Analysis of $MMAP/PH/1$ Classical Retrial Queue with Non-preemptive priority, Second optional service, Differentiate breakdowns, Phase type repair, Single vacation, Emergency vacation, Closedown, Setup and Discouragement

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Abstract

A single server retrial queueing model with non-preemptive priority was examined in this research. The arrival of priority consumers follow a marked Markovian arrival pattern, and both high priority and low priority service times are according to phase type distribution. Matrix analytic method are used to examine the steady state analysis of this model. Various system performance measures, cost analysis and busy period analysis also examined in this model. In additionally, by using some system performance measures we provide the numerical illustration with numerically and graphically.

Keywords: Markovian arrival process, Priority queue, Phase type Repair, Second optional service, Differentiate breakdown, Closedown, Single vacation, Emergency vacation, Setup, Balking.

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1. Introduction

Retrial queues in queueing theory have gained attention recently as a significant topic of study as a result of its numerous applications. System manufacturing, designing of local area communication networks and data communication networks are the most common examples. The customers are bound to be impatient in general. From the real-life experience, we can observe that the customers who require service must form a queue. However, some customers decide not to wait in queue due to time restrictions, and some customers who do wait in queue get impatient and leave the queue before receiving service.

The Markovian Arrival Process (MAP) is considered to be the most significant process tool in this theory. Neuts [26] pioneered the Versatile Markovian Point Process (VMPP). He has used the concept of point process which is Markovian arrival process. Chakravarthy [10] have analysed the MAP which is represented by n-dimensional parameter matrices (D_0, D_1) where D_0 governing the transition for no arrival and D_1 governing arrivals.

There are two types of priority services such as preemptive and non-preemptive. The arrival of priority customers have to wait until the regular customers service completed such as non-preemptive priority. The low priority consumers should be interrupted by the preemptive priority, also known as the high priority customers. Isotupa and Stanford [17] looked into a single server queue that takes connections that arrive from N classes of clients in a non-preemptive priority manner. They found R matrix and waiting time distribution. Numerical results also provided in their model. Baek et al. [9] investigated a single server priority queueing system with two types of customers and consumable additional items. Additionally, they looked at buffer systems with zero buffers for type 1 customers, infinite buffers for type 2 customers, and buffers with K capacities for additional items. Krishnamoorthy and Divya [19] examined a single server queueing model with working vacation, non-preemptive priority, and two distinct N-policies. Their concept states that when the server is on vacation, responses to high priority (type I) clients continue, while responses to low priority (type II) clients must wait until the server resumes normal operations. Also they found busy period analysis, waiting time distribution and numerical illustrations.

A queueing model with two different kinds of clients in which arrival follows Markovian arrival process which was investigated by Chakravarthy and Dudin [13]. The steady state probability vector, waiting time distribution and several numerical illustrations are also found in their model. Sleptchenko et al. [28] has developed a single server queueing model along with arbitrary N client classes, class-dependent service rates, and priority classes. Krishnamoorthy et al. [20] looked into a multi-server queue with self-generated priorities and non-preemptive priority services. The arriving customer to a C-server counter follows MAP and service time follows PH for both priority customers. They found cost analysis and performance measures in their model. Nair et al. [24] analysed a $M/M/1$ queue with priority loss through feedback. They took into consideration the arrival of consumers with different priorities P_1 and P_2 in accordance with a marked Markovian arrival process and phase type distribution is used for service time. They discussed two types of model such as model 1 and 2. In model 1 was considered as non-preemptive service for P_2 customers and in model 2 was considered as preemptive policy of P_2 customers. Also they find waiting time analysis and system performance measures.

In real life situations, every working place and offices vacations are essential. Here we considered single and emergency vacation. The server can take the vacation after completion of service and also during the service time, the server can take emergency vacation. In many working places, the servers may take the vacation during the busy time and continue the remaining service of that customers. A queueing model with priority services was investigated by Ayyappan and Udayageetha [7]. They considered two types of vacation such as modified Bernoulli vacation and emergency vacation. After completing the requested service, the server goes on Bernoulli vacation if there are no high priority clients in the system; otherwise, the server is idle. During the service time of high priority customers, the server take emergency vacation. They presents some numerical examples and performance measures.

The arrival of negative consumers should detract the positive customers who gets the services from the server and they exit the system without the service being completed which is called as negative arrival. $M/M/1$ retrial queue with preemptive priority and a maximum of J vacations has been studied by Yuvarani and Saravanarajan [31]. They considered negative arrival of customer occurs in the busy period of positive customer. Due to the negative arrival, the positive customer spoiled their service and leave the system. Some performance measures and numerical illustrations are also given. A multi server queueing model with negative customer and partial protection of service has been done by Klimenok and Dudin [18]. A non-preemptive priority queue with server's walking process was done by Fukagawa et al. [15]. The stationary probability vector, queue waiting time and evaluation measures of the queue also done by them. A Single-server Discrete-time Retrial G-queue with server Breakdowns and Repairs was done by Wang and Zhang [30].

Ayyappan and Thilagavathy [6] accomplishes a single-server priority retrial queue with stand by server, breakdown, repair, vacation, negative arrival, balking and reneging. They used the concept of negative arrival while the main server is in busy. The negative arrival are affected to the positive customer those who gets the service to be removed completely from the system and server moved into repair process. Busy period analysis, cost analysis and graphical illustrations are all given. Retrial queueing model $MMAP/M_2/1$ with two orbits have been studied by Avrachenkov et al. [8]. They considered two orbits, one is an infinite capacity and another one is finite capacity. Some of the performance measures and numerical illustrations are also provided. A multichannel queueing model with quasi-random input retrial times and phase type services has investigated by Artalejo and Corral [2]. By solving Quasi birth and death process, the stationary probability vector has been evaluated and some performance measures also evaluated.

The concept of optional secondary service is that after completion of primary customer service, the customer may need a secondary service with probability p or the customer leaves the system completely with probability q. A queue with single server subject to second optional service has been done by Madan [22]. They considered two types of service such as essential and optional service. The essential service are given to all the customers in the system while second optional service are provided only for some customers those who need the service once again. Waiting time distribution and particular cased are also derived.

Chakravarthy [11] looked into a $MAP/PH(1), PH(2)/1$ queueing structure whereby services were given on a first come, first served basis subject to vacations and optional secondary services. A single server queueing model with N-policy and second optional services have been evaluated by Das et al. [14]. They presented the cost analysis and various performance measures of their model. A single server queue with setup, closedown, multiple vacation, standby server, breakdown, repair and reneging was studied by Ayyappan and Thilagavathy [5]. Chakravarthy and Agarwal [12] explored a machine repair problem with Unreliable server. In their model, they considered phase type distribution for the service and repair time of server. They also determine the performance measures and some numerical illustrations. A single server retrial queueing model with Bernoulli vacation, feedback, breakdown and repair was analysed by Ayyappan and Gowthami [3]. Also they found the cost analysis, some performance measures and by using the performance measures they evaluate the numerical results.

There are two types of breakdown such as active and passive breakdown. The active breakdown occurs while the busy period of server and the passive breakdown occurs during the server idle period. Gao et al. [16] examined two kinds of breakdown and delayed repairs in an unreliable retry queue. They employed passive and active breakdowns in the periods of idle and busy, respectively. When a passive breakdown happens, the server cannot be repaired immediately and must wait for consumers to arrive from the outside or from orbit because the server lacks a monitoring system during idle times. They provided a few performance measures based on the likelihood that a server would be busy, idle, or undergoing maintenance, among other factors. By using performance measures they find the numerical values. A queueing model with single server subject to working vacation and two type of server breakdown have been analysed by Agarwal et al. [1]. They considered the server breakdown while server is in working vacation or normal busy period. Numerical illustrations are also examined by them.

Niu et al. [27] investigated a vacation queue with Setup and Closedown periods, as well as batch Markovian Arrival Processes. In their model after completion of service, the server closedown the system and setup the system when the server return from vacation. The arrival process follows Markovian arrival process and service time follows phase type distribution with the random variables Bernoulli vacation, setup, Bernoulli feedback, breakdown, repair and impatient customers was investigated by Ayyappan and Gowthami [4]. Now-a-days, in many places, most of the peoples does not prefer to wait a line at long time. Here we consider balking such as the customer does not enter into the system due to impatient. Swathi et al. [29] examined a queueing system with balking and reneging. In their model, they included the concept of customer balking and reneging as a result of the server's unavailability during vacation and breakdown times. The steady state analysis of the system and several performance measures were also derived by them.

The remainder of the article is organised as follows: The narration for our model is located in section 2. Section 3 discusses the matrix generating procedure and some notations. The system stableness, the invariant probability vector, and R matrix are all obtained in section 4. The busy period analysis is presented in section 5. Section 6 contains performance measures. Section 7 presents the cost analysis. Section 8 contains some numerical and graphical outcomes. Section 9 contains the conclusion part.

2. Narration of the Model

- In this article, we analyse a single server classical retrial policy with preemptive priority queue, differentiate breakdown, second optional service, phase type repair, two types of vacation, closedown, setup and balking.
- Arrival of both high priority and low priority (HP and LP) clients in accordance with $MMAP$, which is a generic version of MAP with parameter matrices of order (D_0, D_1, D_2) of order m_2. The D_0 matrix denotes the absence of positive customer arrivals, while the D_1 and D_2 matrix denotes customer arrivals.
- While low priority customers only have a "L" size finite buffer, high priority customers have an infinite capacity. The negative arrival of customers are also follows MAP with representation (C_0, C_1) of order m_1, where C_0 represents to no arrival and C_1 represents to arrival of customers.
- The service offered to the customers in the basis of first come first service. The customers receive the service immediately if server becomes idle. In idle time, the server may struck due to breakdown to starts the service and then moves to repair process.
- During the service period, the server experiences a breakdown owing to a negative arrival and immediately enters the repair process. At the same time, positive customer who receive service from the server will abandon the system totally.
- When a low priority client attempts to join an orbit that is already full, the action is deemed unsuccessful. If any low priority customers retrial from the orbit while the server is idle, the low priority customers will receive service from the server successfully.
- The duration of service time of both (HP/LP) customers which follows PH type distribution with the notations (α, T) of order n_1 where $T^0 + Te = 0$ such that $T^0 = -Te$ and the optional service of HP customers also follows PH type distribution with notation (α_1, T_1) of order n_2 where $T_1^0 + T_1 e = 0$ such that $T_1^0 = -T_1 e$.
- The server repair time follows a PH type distribution with representation (β, S) of order s where $S^0 + Se = 0$ and $S^0 = -Se$.
- During the service time of LP customers, the server takes emergency vacation and the customers those who are receives the service have to join the orbit and will get the service after the vacation completion by server. When the service is finished, the server shuts down the system and goes on vacation.
- The server will startup the system after completion of vacation. When on vacation, the customer may join the system with probability $(1 - b)$ or balk the system because of impatience with probability b.
- Inter-retrial times, emergency vacation, single vacation, breakdown times, closedown and setup times are all based on exponential distribution and its parameters as δ, η_1, η_2, τ, ϕ and ψ respectively. (see Figure 1).

3. The QBD process infinitesimal generation matrix

Let us narrate the few notation of this model which followed by generator matrix of the QBD process as follows:

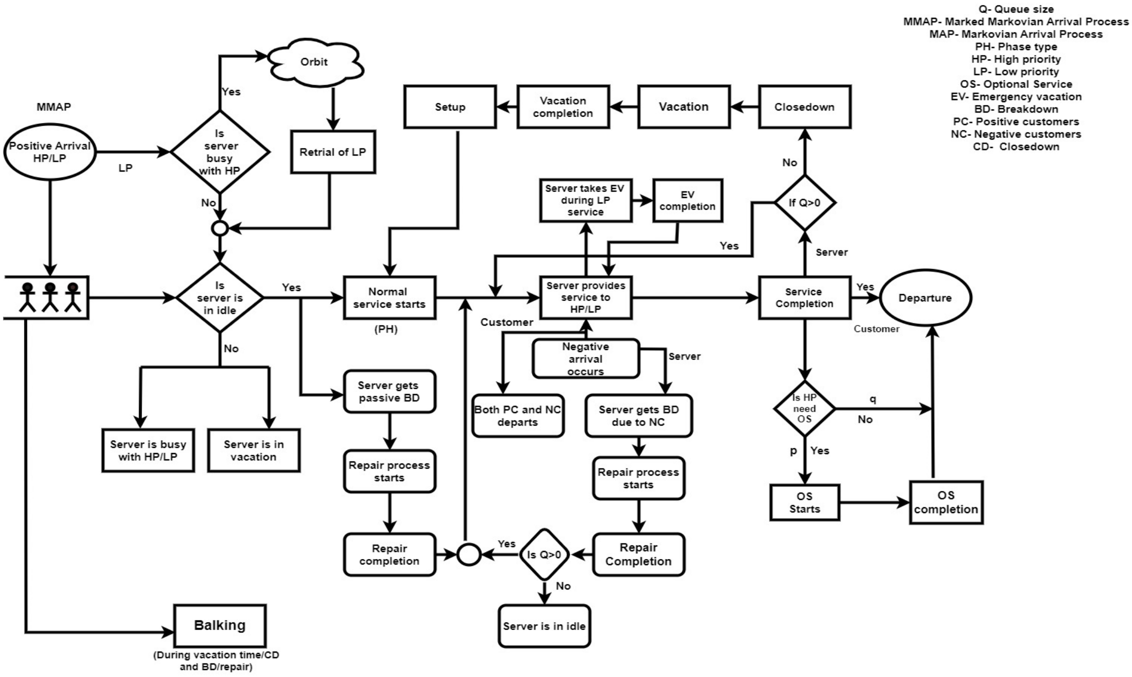

Figure 1: *Representations of our model in schematic form*

Notations:

- $\otimes$ - Two matrices with varying orders are combined in a Kronecker product.
- $\oplus$ - Two matrices with different orders are combined in a Kronecker sum.
- I_m - Represents the identity matrix of order $m \times m$.
- e - Each entry in a column vector has the required dimension, which is 1.
- 0 - It denotes an appropriate order of zero matrices.
- The fundamental arrival rate λ_i, where i=1,2 which is specified as $\lambda_i = \pi D_i e_{m_2}$, π represents the stationary probability vector of the generator matrix $D_0 + D_1 + D_2$ which determines MMAP transitions.
- The negative arrival rate be λ_3 which is specified as $\lambda_3 = \pi_1 C_1 e_{m_1}$, where π_1 is the steady state probability vector of generator matrix $C = C_0 + C_1$.
- The rate of normal service of server is indicated as $\mu = [\alpha(-T^{-1})e_{n_1}]^{-1}$.
- The rate of optional service of server is indicated as $\mu_1 = [\alpha_1(-T_1^{-1})e_{n_2}]^{-1}$.
- The repair rate for normal/optional service of server as represented by $\sigma = [\beta(-S^{-1})e_s]^{-1}$.
- $N_1(t)$ indicates the total number of customers with high priority in the system at time t.
- $N_2(t)$ indicates the total number of customers with low priority in the orbit.
- $C(t)$ stands for the server status at time t.

$$C(t) = \begin{cases} 0, & \text{server is in idle.} \\ 1, & \text{server is busy in HP normal Serivce.} \\ 2, & \text{server is busy in HP optional service.} \\ 3, & \text{server is busy for LP normal service.} \\ 4, & \text{server is in repair process.} \\ 5, & \text{sever is in emergency vacation.} \\ 6, & \text{server is in closedown.} \\ 7, & \text{server is in vacation} \\ 8, & \text{server is in setup.} \end{cases}$$

- $S(t)$ represents the service phase of server.
- $K(t)$ represents the repair phase of server.
- $A_i(t)$ stands for the arrival phase of negative and positive customers, where i=1,2.

Let $\{N_1(t), N_2(t), C(t), S(t), K(t), A_1(t), A_2(t), \quad t \geq 0\}$ is the $CTMC$ with the state space as follows,

$$\Omega = l(0) \bigcup_{i=1}^{\infty} l(i),$$

where

$$\begin{aligned} l(0)=&\{(0, i_2, 0, a_2) : \ 0 \leq i_2 \leq L, \ 1 \leq a_2 \leq m_2\} \\ &\cup\{(0, i_2, 3, k_1, a_1, a_2) : \ 0 \leq i_2 \leq L, \ 1 \leq k_1 \leq n_1, \ 1 \leq a_1 \leq m_1, \ 1 \leq a_2 \leq m_2\} \\ &\cup\{(0, i_2, 4, b, a_2) : \ 0 \leq i_2 \leq L, \ 1 \leq b \leq s, \ 1 \leq a_2 \leq m_2\} \\ &\cup\{(0, i_2, j, a_2) : \ 0 \leq i_2 \leq L, \ j = 5, 6, 7, 8, \ 1 \leq a_2 \leq m_2\}, \end{aligned}$$

and for $i \geq 1$,

$$\begin{aligned} l(i)=&\{(i_1, i_2, 1, k_1, a_1, a_2) : \ i_1 \epsilon \mathbb{Z}^+, \ 0 \leq i_2 \leq L, \ 1 \leq k_1 \leq n_1, \ 1 \leq a_1 \leq m_1, \ 1 \leq a_2 \leq m_2\} \\ &\cup\{(i_1, i_2, 2, k_2, a_1, a_2) : \ i1 \epsilon \mathbb{Z}^+, \ 0 \leq i_2 \leq L, \ 1 \leq k_2 \leq n_2, \ 1 \leq a_1 \leq m_1, \ 1 \leq a_2 \leq m_2\} \\ &\cup\{(i_1, i_2, 3, k_1, a_1, a_2) : \ i \epsilon \mathbb{Z}^+, \ 0 \leq i_2 \leq L, \ 1 \leq k_1 \leq n_1, \ 1 \leq a_1 \leq m_1, \ 1 \leq a_2 \leq m_2\} \\ &\cup\{(i_1, i_2, 4, b, a_2) : \ i \epsilon \mathbb{Z}^+, \ 0 \leq i_2 \leq L, \ 1 \leq b \leq s, \ 1 \leq a_2 \leq m_2\} \\ &\cup\{(i_1, i_2, j, a_2) : \ i \epsilon \mathbb{Z}^+, \ j = 5, 6, 7, 8, \ 0 \leq i_2 \leq L, \ 1 \leq a_2 \leq m_2\}. \end{aligned}$$

3.1. The Infinitesimal Matrix Generation

The quasi birth and death process has the generating matrix Q, is as follows:

$$Q = \begin{bmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots & \dots \end{bmatrix},$$

where

$$B_{00} = \begin{bmatrix} B_{00}^1 & B_{00}^2 & B_{00}^3 & 0 & 0 & 0 & 0 \\ B_{00}^4 & B_{00}^5 & B_{00}^6 & B_{00}^7 & B_{00}^8 & 0 & 0 \\ B_{00}^9 & 0 & B_{00}^{10} & 0 & 0 & 0 & 0 \\ 0 & B_{00}^{11} & 0 & B_{00}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{00}^{13} & B_{00}^{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{00}^{15} & B_{00}^{16} \\ B_{00}^{17} & 0 & 0 & 0 & 0 & 0 & B_{00}^{18} \end{bmatrix},$$

where $B_{00}^1 = diag(D_0 - \tau I_{m_2}, D_0 - (\tau+\delta)I_{m_2}, ..., D_0 - (\tau + k\delta)I_{m_2})$

$$B_{00}^2 = \begin{bmatrix} e' \otimes \alpha \otimes D_2 & 0 & 0 & \dots & 0 \\ e' \otimes \delta\alpha \otimes I_{m_1} & e' \otimes \alpha \otimes D_2 & 0 & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & e' \otimes L\delta\alpha \otimes I_{m_1} & e' \otimes \alpha \otimes D_2 & \dots & 0 \end{bmatrix},$$

$$B_{00}^3 = I_{(L+1)} \otimes e' \otimes \tau I_{m_2},\ B_{00}^4 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & e_{m_1} \otimes T^0 \otimes I_{m_2} & 0 & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & e_{m_1} \otimes T^0 \otimes I_{m_2} \end{bmatrix},$$

$$B_{00}^5 = \begin{bmatrix} f_1 & f_2 & 0 & \dots & 0 \\ 0 & f_1 & f_2 & \dots & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & f_1 + f_2 \end{bmatrix}, \text{ where } f_1 = T \oplus D_0 \oplus C_0 - \eta_1 I_{nm_1m_2},\ f_2 = I_n \otimes I_m \otimes D_2,$$

$$B_{00}^6 = I_{(L+1)} \otimes I_{n_1} \otimes C_1 \otimes e_{m_2}, \quad B_{00}^7 = I_{(L+1)} \otimes e_{nm_1} \otimes \eta_1 \otimes I_{m_2},$$

$$B_{00}^8 = \begin{bmatrix} e_n \otimes T^0 \otimes I_{m_2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B_{00}^9 = I_{(L+1)} \otimes R^0 \otimes I_{m_2},$$

$$B_{00}^{10} = \begin{bmatrix} f_3 & f_4 & 0 & \dots & 0 \\ 0 & f_3 & f_4 & \dots & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & f_3 + f_4 \end{bmatrix}, \text{ where } f_3 = D_0 \oplus R, f_4 = I_s \otimes D_2,$$

$$B_{00}^{11} = I_{(L+1)} \otimes e' \otimes \alpha_1 \eta_1 I_{m_2},$$

$$B_{00}^{12} = \begin{bmatrix} f_5 & f_6 & 0 & \dots & 0 \\ 0 & f_5 & f_6 & \dots & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & f_5 + f_6 \end{bmatrix}, \text{ where } f_5 = (D_0 + b(D_1 + D_2)) - \eta_1 I_{m_2}, f_6 = D_2(1-b)$$

$$B_{00}^{13} = \begin{bmatrix} f_7 & f_8 & 0 & \dots & 0 \\ 0 & f_7 & f_8 & \dots & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & f_7 + f_8 \end{bmatrix}, \text{ where } f_7 = (D_0 + b(D_1 + D_2)) - \phi_1 I_{m_2}, f_6 = f_8,$$

$$B_{00}^{15} = \begin{bmatrix} f_9 & f_{10} & 0 & \dots & 0 \\ 0 & f_9 & f_{10} & \dots & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & f_9 + f_{10} \end{bmatrix}, \text{ where } f_9 = (D_0 + b(D_1 + D_2)) - \eta_2 I_{m_2},\ f_{10} = f_6,$$

$$B_{00}^{14} = I_{(L+1)} \otimes \phi I_{m_2},\ B_{00}^{16} = I_{(L+1)} \otimes \eta_2 I_{m_2},\ B_{00}^{17} = I_{(L+1)} \otimes \psi I_{m_2},$$

$$B_{00}^{18} = \begin{bmatrix} f_{11} & f_{12} & 0 & \dots & 0 \\ 0 & f_{11} & f_{12} & \dots & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & 0 & \dots & f_{11}+f_{12} \end{bmatrix}, \text{ where } f_{11} = D_0 - \psi I_{m_2}, f_{12} = D_2,$$

$$B_{01} = \begin{bmatrix} B_{01}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{01}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{01}^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{01}^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{01}^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{01}^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{01}^7 \end{bmatrix}, \text{ where } B_{01}^1 = I_{(L+1)} \otimes e' \otimes \alpha \otimes D_1,$$

$B_{01}^2 = I_{(L+1)} \otimes I_{n_1} \otimes D_1 \otimes I_{m_1}$, $B_{01}^3 = I_{(L+1)} \otimes I_s \otimes D_1$, $B_{01}^4 = B_{01}^5 = B_{01}^6 = I_{(L+1)} \otimes D_1(1-b)$, $B_{01}^7 = I_{(L+1)} \otimes D_1$,

$$B_{10} = \begin{bmatrix} B_{10}^1 & 0 & B_{10}^2 & 0 & B_{10}^3 & 0 & 0 \\ B_{10}^4 & 0 & B_{10}^5 & 0 & B_{10}^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ where } B_{10}^1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & e_{m_1} \otimes qT^0 \otimes I_{m_2} & 0 & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & e_{m_1} \otimes qT^0 \otimes I_{m_2} \end{bmatrix},$$

$B_{10}^2 = I_{(L+1)} \otimes e_s \otimes I_{m_2}$,

$$B_{10}^3 = \begin{bmatrix} e_{m_1} \otimes qT^0 \otimes I_{m_2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B_{10}^4 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & e_{m_1} \otimes T_1^0 \otimes I_{m_2} & 0 & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & e_{m_1} \otimes T_1^0 \otimes I_{m_2} \end{bmatrix},$$

$$B_{10}^5 = I_{(L+1)} \otimes e_s \otimes C_1 \otimes I_{m_2}, B_{10}^6 = \begin{bmatrix} e_{m_1} \otimes T_1^0 \otimes I_{m_2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} A_0^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_0^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_0^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_0^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_0^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_0^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_0^7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_0^8 \end{bmatrix} \text{ where } A_0^1 = I_{(L+1)} \otimes I_{n_1} \otimes I_{m_1} \otimes D_1,$$

$A_0^2 = I_{(L+1)} \otimes I_{n_2} \otimes I_{m_1} \otimes D_1$, $A_0^3 = I_{(L+1)} \otimes I_{n_1} \otimes I_{m_1} \otimes D_1$, $A_0^4 = I_{(L+1)} \otimes I_s \otimes D_1$,

$A_0^5 = I_{(L+1)} \otimes D_1(1-b)$, $A_0^6 = A_0^7 = A_0^5$, $A_0^8 = I_{(L+1)} \otimes D_1$,

$$A_1 = \begin{bmatrix} A_1^1 & A_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^4 & 0 & A_1^5 & A_1^6 & A_1^7 & 0 & 0 & 0 \\ A_1^8 & 0 & 0 & A_1^9 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1^{10} & 0 & A_1^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_1^{12} & A_1^{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_1^{14} & A_1^{15} \\ A_1^{16} & 0 & 0 & 0 & 0 & 0 & 0 & A_1^{17} \end{bmatrix}, \text{where } A_1^1 = \begin{bmatrix} f_{13} & f_{14} & 0 & \dots & 0 \\ 0 & f_{13} & f_{14} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f_{13}+f_{14} \end{bmatrix},$$

where $f_{13} = (T \oplus D_0) \oplus C0$, $f_{14} = D_2 \otimes I_{m_1 n_1}$.

$$A_1^3 = \begin{bmatrix} f_{15} & f_{16} & 0 & \dots & 0 \\ 0 & f_{15} & f_{16} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f_{15}+f_{16} \end{bmatrix}, \text{where } A_1^2 = I_{(L+1)} \otimes \alpha_1 \otimes pT^0 \otimes I_{m_1 m_2},$$

$f_{15} = (T_1 \oplus D_0) \oplus C0$, $f_{16} = D_2 \otimes I_{m_1 n_2}$, $A_1^4 = I_{(L+1)} \otimes \alpha \otimes T^0 \otimes I_{m_1 m_2}$,

$$A_1^5 = \begin{bmatrix} f_{17} & f_{18} & 0 & \dots & 0 \\ 0 & f_{17} & f_{18} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f_{17}+f_{18} \end{bmatrix} \text{ where } f_{17} = ((T \oplus D_0) \oplus C_0) - \eta_1 I_{n_1 m_1 m_2},$$

$f_{18} = D_2 \otimes I_{m_1 n_1}$,

$A_1^6 = I_{(L+1)} \otimes e_{n_1} \otimes C_1 \otimes I_{m_2}$, $A_1^7 = I_{(L+1)} \otimes e_{n_1} \otimes e_{m_1} \otimes \eta_1 I_{m_2}$, $A_1^8 = I_{(L+1)} \otimes e'_{m_1} \otimes S^0 \alpha \otimes I_{m_2}$,

$$A_1^9 = \begin{bmatrix} f_{19} & f_{20} & 0 & \dots & 0 \\ 0 & f_{19} & f_{20} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots f_{19}+f_{20} & \end{bmatrix} \text{ where } f_{19} = S \oplus D_0, f_{20} = I_s \otimes D_2,$$

$$A_1^{10} = I_{(L+1)} \otimes e'_{n_1} \otimes e'_{m_1} \otimes \eta_1 I_{m_2}, \ A_1^{11} = \begin{bmatrix} f_{21} & f_{22} & 0 & \dots & 0 \\ 0 & f_{21} & f_{22} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f_{21}+f_{22} \end{bmatrix},$$

$$\text{where } f_{21} = (D_0 + b(D_1 + D_2)) - \eta_1 I_{m_2}, f_{22} = D_2(1-b), A_1^{12} = \begin{bmatrix} f_{23} & f_{24} & 0 & \dots & 0 \\ 0 & f_{23} & f_{24} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f_{23}+f_{24} \end{bmatrix},$$

where $f_{23} = (D_0 + b(D_1 + D_2)) - \varphi I_{m_2}, f_{24} = D_2(1-b)$, $A_1^{13} = I_{(L+1)} \otimes \varphi I_{m_2}$,

$$A_1^{14} = \begin{bmatrix} f_{25} & f_{26} & 0 & \dots & 0 \\ 0 & f_{25} & f_{26} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f_{25}+f_{26} \end{bmatrix}, \text{where } f_{25} = (D_0 + b(D_1 + D_2)) - \eta_2 I_{m_2}, f_{26} = D_2(1-b),$$

$$A_1^{15} = I_{(L+1)} \otimes \eta_2 I_{m_2}, \quad A_1^{16} = I_{(L+1)} \otimes e' \otimes \alpha \otimes \psi I_{m_2},$$

$$A_1^{17} = \begin{bmatrix} f_{27} & f_{28} & 0 & \dots & 0 \\ 0 & f_{27} & f_{28} & \dots & 0 \\ & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f_{27}+f_{28} \end{bmatrix}, \text{where } f_{27} = D_0 - \psi I_{m_2}, \quad f_{28} = D_2,$$

$$A_2 = \begin{bmatrix} A_2^1 & 0 & 0 & A_2^2 & 0 & 0 & 0 & 0 \\ A_2^3 & 0 & 0 & A_2^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ where } A_2^1{=}I_{(L+1)} \otimes \alpha \otimes qT^0 \otimes I_{m_1m_2},$$

$A_2^2{=}I_{(L+1)} \otimes e_s \otimes C_1 \otimes I_{m_2}$, $A_2^3{=}I_{(L+1)} \otimes \alpha \otimes T_1^0 \otimes I_{m_1m_2}$, $A_2^4{=}I_{(L+1)} \otimes e_s \otimes C_1 \otimes I_{m_2}$.

4. System Analysis

We evaluate this model, beneath of the certain conditions to ensure that the system to be stable.

4.1. Stability condition for the system

Let A be the matrix , where $A = A_0 + A_1 + A_2$. The invariant probability vector ς, which is referred to as a generator matrix and its satisfying

$$\varsigma A = 0, \quad \varsigma e = 1.$$

The vector ς represents the states which is partitioned by
$\varsigma{=}(\varsigma_0, \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6, \varsigma_7)$ and it is subdivided by $(\varsigma_{00}, \varsigma_{01}, ..., \varsigma_{0K}, \varsigma_{10}, \varsigma_{11}, ..., \varsigma_{1K}, \varsigma_{20}, \varsigma_{21}, ..., \varsigma_{2K}, \varsigma_{30}, \varsigma_{31}, ..., \varsigma_{3K}, \varsigma_{40}, \varsigma_{41}, ..., \varsigma_{4K}, \varsigma_{50}, \varsigma_{51}, ..., \varsigma_{5K}, \varsigma_{60}, \varsigma_{61}, ..., \varsigma_{6K}, \varsigma_{70}, \varsigma_{71}, ..., \varsigma_{7K})$ which is evaluated by the aid of subsequent equation:

$$\begin{aligned}
&\varsigma_{00}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + ((T \oplus D_0) \oplus C_0) + (\alpha \otimes qT^0 \otimes I_{m_1m_2})] + \varsigma_{10}[\alpha \otimes T_1^0 \otimes I_{m_1m_2}] \\
&\qquad + \varsigma_{20}[\alpha \otimes T^0 \otimes I_{m_1m_2}] + \varsigma_{30}[e'_{m_1} \otimes S^0\alpha \otimes I_{m_2}] + \varsigma_{70}[e' \otimes \alpha \otimes \psi I_{m_2}] = 0, \\
&\varsigma_{0(i-1)}[D_2 \otimes I_{m_1n_1}] + \varsigma_{0i}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + ((T \oplus D_0) \oplus C_0) + (\alpha \otimes qT^0 \otimes I_{m_1m_2}] \\
&\qquad + \varsigma_{1i}[\alpha \otimes T_1^0 \otimes I_{m_1m_2}] + \varsigma_{2i}[\alpha \otimes T^0 \otimes I_{m_1m_2}] + \varsigma_{3i}[e'_{m_1} \otimes S^0\alpha \otimes I_{m_2}] \\
&\qquad\qquad + \varsigma_{7i}[e' \otimes \alpha \otimes \psi I_{m_2}] = 0, 1 \le i \le K. \\
&\varsigma_{0K}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + ((T \oplus D_0) \oplus C_0) + (\alpha \otimes qT^0 \otimes I_{m_1m_2}) + D_2 \otimes I_{m_1n_1}] + \varsigma_{1K}[\alpha \otimes T_1^0 \otimes I_{m_1m_2}] \\
&\qquad + \varsigma_{2K}[\alpha \otimes T^0 \otimes I_{m_1m_2}] + \varsigma_{3K}[e'_{m_1} \otimes S^0\alpha \otimes I_{m_2}] + \varsigma_{7K}[e' \otimes \alpha \otimes \psi I_{m_2}] = 0, \\
&\varsigma_{00}[\alpha_1 \otimes pT^0 \otimes I_{m_1m_2}] + \varsigma_{10}[(I_{n_2} \otimes I_{m_1} \otimes D_1) + ((T_1 \oplus D_0) \oplus C_0)] = 0, \\
&\varsigma_{0i}[\alpha_1 \otimes pT^0 \otimes I_{m_1m_2}] + \varsigma_{1(i-1)}[D_2 \otimes I_{m_1n_2}] + \varsigma_{1i}[(I_{n_2} \otimes I_{m_1} \otimes D_1) + ((T_1 \oplus D_0) \oplus C_0)] = 0, 1 \le i \le K-1. \\
&\varsigma_{0K}[\alpha_1 \otimes pT^0 \otimes I_{m_1m_2}] + \varsigma_{1K}[(I_{n_2} \otimes I_{m_1} \otimes D_1) + ((T_1 \oplus D_0) \oplus C_0) + (D_2 \otimes I_{m_1m_2})] = 0, \\
&\varsigma_{20}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + ((T \oplus D_0) \oplus C_0) - \eta_1 I_{n_1m_1m_2}] + \varsigma_{40}[e'_{n_1} \otimes e'_{m_1} \otimes \eta_1 I_{m_2}] = 0, \\
&\varsigma_{2(i-1)}[D_2 \otimes I_{m_1n_1}] + \varsigma_{2i}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + ((T \oplus D_0) \oplus C_0) - \eta_1 I_{n_1m_1m_2}] \\
&\qquad\qquad + \varsigma_{4i}[e'_{n_1} \otimes e'_{m_1} \otimes \eta_1 I_{m_2} = 0, \ 1 \le i \le K-1. \\
&\varsigma_{2K}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + (((T \oplus D_0) \oplus C_0) - \eta_1 I_{n_1m_1n_2}) + D_2 \otimes I_{m_1n_1}] + \varsigma_{4K}[e'_{n_1} \otimes e'_{m_1} \otimes \eta_1 I_{m_2}] = 0, \\
&\varsigma_{20}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + ((T \oplus D_0) \oplus C_0) - \eta_1 I_{n_1m_1m_2}] + \varsigma_{40}[e'_{n_1} \otimes e'_{m_1} \otimes \eta_1 I_{m_2}] = 0, \\
&\varsigma_{2(i-1)}[D_2 \otimes I_{m_1n_1}] + \varsigma_{2i}[(I_{n_1} \otimes I_{m_1} \otimes D_1) + ((T \oplus D_0) \oplus C_0) - \eta_1 I_{n_1m_1m_2}] \\
&\qquad\qquad + \varsigma_{4i}[e'_{n_1} \otimes e'_{m_1} \otimes \eta_1 I_{m_2} = 0, \ 1 \le i \le K-1.
\end{aligned}$$

$$\varsigma_{20}[(I_{n_1}\otimes I_{m_1}\otimes D_1)+((T\oplus D_0)\oplus C_0)-\eta_1 I_{n_1m_1m_2}]+\varsigma_{40}[e'_{n_1}\otimes e'_{m_1}\otimes \eta_1 I_{m_2}]=0,$$
$$\varsigma_{2(i-1)}[D_2\otimes I_{m_1n_1}]+\varsigma_{2i}[(I_{n_1}\otimes I_{m_1}\otimes D_1)+((T\oplus D_0)\oplus C_0)-\eta_1 I_{n_1m_1m_2}]$$
$$+\varsigma_{4i}[e'_{n_1}\otimes e'_{m_1}\otimes \eta_1 I_{m_2}=0,\ 1\le i\le K-1.$$
$$\varsigma_{2K}[(I_{n_1}\otimes I_{m_1}\otimes D_1)+(((T\oplus D_0)\oplus C_0)-\eta_1 I_{n_1m_1n_2})+D_2\otimes I_{m_1n_1}]+\varsigma_{4K}[e'_{n_1}\otimes e'_{m_1}\otimes \eta_1 I_{m_2}]=0,$$
$$\varsigma_{00}[e_s\otimes C_1\otimes I_{m_2}]+\varsigma_{10}[e_s\otimes C_1\otimes I_{m_2}]+\varsigma_{20}[e_{n_1}\otimes C_1\otimes I_{m_2}]+\varsigma_{30}[(I_s\otimes D_1)+(S\oplus D_0)]=0,$$
$$\varsigma_{0i}[e_s\otimes C_1\otimes I_{m_2}]+\varsigma_{1i}[e_s\otimes C_1\otimes I_{m_2}]+\varsigma_{2i}[e_n\otimes C_1\otimes I_{m_2}]$$
$$+\varsigma_{3(i-1)}[I_s\otimes D_2]+\varsigma_{3i}[(I_s\otimes D_1)+(S\oplus D_0)]=0,1\le i\le K-1.$$
$$\varsigma_{0K}[e_s\otimes C_1\otimes I_{m_2}]+\varsigma_{1K}[e_s\otimes C_1\otimes I_{m_2}]+\varsigma_{2K}[e_n\otimes C_1\otimes I_{m_2}]$$
$$+\varsigma_{3K}[(I_s\otimes (D_1+D_2))+(S\oplus D_0)]=0,$$
$$\varsigma_{20}[e_{n_1}\otimes e_{m_1}\otimes \eta I_{m_2}]+\varsigma_{40}[(D_0+D_1)+bD_2-\eta_1 I_{m_2}]=0,$$
$$\varsigma_{2i}[e_{n_1}\otimes e_{m_1}\otimes \eta I_{m_2}]+\varsigma_{4(i-1)}[D_2(1-b)]+\varsigma_{4i}[(D_0+D_1)+bD_2-\eta_1 I_{m_2}]=0,1\le i\le K-1.$$
$$\varsigma_{2K}[e_{n_1}\otimes e_{m_1}\otimes \eta I_{m_2}]+\varsigma_{4K}[(D_0+D_1+D_2)-\eta_1 I_{m_2}]=0,$$
$$\varsigma_{50}[(D_0+D_1)+bD_2-\pi I_{m_2}]=0,$$
$$\varsigma_{5(i-1)}[D_2(1-b)]+\varsigma_{5i}[(D_0+D_1)+bD_2-\pi I_{m_2}]=0,1\le i\le K-1.$$
$$\varsigma_{5K}[(D_0+D_1+D_2)-\pi I_{m_2}]=0,$$
$$\varsigma_{50}[\pi I_{m_2}]+\varsigma_{60}[(D_0+D_1)+bD_2-\eta_2 I_{m_2}]=0,$$
$$\varsigma_{5i}[\pi I_{m_2}]+\varsigma_{6(i-1)}[D_2(1-b)]+\varsigma_{6i}[(D_0+D_1)+bD_2-\eta_2 I_{m_2}]=0,\ 1\le i\le K-1.$$
$$\varsigma_{5K}[\pi I_{m_2}]+\varsigma_{6K}[(D_0+D_1+D_2)-\eta_2 I_{m_2}]=0,$$
$$\varsigma_{60}[\eta_2 I_{m_2}]+\varsigma_{70}[(D_0+D_1)\psi I_{m_2}]=0,$$
$$\varsigma_{6i}[\eta_2 I_{m_2}]+\varsigma_{7(i-1)}[D_2]+\varsigma_{7i}[(D_0+D_1)-\psi I_{m_2}]=0,$$
$$\varsigma_{6K}[\eta_2 I_{m_2}]+\varsigma_{7K}[(D_0+D_1+D_2)-\psi I_{m_2}]=0,$$

subject to

$$[\sum_{i=0}^{K}\varsigma 0i+\sum_{i=0}^{K}\varsigma_{2i}]e_{n_2m_1m_2}+[\sum_{i=0}^{K}\varsigma_{1i}]e_{n_2m_1m_2}+[\sum_{i=0}^{K}\varsigma_{3i}]e_{sm_2}+[\sum_{r=4}^{7}\sum_{i=0}^{K}\varsigma_{ri}]e_{m_2}=1.$$

The necessary and sufficient condition of a QBD process which satisfy the condition $\varsigma A_0 e < \varsigma A_2 e$ that system to be stay in stable.
Therefore,

$$\varsigma_{00}[e_{n_1}\otimes e_{m_1}\otimes D_1 e_{m_2}]+\varsigma_{01}[e_{n_1}\otimes e_{m_1}\otimes D_1 e_{m_2}]+...+\varsigma_{0K}[e_{n_1}\otimes e_{m_1}\otimes D_1 e_{m_2}]$$
$$+\varsigma_{10}[e_{n_2}\otimes e_{m_1}\otimes D_1 e_{m_2}]+\varsigma_{11}[e_{n_2}\otimes e_{m_1}\otimes D_1 e_{m_2}]+...+\varsigma_{1K}[e_{n_2}\otimes e_{m_1}\otimes D_1 e_{m_2}]$$
$$+\varsigma_{20}[e_{n_1}\otimes e_{m_1}\otimes D_1 e_{m_2}]+\varsigma_{21}[e_{n_1}\otimes e_{m_1}\otimes D_1 e_{m_2}]+...+\varsigma_{2K}[e_{n_1}\otimes e_{m_1}\otimes D_1 e_{m_2}$$
$$+\varsigma_{30}[e_s\otimes D_1 e_{m_2}]+\varsigma_{31}[e_s\otimes D_1 e_{m_2}]+...+\varsigma_{3K}[e_s\otimes D_1 e_{m_2}]$$
$$+\varsigma_{40}[D_1(1-b)]+\varsigma_{41}[D_1(1-b)]+...+\varsigma_{4K}[D_1(1-b)]$$
$$+\varsigma_{50}[D_1(1-b)]+\varsigma_{51}[D_1(1-b)]+...+\varsigma_{5K}[D_1(1-b)]$$
$$+\varsigma_{60}[D_1(1-b)]+\varsigma_{61}[D_1(1-b)]+...+\varsigma_{6K}[D_1(1-b)]$$
$$+\varsigma_{70}[D_1]+\varsigma_{71}[D_1]+...+\varsigma_{7K}[D_1]<\varsigma_{00}[qT^0\otimes e_{m_1}\otimes e_{m_2}]+\varsigma_{10}[T_1^0\otimes e_{m_1}\otimes e_{m_2}]$$
$$+\varsigma_{01}[qT^0\otimes e_{m_1}\otimes e_{m_2}]+\varsigma_{11}[T_1^0\otimes e_{m_1}\otimes e_{m_2}]+...+\varsigma_{0K}[qT^0\otimes e_{m_1}\otimes e_{m_2}]$$
$$+\varsigma_{1K}[T_1^0\otimes e_{m_1}\otimes e_{m_2}]+\varsigma_{00}[e_s\otimes C_1 e_{m_1}\otimes e_{m_2}]$$
$$+\varsigma_{10}[e_s\otimes C_1 e_{m_1}\otimes e_{m_2}]+\varsigma_{01}[e_s\otimes C_1 e_{m_1}\otimes e_{m_2}]$$
$$+\varsigma_{11}[e_s\otimes C_1 e_{m_1}\otimes e_{m_2}]+...+\varsigma_{0K}[e_s\otimes C_1 e_{m_1}\otimes e_{m_2}]+\varsigma_{1K}[e_s\otimes C_1 e_{m_1}\otimes e_{m_2}]$$

4.2. The Invariant Probability Vector

Let X represents the infinitesimal generator matrix Q and which is split by $X = (X_0, X_1, X_2, ...)$.

For X_0 is of dimension $((L+1)m_2 + (L+1)n_1m_1m_2 + (L+1)sm_2 + 4(L+1)m_2)$ and $X_i's$ are of dimension $(2(L+1)n_1m_1m_2 + (L+1)n_2m_1m_2 + (L+1)sm_2 + 4(L+1)m_2)$, $i \geq 1$.
As X is a vector of Q satisfies the relation

$$XQ = 0, \quad Xe = 1.$$

After satisfying the stability criterion, use the below equation to find the invariant probability vector X,

$$X_i = X_1R^{i-1}, \qquad i = 2, 3, ...$$

where R is the matrix created by solving the quadratic matrix equation, also known as the rate matrix.

$$R^2A_2 + RA_1 + A_0 = 0.$$

With the aid of succeeding equation we can find the vectors namely X_0, X_1 and X_2,

$$X_0B_{00} + X_1B_{10} = 0,$$
$$X_0B_{01} + X_1[A_1 + RA_2] = 0,$$

subject to normalizing condition

$$X_0e_{((L+1)m_2+(L+1)n_1m_1m_2+(L+1)sm_2+4(L+1)m_2)} + X_1[I-R]^{-1}e_{(2(L+1)n_1m_1m_2+(L+1)n_2m_1m_2+(L+1)sm_2+4(L+1)m_2)} = 1.$$

Therefore, the logarithmic reduction algorithm can be used to find the rate matrix R with the help of Latouche and Ramaswami [21].

5. Busy Period Analysis

- Under the busy period of $MMAP/PH/1$ queuing model, we will understand the epoch of the time interval starts from a new arrival which find the empty system and ends when the system becomes empty again at the completion of service.
- A busy cycle which is defined by the initial passage time of the level between 1 and 0 and the time return to level 0, requiring at least one visit to any other level.
- From level i to level $i-1$, where $i = 2, 3, 4, ...$ which is the initial passage time under the consideration of the QBD process. In the boundary states namely, $i = 0, 1$ which deals separately.
- For all the level i, where $i = 1, 2, 3, ...$, we seen that there are $(2(L+1)n_1m_1m_2 + (L+1)n_2m_1m_2 + (L+1)sm_2 + 4(L+1)m_2)$ states.

Notations:

- Let $G_{j,j'}(k, x)$ represent the conditional probability that the QBD process, starting at time $t = 0$ in the state (i, j) $t = 0$ and ends up in the state (i, j') by making meticulously k left jumps and obtaining both stages at the same period.
- Let the joint transform matrix

$$\tilde{G}_{j,j'}(z, s) = \sum_{k=1}^{\infty} z^k \int_0^{\infty} e^{-sx} dG_{j,j'}(k, x); \quad |z| \leq 1, \quad Re(s) \geq 0$$

.

- The matrix $\tilde{G}(z,s) = \tilde{G}_{j,j'}(z,s)$. [Neuts [25]]
- Except for the boundary states, the matrix $G = G_{jj'} = \tilde{G}(1,0)$ be concerns the initial passage times.
- At time $t = 0$, when returning from stage 1 to stage 0, the conditional probability that described in the first return time and it is denoted as $G^{(1,0)}_{jj'}(k,x)$.
- At time $t = 0$, when returning to stage 0, the conditional probability that described and it is denoted as $G^{(0,0)}_{jj'}(k,x)$.
- At time $t = 0$, let S_{1j} represent the process's average initial passage time between stages i and $i-1$ and in the state (i,j).
- At time $t = 0$, in the initial passage procedure between levels i and $i-1$, which starts in the state (i,j), let S_{2j} be the average number of consumers that received service.
- $\tilde{S}_1$, $\tilde{S}_2$ be the column vectors along with S_{1j} and S_{2j} as their entries respectively.
- The expected first return time between stage 1 and stage 0 is represented by $\tilde{S}^{(1,0)}_1$.
- In the first return period from stage 1 to stage 0, the expected number of services completed and it is represented by $\tilde{S}^{(1,0)}_2$.
- The expected initial return time to stage 0 is represented by $\tilde{S}^{(0,0)}_1$.
- During the initial return time to stage 0, the expected number of services were rendered and it is represented by $\tilde{S}^{(0,0)}_2$.

We evaluate $\tilde{G}(z,s)$ matrix which satisfies the equation

$$\tilde{G}(z,s) = z(sI - A_1)^{-1}A_2 + (sI - A_1)^{-1}A_0\tilde{G}^2(z,s).$$

After found the rate matrix R, we can evaluate G matrix by using logarithmic reduction algorithm method which is given by Latouche and Ramaswami [21]

$$G = -(A_1 + RA_2)^{-1}A_2.$$

In the boundary states namely 1 and 0 and the equations represented by $\tilde{G}^{(1,0)}(z,s)$ and $\tilde{G}^{(0,0)}(z,s)$.

$$\tilde{G}^{(1,0)}(z,s) = z(sI - B_{11})^{-1}B_{10} + (sI - B_{11})^{-1}B_{12}\tilde{G}^{(2,1)}(z,s)\tilde{G}^{(1,0)}(z,s),$$
$$\tilde{G}^{(0,0)}(z,s) = (sI - B_{00})^{-1}B_{01}\tilde{G}^{(1,0)}(z,s).$$

The matrices $G, \tilde{G}^{(1,0)}(1,0)$ and $\tilde{G}^{(0,0)}(1,0)$ are stochastic.
The instants can be calculated as obeys:

$$\tilde{S}_1 = -\frac{\partial \tilde{G}(z,s)}{\partial s}\Big|_{s=0,z=1} e = -[A_0(G+I) + A_1]^{-1}e,$$
$$\tilde{S}_2 = \frac{\partial \tilde{G}(z,s)}{\partial z}\Big|_{s=0,z=1} e = -[A_0(G+I) + A_1]^{-1}]A_2 e,$$
$$\tilde{S}^{(1,0)}_1 = -\frac{\partial \tilde{G}^{(1,0)}(z,s)}{\partial s}\Big|_{s=0,z=1} = -[B_{11} + B_{12}\tilde{G}^{(2,1)}(1,0)]^{-1}[e + B_{12}\tilde{S}^{(2,1)}_1],$$
$$\tilde{S}^{(1,0)}_2 = \frac{\partial \tilde{G}^{(1,0)}(z,s)}{\partial z}\Big|_{s=0,z=1} e = -[B_{12}\tilde{G}^{(2,1)}(1,0) + B_{11}]^{-1}[B_{12}\tilde{S}^{(2,1)}_2 + B_{10}e],$$
$$\tilde{S}^{(0,0)}_1 = -\frac{\partial \tilde{G}^{(0,0)}(z,s)}{\partial s}\Big|_{s=0,z=1} e = -B^{-1}_{00}[B_{01}\tilde{S}^{(1,0)}_1 + e],$$
$$\tilde{S}^{(0,0)}_2 = \frac{\partial \tilde{G}^{(0,0)}(z,s)}{\partial z}\Big|_{s=0,z=1} e = -B^{-1}_{00}[B_{01}\tilde{S}^{(1,0)}_2].$$

6. Performance Measure

- Probability of server being idle:

$$P_{SI} = \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{0i_2 1a_2}.$$

- Probability of server to be busy with HP customers:

$$P_{BH} = \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{k_1=1}^{n_1} \sum_{a_1=1}^{m_1} \sum_{a_2=1}^{m_2} X_{i_1 i_2 1k_1 a_1 a_2}.$$

- Probability of server to be busy with LP customers:

$$P_{BL} = \sum_{i_2=0}^{L} \sum_{k_1=1}^{n_1} \sum_{a_1=1}^{m_1} \sum_{a_2=1}^{m_2} X_{0i_2 3k_1 a_1 a_2} + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{k_1=1}^{n_1} \sum_{a_1=1}^{m_1} \sum_{a_2=1}^{m_2} X_{i_1 i_2 3k_1 a_1 a_2}.$$

- Probability of server to be busy with optional service of HP customers:

$$P_{BHOS} = \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{k_2=1}^{n_2} \sum_{a_1=1}^{m_1} \sum_{a_2=1}^{m_2} X_{i_1 i_2 2k_2 a_1 a_2}.$$

- Probability of server being emergency vacation:

$$P_{EV} = \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{0i_2 5a_2} + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{i_1 i_2 5a_2}.$$

- Probability of server being normal vacation:

$$P_{NV} = \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{0i_2 7a_2} + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{i_1 i_2 7a_2}.$$

- Probability of server being closedown:

$$P_{CD} = \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{0i_2 6a_2} + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{i_1 i_2 6a_2}.$$

- Probability of server being setup:

$$P_{SU} = \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{0i_2 8a_2} + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{a_2=1}^{m_2} X_{i_1 i_2 8a_2}.$$

- Expected number of HP customers in the system:

$$\begin{aligned} E_{System} &= \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{L} \sum_{k_1=1}^{n_1} i_1 X_{i_1 i_2 1k_1 a_1 a_2} + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{k_2=1}^{n_2} \sum_{a_1=1}^{m_1} i_1 X_{i_1 i_2 2k_2 a_1 a_2} \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{k_2=1}^{n_2} \sum_{a_1=1}^{m_1} \sum_{a_2=1}^{m_2} i_1 X_{i_1 i_2 3k_2 a_1 a_2} + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{L} \sum_{j=4}^{8} \sum_{a_2=1}^{m_2} i_2 X_{i_1 i_2 j a_2} \\ &= X_1 (I-R)^{-2} e_{2(L+1)n_1 m_1 m_2 + (L+1)n_2 m_1 m_2 + (L+1)s m_2 + 4(L+1)m_2}. \end{aligned}$$

- Expected number of LP customers in the orbit:

$$
\begin{aligned}
E_{Orbit} = & \sum_{i_2=1}^{L}\sum_{a_2=1}^{m_2} i_2 X_{0 i_2 0 a_2} + \sum_{i_2=1}^{L}\sum_{k_1=1}^{n_1}\sum_{a_1=1}^{m_1}\sum_{a_2=1}^{m_2} i_2 X_{0 i_2 3 k_1 a_1 a_2} + \sum_{i_2=1}^{L}\sum_{b=1}^{s}\sum_{a_2=1}^{m_2} i_2 X_{0 i_2 4 b a_2} \\
& + \sum_{i_2=1}^{L}\sum_{j=5}^{8}\sum_{a_2=1}^{m_2} i_2 X_{0 i_2 j a_2} + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{L}\sum_{k_1=1}^{n_1}\sum_{a_1=1}^{m_1}\sum_{a_2=1}^{m_2} i_2 X_{i_1 i_2 1 k_1 a_1 a_2} \\
& + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{L}\sum_{k_2=1}^{n_2}\sum_{a_1=1}^{m_1}\sum_{a_2=1}^{m_2} i_2 X_{i_1 i_2 k_2 a_1 a_2} + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{L}\sum_{k_1=1}^{n_1}\sum_{a_1=1}^{m_1}\sum_{a_2=1}^{m_2} i_2 X_{i_1 i_2 3 k_1 a_1 a_2} \\
& + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{L}\sum_{b=1}^{s}\sum_{a_2=1}^{m_2} i_2 X_{i_1 i_2 4 b a 2} + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{L}\sum_{j=5}^{8}\sum_{a_2=1}^{m_2} i_2 X_{i_1 i_2 j a_2}.
\end{aligned}
$$

7. Analysis of cost model

In this section, we introduce a cost function TC with the following assumption:

- TC - Total cost per unit time.
- C_{H_h} - Holding cost of each HP customer in the system at per unit time.
- C_{H_l} - Holding cost of each LP customer in the orbit at per unit time.
- C_{SI} - Per unit time cost during the server is in idle period.
- C_{BH} - Per unit time cost during the server is busy with HP customers.
- C_{BL} - Per unit time cost during the server is busy with LP customers.
- C_{BHOS} - Per unit time cost during the server is busy with optional service of HP customers.
- C_R - Per unit time cost during the server is in under repair process.
- C_{EV} - Per unit time cost during the server is in emergency vacation.
- C_{NV} - Per unit time cost during the server is in normal vacation.
- C_{CD} - Per unit time cost during the server is in closedown.
- C_{SU} - Per unit time cost during the server is in setup.
- C_1 - Cost obtained by the server in carrying out the normal service to HP/LP customers.
- C_2 - Cost obtained by the server in carrying out the optional service to HP customers.
- C_3 - Cost obtained by the server in carrying out the repair process.
- C_4 - Cost obtained by the server in carrying out the breakdown.
- C_5 - Cost obtained for the arrival of negative customers.
- C_6 - Cost obtained by the server in carrying out the emergency vacation.
- C_7 - Cost obtained by the server in carrying out the normal vacation.
- C_8 - Cost obtained in carrying out the closedown process.
- C_9 - Cost obtained by the server carrying out the setup process.

The total average cost per unit time is given by

$$
\begin{aligned}
TC = & C_{H_h} E_{System} + C_{H_l} E_{Orbit} + C_{SI} P_{SI} + C_{BH} P_{BH} + C_{BL} P_{BL} + C_{BHOS} P_{BHOS} \\
& + C_R P_R + C_{EV} P_{EV} + C_{NV} P_{NV} + C_{CD} P_{CD} + C_{SU} P_{SU} + \mu C_1 + \mu_1 C_2 \\
& + \sigma C_3 + \tau C_4 + \lambda_3 C_5 + \eta_1 C_6 + \eta_2 C_7 + \phi C_8 + \psi C_9.
\end{aligned}
$$

8. Numerical Results

From this part, we determine the results of our model by representing numerically and graphically. The representations of MAP are distinct with the following variance and correlation structures, each of which has a mean value of 1. The arrival process as $ERL-A$, $EXP-A$ and $HYP-A$ correspond with renewal process, thus correlation is zero. This values are taken from Chakravarthy [10].

Positive Arrival in Erlang of order 2 (ERL-A):

$$D_0 = \begin{bmatrix} -4 & 4 \\ 0 & -4 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 2.8 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 1.2 & 0 \end{bmatrix}$$

Positive Arrival in Exponential (EXP-A):

$$D_0 = [-1], \quad D_1 = [0.6], \quad D_2 = [0.4]$$

Positive Arrival in Hyper exponential (HYP-EXP-A):

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.026 & 0.114 \\ 0.1026 & 0.0114 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.684 & 0.076 \\ 0.0684 & 0.0076 \end{bmatrix}$$

Negative Arrival in Erlang of order 2 (ERL-A):

$$C_0 = \begin{bmatrix} -0.5 & 0.5 \\ 0 & -0.5 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}$$

Negative Arrival in Exponential (EXP-A):

$$C_0 = [-0.1], \qquad C_1 = [0.1]$$

Negative Arrival in Hyper exponential (HYP-EXP-A):

$$C_0 = \begin{bmatrix} -0.190 & 0 \\ 0 & -0.019 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.1710 & 0.0190 \\ 0.0171 & 0.0019 \end{bmatrix}$$

Let us consider the service and repair process as PH-distributions and these values are incurred from Chakravarthy [10] which are as follows:

ERL-S (Normal Service in Erlang of order 2):

$$\alpha = (1, 0), \qquad T = \begin{bmatrix} -25 & 5 \\ 8 & -25 \end{bmatrix}$$

ERL-S (Optional Service in Erlang of order 2):

$$\alpha = (1, 0), \qquad T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

ERL-R (Repair in Erlang of order 2):

$$\beta = (1, 0), \qquad S = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

EXP-S (Normal Service in Exponential):

$$\alpha = (1), \qquad T = [-1]$$

EXP-S (Optional Service in Exponential):

$$\alpha_1 = (1), \qquad T_1 = [-1]$$

EXP-R (Repair in Exponential):

$$\beta = (1), \qquad S = [-1]$$

HYP-EXP-S (Normal Service in Hyper exponential):

$$\alpha = (0.3, 0.7), \qquad T = \begin{bmatrix} -9 & 7 \\ 8 & -10 \end{bmatrix}$$

HYP-EXP-S (Optional Service in Hyper exponential):

$$\alpha_1 = (0.4, 0.6), \qquad T_1 = \begin{bmatrix} -12 & 6 \\ 5 & -10 \end{bmatrix}$$

HYP-EXP-R (Repair in Hyper exponential):

$$\beta = (0.4, 0.6), \qquad S = \begin{bmatrix} -6 & 4 \\ 3 & -4 \end{bmatrix}$$

8.1. Illustration 1

In tables 1,2 and 3, we determine the outcome of the repair rate of server (σ) on the expected system size (ES).
Fix $\lambda_1 = 0.8$, $\lambda_2 = 0.2$, λ_3, $\mu = 45$, $\mu_1 = 40$, $\eta_1 = 4$, $\eta_2 = 3$, $\varphi = 12$, $\psi = 12$, $\tau = 2$, $p = 0.5$, $q = 0.5$, $b = 0.05$, $\delta = 3$, $L = 3$.

Table 1: *Repair rate (σ) vs ES - **EXP-S***

σ	EXP-A	ERL-A	HYP-A
10	0.351226	0.119414	0.000351
10.5	0.351418	0.119473	0.000358
11	0.351632	0.119539	0.000365
11.5	0.351862	0.119609	0.000371
12	0.352100	0.119681	0.000376
12.5	0.352343	0.119756	0.000381
13	0.352588	0.119831	0.000385
13.5	0.352832	0.119906	0.000389
14	0.353074	0.119980	0.000392
14.5	0.353312	0.120054	0.000396

Table 2: *Repair rate (σ) vs ES - **ERL-S***

σ	EXP-A	ERL-A	HYP-A
10	0.211804	0.350087	0.042376
10.5	0.211885	0.350782	0.042475
11	0.211978	0.351445	0.042566
11.5	0.212078	0.352002	0.042650
12	0.212182	0.352487	0.042728
12.5	0.212289	0.352955	0.042800
13	0.212397	0.353404	0.042867
13.5	0.212504	0.353836	0.042930
14	0.212612	0.354250	0.042989
14.5	0.212717	0.354647	0.043044

Table 3: *Repair rate (σ) vs ES -* ***HYP-EXP-S***

σ	EXP-A	ERL-A	HYP-A
10	0.211852	0.112162	0.137225
10.5	0.211920	0.112284	0.137312
11	0.211995	0.112400	0.137393
11.5	0.212073	0.112509	0.137468
12	0.212154	0.112613	0.137538
12.5	0.212236	0.112712	0.137604
13	0.212319	0.112805	0.137666
13.5	0.212401	0.112893	0.137724
14	0.212481	0.112977	0.137779
14.5	0.212561	0.113057	0.137831

We observe that from the following tables 1,2, and 3.

- ES values rise for various combinations of arrival and service times as the server repair rate (σ) increases.
- When comparing the values of various service times, it can be seen that the expected system size increases more quickly for hyper exponential service times and slowly for Erlang service times.

8.2. Illustration 2

Using the two-dimensional graphs $2-10$, we investigate the impact of the normal service rate (μ) on the possibility that the server will be busy with high-priority customers (P_{BH}). Fix $\lambda_1 = 0.8$, $\lambda_2 = 0.2$, λ_3, $\mu_1 = 40$, $\sigma = 10$, $\eta_1 = 4$, $\eta_2 = 3$, $\tau = 2$, $\varphi = 12$, $\psi = 12$, $p = 0.5$, $q = 0.5$, $b = 0.05$, $\delta = 3$, $L = 3$.

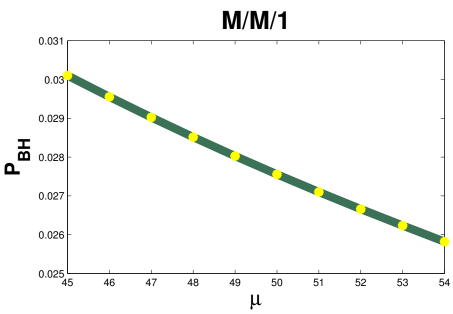

Figure 2: *Normal service rate (μ) vs. P_{BH}*

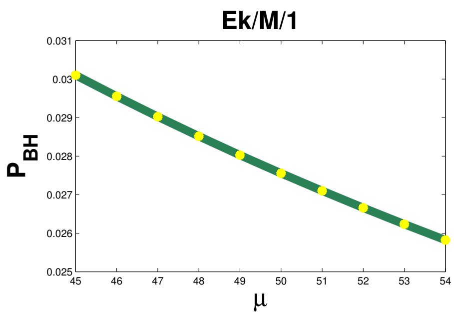

Figure 3: *Normal service rate (μ) vs. P_{BH}*

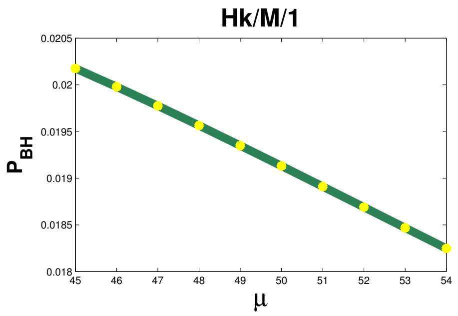

Figure 4: *Normal service rate (μ) vs. P_{BH}*

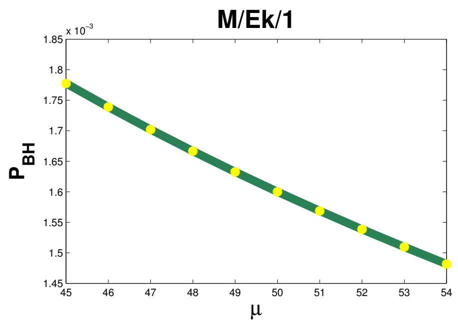

Figure 5: *Normal service rate (μ) vs. P_{BH}*

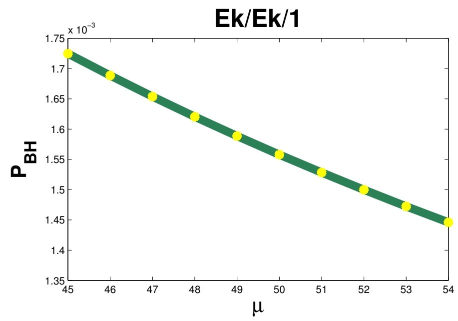

Figure 6: *Normal service rate (μ) vs. P_{BH}*

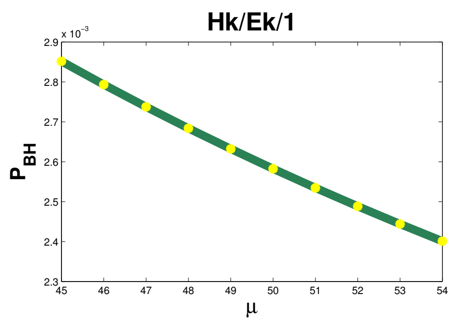

Figure 7: *Normal service rate (μ) vs. P_{BH}*

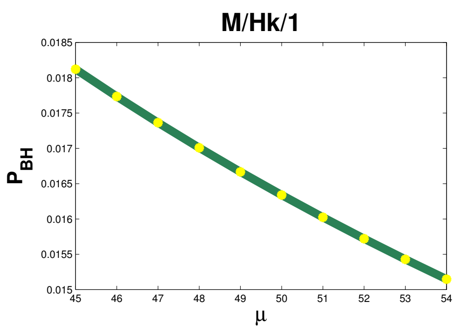

Figure 8: *Normal service rate (μ) vs. P_{BH}*

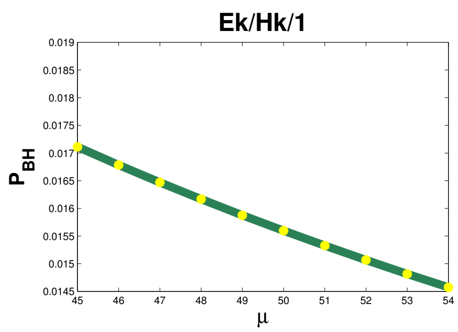

Figure 9: *Normal service rate (μ) vs. P_{BH}*

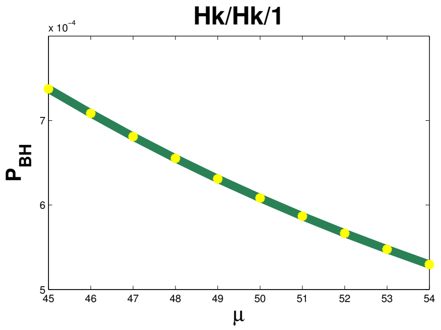

Figure 10: *Normal service rate (μ) vs. P_{BH}*

According to Figures $2-10$, as the normal service rate (μ) is raised, the probability that the server is busy with service (P_{BH}) increases for different arrival and service patterns. When increasing normal service rate (μ) on P_{BH} size increases much slower in Erlang arrival rather than hyper-exponential arrival.

8.3. Illustration 3

We investigate the impact of the normal service rate (μ) and breakdown rate (τ) on the probability that the server is busy with the normal service of high priority clients (P_{BH}) by using the three-dimensional graphs $11-19$. Fix $\lambda_1 = 0.8$, $\lambda_2 = 0.2$, λ_3, $\mu_1 = 40$, $\sigma = 10$, $\eta_1 = 4$, $\eta_2 = 3$, $\varphi = 12$, $\psi = 12$, $p = 0.5$, $q = 0.5$, $b = 0.05$, $\delta = 3$, $L = 3$.

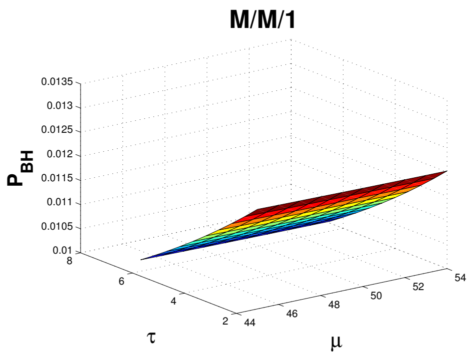

Figure 11: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

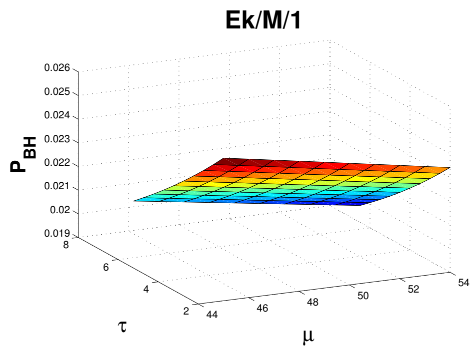

Figure 12: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

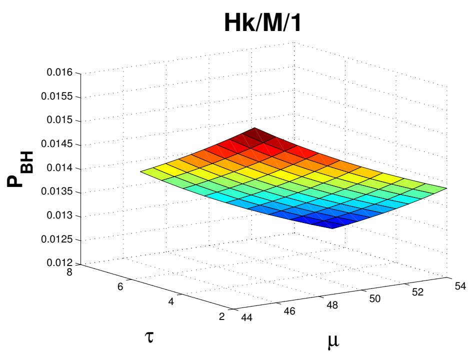

Figure 13: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

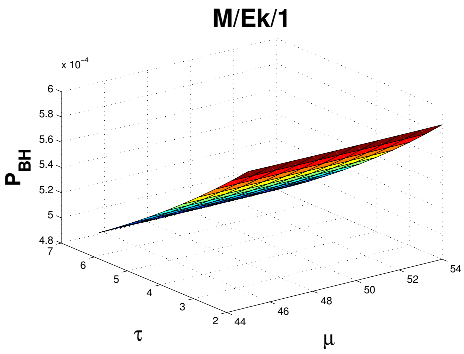

Figure 14: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

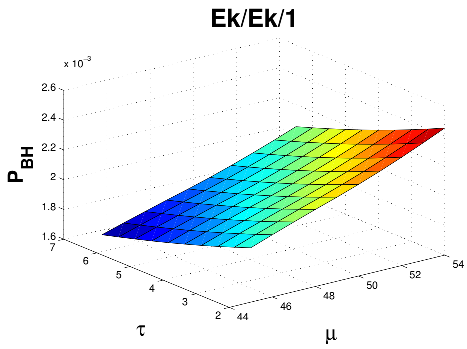

Figure 15: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

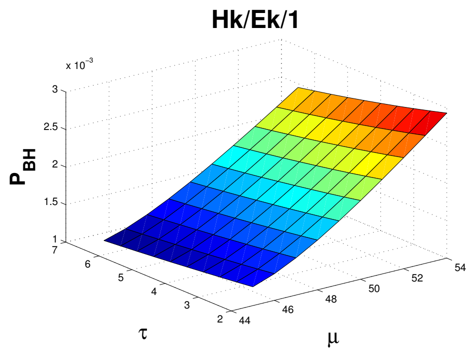

Figure 16: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

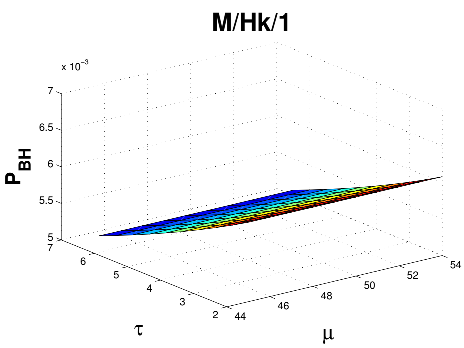

Figure 17: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

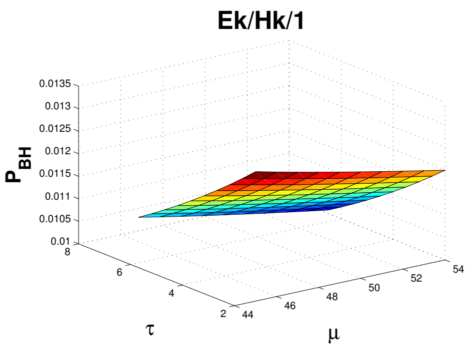

Figure 18: *(Service (normal) (μ) and Breakdown (τ) rates) vs. P_{BH}*

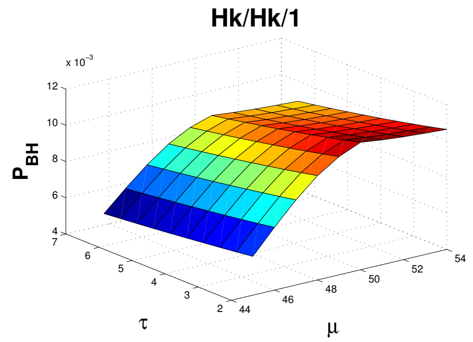

Figure 19: *(Service (normal) (μ) and Breakdown (τ) rates) vs.* P_{BH}

The probability that the server is busy with the normal service (P_{BH}) reduces for different arrival and service patterns when both the normal service rate (μ) and the breakdown rate (τ) are raised, as shown in Figures $11 - 19$. Rather than increasing at a hyper-exponential arrival, the Erlang arrival grows quickly. Similar to hyper-exponential services, the increment rate decreases for Erlang services.

9. Conclusion

In this paper, we have developed the queueing model with non preemptive priority queue, optional service, negative arrival, single vacation, emergency vacation, differentiate breakdown, repair, closedown , setup and balking. A queue with two categories of consumers with positive arrivals following $MMAP$, while negative arrival follows MAP and service times follows to be phase type distribution. By using matrix analytic method, we found the stationary probability since the queueing systems are Quasi Birth-Death process. The stability condition for the $MMAP/PH/1$ queuing system has analyzed and some performance measures for queueing system was selected and implemented in numerical illustrations by using three dimensional graphs. For further work, the model can be investigate with batch arrival and batch service which follows Markovian arrival process and various service rates with N-policy.

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