of utilizing the known information of the population parameters of the auxiliary variables is through ratio, product, and regression method of estimations using different probability sampling designs such as simple random sampling, stratified random sampling, cluster sampling, systematic sampling, and double sampling. In the present paper we will use knowledge of the auxiliary variables under the framework of systematic sampling. Due to its simplicity, systematic sampling is the most commonly used probability design in survey of finite populations; see W. G. Madow and L. H. Madow [23]. Apart from its simplicity, systematic sampling provides estimators which are more efficient than simple random sampling or stratified random Sampling for certain types of population; see Cochran [24], Gautschi [25], and Hajeck [9]. later on the problem of estimating the population mean using information on auxiliary variable has also been discussed by various authors including Quenouille [15], Hansen et al. [12], Swain [1], Singh [14], Shukla [16], Srivastava and Jhajj [21], Kushwaha and Singh [10], Bahl and Tuteja [20], Banarasi et al. [2], H. P. Singh and R. Singh [5], Kadilar and Cingi [3], Koyuncu and Kadilar [17], Singh et al. [8], Singh and Solanki [6], Singh and Jatwa [7], Tailor et al. [22], Khan and Singh [13], and Khan and Abdullah [11], R. Singh et al. [18], R. Singh et al. [19], D. S. Robson [4].

Consider a finite population $U = U_1, U_2, U_3 \dots \dots .. U_N$ of size N units numbered from 1 to N in some order .A sample of size n is taken size n units is taken at random from the first k units and every kth subsequent unit; then, $= nk$ where n and k are positive integers; thus, there will be k samples (clusters) each of size n and observe the study variate y and auxiliary variate x for each and every unit selected in the sample

Let $(y_{ij,} x_{ij})\ for\ i = 1.2 \dots .. k,\quad j = 1,2 \dots \dots .. n$ denote the value of jth unit in the ith sample. Then, the systematic sample means are defined as follows:

$\bar{y}_{st} = t_0 = 1/n \sum_{j=1}^{n} y_{ij,}$ and $\bar{x}_{st} = t_0 = 1/n \sum_{j=1}^{n} y_{ij,}$ are the unbiased estimators of the population means

$\bar{Y} = 1/n \sum_{j=1}^{n} y_{ij,}$ and $\bar{X} = 1/n \sum_{j=1}^{n} x_{ij,}\ of\ \ y\ \ on\ x$

To obtain the properties of the estimators up to first order of approximation, we use the following errors terms:

$$e_0 = \bar{y}_{sys} - \bar{Y}/\bar{Y}\ , e_1 = \bar{x}_{sys} - \bar{X}/\bar{X}\ , e_2 = \bar{z}_{sys} - \bar{Z}/\bar{Z}\ , Such\ that\ E(e_1) = 0\ for\ i = 0,1\ and\ 2$$

and

$$\rho_{yx} = \frac{s_{yx}}{s_y s_x}$$

$$\rho_{yz} = \frac{s_{yz}}{s_y s_z}$$

$$\rho_{xz} = \frac{s_{xz}}{s_x s_z}$$

$$k = \frac{\rho_{yx} C_y}{C_x}$$

$$k^* = \frac{\rho_{yz} C_y}{C_z}$$

$$\rho^*{}_y = \{1 + (n-1)\rho_y\}$$

$$\rho^*{}_x = \{1 + (n-1)\rho_x\}$$

$$\rho^*{}_z = \{1 + (n-1)\rho_z\}$$

$$\rho^{**} = \frac{\rho^*{}_y}{\rho^*{}_x}$$

$$\rho_2{}^{**} = \frac{\rho^*{}_y}{\rho^*{}_z}$$

$$\rho_1{}^{**} = \frac{\rho^*{}_y}{\rho^*{}_z}$$

Where

ρ_y , ρ_x, ρ_z are intra class correlation among the pair of units for the variables y,x and z.

2. Estimators in Literature:

In this part, we consider some estimators of the finite population mean in the sampling literature. The variance and MSE's of all the estimators computed here are obtained in the first order of approximation.

The variance of the unbiased estimator for population mean is

$$var(t_0) = \lambda \bar{Y}^2 \varphi_0 \tag{1.1}$$

Swain [14] and Shukla [16] suggested the classical ratio and product estimators for finite population mean by are given by

$$t_1 = \bar{y}_{sy}\left(\frac{\bar{X}}{\bar{x}_{sy}}\right) \tag{1.2}$$

$$t_2 = \bar{y}_{sy} exp\left(\frac{\bar{z}_{sy}}{\bar{Z}}\right) \tag{1.3}$$

The mean square errors of the estimators to the first order of approximation are given as follows:

$$MSE(t_1) = \lambda \bar{Y}^2\left[\varphi_0 + \varphi_2\left(1 - 2k\sqrt{\rho^{**}}\right)\right] \tag{1.4}$$

$$\text{MSE}(t_2) = \lambda \bar{Y}^2\left[\varphi_0 + \varphi_3\left(1 + 2k\sqrt{\rho_2{}^{**}}\right)\right] \tag{1.5}$$

Singh et al. [20] utilizing the known knowledge of the auxiliary variable, suggested the following ratio and product type exponential estimators:

$$t_3 = \bar{y}_{sy} exp\left(\frac{\bar{X} - \bar{x}_{sy}}{\bar{X} + \bar{x}_{sy}}\right) \tag{1.6}$$

$$t_4 = \bar{y}_{sy} exp\left(\frac{\bar{x}_{sy} - \bar{X}}{\bar{x}_{sy} + \bar{X}}\right) \tag{1.7}$$

$$\text{MSE}(t_3) = \lambda \bar{Y}^2\left[\varphi_0 + \frac{\varphi_2}{4}\left(1 - 4k\sqrt{\rho^{**}}\right)\right] \tag{1.8}$$

$$\text{MSE}(t_4) = \lambda \bar{Y}^2\left[\varphi_0 + \frac{\varphi_2}{4}\left(1 + 4k\sqrt{\rho^{**}}\right)\right] \tag{1.9}$$

Tailor et al. [25] define the following ratio-cum product estimator for the population mean $\bar{Y}$:

$$t_5 = \bar{y}_{sy}\left(\frac{\bar{X}}{\bar{x}_{sy}}\right)\left(\frac{\bar{z}_{sy}}{\bar{Z}}\right) \tag{2.0}$$

The mean square error of the estimator t_6 up to first order of approximation, is given by

$$\mathrm{MSE}(t_5) = \lambda \bar{Y}^2\left[\varphi_0 + \varphi_2\left(1 - 2k\sqrt{\rho^{**}}\right) + \varphi_3\left(1 - 2k^{**}\sqrt{\rho_1{}^{**}}\right) + 2\varphi_4\sqrt{\rho_y{}^*\rho_z{}^*}\right] \tag{2.1}$$

Where,

$\varphi_0 = \rho_y{}^* C_y{}^2$

$\varphi_1 = 2C_x{}^2\sqrt{\rho_y{}^*\rho_x{}^*}$

$\varphi_2 = \rho_x{}^* C_x{}^2$

$\varphi_3 = \rho_z{}^* C_z{}^2$

$\varphi_4 = k^* C^2{}_z$

3. Proposed estimator

In this section, we have proposed an exponential ratio type estimator for population mean of the study variable y under systematic sampling as given by:

$$t_{RK} = \bar{\mathrm{y}}_{\mathrm{sy}}\left(\frac{\bar{\mathrm{x}}_{\mathrm{sy}}}{\bar{\mathrm{X}}}\right)^{\alpha} \exp\left(\frac{\bar{\mathrm{X}} - \bar{\mathrm{x}}_{\mathrm{sy}}}{\bar{\mathrm{X}} + \bar{\mathrm{x}}_{\mathrm{sy}}}\right) \tag{2.2}$$

The first order of approximation of the above error terms is given by

$E(e_0{}^2) = \lambda\rho_y{}^* C_y{}^2$,

$E(e_1{}^2) = \rho_x{}^* C_x{}^2$

$E(e_0 e_1) = \lambda C^2{}_x\sqrt{\rho_y{}^*\rho_x{}^*}$

Where, $\lambda = \left(\frac{N-1}{nN}\right)$

Expressing (2.2) in terms of e's

$$t_{RK} = \bar{Y}(1+\varepsilon_0)(1+\varepsilon_1)^{\alpha} exp\left(\frac{\bar{X} - \bar{X}(1+\varepsilon_1)}{\bar{X} + \bar{X}(1+\varepsilon_1)}\right)$$

$$t_{RK} = \bar{Y}(1+\varepsilon_0)(1+\varepsilon_1)^{\alpha} exp\left[\frac{\varepsilon_1}{2}\left(1+\frac{\varepsilon_1}{2}\right)^{-1}\right] \tag{2.3}$$

$$t_{RK} = \bar{Y}(1+\varepsilon_0)(1+\varepsilon_1)^{\alpha} exp\left[\frac{\varepsilon_1}{2}\left(1+\frac{\varepsilon_1}{2}+\frac{\varepsilon_1{}^2}{4}+\cdots\right)\right]$$

$$t_{RK} = \bar{Y}(1+\varepsilon_0)\left(1+\alpha\varepsilon_1+\frac{\alpha(\alpha-1)}{2}\varepsilon_1{}^2+\cdots\right) exp\left[\frac{\varepsilon_1}{2}\left(1+\frac{\varepsilon_1}{2}+\frac{\varepsilon_1{}^2}{4}+\cdots\right)\right] \tag{2.4}$$

From (2.4)

$$t_{RK} - \bar{Y} \cong \bar{Y}\left[\alpha\varepsilon_1 + \frac{\alpha(\alpha-1)}{2}\varepsilon_1{}^2 + \frac{\varepsilon_1}{2} + \frac{\alpha\varepsilon_1{}^2}{2} + \frac{3\varepsilon_1{}^2}{8} + \varepsilon_0 + \alpha\varepsilon_1\varepsilon_0 + \frac{\varepsilon_0\varepsilon_1}{2}\right] \tag{2.5}$$

Squaring (2.5) and then taking expectation on both sides, the MSE of the estimator $\bar{y}_{RK}$ is

$$MSE(t_{RK}) = \lambda \bar{Y}^2 \left[\varphi_0 + \frac{K\varphi_1}{2} + \alpha K\varphi_1 + (\alpha^2 + \alpha)\frac{\varphi_2}{2} + \frac{\varphi_2}{8}\right] \quad (2.6)$$

Obtain the optimum α to minimize $MSE(\bar{y}_{RK})$. Differentiating $MSE(\bar{y}_{RK})$ w.r.t α and equating the derivative to zero. Optimum value of α is given by:

$$\alpha = -\frac{(K\varphi_1 + \varphi_2)}{2\varphi_2}$$

Substituting the value of α_{opt} in (2.6), we get the minimum value of $\bar{y}_{RK}$ as:

$$MSE_{min}(t_{RK}) = \lambda \bar{Y}^2 \varphi_0 \left[1 - {\rho_{yx}}^2\right] \quad (2.7)$$

It follows from (2.7) that the proposed estimator $\bar{y}_{RK}$ at its optimum condition is equal efficient as that of the usual linear regression estimator.

4. Efficiency Comparisons

In this section, the MSE of traditional estimatorst_0,t_1, t_2, t_3, $t_4$and $t_5$are compared with the MSE of proposed estimator $\bar{y}_{RK}$.

From (1.1) - (2.0) and (2.1)

$$[var(t_0) - MSE_{min}(\bar{y}_{RK})] > 0$$

$$\left[\lambda \bar{Y}^2 \varphi_0 {\rho_{yx}}^2\right] > 0 \quad (2.8)$$

$$[MSE(t_1) - MSE_{min}(\bar{y}_{RK})] > 0$$

$$\lambda \bar{Y}^2 \left[\varphi_0\left(1 - 2k\sqrt{\rho^{**}}\right)\right] - \left[\lambda \bar{Y}^2 \varphi_0 {\rho_{yx}}^2\right] > 0 \quad (2.9)$$

$$[MSE(t_2) - MSE_{min}(\bar{y}_{RK})] > 0$$

$$\lambda \bar{Y}^2 \left[\varphi_0\left(1 + 2k\sqrt{\rho^{**}}\right)\right] - \left[\lambda \bar{Y}^2 \varphi_0 {\rho_{yx}}^2\right] > 0 \quad (3.0)$$

$$[MSE(t_3) - MSE_{min}(\bar{y}_{RK})] > 0$$

$$\lambda \bar{Y}^2 \left[\frac{\varphi_0}{4}\left(1 - 4k\sqrt{\rho^{**}}\right)\right] - \left[\lambda \bar{Y}^2 \varphi_0 {\rho_{yx}}^2\right] > 0 \quad (3.1)$$

$$[MSE(t_4) - MSE_{min}(\bar{y}_{RK})] > 0$$

$$\lambda \bar{Y}^2 \left[\frac{\varphi_0}{4}\left(1 + 4k\sqrt{\rho^{**}}\right)\right] - \left[\lambda \bar{Y}^2 \varphi_0 {\rho_{yx}}^2\right] > 0 \quad (3.2)$$

$$[MSE(t_5) - MSE_{min}(\bar{y}_{RK})] > 0$$

$$\lambda \bar{Y}^2 \left[\varphi_0\left(1 - 2k\sqrt{\rho^{**}}\right) + \varphi_3\left(1 - 2k^{**}\sqrt{{\rho_1}^{**}}\right) + 2\varphi_4\sqrt{{\rho_y}^{*}{\rho_z}^{*}}\right] - \left[\lambda \bar{Y}^2 \varphi_0 {\rho_{yx}}^2\right] > 0 \quad (3.3)$$

5. Empirical Study

To examine the merits of the proposed estimator over the other existing estimators at optimum conditions, we have considered natural population data sets from the literature. The sources of population are given as follows.

(Source: Tailor et al. [20]). Consider Population

$N = 15$, $n = 3, \bar{X} = 44.47, \bar{Y} = 80, \bar{Z} = 48.40, C_y = 0.56, C_x = 0.28, C_z = 0.43$

$S_y^2 = 2000, S_x^2 = 149.55, S_z^2 = 427.83, S_{yx} = 538.57, S_{yz} = -902.86, S_{xz} = -241.06,$

$\rho_{yx} = 0.9848, \rho_{yz} = -0.9760 \rho_{xz} = -0.9530, \rho_y = 0.6652, \rho_x = 0.707, \rho_z = 0.5487.$

In order to find (PREs) of the estimator, we use the following formula and $PRE(t_\alpha, t_0) = MSE(t_0)/MSE(t_\alpha) \times 100$, For $\alpha = 0,1,2,3,4,5$ and RK

Table 1: *The percent relative efficiency of different estimators with respect to* t_0.

Population		
Estimator	MSE(t_α)	$PRE(t_\alpha, t_0)$
t_0	1455.08	100.00
t_1	373.32	389.62
t_2	768.06	189.45
t_3	820.09	177.43
t_4	1044.42	139.32
t_5	187.08	777.79
t_{RK}	43.88	3316.04

Fig 1: Estimator V/s MSE(t_α)

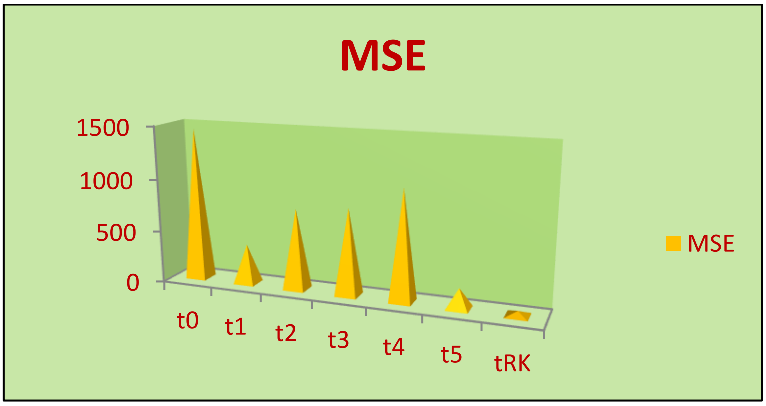
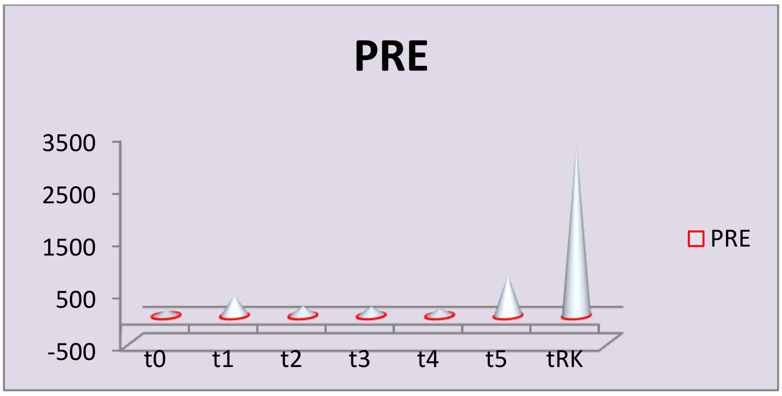


Fig 2: Estimator V/s $PRE(t_\alpha, t_0)$



6. Conclusion

In this article, an exponential ratio type estimator has been proposed under systematic design. The mathematical form of the estimator has been derived and its condition of efficiencies has been formulated with respect to some existing estimators from literature. The properties of the proposed estimator are obtained up to first order of approximation .it has been seen that the suggested estimator performed better than the existing estimator both theoretically and empirically.

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