DISTRIBUTIONAL PROPERTIES OF ORDER STATISTICS AND RECORD STATISTICS FROM ERLANG-TRUNCATED EXPONENTIAL FAMILY OF DISTRIBUTION AND ITS CHARACTERIZATIONS

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Abstract

Erlang Truncated Exponential Distributions are characterized by distributional properties of order statistics. These characterizations include known results for ordinary order statistics based on two non-adjacent order statistics coming from two independent Erlang truncated exponential distributions. Using this method and compared with an efficient recent method given by [20], three examples of real lifetime data-sets are analyzed by that deals with non-random samples. Such type of examples predicts the accumulative new cases per million foe infection of the new COVID-19. Corollaries for Pareto and power function distributions are also derived.

Keywords: Order statistics; characterization of distributions; reliability characteristics; Erlang truncated exponential; random translation

1. Introduction

Various characterizations of Erlang truncated exponential distributions based on distributional properties of order statistics are found in the literature. Let $X_{1,n}, X_{2,n}, \cdots X_{n,n}$ denote the order statistics of a identically independent distributed (i.i.d) random variables $X_1, X_2, \cdots, X_n$, $n \geq 2$, each with distribution function $\mathrm{F_X}(x)$. Furthermore, a variety of other models of ordered random variables are contained in this concept. For a detailed discussion of several of these models, such as sequential order statistics, k^{th} record values and Pfeifer's record model.

In this paper we present characterizations of Erlang truncated exponential distributions DF exp($\beta\alpha_\lambda$), with mean $\frac{1}{(\beta\alpha_\lambda)}$, $\beta > 0, \propto > 0$, $\lambda > 0$. via distributional properties of generalized order statistics including the known results for ordinary order statistics.

Consider a sequence of real numbers $X_1, X_2, \cdots, X_n$ which are independently and identically, distributed with common cumulative distribution (DF) $F_X(x)$ and the probability density function PDF $f_X(x)$ and the distribution function Then the PDF and DF of $X_{U(r)}$, r^{th} upper record is [5] and [9].

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}\,[R(x)]^{r-1} f(x) \tag{1}$$

and

$$\bar{F}_{X_{U(r)}}(x) = 1 - F_{X_{U(r)}}(x) = e^{-R(x)} \sum_{j=0}^{r-1} \frac{[R(x)]^j}{j!} \tag{2}$$

where

$$R(x) = -\ln\bar{F}(x),\ \bar{F}(x) = 1 - F(x) \tag{3}$$

The PDF and DF of $X_{r:n}$, the r^{th} order statistic from a sample of size n is given as [8] and [13].

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}\,[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \tag{4}$$

and

$$F_{X_{r:n}}(x) = \sum_{j=r}^{n}\binom{n}{j}[F(x)]^{j}[1-F(x)]^{n-j} \tag{5}$$

2. Model

The cumulative distribution function CDF $F_X(x)$and probability density function PDF $f_X(x)$ of the Extended Erlang-Truncated Exponential (EETE) distribution are given by

$$F_X(x) = [1 - e^{-\beta(\alpha_\lambda)x}]^{\alpha}\ ,\ 0 \le x < \infty,\ \alpha,\ \beta,\ \lambda > 0, \tag{6}$$

and

$$f_X(x) = \alpha\,\beta\,(\alpha_\lambda)\,e^{-\beta(\alpha_\lambda)x}[1 - e^{-\beta(\alpha_\lambda)x}]^{\alpha-1}\ ,\ 0 \le x < \infty,\ \alpha,\beta,\ \lambda > 0 \tag{7}$$

where α and β are the shape parameters and λ is the scale parameter.

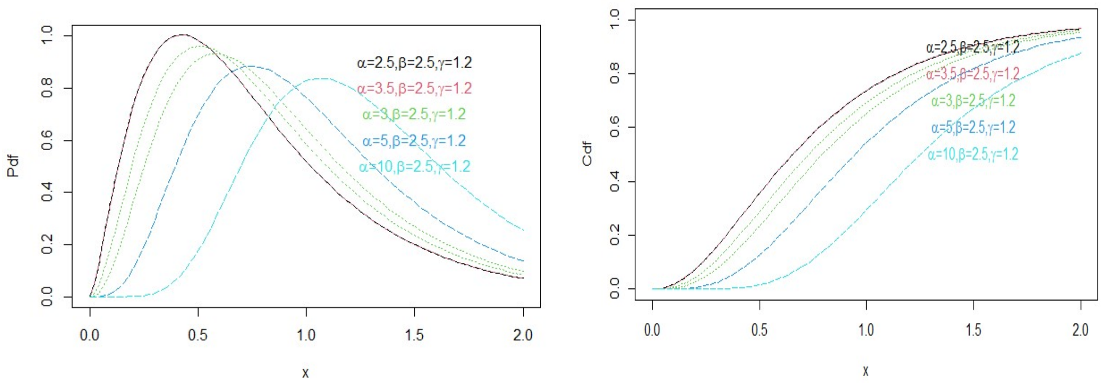

Figure 1. *Possible shapes of the probability density function $f(x)$ (left) and cumulative distribution function $F(x)$ (right) of the Extended Erlang-Truncated Exponential (EETE) distribution for fixed parameter values of β and λ.*

The Extended Erlang-Truncated Exponential (EETE) distribution reduces to Erlang-Truncated Exponential (ETE) when α = 1.

Erlang-Truncated Exponential (ETE) distribution was originally introduced by [15] as an extension of the standard one parameter exponential distribution. The Erlang-Truncated Exponential (ETE) distribution results from the mixture of Erlang distribution and the left truncated one-parameter exponential distribution. The cumulative distribution function CDF $F_X(x)$, and probability density function PDF$f_X(x)$ of the Erlang-Truncated Exponential (ETE) distribution are given by

$$F_X(x) = [1 - e^{-\beta(\alpha_\lambda)x}],\quad 0 \le x < \infty,\ \beta,\ \lambda > 0, \tag{8}$$

where $\alpha_\lambda = \left(1 - e^{-\lambda}\right)$

and

$$f_X(x) = \beta\,(\alpha_\lambda)\,e^{-\beta(\alpha_\lambda)x}\ ,\qquad 0 \le x < \infty,\ \beta,\ \lambda > 0 \tag{9}$$

respectively, where β is the shape parameter and λ is the scale parameter. The Erlang-Truncated Exponential (ETE) distribution collapses to the classical one-parameter exponential distribution with parameter β and $\lambda \to \infty$.

$X \sim Par(\beta(\alpha_\lambda))$

if X has a Pareto distribution with the DF

$$F(x) = [1 - x^{-\beta(\alpha_\lambda)}], \quad 1 < x < \infty, \ \beta > 0, \alpha_\lambda > 0 \tag{10}$$

$X \sim pow\ (\beta(\alpha_\lambda))$

if X has a power function distribution with the DF

$$F(x) = x^{\beta(\alpha_\lambda)}, \quad 0 < x < 1, \ \beta > 0, \alpha_\lambda > 0 \tag{11}$$

It may further be noted that

if $\log X \sim$ Erlang-truncated exp $(\beta(\alpha_\lambda))$ then $X \sim$ Par $(\beta(\alpha_\lambda))$ (12)

if $-\log X \sim$ Erlang-truncated exp $(\beta(\alpha_\lambda))$ then $X \sim$ pow $(\beta(\alpha_\lambda))$ (13)

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3. RELIABILITY CHARACTERISTICS

The reliability function R(x) is an important tool for characterizing life phenomenon. R(x) is analytically expressed as $R(x) = 1 - F(x)$. Under certain predefined conditions, the reliability function R(x) gives the probability that a system will operate without failure until a specified time x. The reliability function of the Extended Erlang-Truncated Exponential (EETE) distribution is given by

$$R(x) = 1 - \left(1 - e^{-\beta(\alpha_\lambda)x}\right)^{\alpha}, 0 \le x < \infty, \ \alpha, \ \beta, \ \lambda > 0 \tag{14}$$

Another important reliability characteristics is the failure rate function. The failure rate function gives the probability of failure for a system that has survived up to time x. The failure rate function h(x) is mathematically expressed $h(x) = f(x)/R(x)$. The failure rate function the Extended Erlang-Truncated Exponential (EETE) distribution is given by:

$$h(x) = \frac{\alpha\, \beta\,(\alpha_\lambda)\, e^{-\beta(\alpha_\lambda)x}[1-e^{-\beta(\alpha_\lambda)x}]^{\alpha-1}}{1-[1-e^{-\beta(\alpha_\lambda)x}]^{\alpha}}, \ 0 \le x < \infty, \ \alpha, \beta, \ \lambda > 0$$

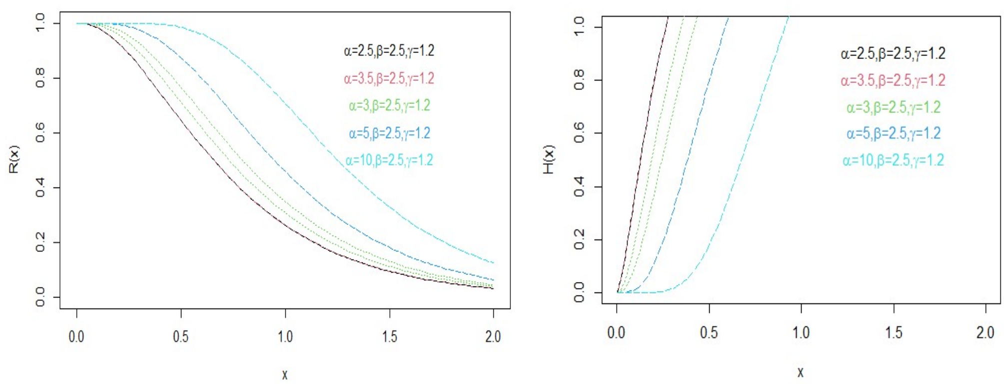

Figure 2. *Possible shapes of the reliability function $R(x)$ (left) and failure rate function $h(x)$ (right) of the Extended Erlang-Truncated Exponential (EETE) distribution for fixed parameter values of β and λ*

4. CHARACTERISTION RESULTS BASED ON UPPER RECORDS

In this section we consider a relation characterizing the Erlang-Truncated Exponential distribution based on order statistics and record statistics. This generalizes some previous characterization results and uses upper as well as lower order statistics. It has been assumed here throughout that the *df* is differentiable *w.r.t.* its argument.

THEOREM 4.1 :-

A random variable $X_{U(r)}$ be a sequence of i.i.d. non-negative random variables with an absolutely continuous distribution having the r^{th} upper statistic from a sample of size n drawn from a continuous DF F_X (x) with PDF $f_X(x)$. Furthermore, let $Y_{U(j)}$ be the r^{th} upper statistic based on a sample of size n, which is drawn from a continuous DF $F_z(z) = P(Z \leq z)$, where Y is independent of X. Finally, let the relation

$$X_{U(N_o)} \stackrel{d}{=} X_{U(R)} + \tilde{Z} \tag{15}$$

be satisfied for all $1 \leq R < N_o \leq n$, Then, $\tilde{z} \stackrel{d}{=} X_{U(N_o-R)}$ and Z ~ Erlang truncated exponential $(\beta\alpha_\lambda)$ if and if Y ~ Erlang truncated exponential$(\beta\alpha_\lambda)$, $\beta > 0,\ \alpha > 0, \lambda > 0$.

Proof. We first prove the necessary part. Let the moment generating function (MGF) of $X_{U(N_o)}$ be $M_{X_{U(N_o)}}(t)$. Then, (15) implies that

$$M_{X_{U(N_o)}}(t) = M_{X_{U(R)}}(t) \cdot M_{\tilde{Z}}(t) \tag{16}$$

Let us now derive the MGF of the $X_{U(R)}$based on Erlang truncated $\exp(\beta\alpha_\lambda)$. Clearly, in view of (15), we get

$$M_{X_{U(R)}}(t) = \frac{\beta(\alpha_\lambda)}{(r-1)!}\int_0^\infty e^{-x((\beta\alpha_\lambda - t)}x^{R-1}dx = \left(\frac{\propto}{\propto - t}\right)^R \tag{17}$$

Where $\Gamma(.)$ is the usual gamma function. On the other hand, in view of (16)

$$M_{\tilde{Z}}(t) = \frac{M_{X_{U(N_o)}}(t)}{M_{X_{U(R)}}(t)} = \left(\frac{\propto}{\propto - t}\right)^{N_o - R} \tag{18}$$

On comparing (18) with (17), we deduce that $M_{\tilde{Z}}(t)$ is the MGF of $Y(N_o - R)$, i.e., the $(N_o - \mathrm{R})^{th}$ upper record statistics from a sample of size R and is independent of $X_{U(R)}$drawn from the DF Erlang truncated $\exp(\beta(\alpha_\lambda)$. Hence the proved Necessity part.

W To prove the sufficiency part. In view of (15) be satisfied with $\tilde{Z} \stackrel{d}{=} Y_{U(N_o-R)}$ and Y ~ $\exp(\beta(\alpha_\lambda))$. Furthermore, let $X_{U(N_o)}$and $X_{U(R)}$ in (15) be upper statistic, which are based on an unknown DF $F_X(x)$ and they are independent of $Y_{U(N_o)}$. Therefore, the convolution relation (3.1) implies that

$$\begin{aligned} f_{X_{U(N_o)}}(x) &= \int_0^\infty f_{X_{U\,(R)}}(y) f_{Y_{U(N_o)}}(x-y)dy \\ &= \frac{(\beta(\alpha_\lambda))^{N_o-R}}{(N_o-R-1)!}\int_0^\infty e^{-\beta(\alpha_\lambda)(x-y)} \times [x-y]^{N_o-R-1} f_{X_{U(R)}}(y)dy \end{aligned} \tag{19}$$

Differentiating both the sides of (19) $w.r.t.$ x, we get

$$\begin{aligned} \frac{d}{dx} f_{X_{U(N_o)}}(x) &= \frac{(\beta(\alpha_\lambda))^{N_o-R}}{(N_o-R-2)!}\int_0^\infty e^{-\beta(\alpha_\lambda)(x-y)} \times [x-y]^{N_o-R-2} f_{X_{U(R)}}(y)dy \\ &\quad - \frac{(\beta(\alpha_\lambda))^{N_o-R+1}}{(N_o-R-1)!}\int_0^\infty e^{-\beta(\alpha_\lambda)(x-y)} \times [x-y]^{N_o-R-1} f_{X_{U(R)}}(y)dy \end{aligned} \tag{20}$$

and by using the representation (19), we get

$$f_{X_{U(N_o-1)}}(x) = \frac{(\beta(\alpha_\lambda))^{N_o-R-1}}{(N_o-R-2)!}\int_0^\infty e^{-\beta(\alpha_\lambda)(x-y)} \times [x-y]^{N_o-R-1} f_{X_{U(r)}}(y)dy \tag{21}$$

and by combing (20) and (21), weget

$$\frac{d}{dx} f_{X_{U(N_o)}}(x) = \beta(\alpha_\lambda)[f_{X_{U(N_o-1)}}(x) - f_{X_{U(N_o)}}(x)]$$

or equivalently, by integrating from 0 to x

$$f_{X_{U(N_o)}}(x) = \beta(\alpha_\lambda)[F_{X_{U(N_o-1)}}(x) - F_{X_{U(N_o)}}(x)] \tag{22}$$

Now, by using the relation (II) of [5] and [9] on page 75, we get

$$F_{X_{U(N_o-1)}}(x) - F_{X_{U(N_o)}}(x) = \frac{[R(x)]^{N_o-1}}{(N_o-1)!}\,[\bar{F}_X(x)] \quad (23)$$

Therefore, by combing (1), (22) and (323), we have

$$\frac{f_X(x)}{\bar{F}_X(x)} = \beta(\alpha_\lambda)$$

Hence, the complete sufficient part, $F_X(x) = [1 - e^{-\beta(\alpha_\lambda)x}]\,, x > 0, \beta > 0, \propto > 0, \lambda > 0$.

Remark 4.1. ([7], Remark 1) have shown that for two adjacent upper records

$$X_{U(2)} \overset{d}{=} X_{U(1)} + \tilde{Y}$$

Then, $\tilde{Y} \overset{d}{=} X_{U(1)}$ and Y ~ exp($\beta\alpha_\lambda$) if and if $X \sim \exp(1)$, $\beta > 0, \propto > 0, \lambda > 0$.

Remark 4.2. [14] and [6] have shown that

$$X_{U(R+1)} \overset{d}{=} X_{U(R)} + \tilde{Z}$$

Then, $\tilde{Z} \overset{d}{=} X_{U(1)}$ and Y ~ exp($\beta\alpha_\lambda$) if and if $X \sim \exp(1)$, $\beta > 0, \propto > 0, \lambda > 0$.

Remark 4.3. [11] have shown

$$X_{U(N_o)} \overset{d}{=} X_{U(R)} + \tilde{Z}$$

Then, $\tilde{Z} \overset{d}{=} X_{U(R)}$ and Y ~ Ga($N_o - R, 1$) if and if $X \sim \exp(1)$, $\beta > 0, \propto > 0, \lambda > 0$.

Corollary 4.1. Assume that the RVs X and Y are independent, as we assumed in Theorem 4.1. By replacing the additive relation (15) by the multiplication relation

$$X_{U(N_o)} \overset{d}{=} X_{U(R)} + \tilde{Z} \quad (24)$$

be satisfied for all $1 \leq R < N_o \leq n$, Then, $\tilde{Z} \overset{d}{=} X_{U(N_o-R)}$ and Y ~ exp($\beta\alpha_\lambda$) if and if $X \sim Par(\beta\alpha_\lambda)$, $\beta > 0, \propto > 0, \lambda > 0$.

Proof. Here the proof immediately follows, by noting that if X ~ $Pareto(\beta(\alpha_\lambda))$, then $\log X \sim exp\ (\beta(\alpha_\lambda))$ and

$$\log X_{U(N_o)} \overset{d}{=} \log X_{U(R)} + \log \tilde{Z}$$

which implies

$$X_{U(N_o)} \overset{d}{=} X_{U(R)} + \tilde{Z}$$

Corollary 4.2. Assume that the RVs X and Y are independent, as we assumed in Theorem 4.1. By replacing the additive relation (15) by the multiplication relation

$$X_{L(N_o)} \overset{d}{=} X_{L(R)} + \tilde{Z} \quad (25)$$

be satisfied for all $1 \leq R < N_o \leq n$, Then, $\tilde{Z} \overset{d}{=} X_{L(N_o-R)}$ and Y ~ exp($\beta\alpha_\lambda$) if and if $X \sim Pow(\beta\alpha_\lambda)$, $\beta > 0, \propto > 0, \lambda > 0$.

Proof. The Corollary can be proved by considering if X $\sim Power(\beta(\alpha_\lambda))$, then $-\log X \sim exp(\beta(\alpha_\lambda))$ and

$$-log X^*_{L(N_o)} \overset{d}{=} - log\ X^* X^*_{L(R)} - \log Y^*$$

which implies

$$X^*_{L(N_o)} \overset{d}{=} X^*_{L(R)} \cdot Y^*$$

5. CHARACTERISTION RESULTS BASED ON ORDER STATISTICS

THEOREM 5.1 :-

A random variable $X_{R:n}$ be a sequence of *i.i.d.* non-negative random variables with an absolutely continuous distribution having the R^{th} order statistics from a sample of size n drawn from a continuous DF F_X (x) with PDF $f_X(x)$. Furthermore, let $Y_{r:n}$ be the r^{th} order statistics based on a sample of size n , which is drawn from a continuous DF $F_y(y)$, where Y is independent of X. Finally, let the relation

$$X_{N_o:n} \overset{d}{=} X_{R:n} + \tilde{Z}, \quad (26)$$

be satisfied for all $1 \leq R < N_o$, Then, $\tilde{Z} \overset{d}{=} X_{N_o-R:n-R}$ and Y ~ exp($\beta\alpha_\lambda$) if and if $X \sim \exp(\beta\alpha_\lambda)$, $\beta > 0, \propto > 0, \lambda > 0$.

Proof. The necessary part can be proved easily using mgf. Namely, let in view of (26) be satisfied with $X_{R:n}$ be $M_{X_{R:n}}(t)$. Then, (26) implies that

$$M_{X_{N_o:n}}(t) = M_{X_{N_o:n}}(t) \cdot M_{\tilde{Z}}(t) \tag{27}$$

Let us now derive the MGF of the $X_{X_{R:n}}$ based on Erlang truncated exp($\beta\alpha_\lambda$). Clearly, in view of (26), we get

$$M_{X_{R:n}}(t) = \frac{\beta(\alpha_\lambda)\,\Gamma(n+1)}{(R-1)!\,\Gamma(n-R+1)} \int_0^\infty [e^{-x(\beta\alpha_\lambda)}]^{n-R+\frac{t}{\alpha}}\ [1-e^{-x(\beta\alpha_\lambda}]^{R-1}\ e^{-x(\beta\alpha_\lambda)}dx \tag{28}$$

Which by using the transformation y $= e^{-x(\beta\alpha_\lambda)}$ takes the form

$$M_{X_{R:n}}(t) = \frac{\Gamma(n+1)\,\Gamma(n-R-\frac{t}{\propto}+1)}{\Gamma(n-R+1)\,\Gamma(n-\frac{t}{\propto}+1)} \tag{29}$$

Where $\Gamma(.)$ is the usual gamma function. On the other hand, in view of (28)

$$M_{\tilde{Z}}(t) = \frac{M_{X_{N_o:n}}(t)}{M_{X_{R:n}}(t)} = \frac{\Gamma(n-R+1)\,\Gamma(n-N_o-\frac{t}{\propto}+1)}{\Gamma(n-N_o+1)\,\Gamma(n-R-\frac{t}{\propto}+1)} \tag{30}$$

On comparing (30) with (29), we deduce that $M_{\tilde{Z}}(t)$ is the MGF of $Y_{N_o-R:n-R}$, i.e., the $(N_o - \text{R})^{th}$ order statistics from a sample of size (n − R)drawn from the DF Erlang truncated exp($\beta(\alpha_\lambda)$) and is independent of $X_{R:n}$ drawn from . This completes the proof of the necessity part.
while the proof of the sufficiency part follows closely as the sufficiency part of Theorem 5.1. Namely, let the representation (26) be satisfied with $\tilde{Y} \overset{d}{=} X_{N_o-R:n-R}$ and Y ~ exp($\beta\alpha_\lambda$). Furthermore, let $X_{N_o:n}$ and $X_{N_o-R:n-R}$ in (26) be order statistics, which are based on an unknown DF $F_X(x)$ and they are independent of $X_{R:n}$. Therefore, the convolution relation (26) implies that

$$f_{X_{N_o:n}}(x) = \int_0^x f_{X_{R:n}}(y) f_{X_{N_o-R:n-R}}(x-y)dy$$

$$= \frac{\beta(\alpha_\lambda)\,(n-R)!}{(N_o-R-1)!\,(n-N_o)!} \int_0^x e^{-\beta(\alpha_\lambda)(x-y)} \times [1-(e^{-\beta(\alpha_\lambda)(x-y)})]^{n-N_o+1} f_{X_{R:n}}(y)dy \tag{31}$$

By differentiating both the sides of (31) with respect to x, we get

$$\frac{df_{X_{N_o:n}}(x)}{dx} = \frac{(\beta(\alpha_\lambda))^2\,(N_o-R-1)\,(n-R)!}{(N_o-R-1)!\,(n-N_o)!} \int_0^x [e^{-\beta(\alpha_\lambda)\,(x-y)}]^{(n-N_o+2)} \times [1-e^{-\beta(\alpha_\lambda)(x-y)}]^{N_o-R-2} f_{X_{R:n}}(y)d$$

$$-\frac{(\beta(\alpha_\lambda))^2\,(n-N_o+1)\,(n-r)!}{(N_o-R-1)!\,(n-N_o)!} \int_0^x [e^{-\beta(\alpha_\lambda)(x-y)}]^{n-N_o+1} \times [1-(e^{-\beta(\alpha_\lambda)(x-y)})^{m+1}]^{N_o-R-1} f_{X_{R:n}}(y)dy$$

$$= \beta(\alpha_\lambda)\,(n-N_o+1)\,[f_{X_{N_o-1:n}}(x) - f_{X_{N_o:n}}(x)]$$

Or equivalently, by integrating from 0 to x,

$$f_{X(N_o,n)}(x) = \beta(\alpha_\lambda)(n-N_o+1)[F_{X(N_o-1,n-1)}(x) - F_{X(N_o,n)}(x)] \tag{32}$$

Now, by using the relation of [13],

$$\frac{f_X(x)}{\bar{F}_X(x)} = \beta(\alpha_\lambda)$$

which implies that

$$\mathrm{F_X}(x) = [1-e^{-\beta(\alpha_\lambda)y}],\ \beta > 0, \propto > 0, \lambda > 0, x > 0$$

This complete the proof of the sufficiency part, as well as the proof of Theorem 4.1.

Corollary 5.1. A random variables (RVs) X and Y are independent, as we assumed in Theorem 5.1. By replacing the additive relation (26) by the multiplicative relation

$$X_{N_o:n} \overset{d}{=} X_{R:n}\ .\tilde{Z} \tag{33}$$

Then, $\tilde{Z} \overset{d}{=} Y_{N_o-R:n-rR}$ and Y ~ $Pareto(\beta(\alpha_\lambda))$ if and only if X ~ $Pareto(\beta(\alpha_\lambda))$
Proof. The proof follows exactly as the proof of Corollary 4.1.

Remark 5.1. [7] have proved that

$$X_{R:n} \overset{d}{=} X_{R-1:n} + U$$

Where U ~ exp (n − R + 1) if and only if X_1~exp (1).

Remark 5.2. [11] have shown that

$$X_{N_o:n} \overset{d}{=} X_{R:n} + U$$

Where U $\overset{d}{=} -log\, M$ W ~ Be (n − R + 1, N_o − R if and only if X_1~exp (1).

Corollary 5.2. A random variables (RVs) X and Y are independent, as we assumed in Theorem 5.1. By replacing the additive relation (26) by the multiplicative relation

$$X^*_{R:n} \overset{d}{=} X^*_{N_o:n} \cdot Z^* , \quad (34)$$

Then, $Z^* \overset{d}{=} Y^*_{r:s-1}$ and $Y^* \sim Power(\beta(\alpha_\lambda))$ if and if $X^* \sim Power(\beta(\alpha_\lambda)$, $\beta > 0, \propto > 0, \lambda > 0$.

Proof. To prove the corollary, we note that

$$-\log X_{N_o:n} \overset{d}{=} -log\, X_{R:n} - log X_{N_o-R:n-R}$$

implies

$$X_{n-N_o+1:n} \overset{d}{=} X_{n-R+1:n} + X_{n-N_o+1:n-R}$$

Or,

$$X_{n-N_o+1:n} \overset{d}{=} X_{n-R+1:n} + X_{n-N_o+1:n-R}$$

THEOREM 5.2 :-

A random variable $X_{R:n}$ be a sequence of i.i.d. non-negative random variables with an absolutely continuous distribution having the R^{th} order statistics from a sample of size n drawn from a continuous DF F_X (x) with PDF $f_X(x)$. Furthermore, let $Y_{R:n}$ be the R^{th} order statistics based on a sample of size n , which is drawn from a continuous DF $F_z(z)$, where Z is independent of X. Finally, let the relation

$$X_{N_o:n} \overset{d}{=} X_{N_o-R:n-R} + \tilde{Z}, \quad (35)$$

be satisfied for all $1 \le R < N_o$, Then, $\tilde{Z} \overset{d}{=} X_{R:n}$ and Y ~ exp($\beta\alpha_\lambda$) if and if $X \sim$ exp($\beta\alpha_\lambda$), $\beta > 0$, $\propto > 0, \lambda > 0$.

Proof. We first prove the necessary part. Let the moment generating function (MGF) of $X_{R:n}$ be $M_{X_{R:n}}(t)$. Then, (38) implies that

$$M_{X_{N_o:n}}(t) = M_{X_{N_o:n}}(t) \cdot M_{\tilde{Z}}(t) \quad (36)$$

Let us now derive the MGF of the $X_{X_{R:n}}$ based on Erlang truncated exp($\beta\alpha_\lambda$). Clearly, in view of (26), we get

$$M_{X_{R:n}}(t) = \frac{\beta(\alpha_\lambda)\, \Gamma(n+1)}{(R-1)!\, \Gamma(n-R+1)} \int_0^\infty [e^{-x(\beta\alpha_\lambda)}]^{n-R+\frac{t}{\alpha}} \; [1 - e^{-x(\beta\alpha_\lambda}]^{R-1} \; e^{-x(\beta\alpha_\lambda)} dx \quad (37)$$

Which by using the transformation $y = e^{-x(\beta\alpha_\lambda)}$ takes the form

$$M_{X_{R:n}}(t) = \frac{\Gamma(n+1)\, \Gamma(n-R-\frac{t}{\propto}+1)}{\Gamma(n-R+1)\, \Gamma(n-\frac{t}{\propto}+1)} \quad (38)$$

Where Γ(.) is the usual gamma function. On the other hand, in view of (3.14)

$$M_{\tilde{Z}}(t) = \frac{M_{X_{N_o:n}}(t)}{M_{X_{R:n}}(t)} = \frac{\Gamma(n-R+1)\, \Gamma(n-s-\frac{t}{\propto}+1)}{\Gamma(n-N_o+1)\, \Gamma(n-R-\frac{t}{\propto}+1)} \quad (39)$$

On comparing (39) with (38), we deduce that $M_{\tilde{Z}}(t)$ is the MGF of $Y_{N_o-R:n-R}$, i.e., the $(N_o - R)^{th}$ order statistics from a sample of size (n − R)drawn from the DF Erlang truncated exp($\beta(\alpha_\lambda)$) and is independent of $X_{R:n}$ drawn from . This completes the proof of the necessity part.

while the proof of the sufficiency part follows closely as the sufficiency part of Theorem 4.1. Namely, let the representation (26) be satisfied with $\tilde{Z} \overset{d}{=} X_{N_o-R:n-R}$ and Y ~ exp($\beta\alpha_\lambda$). Furthermore, let $X_{N_o:n}$ and $X_{N_o-R:n-R}$ in (26) be order statistics, which are based on an unknown DF $F_X(x)$ and they are independent of $X_{R:n}$. Therefore, the convolution relation (26) implies that

$$f_{X_{N_o:n}}(x) = \int_0^x f_{X_{N_o-R:n-R}}(y) f_{X_{r:n}}(x-y)dy$$

$$= \frac{\beta(\alpha_\lambda)}{B(n-R+1,R)} \int_0^x e^{-\beta(\alpha_\lambda)(x-y)^{n-R+1}} \times [1-(e^{-\beta(\alpha_\lambda)(x-y)})]^{R-1} f_{X_{N_o-R:n-R}}(y)dy \quad (40)$$

By differentiating both the sides of (40) with respect to x, we get

$$\frac{df_{X_{N_o:n}}(x)}{dx} = \frac{(\beta(\alpha_\lambda))^2\ (R-1)}{B(n-R+1,R)} \int_0^x [e^{-\beta(\alpha_\lambda)\,(x-y)}]^{(n-R+2)} \times [1-e^{-\beta(\alpha_\lambda)(x-y)}]^{R-2} f_{X_{N_o-R:n-R}}(y)dy$$

$$-\frac{(\beta(\alpha_\lambda))^2\ (n-R+1)}{B(n-R+1,R)} \int_0^x [e^{-\beta(\alpha_\lambda)(x-y)}]^{n-R+1} \times [1-(e^{-\beta(\alpha_\lambda)(x-y)})]^{R-1} f_{X_{N_o-R:n-R}}(y)dy$$

$$= \beta(\alpha_\lambda)\ (n)\ [f_{X_{N_o-1:n-1}}(x) - f_{X_{N_o:n}}(x)]$$

Or equivalently, by integrating from 0 to x,

$$f_{X_{N_o:n}}(x) = \beta(\alpha_\lambda)(n)[F_{X(N_o-1,n-1)}(x) - F_{X(N_o,n)}(x)] \quad (41)$$

Now, by using the relation of [13], we get

$$F_{X(N_o-1,n-1)}(x) - F_{X(N_o,n)}(x) = \binom{n-1}{N_o-1} [F_X(x)]^{N_o-1}[1-F_X(x)]^{n-N_o+1} \quad (42)$$

Therefore, by combing (1), (41) and (42), we get

$$\frac{f_X(x)}{\overline{F}_X(x)} = \beta(\alpha_\lambda)$$

which implies that

$\mathrm{F_X}(x) = [1-e^{-\beta(\alpha_\lambda)y}]$, $\beta > 0, \propto > 0, \lambda > 0, x > 0$

This complete the proof of the sufficiency part, as well as the proof of Theorem 4.1.

Corollary 5.1. A random variables (RVs) X and Y are independent, as we assumed in Theorem 5.2. By replacing the additive relation (35) by the multiplicative relation

$$X_{N_o:n} \overset{d}{=} X_{R:n}\ .\tilde{Z} \quad (43)$$

Then, $\tilde{Z} \overset{d}{=} Y_{N_o-R:n-R}$ and $Y \sim Pareto(\beta(\alpha_\lambda))$ if and only if $X \sim Pareto(\beta(\alpha_\lambda))$.

Proof. The proof follows exactly as the proof of Corollary 4.1.

Corollary 5.2. A random variables (RVs) X and Y are independent, as we assumed in Theorem 3.2. By replacing the additive relation (26) by the multiplicative relation

$$X^*_{R:n} \overset{d}{=} X^*_{N_o:n} \cdot Z^*, \quad (44)$$

Then, $Z^* \overset{d}{=} Y^*_{r:s-1}$ and $Y^* \sim Power(\beta(\alpha_\lambda))$ if and if $X^* \sim Power(\beta(\alpha_\lambda)$, $\beta > 0, \propto > 0, \lambda > 0$.

Proof. To prove the corollary, in view of (11) and (20).

6. APPLICATIONS

Many authors have considered prediction problems based on samples of random sizes, The importance of the order statistics in the reliability theory is attributed to the fact that the r^{th} order statistics $(n-r+1)$ out-of-n system made up of n identical components with independent life lengths. On the other hand, in dealing with censored samples, where the life-test is terminated after observing the r^{th} failure (Type II censoring), or the termination of the test occurs after a given time lapse (Type I censoring), the complete survival times can not usually be observed (due to time or cost). In many biological and agriculture problems, we often come across a situation where the sample size is not deterministic because either some observations get lost for various reasons, or the size of the target population and its representative sample cannot be determined well. For example, assume that the inhabitants of a populous town are exposed to a dose of radiation resulting from an atomic accident, or exposed to an infection of an unknown epidemic. Furthermore, assume that our interest focuses on the time at which r persons would die among a big random sample of size n that is drawn from the residents of this town. Since the number of infected people in this town is unknown and changes randomly with time, the drawn sample contains a random number of infected and non-infected people. Accordingly, the sample size of the sub-sample of the infected people will be a non-negative integer valued RV, e.g.N, and it will be described by a sequence of independent and identically distributed RVs $X_1, X_2, \cdots, X_N$. Therefore, the r^{th} smallest order statistic will be denoted by $X_{r:N}$, which represents the time at which r persons will die.

7. CONCLUSIONS

In this paper we consider the equality by distribution of the form $Y \stackrel{d}{=} XV$, where X and V are two independent random variables. It should be noted that the random contraction–dilation schemes have important applications in many areas such as economic modeling and reliability. The characterization results given in Section 4 can be used in developing goodness-of-fit tests for the corresponding probability distributions. This paper deals with the generalized order statistics and dual generalized order statistics within a class of Erlang-Truncated Exponential distribution. Two theorems for characterizing the general form of distribution based on generalized order statistics dual generalized order statistics are given. Special cases are also deduced.

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