ON SOME STATISTICAL PROPERTIES AND APPLICATIONS OF THREE-PARAMETER SUJATHA DISTRIBUTION

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Abstract

In this paper some important statistical properties of three-parameter Sujatha distribution including descriptive measures based on moments, reliability properties, mean deviations, stochastic ordering and Bonferroni and Lorenz curves have been discussed. The estimation of parameters using maximum likelihood estimation has been discussed. Finally, the goodness of fit of the distribution has been presented for two real lifetime datasets and compared with several one and two-parameter well-known lifetime distributions.

Keywords: Sujatha distribution, Extended Sujatha distributions, Statistical Properties, Estimation, Applications.

1. Introduction

Due to stochastic nature of lifetime data, it is really very challenging to search a suitable distribution to model lifetime data. The search for a suitable distribution for modeling of lifetime data is very challenging because the lifetime data are stochastic in nature. The analysis and modeling of lifetime data are essential in almost every field of knowledge including medical science, engineering, physical sciences, finance, insurance, demography, social sciences, literature, etc. and during recent eras several researchers in mathematics and statistics tried to introduce lifetime distributions. Recently, Sharma et al. [1] studied comparative study of several one parameter lifetime distributions and observed that there are some datasets which are extreme skewed to the right where these distributions were not giving well fit.

Recently, Nwikpe and Iwok [2] proposed a three-parameter generalization of Sujatha distribution (ATPSD) and studied few of its properties including hazard rate function, moment generating function, moments about origin, distribution of order statistics, and application on a dataset. The probability density function (pdf) and the cumulative distribution function (cdf) of ATPSD are given by

$$f\left(x;\theta,\lambda,\alpha\right)=\frac{\theta^{2}}{2\left(\theta^{2}+\lambda+\alpha\right)}\left(2\theta+2\lambda x+\theta\alpha x^{2}\right)e^{-\theta x};x>0,\left(\theta,\lambda,\alpha\right)>0 \tag{1}$$

$$F\left(x;\theta,\lambda,\alpha\right)=1-\left[1+\frac{\theta^{2}\alpha x^{2}+2\theta\left(\lambda+\alpha\right)x}{2\left(\theta^{2}+\lambda+\alpha\right)}\right]e^{-\theta x};x>0,\left(\theta,\lambda,\alpha\right)>0 \tag{2}$$

The survival function of ATPSD is given by

$$S\left(x;\theta,\lambda,\alpha\right)=1-F\left(x;\theta,\lambda,\alpha\right)=\frac{\left\{\theta^{2}\alpha x^{2}+2\theta\left(\lambda+\alpha\right)x+2\left(\theta^{2}+\lambda+\alpha\right)\right\}e^{-\theta x}}{2\left(\theta^{2}+\lambda+\alpha\right)};x>0,\theta>0,\alpha>0 \tag{3}$$

It has been observed that there are several interesting properties of ATPSD including central moments and moments based descriptive measures, reliability properties, mean deviations, stochastic ordering and Bonferroni and Lorenz curves have not been studied. In this paper an attempt has been made to discuss these statistical properties of ATPSD and propose some areas of applications.

The distributions which are particular case of ATPSD are summarized in table 1 along with its introducers.

Table 1: *Some particular distributions of ATPSD*

Parameter values(distributions)	**Pdf of distribution**	**Introducer**
$\left(\alpha=2,\lambda=\theta\right)$ Sujatha distribution	$f\left(x,\theta\right)=\frac{\theta^{3}}{\theta^{2}+\theta+2}\left(1+x+x^{2}\right)e^{-\theta x}$	Shanker [3]
$\left(\alpha=0,\lambda=\theta\right)$ Lindley distribution	$f\left(x,\theta\right)=\frac{\theta^{2}}{\theta+1}\left(1+x\right)e^{-\theta x}$	Lindley [4]
$\left(\alpha=2,\lambda=0\right)$ Akash distribution	$f\left(x,\theta\right)=\frac{\theta^{3}}{\theta^{2}+2}\left(1+x^{2}\right)e^{-\theta x}$	Shanker [5]
$\left(\alpha=0,\lambda=1\right)$ Shanker distribution	$f\left(x,\theta\right)=\frac{\theta^{2}}{\theta^{2}+1}\left(\theta+x\right)e^{-\theta x}$	Shanker [6]
$\alpha=0$ (Quasi Lindley distribution)	$f\left(x,\theta\right)=\frac{\theta}{\alpha+1}\left(\alpha+\theta x\right)e^{-\theta x}$	Shanker and Mishra [7]
$\left(\alpha=0,\lambda=0\right)$ Exponential distribution	$f\left(x,\theta\right)=\theta e^{-\theta x}$	

The behavior of the pdf and the cdf of ATPSD are presented in figures 1 and 2 respectively.

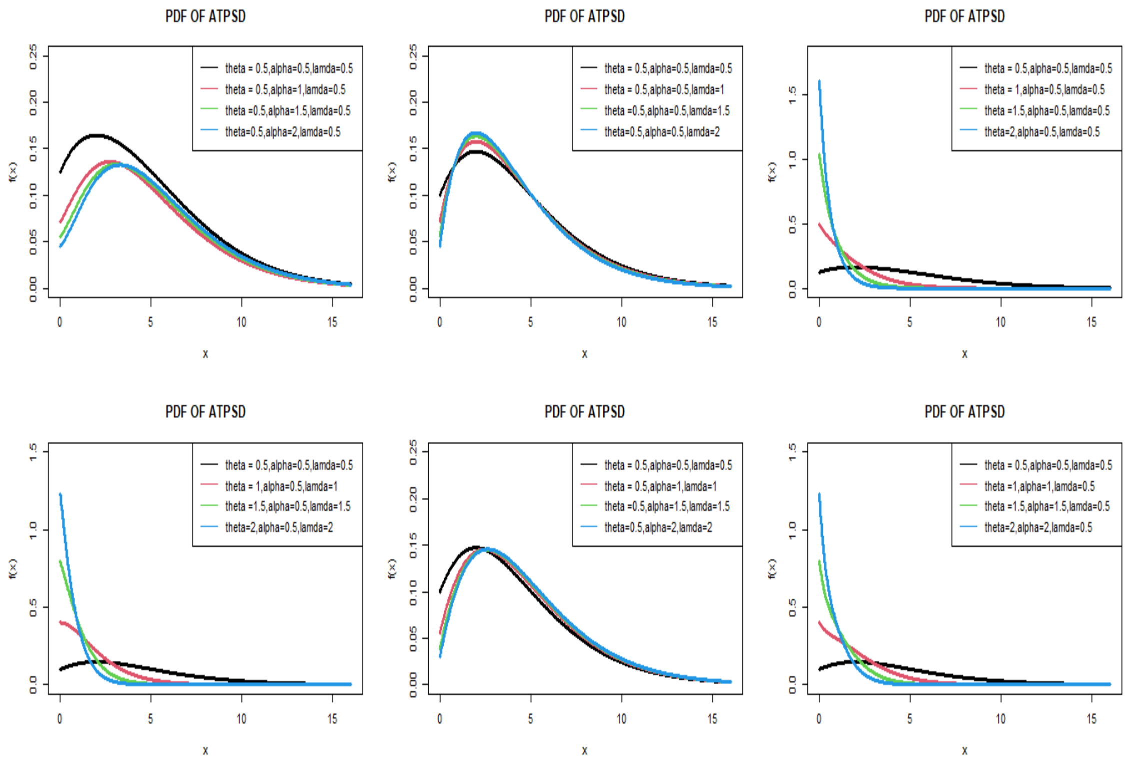

Figure 1: *The graphs of the pdf of ATPSD for different values of* θ, α *and* λ

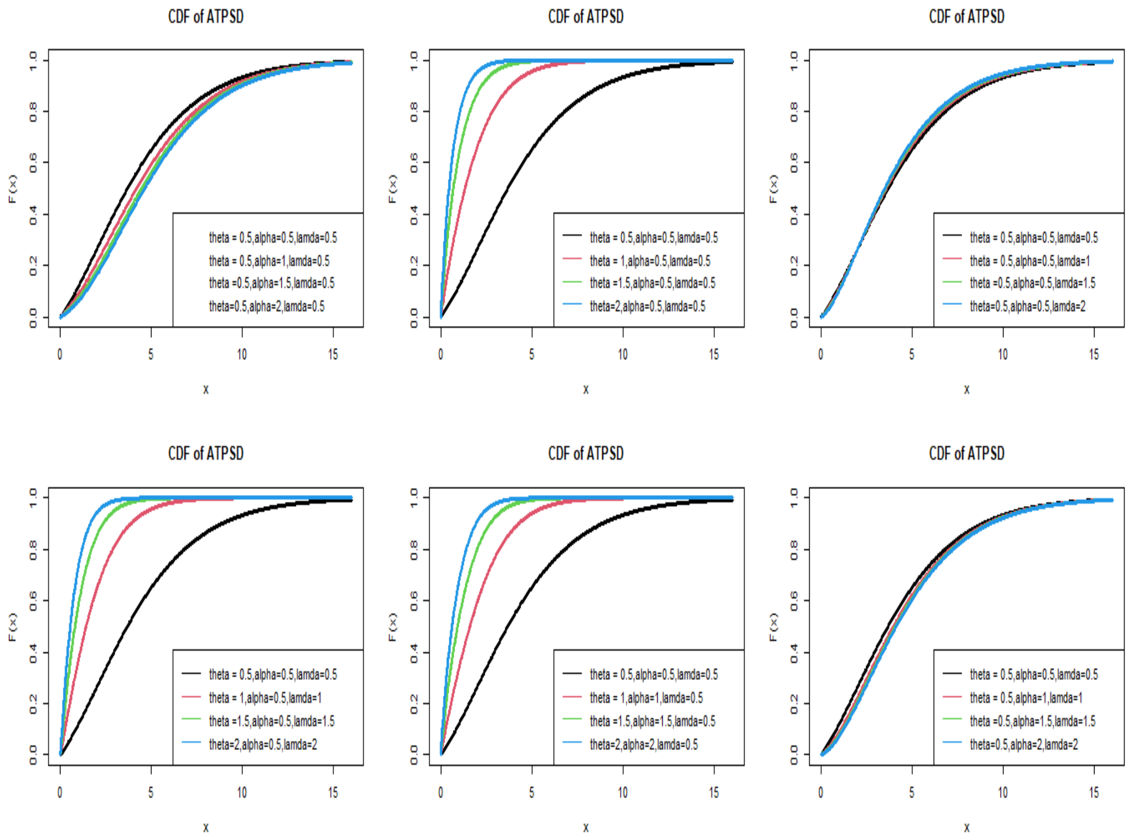

Figure 2: *The graphs of the cdf of ATPSD for different values of* θ, α *and* λ

2. Descriptive Properties Based On moments

The r th moment about origin, μ_r', of ATPSD is given by

$$\mu_r' = \frac{r!\left\{2\theta^2 + 2(r+1)\lambda + (r+1)(r+2)\alpha\right\}}{2\theta^r\left(\theta^2+\lambda+\alpha\right)}; r = 1, 2, 3, \cdots \quad (4)$$

Thus, the first four moments about origin are obtained as

$$\mu_1' = \frac{\theta^2+2\lambda+3\alpha}{\theta\left(\theta^2+\lambda+\alpha\right)}, \mu_2' = \frac{2\left(\theta^2+3\lambda+6\alpha\right)}{\theta^2\left(\theta^2+\lambda+\alpha\right)}, \mu_3' = \frac{6\left(\theta^2+4\lambda+10\alpha\right)}{\theta^3\left(\theta^2+\lambda+\alpha\right)}, \mu_4' = \frac{24\left(\theta^2+5\lambda+15\alpha\right)}{\theta^4\left(\theta^2+\lambda+\alpha\right)}.$$

Now using the relationship between moments about mean and the moments about origin, the moments about the mean of ATPSD are obtained as

$$\mu_2 = \frac{\theta^4+(8\alpha+4\lambda)\theta^2+\left(2\lambda^2+3\alpha^2+6\lambda\alpha\right)}{\theta^2\left(\theta^2+\lambda+\alpha\right)^2}$$

$$\mu_3 = \frac{2\left\{\theta^6+(15\alpha+6\lambda)\theta^4+\left(9\alpha^2+6\lambda^2+21\lambda\alpha\right)\theta^2+\left(2\lambda^3+3\alpha^3+9\lambda^2\alpha+9\lambda\alpha^2\right)\right\}}{\theta^3\left(\theta^2+\lambda+\alpha\right)^3}$$

$$\mu_4 = \frac{3\left\{\begin{array}{l}3\theta^8+(64\alpha+24\lambda)\theta^6+\left(102\alpha^2+172\lambda\alpha+44\lambda^2\right)\theta^4\\+\left(72\alpha^3+32\lambda^3+160\lambda^2\alpha+192\lambda\alpha^2\right)\theta^2+\left(8\lambda^4+15\alpha^4+48\lambda^3\alpha+60\lambda\alpha^3+84\lambda^2\alpha^2\right)\end{array}\right\}}{\theta^4\left(\theta^2+\lambda+\alpha\right)^4}$$

The coefficients of variation (C.V), skewness $\left(\sqrt{\beta_1}\right)$, kurtosis $\left(\beta_2\right)$ and index of dispersion (γ) of ATPSD are thus obtained as

$$C.V = \frac{\sqrt{\mu_2}}{\mu_1'} = \frac{\sqrt{\theta^4+(8\alpha+4\lambda)\theta^2+\left(2\lambda^2+3\alpha^2+6\lambda\alpha\right)}}{\theta^2+2\lambda+3\alpha}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{2\left\{\theta^6+(15\alpha+6\lambda)\theta^4+\left(9\alpha^2+6\lambda^2+21\lambda\alpha\right)\theta^2+\left(2\lambda^3+3\alpha^3+9\lambda^2\alpha+9\lambda\alpha^2\right)\right\}}{\left\{\theta^4+(8\alpha+4\lambda)\theta^2+\left(2\lambda^2+3\alpha^2+6\lambda\alpha\right)\right\}^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\left\{\begin{array}{l}3\theta^8+(64\alpha+24\lambda)\theta^6+\left(102\alpha^2+172\lambda\alpha+44\lambda^2\right)\theta^4\\+\left(72\alpha^3+32\lambda^3+160\lambda^2\alpha+192\lambda\alpha^2\right)\theta^2+\left(8\lambda^4+15\alpha^4+48\lambda^3\alpha+60\lambda\alpha^3+84\lambda^2\alpha^2\right)\end{array}\right\}}{\left\{\theta^4+(8\alpha+4\lambda)\theta^2+\left(2\lambda^2+3\alpha^2+6\lambda\alpha\right)\right\}^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^4 + 4(2\alpha+\lambda)\theta^2 + (2\lambda^2 + 3\alpha^2 + 6\alpha\lambda)}{\theta(\theta^2+\lambda+\alpha)(\theta^2+2\lambda+3\alpha)}$$

The nature of the coefficient of variation, skewness, kurtosis and index of dispersion of ATPSD are shown graphically in figure 3.

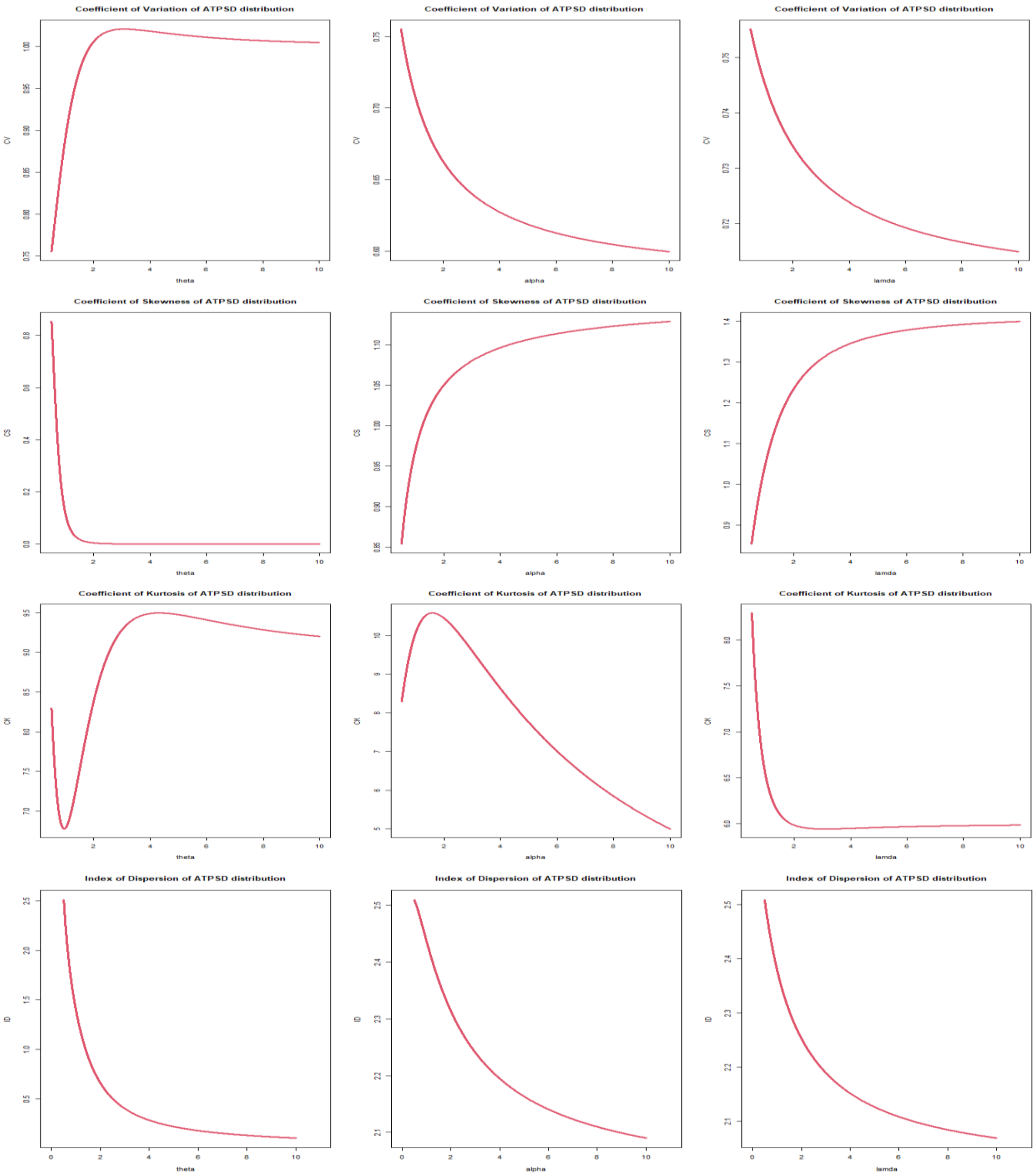

Figure 3: *The nature of the coefficient of variation, skewness, kurtosis and index of dispersion of ATPSD*

When α and λ are fixed and θ increases, the value of the CV is increases till $\theta \le 3$ and when $\theta > 3$, then CV starts decreasing slowly increasing values of θ. When θ and λ is fixed, then CV decreases for increasing values of α. Similarly, for, fixed values of θ and α and increasing values of λ, CV decreases.

When α and λ are fixed and θ increases, skewness decreases speedily till $\theta \le 1$ and when $\theta > 1$, it becomes constant. When θ and λ is fixed, then skewness decreases for increasing values of α. Similarly, for, fixed values of θ and α and increasing values of λ, skewness increases.

For fixed values of (α, λ) and increasing values of θ, the kurtosis is decreasing, increasing and again decreasing. For fixed values of (θ, λ) and increasing values of α, the kurtosis is increasing and then decreasing. And for fixed values of (θ, α) and increasing values of λ, the kurtosis is decreasing speedily till $\lambda \le 2$ and for $\lambda > 2$, it starts increasing very slowly.

For the nature of index of dispersion, it is always decreasing for increasing values of one parameter and fixed values of another two-parameter.

3. Reliability Properties

3.1. Hazard Rate Function

The hazard rate function of ATPSD is obtained as

$$h(x;\theta,\lambda,\alpha) = \frac{f(x;\theta,\lambda,\alpha)}{S(x;\theta,\lambda,\alpha)} = \frac{\theta^2\left(2\theta + 2\lambda x + \theta\alpha x^2\right)}{\theta^2\alpha x^2 + 2\theta(\lambda+\alpha)x + 2\left(\theta^2+\lambda+\alpha\right)}\,; x>0, (\theta,\lambda,\alpha)>0 \qquad (5)$$

It can be easily verified that $h(0;\theta,\lambda,\alpha) = f(0;\theta,\lambda,\alpha) = \dfrac{f(x;\theta,\lambda,\alpha)}{S(x;\theta,\lambda,\alpha)} = \dfrac{\theta^3}{\left(\theta^2+\lambda+\alpha\right)}$.

The behaviors of the hazard rate function of ATPSD are explained in the following figure 4.

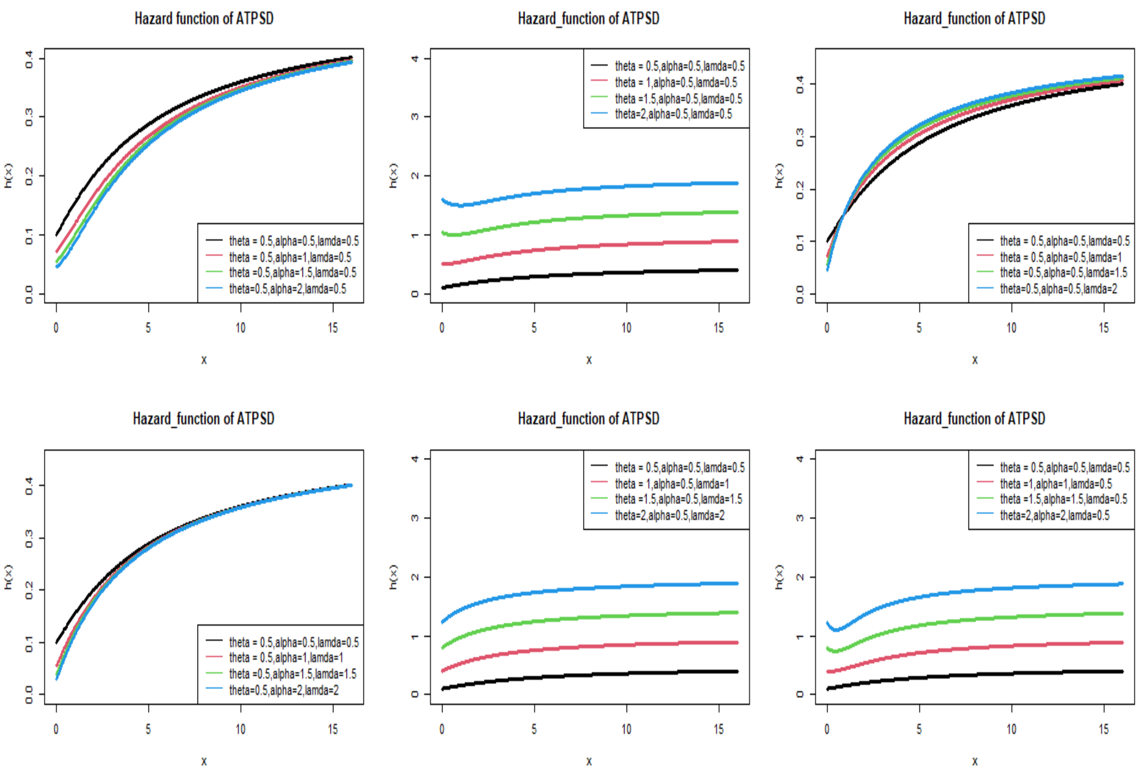

Figure 4: *The graphs of the hazard rate function of ATPSD for different values of* θ, α *and* λ

3.2. Mean Residual Life Function

The mean residual life function of ATPSD can be obtained as

$$m(x;\theta,\lambda,\alpha)=\frac{1}{1-F(x;\theta,\lambda,\alpha)}\int_x^{\infty}\left[1-F(t;\theta,\lambda,\alpha)\right]dt$$

$$=\frac{1}{\left\{\theta^2\alpha x^2+2\theta(\lambda+\alpha)x+2\left(\theta^2+\lambda+\alpha\right)\right\}e^{-\theta x}}\int_x^{\infty}\left[\theta^2\alpha t^2+2\theta(\lambda+\alpha)t+2\left(\theta^2+\lambda+\alpha\right)\right]e^{-\theta t}dt$$

$$=\frac{\alpha\left(\theta^2x^2+2\theta x+2\right)+2(\lambda+\alpha)(\theta x+1)+2\left(\theta^2+\lambda+\alpha\right)}{\theta\left[\theta^2\alpha x^2+2\theta(\lambda+\alpha)x+2\left(\theta^2+\lambda+\alpha\right)\right]} \quad (6)$$

It can be easily verified that $m(0;\theta,\lambda,\alpha)=\dfrac{\theta^2+2\lambda+3\alpha}{\theta\left(\theta^2+\lambda+\alpha\right)}=\mu_1'$ .The behavior of mean residual life function is explained in the following figure 5.

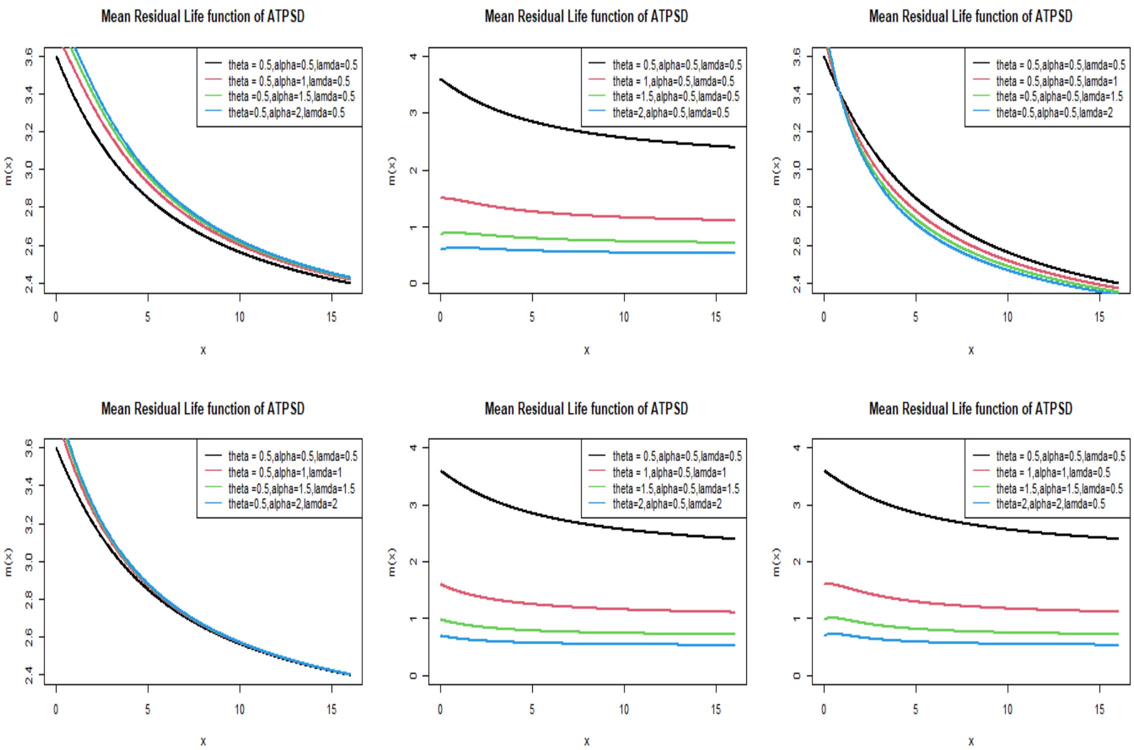

Figure 5: *The graphs of the mean residual life function of ATPSD for different values of* θ,α *and* λ

4. Stochastic Ordering

In probability theory and statistics, a stochastic order quantifies the concept of one random variable being bigger than another. A random variable X is said to be smaller than a random variable Y in the:

i. Stochastic order $(X\le_{st}Y)$ if $F_X(x)\ge F_Y(y)$ for all x

ii. Hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X\left(x\right) \ge h_Y\left(y\right)$ for all x

iii. Mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X\left(x\right) \ge m_Y\left(y\right)$ for all x

iv. Likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(y\right)}$ decrease in x

The following results due to Shaked and Shantikumar [8] are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{mrl} Y \\ \Downarrow \\ X <_{st} Y \end{array}$$

Theorem: Let $X \sim$ATPSD $\left(\theta_1, \lambda_1, \alpha_1\right)$ and $Y \sim$ ATPSD $\left(\theta_2, \lambda_2, \alpha_2\right)$. If $\lambda_1 = \lambda_2, \alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2, \alpha_1 = \alpha_2$ and $\lambda_1 < \lambda_2$ or $\theta_1 = \theta_2, \lambda_1 = \lambda_2$ and $\alpha_1 < \alpha_2$ then $X <_{lr} Y$ hence $X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^3(\theta_2^2 + \lambda_2 + \alpha_2)}{\theta_2^3(\theta_1^2 + \lambda_1 + \alpha_1)}\left(\frac{\theta_1\alpha_1 x^2 + 2\lambda_1 x + 2\theta_1}{\theta_2\alpha_2 x^2 + 2\lambda_2 x + 2\theta_2}\right)e^{-\left(\theta_1 - \theta_2\right)x}\ ; x > 0$$

Now

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log\left(\frac{\theta_1^3(\theta_2^2 + \lambda_2 + \alpha_2)}{\theta_2^3(\theta_1^2 + \lambda_1 + \alpha_1)}\right) + \log\left(\frac{\theta_1\alpha_1 x^2 + 2\lambda_1 x + 2\theta_1}{\theta_2\alpha_2 x^2 + 2\lambda_2 x + 2\theta_2}\right) - \left(\theta_1 - \theta_2\right)x$$

Therefore

$$\frac{d}{dx}\left(\log\frac{f_X(x)}{f_Y(x)}\right) = \frac{2\left(\alpha_1\theta_1\lambda_2 - \alpha_2\theta_2\lambda_1\right)x^2 + 4\theta_1\theta_2(\alpha_1 - \alpha_2)x + 4(\lambda_1\theta_2 - \lambda_2\theta_1)}{\left(\theta_1\alpha_1 x^2 + 2\lambda_1 x + 2\theta_1\right)\left(\theta_2\alpha_2 x^2 + 2\lambda_2 x + 2\theta_2\right)} - \left(\theta_1 - \theta_2\right)$$

Thus, if $\lambda_1 = \lambda_2, \alpha_1 = \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2, \alpha_1 = \alpha_2$ and $\lambda_1 < \lambda_2$ or

$\theta_1 = \theta_2, \lambda_1 = \lambda_2$ and $\alpha_1 < \alpha_2$, $\dfrac{d}{dx}\left(\log\dfrac{f_X(x)}{f_Y(x)}\right) < 0$. This means that $X <_{lr} Y$ and hence

$X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

5. Deviation from Mean and Median

The amount of dispersion in a population is an evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and mean deviation about median and are defined by

$$\delta_1(x) = \int_0^\infty |\, x - \mu \,|\ f(x)dx \quad \text{and} \quad \delta_2(x) = \int_0^\infty |\, x - M \,|\ f(x)dx \ \text{ respectively,}$$

where $\mu = E(X)$ and $M = Median(X)$.

The measures $\delta_1(x)$ and $\delta_2(x)$ can be calculated using the following relationships

$$\delta_1(x) = 2\mu F(\mu) - 2\int_0^{\mu} xf(x)dx$$

and
$$\delta_2(x) = -\mu + 2\int_M^{\infty} xf(x)dx$$

Thus, the mean deviation about the mean $\delta_1(x)$, and the mean deviation about the median $\delta_2(x)$ of ATPSD are obtained as

$$\delta_1(x) = \frac{\left[\alpha\theta^2\mu^2 + 2\theta\lambda\mu + 4\alpha\theta\mu + 2\theta^2 + 4\lambda + 6\alpha\right]e^{-\theta\mu}}{\theta(\theta^2 + \lambda + \alpha)} \tag{7}$$

$$\delta_2(x) = \frac{2\left[\alpha\theta^3M^3 + \theta^2(2\lambda + 3\alpha)M^2 + (2\theta^3 + 4\theta\lambda + 6\alpha\theta)M + 2\theta^2 + 4\lambda + 6\alpha\right]e^{-\theta M}}{2\theta(\theta^2 + \lambda + \alpha)} - \mu \tag{8}$$

6. Bonferroni and Lorenz Curves and Indices

The Bonferroni and Lorenz curves by Bonferroni [9] and Bonferroni and Gini indices have wide applications in economics to study income and poverty, but it also used in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$\mathrm{B(p)} = \frac{1}{p\mu}\int_0^q x\,f(x)\,dx = \frac{1}{p\mu}\left[\int_0^{\infty} x\,f(x)dx - \int_q^{\infty} x\,f(x)\,dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^{\infty} x\,f(x)\,dx\right]$$

and
$$\mathrm{L(p)} = \frac{1}{\mu}\int_0^q x\,f(x)\,dx = \frac{1}{\mu}\left[\int_0^{\infty} x\,f(x)\,dx - \int_q^{\infty} x\,f(x)\,dx\right] = \frac{1}{\mu}\left[\mu - \int_q^{\infty} x\,f(x)\,dx\right]$$

respectively.

The Bonferroni and Gini indices are obtained as

$$B = 1 - \int_0^1 B(p)\,dp \quad \text{and} \quad G = 1 - 2\int_0^1 \mathrm{L}(p)\,dp, \text{ respectively.}$$

Using pdf of ATPSD and little algebraic simplification, we get

$$B(p) = \frac{1}{p}\left[1 - \frac{\left[\alpha\theta^3q^3 + \theta^2(2\lambda + 3\alpha)q^2 + (2\theta^3 + 4\theta\lambda + 6\alpha\theta)q + 2\theta^2 + 4\lambda + 6\alpha\right]e^{-\theta q}}{2(\theta^2 + 2\lambda + 3\alpha)}\right] \tag{9}$$

$$\mathrm{L}(p) = 1 - \frac{\left[\alpha\theta^3q^3 + \theta^2(2\lambda + 3\alpha)q^2 + (2\theta^3 + 4\theta\lambda + 6\alpha\theta)q + 2\theta^2 + 4\lambda + 6\alpha\right]e^{-\theta q}}{2(\theta^2 + 2\lambda + 3\alpha)} \tag{10}$$

Now using equations and after some simple algebraic simplifications, the Bonferroni and Gini indices of ATPSD are obtained as

$$B = 1 - \frac{\left[\alpha\theta^3q^3 + \theta^2(2\lambda + 3\alpha)q^2 + (2\theta^3 + 4\theta\lambda + 6\alpha\theta)q + 2\theta^2 + 4\lambda + 6\alpha\right]e^{-\theta q}}{2(\theta^2 + 2\lambda + 3\alpha)} \tag{11}$$

$$G = \frac{2\left[\alpha\theta^3q^3 + \theta^2(2\lambda + 3\alpha)q^2 + (2\theta^3 + 4\theta\lambda + 6\alpha\theta)q + 2\theta^2 + 4\lambda + 6\alpha\right]e^{-\theta q}}{2(\theta^2 + 2\lambda + 3\alpha)} - 1. \tag{12}$$

7. Maximum Likelihood Estimation

Let $(x_1, x_2, x_3, ..., x_n)$ be a random sample of size n from ATPSD$(\theta, \lambda, \alpha)$. Then the likelihood function is given by

$$L = \left(\frac{\theta^2}{2\left(\theta^2 + \lambda + \alpha\right)} \right)^n \prod_{i=1}^{n} \left(2\theta + 2\lambda x_i + \theta\alpha\, x_i^2 \right) e^{-n\,\theta\,\bar{x}}$$, where $\bar{x}$ is the sample mean.

The log-likelihood function is thus obtained as

$$\log L = n\left[2\log\theta - \log 2 - \log\left(\theta^2 + \lambda + \alpha\right) \right] + \sum_{i=1}^{n} \log\left(2\theta + 2\lambda x_i + \theta\alpha x_i^2 \right) - n\theta\bar{x}\,. \qquad (13)$$

The maximum likelihood estimates $(\hat{\theta}, \hat{\lambda}, \hat{\alpha})$ of parameters $(\theta, \lambda, \alpha)$ are the solution of the following log-likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} + \frac{2n\theta}{\theta^2 + \lambda + \alpha} + \sum_{i=1}^{n} \frac{2 + \alpha x_i^2}{2\theta + 2\lambda x_i + \theta\alpha x_i^2} - n\,\bar{x} = 0$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{-n}{\theta^2 + \lambda + \alpha} + \sum_{i=1}^{n} \frac{2x_i}{2\theta + 2\lambda x_i + \theta\alpha x_i^2} - n\,\bar{x} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{-n}{\theta^2 + \lambda + \alpha} + \sum_{i=1}^{n} \frac{\theta x_i^2}{2\theta + 2\lambda x_i + \theta\alpha x_i^2} = 0$$

These three log-likelihood equations do not seem to be solved directly. We have to use Fisher's scoring method for solving these three log-likelihood equations. We have

$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{-2n}{\theta^2} + \frac{2n\left(\theta^2 - \lambda - \alpha\right)}{\left(\theta^2 + \lambda + \alpha\right)^2} - \sum_{i=1}^{n} \frac{\left(2 + \alpha x_i^2\right)}{\left(2\theta + 2\lambda x_i + \theta\alpha x_i^2\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{n}{\left(\theta^2 + \lambda + \alpha\right)^2} - \sum_{i=1}^{n} \frac{4x_i^2}{\left(2\theta + 2\lambda x_i + \theta\alpha x_i^2\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n}{\left(\theta^2 + \lambda + \alpha\right)^2} - \sum_{i=1}^{n} \frac{\theta^2 x_i^4}{\left(2\theta + 2\lambda x_i + \theta\alpha x_i^2\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \lambda} = \frac{2n\theta}{\left(\theta^2 + \lambda + \alpha\right)^2} - \sum_{i=1}^{n} \frac{4x_i + 2\alpha x_i^3}{\left(2\theta + 2\lambda x_i + \theta\alpha x_i^2\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{2n\theta}{\left(\theta^2 + \lambda + \alpha\right)^2} + \sum_{i=1}^{n} \frac{2\lambda x_i^2}{\left(2\theta + 2\lambda x_i + \theta\alpha x_i^2\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{-12n\theta}{\left(\alpha\theta^2 + 6\right)^2} - \sum_{i=1}^{n} \frac{x_i^3}{\left(\alpha + \theta x_i^3\right)^2} = \frac{\partial^2 \log L}{\partial \alpha \partial \theta}\,.$$

$$\frac{\partial^2 \log L}{\partial\lambda\,\partial\alpha} = \frac{n}{\left(\theta^2+\lambda+\alpha\right)^2} - \sum_{i=1}^{n}\frac{2\theta x_i^3}{\left(2\theta+2\lambda x_i+\theta\alpha x_i^2\right)^2} = \frac{\partial^2 \log L}{\partial\alpha\,\partial\lambda}$$

The following equations can be solved for MLEs $\left(\hat{\theta},\hat{\lambda},\hat{\alpha}\right)$ of $\left(\theta,\lambda,\alpha\right)$ for ATPSD

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial\theta^2} & \frac{\partial^2 \log L}{\partial\theta\,\partial\lambda} & \frac{\partial^2 \log L}{\partial\theta\,\partial\alpha} \\ \frac{\partial^2 \log L}{\partial\lambda\,\partial\theta} & \frac{\partial^2 \log L}{\partial\lambda^2} & \frac{\partial^2 \log L}{\partial\lambda\,\partial\alpha} \\ \frac{\partial^2 \log L}{\partial\alpha\,\partial\theta} & \frac{\partial^2 \log L}{\partial\alpha\,\partial\lambda} & \frac{\partial^2 \log L}{\partial\alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\lambda}=\lambda_0\\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\lambda}-\lambda_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial\theta} \\ \frac{\partial \log L}{\partial\lambda} \\ \frac{\partial \log L}{\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\lambda}=\lambda_0\\ \hat{\alpha}=\alpha_0}} \quad (14)$$

where $\left(\theta_0,\lambda_0,\alpha_0\right)$ are the initial values of $\left(\theta,\lambda,\alpha\right)$ respectively. These equations are solved iteratively till sufficiently close values of $\left(\hat{\theta},\hat{\lambda},\hat{\alpha}\right)$ are obtained.

8. Applications to Lifetime Data

The following real lifetime datasets have been considered for testing the goodness of fit of ATPSD over other one parameter and two-parameter lifetime distributions.

Data Set 1: The real data discussed by Almongy et al [10] that represents a COVID 19 mortality rate data belongs to Mexico of 108 days that is recorded from 4 March to 20 July 2020. This data formed of rough mortality rate. The data are as follows:
8.826, 6.105, 10.383, 7.267, 13.220, 6.015, 10.855, 6.122, 10.685, 10.035, 5.242, 7.630, 14.604, 7.903, 6.327, 9.391, 14.962, 4.730, 3.215, 16.498, 11.665, 9.284, 12.878, 6.656, 3.440, 5.854, 8.813, 10.043, 7.260, 5.985, 4.424, 4.344, 5.143, 9.935, 7.840, 9.550, 6.968, 6.370, 3.537, 3.286, 10.158,8.108,6.697, 7.151, 6.560, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 3.499, 3.751, 7.486, 6.625, 6.140, 4.909, 4.661, 1.867, 2.838, 5.392, 12.042, 8.696, 6.412, 3.395, 1.815, 3.327, 5.406, 6.182,4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923.

Data set-2: The following bi-modal dataset, discussed by Ghitany et al. [11], is obtained from the banking sector discusses the waiting time (in minutes) before the customer received service in a bank. The values are:
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3,4.3, 4.4, 4.4, 4.6, 4.7, 4.7,4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9,7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7,10.9, 11.0, 11.0, 11.1 ,11.2, 11.2, 11.5,11.9, 12.4, 12.5, 12.9, 13.0, 13.1,13.3, 13.6, 13.7,13.9,14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9,23.0,27.0, 31.6, 33.1, 38.5.

In order to compare lifetime distributions, values of $-2\log L$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), K-S Statistics (Kolmogorov-Smirnov Statistics) and the corresponding probability value (p-value) for the above data set has been computed. The formulae for computing AIC, AICC and K-S Statistics are as follows:

$$AIC = -2logL + 2k \text{ , } AICC = AIC + \frac{2k(k+1)}{n-k-1} \text{ , } D = \underset{x}{Sup} \mid F_n(x) - F_0(x) \mid$$

where $k =$ number of parameter, $n =$ sample size

The distribution corresponding to the lower values of $-2\log L$, AIC, AICC, and K-S Statistics is the best fit distribution. These statistical values for the two datasets have been computed and presented in tables 2 and 3 respectively. It is obvious from the goodness of fit of distributions given in tables 2 and 3 that ATPSD gives much closure fit as compared to other one parameter and two-parameter distributions and hence it can be considered as a suitable model for the given dataset.

Table 2: *ML estimates, -2logL, AIC, AICC, K-S value and p-value of the distribution for the data set-1*

Distributions	MLE			-2logL	AIC	AICC	K-S	p-value
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$					
ATPSD	0.5209	2277.6180	0.1000	**533.29**	**539.29**	**539.52**	**0.0584**	**0.9246**
TPSD	0.4867	0.0100	…	536.45	540.45	540.56	0.0682	0.8029
NTPSD	0.4842	0.0100	…	542.75	546.75	546.86	0.0869	0.5153
ANTPSD	0.4869	931.2583		536.37	540.37	540.48	0.0684	0.7975
QSD	0.4825	0.1000	…	537.97	541.97	542.08	0.0671	0.8205
NQSD	0.4868	137.8985	…	536.39	540.39	540.50	0.0626	0.8758
WSD	0.9828	3.7285	…	510.77	514.77	515.47	0.0845	0.5561
PSD	0.3491	1.1648	…	537.06	541.06	541.76	0.0937	0.4454
Sujatha	0.4631	…	…	543.36	545.36	545.39	0.0950	0.3961

Table 3: *ML estimates, -2logL, AIC, AICC, K-S value and p-value of the distribution for the data set-2*

Distributions	MLE			-2logL	AIC	AICC	K-S	p-value
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$					
ATPSD	0.2025	0.1270	108.8797	**634.60**	**640.6**	**640.85**	**0.0564**	**0.9539**
TPSD	0.2769	2.4379	…	639.25	643.25	644.05	0.0764	0.7146
NTPSD	0.2316	20.3400	…	635.03	639.03	639.15	0.0699	0.8086
ANTPSD	0.2769	0.4102	…	639.25	643.25	644.05	0.0750	0.7361
QSD	0.2769	0.6752	…	639.25	643.25	644.05	0.0897	0.5133
NQSD	0.2769	0.1136	…	639.25	643.25	644.05	0.0908	0.4959
WSD	0.1958	0.0100	…	602.44	606.44	607.24	0.1079	0.2627
PSD	0.3571	0.9012	…	636.48	640.48	641.28	0.0682	0.8206
Sujatha	0.2846	…	…	639.63	641.63	641.88	0.0949	0.4447

The fitted plot of the considered distributions of the data set-1 and data set-2 are presented in figure 6.

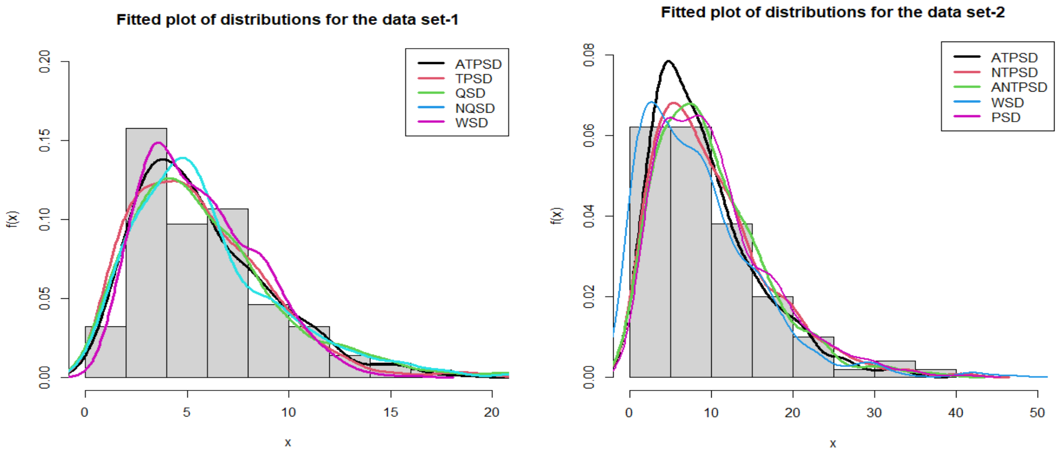

Figure 6: *Fitted plot of distributions of the data set-1 and data set-2.*

9. Conclusion

Some important and useful statistical properties of a three-parameter Sujatha distribution (ATPSD) including descriptive measures based on moments, reliability properties, mean deviations, stochastic ordering and Bonferroni and Lorenz curves have been derived and discussed. Maximum likelihood estimation has been discussed for estimating the parameters. Applications and goodness of fit of the ATPSD have been demonstrated with two real lifetime datasets and it shows better fit over several one parameter and two-parameter lifetime distributions.

10. Conflict of Interest

The Authors declare that there is no conflict of interest.

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