A STUDY ON STATISTICAL PROPERTIES OF A NEW CLASS OF Q-EXPONENTIAL-WEIBULL DISTRIBUTION WITH APPLICATION TO REAL-LIFE FAILURE TIME DATA

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Abstract

This article introduces a new four-parameter probability distribution called the q-Exponential-Weibull distribution based on the q-Exponential-G family of distribution. The proposed new distribution has to decrease and increase failure rates which are more common in reliability scenarios and can be used instead of Weibull and the exponential distribution. It also includes some sub-models like q-Exponential-Exponential, q-Exponential-Rayleigh, Exponential-Weibull, Exponential-Exponential and Exponential-Rayleigh lifetime distributions. Various Mathematical and statistical Properties are investigated, which include Limiting behavior, Moments and Moment Generating functions, Quantile function and Order Statistics. The Maximum Likelihood estimator is used for estimating the model parameters. This new distribution is compared with other lifetime distributions using different kinds of real-life failure time data.

Keywords: q-Exponential-Weibull, Quantile, Reliability Measures, Maximum Likelihood Estimation, failure time data.

1. Introduction

Lifetime distributions are very useful statistical tool for analyzing the various characteristics of lifetime data. The developments and applications of lifetime distribution are essential in numerous fields. Hence, the major aspects of generating new families of probability distributions are they offer greater flexibility and a better fit at the expense of one or more extra parameters.

The non-extensive statistical mechanism plays a vital growth in the past few years. This new formulation is not based on the usual statistical mechanism, provided that will give a better description of the complex system developed by [26]. In the recent decade's probability distribution, which emerge from the non-extensive statistical mechanism called q-type distribution attracted several statisticians to develop new distribution [11], [20] and [23]. Studying this type of distribution is quite interesting because of its complex system and power-law behavior. The application of this type of distribution has been found in many research areas like Physics, Biology, Mathematics, Chemistry, Economics, Medicine etc.

The q-Exponential distribution emerged from maximizing the non-extensive statistical mechanism under appropriate constraints [26]. This theory is a generalization of the classical Boltzmann-Gibbs (BG) statistical mechanism. So, the q-Exponential distribution has found varieties of applications in the research field including in the field of complex systems. This article introduces a new four-parameter probability distribution called the q-Exponential Weibull distribution.

The well-known q-distributions are q-Exponential distributions discussed by Malacarne et al. [15], q-Gamma distribution due to Duarte et al. [7], q-Weibull distribution due to Picoli et al. [21], q-Gaussian distribution due to Adrian et al. [1]. Picoli J.R. et al. [22] discussed q-distribution in complex systems. Ana Claudia souza [3] studied the reliability data analysis of systems in the wear-out phase using q-Exponential likelihood. Fode Zhang et al. [9] discovered the information geometry on the curved q-Exponential family with application to survival analysis. Shalizi [25] express the Maximum Likelihood Estimation for q-Exponential distribution. The geometry of q-exponential distribution with dependent competing risk and accelerated life testing is given by Fode Zhang et al. [10]. Keith Briggs [12] demonstrates the modelling train delay with the q-Exponential distribution. The reliability of stress strength and its estimation of exponentiated q-Exponential distribution is given by Mohammed et al [18]. Modelling censored survival data with q-Exponential distribution discussed by Sundaram [19].

In reliability and survival analysis most commonly, used distributions are Exponential and Weibull distributions [16], q-Exponential is an alternative one. The q-Exponential distribution is a higher version of an Exponential distribution. It has two parameters: q and α, where q is the shape parameter (entropy index/control parameter) and α is the scale parameter. As compared to the Exponential distribution that has just one parameter (α), the q-Exponential distribution has more flexibility regarding the decay of the pdf [3]. Indeed, the Exponential probability distribution is a special case of the q-Exponential when $q \to 1$. Another feature of this distribution is that it does not have the limitation of a constant hazard rate as the Exponential one, thus allowing the modelling of either system improvement ($1 < q < 2$) or degradation ($q < 1$). The pdf of q-Exponential distribution [26], is given by

$$f_q(x) = (2-q)\,\alpha\,[1-(1-q)\,\alpha x]^{1/(1-q)} \qquad \text{for } x, \alpha > 0, q<2 \tag{1}$$

This can also be rewritten as

$$f_q(x) = (2-q)\,\alpha\, e_q(-\alpha x)$$

$$\text{Where} \quad e_q(x) = [1+(1-q)\,x]^{\frac{1}{1-q}}$$

Which is the q-exponential if $q \neq 1$. When $q = 1$, $e_q(x)$ is just $\exp(x)$.

The cumulative distribution function (cdf) of the q-Exponential-generated family is given by.

$$F(x) = \int_0^{G(x)/(1-G(x))} (2-q)\alpha[1-(1-q)\alpha x]^{\frac{1}{1-q}} \tag{2}$$

The simplified form of (2) is.

$$F(X) = 1-\left[1-(1-q)\,\alpha\,\frac{G(X)}{1-G(X)}\right]^{\frac{2-q}{1-q}} \qquad x, \alpha>0, q<2 \tag{3}$$

where q is the shape parameter (entropy index) and α is the scale parameter. The corresponding probability density function is given by

$$f(x) = (2-q)\,\alpha\,\frac{g(x)}{[1-G(x)]^{\wedge}2}\left[1-(1-q)\,\alpha\,\frac{G(X)}{1-G(X)}\right]^{\frac{1}{1-q}} \tag{4}$$

where $X>0$, $\alpha>0$, $q<2$.

The rest of the paper are as follows. In Section 2, The new class q-Exponential-Weibull distribution is introduced and presented its particular cases. The mathematical and statistical properties are discussed in section 3 and in section 4, the maximum likelihood estimation method and their asymptotic behaviors have been discussed. Simulation techniques has been explained in section 5. Real life failure time data has been applied and the results are presented in section 6. In section 7, we have discussed the conclusion of the new class of q-Exponential Weibull distribution.

2. The q-Exponential-Weibull Distribution

The q-Exponential distribution combined with Weibull distribution gives the q-Exponential Weibull distribution. Here the q-exponential is the generator distribution, and the two-parameter Weibull distribution (Waloddi Weibull, 1951) is a parent distribution whose pdf and cdf are given by

$$g(x, \lambda, \gamma) = \lambda\gamma x^{\gamma-1} e^{-\lambda x^{\gamma}} \qquad x, \gamma, \lambda > 0 \tag{5}$$

$$G(x, \lambda, \gamma) = 1 - e^{-\lambda x^{\gamma}} \tag{6}$$

using (6) in (3), we get the new cdf of q-Exponential-Weibull distribution. The simplified form of q-Exponential-Weibull distribution is

$$F(x, \Omega) = 1-[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}} - 1)]^{\frac{2-q}{1-q}} \tag{7}$$

where $\Omega = \{q, \alpha, \lambda, \gamma\}$ be the set of parameters, here q and γ are the shape parameters and α, λ are the scale parameters. The equation (7) is called the cdf of q-Exponential-Weibull distribution.
Substituting (5) and (6) in (4), we get the new pdf. The new pdf is,

$$f(x, \Omega) = (2-q)\,\alpha\lambda\gamma\, e^{\lambda x^{\gamma}} x^{\gamma-1}[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}} - 1)]^{\frac{1}{1-q}} \tag{8}$$

Rewriting the above equation (8), we get

$$f(x, \Omega) = (2-q)\,\alpha\lambda\gamma\, e^{\lambda x^{\gamma}} x^{\gamma-1}\, e_q[-\alpha\,(e^{\lambda x^{\gamma}} - 1)] \tag{9}$$

The equation (8) and (9) are called the pdf of q-Exponential-Weibull distribution (q-EW). The particular case of our new q-Exponential-Weibull distribution is presented in Table 1.

Table 1: *The particular case of q-Exponential-Weibull distribution*

Model	q	α	λ	γ	Cdf	References
q-Exponential-Exponential	q	α	λ	1	1- [1− (1−q) α ($e^{\lambda x}$− 1)] ^$\frac{2-q}{1-q}$	New
q- Exponential-Rayleigh	q	α	$\frac{\lambda}{2}$	2	1- [1− (1−q) α ($e^{\frac{\lambda x^2}{2}}$− 1)] ^ $\frac{2-q}{1-q}$	New
Exponential – Weibull	1	α	λ	γ	$1\text{-}e^{-\alpha(e^{\lambda x^{\gamma}}-1)}$	New
Exponential – Exponential	1	α	λ	1	$1\text{-}e^{-\alpha(e^{\lambda x}-1)}$	Elgarhy et al. (2017)
Exponential- Rayleigh	1	α	$\frac{\lambda}{2}$	2	$1\text{-}e^{-\alpha(e^{\frac{\lambda}{2}x^2}-1)}$	Kawsar Fatima and, S.P Ahmad (2017)

2.1 Reliability Measures: Survival function: (survivor function)

The survivor function for the new distribution S(x) is defined to be the probability that the survival time is greater than or equal to t, and it is given by

$$S(x) = P\,(X \geq t) = 1 - F(x)$$

$$S\,(x, \Omega) = [1 - (1-q)\,\alpha\,(e^{\lambda x^{\gamma}} - 1)]^{\frac{2-q}{1-q}} \tag{10}$$

2.2 Hazard function:

The hazard function is used to express the risk or hazard of an event such as death occurring at some time t, and it is given by

$$h\,(x) = \frac{f(x)}{s(x)}$$

Substituting (8) and (10) we get the hazard function of q-Exponential-Weibull distribution. which is defined below.

$$h\,(x, \Omega) = (2-q)\,\alpha\lambda\gamma\, e^{\lambda x^{\gamma}} x^{\gamma-1}[1 - (1-q)\,\alpha\,(e^{\lambda x^{\gamma}} - 1)]^{\frac{-(1+q)}{(1-q)}} \tag{11}$$

2.3 Reverse hazard rate function

The reverse hazard function of q-Exponential-Weibull distribution is defined by

$$r\,(x) = \frac{f(x)}{F(x)}$$

$$r\,(x, \Omega) = \frac{(2-q)\,\alpha\lambda\gamma\, e^{\lambda x^{\gamma}} x^{\gamma-1}[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}} - 1)]^{\frac{1}{1-q}}}{1-[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}} - 1)]^{\frac{2-q}{1-q}}} \tag{12}$$

2.4 Cumulative hazard function

Cumulative hazard function is presented below,

$$H\,(x, \Omega) = -\log\,(1-F(x))$$

$$H\,(x, \Omega) = -\ln\,(s\,(x, \Omega)) = -\ln\,[[1 - (1-q)\,\alpha\,(e^{\lambda x^{\gamma}} - 1)]^{\frac{2-q}{1-q}}] \tag{13}$$

The above equation is known as cumulative hazard function of q-Exponential-Weibull distribution.

2.5 Graphical Study of q-Exponential Weibull distribution under various functions:

In this section, we studied the structure of the cdf, pdf, S(x) and h(x) of q-Exponential-Weibull distribution using different values of the parameters. The illustrative figures are presented below.

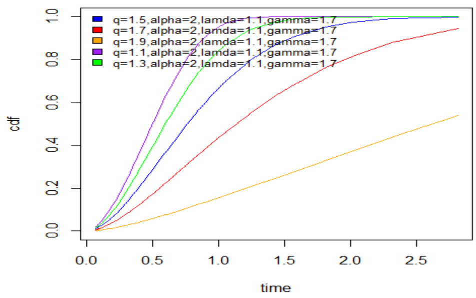

Figure 2.a: *The graph of the cdf of the q-EW distribution with different values of the parameter*

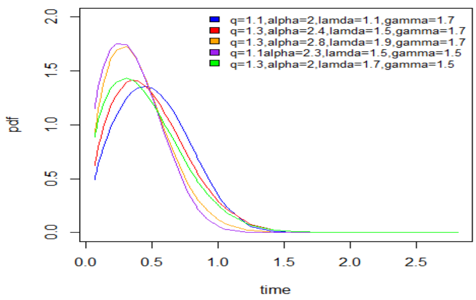

Figure 2.b1: *Graph of the pdf of the q-EW distribution when all the parameters are changed*

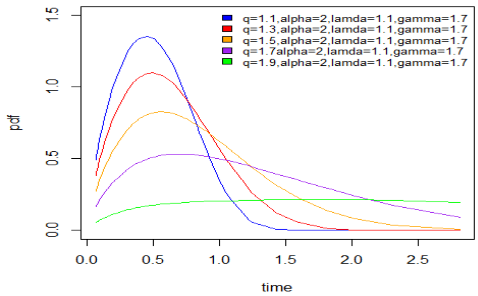

Figure 2.b2:*The graph of the pdf of the q-EW distribution when changing first shape parameter (q) values and other parameters are fixed*

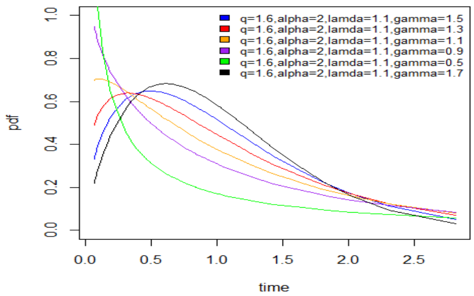

Figure 2.b3: *The graph of the pdf the of q-EW distribution when changing second shape parameter (γ)values and other parameters are fixed*

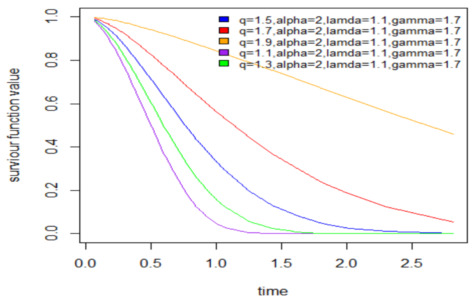

Figure 2.c: *The graph of the survival function of the q-EW distribution with different parameter values*

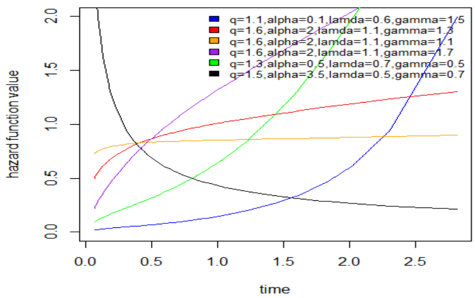

Figure 2.d: *The graph of the Hazard rate of the q-EW distribution for with different parameter values*

Figure 2.a cumulative density plot demonstrates the validity of the distribution as a probability distribution. The probability density function graphs (2.b1,2. b2 and 2.b3) shows that it is skewed and more adaptable for various parameter values. The graph of the hazard function (2.d) demonstrates that it can take on various shapes, including constant, increasing, and decreasing. As a result, fitting data sets of different forms may be done and which was quite well using the q-Exponential Weibull distribution.

3.Properties

In this section we study some mathematical and statistical properties of q-Exponential-Weibull distribution.

3.1 Mixture Representation:

Several properties of the new distribution can be derived using the concept of exponentiated distribution. The mixture representation of q-exponential-Weibull distribution is derived in the following sections.
Using the generalized binomial theorem, where β>0 is real non integer and |z|<1,

$$(1-z)^{\beta-1} = \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k}(z)^k$$

$$f(x) = (2-q)\,\alpha \frac{g(x)}{[1-G(x)]^{2}}\left[1-(1-q)\,\alpha \frac{G(X)}{1-G(X)}\right]^{\frac{2-q}{1-q}-1} \qquad \text{since } \beta = \frac{2-q}{1-q}$$

$$= (2-q)\,\alpha \frac{g(x)}{[1-G(x)]^{2}}\sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k}(1-q)^k \alpha^k \left[\frac{G(X)}{1-G(X)}\right]^k$$

$$= (2-q)\,\alpha \frac{g(x)}{[1-G(x)]^{2}}\sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k}(1-q)^k \alpha^k \frac{[G(x)]^k}{[1-G(x)]^k}$$

$$= \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k}(1-q)^k \alpha^{k+1} (2-q) \frac{g(x)\,[G(x)]^k}{[1-G(x)]^{k+2}}$$

Generalized binomial theorem

$$[1-G(x)]^{-(k+2)} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j+2)}{j!\,\Gamma(k+2)}[G(x)]^j$$

$$= \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k}(1-q)^k \alpha^{k+1} (2-q) \sum_{j=0}^{\infty}\frac{\Gamma(k+j+2)}{j!\,\Gamma(k+2)} g(x)\,[G(x)]^{k+j+1-1}$$

$$= \sum_{k=0}^{\infty}(-1)^k \binom{\beta-1}{k}(1-q)^k \alpha^{k+1} (2-q) \sum_{j=0}^{\infty}\frac{\Gamma(k+j+2)}{j!(k+j+1)\,\Gamma(k+2)}(k+j+1)\, g(x)\,[G(x)]^{k+j+1-1}$$

$$f(x,\Omega) = \sum_{j,k=0}^{\infty} W_{j,k}\, h_{(k+j+1)}(x,\Omega) \qquad (14)$$

$$\text{where } W_{j,k} = (-1)^k \binom{\beta-1}{k}(1-q)^k \alpha^{k+1} (2-q)\frac{\Gamma(k+j+2)}{j!(k+j+1)\,\Gamma(k+2)}$$

$$h_a(x,\Omega) = a\, g(x,\Omega)\,[G(x,\Omega)]^{a-1}$$

The q-Exponential Weibull density can be expressed as an infinite linear combination of exponentiated – G density function.

Then, $$[F(x)]^R = \left[1-\left[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)\right]^{\frac{2-q}{1-q}}\right]^R$$

Using the generalized binomial theorem, where β>0 is real non integer and |z|<1,

$$(1-z)^{\beta-1} = \sum_{l=0}^{\infty}(-1)^l \binom{\beta-1}{l}(z)^l$$

$$[F(x)]^R = \sum_{l=0}^{\infty}(-1)^l \binom{R}{l}\left[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)\right]^{\frac{(2-q)l}{(1-q)}}$$

Which can also be written as,

$$[F(x)]^R = \sum_{l,m=0}^{\infty}(-1)^{l+m}\binom{R}{l}\binom{\vartheta}{m}(1\text{-q})^{m}\,\alpha^{m}\,(e^{\lambda x^{\gamma}}-1)^m \text{ where } \vartheta = \frac{(2-q)l}{(1-q)}$$

$$[F(x)]^R = \sum_{l,m=0}^{\infty}(-1)^{l+m}(1\text{-q})^{m}\,\alpha^{m}\binom{R}{l}\binom{\vartheta}{m}\left[\frac{1-e^{-\lambda x^{\gamma}}}{1-(1-e^{-\lambda x^{\gamma}})}\right]^m$$

Using Generalized binomial theorem, the above equation can be written as,

$$[F(x)]^R = \sum_{l,m,n=0}^{\infty}(-1)^{l+m}(1\text{-q})^{m}\,\alpha^{m}\binom{R}{l}\binom{\vartheta}{m}\binom{m+n-1}{n}\left[1-e^{-\lambda x^{\gamma}}\right]^{m+n}$$

Simply further we get,

$$[F(x)]^R = \sum_{l,m,n,r=0}^{\infty}(-1)^{l+m+r}(1\text{-q})^{m}\,\alpha^{m}\binom{R}{l}\binom{\vartheta}{m}\binom{m+n-1}{n}\binom{m+n}{r}(e^{-\lambda x^{\gamma}})^r$$

$$[F(x)]^R = \sum_{l,m,n,r=0}^{\infty} W_{l,m,n,r}(e^{-\lambda x^{\gamma}})^r \quad (15)$$

Where $W_{l,m,n,r} = (-1)^{l+m+r}(1\text{-q})^{m}\,\alpha^{m}\binom{R}{l}\binom{\vartheta}{m}\binom{m+n-1}{n}\binom{m+n}{r}$

3.2 Limiting Behavior:

Lemma 1: The limit of the cdf of the q-Exponential-Weibull, F(x) as X approaches infinity, $x\to\infty$ is equal to one and limit of the cdf of the q-Exponential-Weibull, F(x) as X tends to zero, $x \to 0$ is equal to zero.

$$\lim_{x\to\infty} F(x) = 1$$

Proof: The cdf of the q-Exponential Weibull F(x) as X approaches infinity ($x \to \infty$), from 7 we get Using equation (9)

$$\lim_{x\to\infty} F(x,\Omega) = \lim_{x\to\infty} 1\text{-}[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)]^{\frac{2-q}{1-q}}$$

$$= 1\text{-}[1-(1-q)\,\alpha\,(e^{\lambda(\infty)^{\gamma}}-1)]^{\frac{2-q}{1-q}}$$

$$= [1-0] = 1$$

Hence, the lemma is proved under limiting property.

$$\lim_{x\to 0} F(x) = 0$$

$$\lim_{x\to 0} F(x,\Omega) = \lim_{x\to 0} 1\text{-}[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)]^{\frac{2-q}{1-q}}$$

$$= 1\text{-}[1-(1-q)\,\alpha\,(e^{\lambda(0)^{\gamma}}-1)]^{2-q/(1-q)}$$

$$= 1\text{-}[1\text{-}0] = 0$$

Lemma 2: In probability theory, of a continuous random variable has the following property

(i) $f(x) \geq 0$; where $-\infty<x<\infty$
(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Using above definition, the validity of the model f(x) is checked. In our survival model the range of x is $0<x<\infty$.

$$\int_0^{\infty} f(x)dx = 1$$

$$= \int_0^{\infty}(2-q)\,\alpha\lambda\gamma\, e^{\lambda x^{\gamma}} x^{\gamma-1}[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)]^{\frac{1}{1-q}}dx$$

$$= (2-q)\,\alpha\lambda\gamma \int_0^{\infty} e^{\lambda x^{\gamma}} x^{\gamma-1}[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)]^{\frac{1}{1-q}}dx$$

Now y = $[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)]$

$$\frac{dy}{dx} = 0 - (1-q)\alpha\lambda\gamma e^{\lambda x^{\gamma}} x^{\gamma-1}$$

$$\frac{dy}{(1-q)\alpha\lambda\gamma e^{\lambda x^{\gamma}} x^{\gamma-1}} = dx$$

$$= (2-q)\,\alpha\lambda\gamma \int_0^{\infty} \frac{e^{\lambda x^{\gamma}} x^{\gamma-1}}{[1-(1-q)\,\alpha\,(e^{\lambda x^{\gamma}}-1)]^{\frac{1}{1-q}}} * \frac{dy}{(1-q)\alpha\lambda\gamma e^{\lambda x^{\gamma}} x^{\gamma-1}}$$

$$= -\frac{2-q}{1-q}\int_0^{\infty} y^{1/(1-q)}\, dy$$

$$\int_0^{\infty} f(x)dx = -[y^{(2-q)/(1-q)}]_0^{\infty}$$

$$= -\{\,[[1-(1-q)\alpha\left(e^{\lambda x^{\gamma}}-1\right)]^{(2-q)/(1-q)}]_{x=\infty} - [[1-(1-q)\alpha\left(e^{\lambda x^{\gamma}}-1\right)]^{(2-q)/(1-q)}]_{x=0}\,\}$$

$$\int_0^{\infty} f(x)dx = -[\,0-1\,] = 1$$

Hence q-Exponential-Weibull distribution is a valid pdf.

3.3 Quantile Function:

The quantile function of X= Q(u) = $F^{-1}(u)$ can be obtained by inverting equation (7) as follows,

$$Q(u) = [\,\frac{1}{\lambda}\ln\left[1+\frac{1}{(1-q)\alpha}\left[1-(1-u)^{\frac{1-q}{2-q}}\right]\right]]^{\frac{1}{\gamma}} \qquad (16)$$

Simulation of q-Exponential-Weibull random variable is straightforward. Let u be the uniform variable on the interval [0,1], then the random variable X = $F^{-1}(u)$ follows q-Exponential-Weibull distribution given in equation (8) with the parameters (q, α, λ, γ). By using equation (20), we can obtain the first, second and third quantiles by replacing u as 0.25, 0.5 and 0.75, respectively.

3.4 Moments:

This section provides the moment and moment generating function of q-Exponential-Weibull distribution. The moments of the functions are quantitative measures related to the shape of the function. The first four moments, skewness and kurtosis of q-Exponential-Weibull distribution can be obtained as

$$\mu_r' = E[\,x^r] = \int_{-\infty}^{\infty} x^r f(x,\Omega)dx$$

Using equation (13) we have,

$$\mu_r' = \int_{-\infty}^{\infty} x^r \sum_{j,k=0}^{\infty} W_{j,k}\, h_{(k+j+1)}(x,\Omega)\, dx$$

$$= \sum_{j,k=0}^{\infty} W_{j,k} \int_{-\infty}^{\infty} x^r h_{(k+j+1)}(x,\Omega)dx$$

Where $I_{j,k}(x,\Omega) = \int_{-\infty}^{\infty} x^r h_{(k+j+1)}(x,\Omega)dx$

$$\mu_r' = E(x^r) = \sum_{j,k=0}^{\infty} W_{j,k}\ I_{j,k}(x, \Omega) \tag{17}$$

The mean, variance, skewness and kurtosis can be obtained from equation (14).

when r =1 gives mean = E(x)

variance = $E(x^2) - [E(x)]^2$

skewness = $\frac{\mu_3(\theta) - 3\,\mu_3(\theta)\mu_2(\theta) + 2\,\mu_1^3(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^{3/2}}$

kurtosis = $\frac{\mu_4(\theta) - 4\,\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^2}$

Generally, the moment generating function of q-Exponential-Weibull distribution is obtained through the following relation

$$M_x(t, \Omega) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r) = \sum_{r,j,k=0}^{\infty} \frac{t^r}{r!} W_{j,k}\ I_{j,k}(x, \Omega) \tag{18}$$

The Characteristic function of q-Exponential-Weibull distribution is obtained through the following relation

$$\phi_x(t, \Omega) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r f(x)dx = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r) = \sum_{r,j,k=0}^{\infty} \frac{(it)^r}{r!} W_{j,k}\ I_{j,k}(x, \Omega) \tag{19}$$

The cumulant generating function of q-Exponential-Weibull distribution is given by

$$k_x(t, \Omega) = log[[\sum_{r=0}^{\infty} \frac{(t)^r}{r!} \int_0^{\infty} x^r f(x)dx] = \log [\sum_{r=0}^{\infty} \frac{(t)^r}{r!} E(x^r)]$$

$$= log[\sum_{r,j,k=0}^{\infty} \frac{(t)^r}{r!} W_{j,k}\, I_{j,k}(x, \Omega)] \tag{20}$$

3.5 Order Statistics:

Let $X_{1:n} < X_{2:n} < X_{3:n} < \ldots < X_{n:n}$ be the order statistics of a random sample of size n following q-Exponential-Weibull distribution with the parameter α, q, λ, γ then the probability density function of p^{th} order statistic can be written as,

$$f_{(x_p)}[x_{(p)}] = \frac{f(x_p)}{B(p, n-p+1)} \sum_{v=0}^{n-p} (-1)^v \binom{n-p}{v} [F(x_p)]^{v+p-1} \tag{21}$$

Substituting (13) and (14) in (18) and replacing R= $(v + p - 1)$ we get

$$f_{(x_p)}[x_{(p)}] = \frac{\sum_{j,k=0}^{\infty} W_{j,k} h_{(k+j+1)}(X, \Omega)}{B(p, n-p+1)} \sum_{v=0}^{n-p} (-1)^v \binom{n-p}{v} \sum_{l,m,n,r=0}^{\infty} W_{l,m,n,r} (e^{-\lambda x^{\gamma}})^r$$

$$f_{(x_p)}[x_{(p)}] = \frac{1}{B(p, n-p+1)} \sum_{j,k=0}^{\infty} \sum_{v=0}^{n-p} \sum_{l,m,n,r=0}^{\infty} \omega^* \, h_{(k+j+1)}(X, \Omega)(e^{-\lambda x^{\gamma}})^r \tag{22}$$

Where $\omega^* = (-1)^v \binom{n-p}{v} W_{j,k} W_{l,m,n,r}$

4. Method of Estimation

In this section, the maximum likelihood estimates (MLE) of the unknown parameters for the q-Exponential-Weibull distribution are determined based on complete samples. Let $x_1, x_2 \ldots x_n$ be a random sample from q-Exponential-Weibull distribution with unknown parameter vector Ω = {q, α, λ, γ}. The likelihood function for the proposed distribution $\mathcal{L}$ is given by

$$\mathcal{L}(x, \Omega) = (2-q)^n \alpha^n \lambda^n \gamma^n \prod_{i=1}^{n} e^{\lambda x_i^{\gamma}} x_i^{\gamma-1} [1 - (1-q)\, \alpha\, (e^{\lambda x_i^{\gamma}} - 1)]^{\frac{1}{1-q}}$$

Then the log likelihood of the equation is

$$\ell\,(\Omega) = \log \mathcal{L}(t, \Omega) = n \log (2\text{-}q) + n \log \alpha + n \log \lambda + n \log \gamma + \lambda \sum_{i=1}^{n} x_i^{\gamma} + (\gamma -1) \sum_{i=1}^{n} \log x_i + \left(\frac{1}{1-q}\right) \sum_{i=1}^{n} \log [1 - (1-q)\, \alpha\, (e^{\lambda x_i^{\gamma}} - 1)] \tag{23}$$

The maximum likelihood estimates of the parameters (q, α, λ, γ) are found by taking a partial derivative of $\ell\,(\Omega)$ with respect to q, α, λ, γ, equating the derivatives to zero, and evaluating them at $\hat{q}$, $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\gamma}$.

$$\frac{\partial \ell\,(\Omega)}{\partial q} = \frac{n}{2-q} + \sum_{i=1}^{n} \frac{1}{1-q} \left[\frac{\alpha (e^{\lambda x_i^{\gamma}} - 1)}{1 - (1-q)\, \alpha\, (e^{\lambda x_i^{\gamma}} - 1)}\right] - \sum_{i=1}^{n} \log [1 - (1-q) \alpha \left(e^{\lambda x_i^{\gamma}} - 1\right) \tag{24}$$

$$\frac{\partial \ell\,(\Omega)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{1}{1-q} \left[\frac{(1-q)(e^{\lambda x_i^{\gamma}} - 1)}{1 - (1-q)\, \alpha\, (e^{\lambda x_i^{\gamma}} - 1)}\right] \tag{25}$$

$$\frac{\partial \ell\,(\Omega)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i^{\gamma} - \sum_{i=1}^{n} \frac{1}{1-q} \left[\frac{\alpha (1-q)\, x_i^{\gamma} e^{\lambda x_i^{\gamma}}}{1 - (1-q)\, \alpha\, (e^{\lambda x_i^{\gamma}} - 1)}\right] \tag{26}$$

$$\frac{\partial \ell\,(\Omega)}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \lambda\, x^{\gamma} \log x_i - \sum_{i=1}^{n} \frac{1}{1-q} \left[\frac{\alpha \lambda (1-q)\, x_i^{\gamma} e^{\lambda x_i^{\gamma}} * \log x_i}{1 - (1-q)\, \alpha\, (e^{\lambda x_i^{\gamma}} - 1)}\right] \tag{27}$$

For solving these non-linear equation's, we can use any iteration method such as Newton-Raphson technique.

5. Generating random samples from q-Exponential Weibull distribution

The Inverse CDF method is used for generating random numbers from a particular distribution. In this method, random numbers from a particular distribution are generated by solving the equation obtained on equating CDF of a distribution to a number u. The number u is itself being generated from u$\sim U(0,1)$. In this section we made an attempt to q-Exponential-Weibull distribution to generate the random number using equation 16 at a fixed values of parameters $(q, \alpha, \lambda, \gamma)$.

$$X = F^{-1}(u)$$

$$X = \left[\frac{1}{\lambda} \ln \left[1 + \frac{1}{(1-q)\alpha} \left[1 - (1-u)^{\frac{1-q}{2-q}}\right]\right]\right]^{\frac{1}{\gamma}} \tag{28}$$

For uniform over (0,1) then x$\sim q - EW(1.2,2,1.1,1.7)$ can be generated random sample of size 50 are presented below.

0.2471, 0.8991, 0.9387, 1.5307, 0.6133, 0.8110, 0.8077, 0.7611, 0.8320, 1.5590,
1.0539, 1.7640, 1.4155, 0.8662, 1.2408, 1.9081, 1.0625, 0.6137, 0.5943, 0.7125,
0.9593, 0.3809, 0.1623, 0.2987, 0.9664, 1.31036, 0.6269, 1.3524, 0.6302, 1.0810,
2.1260, 1.4057, 1.1020, 0.6074, 1.7022, 1.1539, 1.1613, 0.5775, 0.1133, 0.9533,
1.1283, 1.2516, 1.6930, 0.9185, 1.3880, 0.8035, 0.9471, 0.1955, 2.4077, 0.7141

Here we have used one of the goodness of fit tests "Kolmogorov-Smirnov (KS)" test for the above-generated data for testing the q-Exponential-Weibull distribution. The null hypothesis is that the samples are drawn from the q-Exponential-Weibull distribution against the alternative hypothesis is that the samples are not drawn from the q-Exponential-Weibull distribution. The test statistic value of the KS test for the generated samples is (D value) 0.097 at 5% level of significance with the p-value of 0.76. Since the p-value is greater than 0.05, the null hypothesis is accepted. Hence, the samples are drawn from the proposed distribution. Therefore, the q-Exponential-Weibull distribution has satisfied the goodness of fit test.

6. Application to real life data

In this section, we have used different kinds of real-life failure time data to show the suitability of the q-Exponential Weibull distribution, also we have compared to some other related distributions namely Exponentiated Weibull-Exponential (EWE) and Generalized Exponential-Weibull (GEW) distributions. The pdf of the respective distributions is represented below:

- The Exponentiated Weibull-Exponential (EWE) distribution introduced by Elgarhy et.al [8], with pdf
$$f(x) = q\alpha\gamma\lambda[e^{\lambda x}-1]^{\gamma-1}\exp\left[-\left\{\alpha\left[e^{\lambda x}-1\right]^{\gamma}-\lambda x\right\}\right]\left[1-exp\left(-\alpha\left[e^{\lambda x}-1\right]^{\gamma}\right)\right]^{q-1} \quad x, q, \alpha, \lambda, \gamma > 0 \qquad (29)$$
- The Generalized Exponential-Weibull (GEW) distribution introduced by Dikko and Faisal [6], with the pdf
$$f(x) = q(\alpha+\gamma\lambda x^{\lambda-1})e^{-(\alpha x+\gamma x^{\lambda})}[1-e^{-(\alpha x+\gamma x^{\lambda})}]^{q-1} \quad x, \alpha, \gamma, \lambda, q > 0 \qquad (30)$$

In order to assess the flexibility of the proposed distribution, we have considered some model selection criteria like, -2loglikelihood and AIC (Akaike Information Criterion) are used and analyses performed with the aid of R software.

Dataset1: The first data set is the failure times of 84 aircraft windshields. This failure time data set is available in Murthy et al's book "Weibull Models" (2004, page 297). A large aircraft's windscreen is a sophisticated piece of equipment made up of multiple layers of material, including a very touchy outer skin with a heated layer just behind it, all laminated under high temperature and pressure. These failures do not cause aircraft damage, but they do require the repair of the windscreen. The failure times of 84 aircraft windshields are given below:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

Table 2: *Estimates of fitted distribution for aircraft windshield failure data*

Model	Estimated Parameters				Model Selection	
	$\hat{q}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	-2LL	AIC
q-EW	1.729287	4.629657	0.006168	1.539852	**251**	**259**
EWE	15.46262	1.38606	4.08592	0.07846	253	261
GEW	0.04796	0.31873	0.43050	0.68102	419	427

Dataset 2: The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal. The data is presented below:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 3: *Estimates of fitted distribution for guinea pig failure data*

Model	Estimated Parameters				Model Selection	
	$\hat{q}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	-2LL	AIC
q-EW	1.54493	33.49094	0.01988	2.98777	**184**	**192**
EWE	1.08287	83.46078	0.01628	2.99604	187	195
GEW	0.66675	0.07983	0.27917	0.45848	242	250

We observed from the above tables 2 and 3, the -2LL and AIC values of the q-Exponential-Weibull distribution have the smallest among the other distributions. Therefore, the q-Exponential-Weibull distribution has performed well than the other distributions. So, we conclude from this section, the q-Exponential-Weibull distribution has achieved the goal of the suitability of the different kinds of real-life failure time data.

8. Conclusion

In this research article, we have introduced a new class of four-parameter distribution referred to as "q-Exponential-Weibull distribution" by taking the Weibull distribution as the base distribution and the q-Exponential distribution as the generator distribution by using the generator technique. The q-Exponential-Weibull density can be expressed as a linear combination of exponentiated - G densities. For checking the model properties, we have derived survival, hazard, cumulative hazard and reverse hazard functions from q-Exponential-Weibull distribution, and also studied graphically. In the graphical study of the q-Exponential-Weibull distribution under various functions with different parameter values, the proposed distribution has achieved the properties of the density function. The mathematical and statistical properties are applied to q-Exponential-Weibull distribution. The q-Exponential-Weibull distribution has satisfied the above said properties. The parameters of the q-Exponential-Weibull distributions are estimated using the maximum likelihood estimation method. The random samples have been generated from the q-Exponential-Weibull distribution and the goodness of fit test has been verified using Kolmogorov-Smirnov (KS) test, also we have studied the application of real-time failure time data to q-Exponential-Weibull distribution. The proposed distribution performed well than the other distribution based on the model selection criteria. Based on the above-said results, the q-Exponential-Weibull distribution is more adaptable and more flexible to fit the real-life failure time data. We hope that the proposed distribution would draw more widespread applications in different areas of research such as reliability analysis, medicine engineering and economics etc.

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