RT&A, No.3 (74)
Volume 18, September 2023

RELIABILITY AND SENSITIVITY ANALYSIS OF A SYSTEM WITH CONDITIONAL AND EXTENDED WARRANTY

Kajal Sachdeva

•

Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana, India
kajal.rs.maths@mdurohtak.ac.in

Gulshan Taneja

•

Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana, India
drgtaneja@gmail.com

Amit Manocha*

•

Department of Applied Sciences, TITS Bhiwani, Haryana, India
amitmanocha80@yahoo.com

*Corresponding Author

Abstract

System reliability and maintenance cost are the most crucial and decisive factors influencing consumers' buying behaviour. The manufacturer attempts to address the consumer concern by offering a warranty in accordance with the reliability of the system and maintenance costs. This study aims to examine the stochastic behaviour of a single unit system operating in three different time frames, namely normal, extended and expired warranty time duration. The system user can prolong the normal warranty period at an extra cost. This prolonged warranty is termed an 'Extended Warranty'. However, the manufacturer provides a warranty on a system with certain conditions. If the failures are covered under the warranty conditions, the repair/replacement is done free of cost; otherwise, all charges are borne by the system user. Markov and regenerative processes are used to derive the system's reliability and other performability measures. Time distributions used in the study are taken as arbitrary. The profit function for the manufacturer and the user is formulated and analysed. Sensitivity analysis for system availabilities in different time zones and profit functions is also done. Numerical examples for exponential, Weibull and Erlang time distributions are discussed to illustrate the derived measures.

Keywords: Reliability; Extended warranty; Regenerative process; Profit function; Sensitivity analysis.

1. Introduction

The recent advancement in various technological aspects has paved the way for endless technologies and innovations to hit the mainstream, forcing manufacturers to offer a gamut of consumer options. Despite the numerous advantages of technological development, the flip side is the system's complexity. Consequently, the consumers are apprehensive of system reliability which may adversely affect the sales of the developed product. Hence, in the pursuit of ensuring system

reliability and addressing the consumer's concerns and dilemmas, the manufacturer offers a warranty and extended warranty. System reliability and warranty may be considered as interrelated concepts. Warranty is the written agreement provided to system users for cumulative product acceptance without any strain of product manufacturing or faults. Ives and Vitale [1], Ritchken et al. [2], Singpurwalla and Wilson [3] proposed that the warranty on various products caters to risk reduction, quality bench-marking and enhanced market competitiveness. Different types of warranty policies were introduced to optimize profit. Free Replacement Warranty, Full-Service Warranty and Renewing Pro-Rata Warranty were described by Blischke, and Murthy [4], Jain and Maheshwari [5], Bai and Pham [6] respectively. Huang et al. [7] discussed the future problems and challenges in reliability and warranty. Kadyan and Ramniwas [8] proposed a probabilistic model for a single-unit system protected by warranty conditions. Rahman and Chattopadhyay [9] developed cost models on long-term/service contract policies. Pham and Bai [10] discussed warranty costs, compared different warranty policies and evaluated the warranty benefits. The warranties offered to consumers can safeguard them from exorbitant maintenance costs considering the ever-increasing maintenance cost resulting from advancements. A conditional warranty covers the cost associated with defects stipulated in the agreement at the time of purchase.
Taneja [11] developed a stochastic model in which repair/replacement is done by the manufacturer on predefined warranty conditions. Lei et al. [12] discussed the characterization of warranty price policies and optimized the product price, which is profitable to the user. Further, Solkhe and Taneja [13] considered the system with conditional warranty and compared the performability measures before and after the expiry of warranty periods. Niwas [14] analysed a warranted system with waiting time for repair and cost paid by the user if failures happen due to unauthorised modifications. Hooti et al. [15] optimised the warranty duration and the repair. Another type of warranty, known as an extended warranty, is worth considering these days. There are systems used in daily life for which manufacturers, dealers or third parties provide extended warranties, such as home appliances and electronic equipment, particularly in the automotive industry. Though optional, it offers users a sense of security regarding system reliability, maintenance costs, etc. Suiter and Lorson [16] described the pros and cons of an extended warranty for the system user. Huang [17] considered a system with minor, degraded and catastrophic failures where warranty was provided for either one year at a fixed lump sum price or monthly warranty plan which may be extended for another month. Salmasnia and Hatami [18] considered a model with an extended warranty where the failures are controlled using technology and non-periodic maintenance activities.
However, stochastic modelling of a system with prolonged conditional warranty and identification of key parameters influencing the most on the manufacturer/user profit is yet to be reported in the literature. So, the objective of our study is to stochastically analyze a system functioning in normal, extended and beyond warranty periods with a focus on its reliability characteristics and economic viability. The study also identifies the parameter that significantly impacts system profitability from the manufacturer and user perspectives. This article is structured as follows. System description and assumptions made are described in Section 2. Various notations used in the study are cited in Section 3. A probabilistic model for the described system is developed in Section 4. Transition probabilities and mean sojourn times are also evaluated in this Section. Expressions for reliability indices and metrics impacting the system's profitability are derived in Sections 5, 6 and 7, respectively. In Section 8, the profit functions for the manufacturer and the user are established. Section 9 focused on the sensitivity analysis of availabilities and profit functions. To illustrate the developed model, numerical examples for different density functions are discussed in Section 10. Conclusions regarding reliability, system availabilities, the profitability of the manufacturer/user and their sensitivity are also drawn in this Section. Finally, Section 11 provides several concluding insightful interpretations.

2. System Descriptions and Assumptions

Descriptions for the system under consideration and assumptions made for the analysis are as follows:

1. System has a single unit that operates in three different warranty periods named normal, extended and expiry warranty periods.
2. The failed system is inspected by the manufacturer or external source to ensure
   (a) Whether faults occur in a system comes under warranty claims or not.
   (b) Whether the system is repairable or needs replacement.
3. If an inspection reveals that a fault falls inside the purview of a normal or extended warranty conditions, the manufacturer is liable for paying the cost of repair/replacement. Otherwise, all the charges are borne by the user itself.
4. In the expiry (beyond warranty) period, the user manages the expenditure for repair/replacement.
5. Transition time distributions have been taken general.

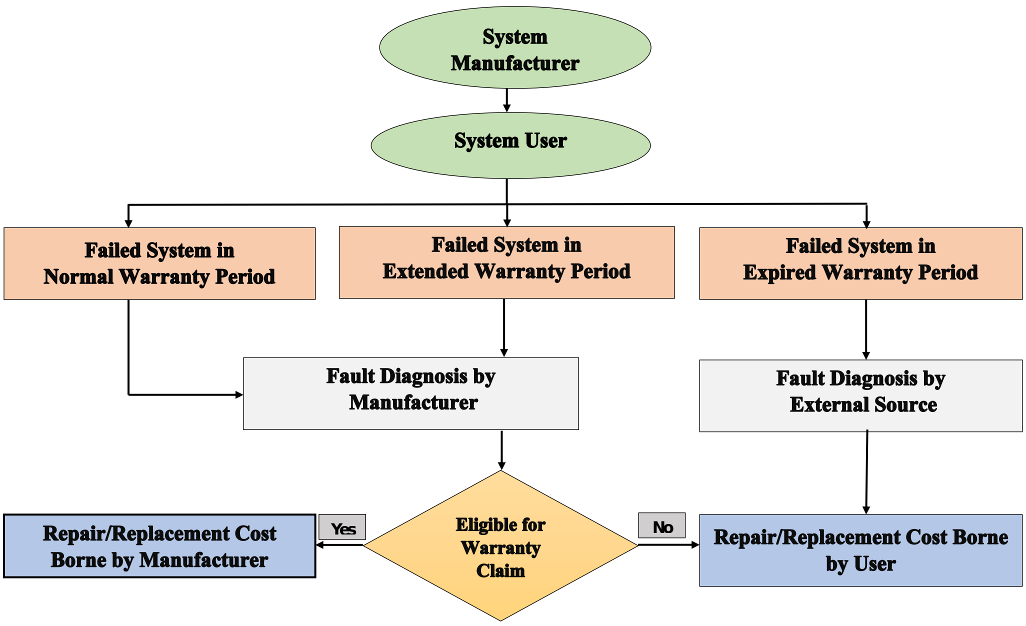

Figure 1: *System description.*

Figure 1 shows the description of the system. Markov and regenerative processes are applied to develop a stochastic model for the system defined. Mathematical expressions for reliability, Mean time to failure (MTTF), availabilities, expected busy period of repairman, and the number of replacements are derived. Profit functions are formulated. Sensitivity analysis is also done for availabilities in three different time zones and for the profit function of the manufacturer and the user. Exponential, Weibull and Erlang distributions are used for numerical calculations. Various conclusions on reliability indices, profitability and sensitivity, are drawn.

3. Nomenclature

The notations for various probabilities/transition densities are:

S_0	state of system at t=0
$\odot$	symbol for Laplace transform
p/q	probability that fault is covered/not covered by warranty terms.
$p_1/p_2/p_3$	probability that system gets failed during normal/extended/ expired warranty period.
r_1/r_2	probability that fault is repairable/ irreparable and only to be replaced.
$f(t)$	p.d.f. of failure time.
$i_m(t)/i_u(t)$	p.d.f. of inspection time by the repairman engaged by manufacturer/ user itself.
$g_k(t)/h_k(t)$	p.d.f. of repair/replacement time within warranty period 'k=n/et/ex'
$I_i^k(t)$	P{repairman of manufacturer is engaged in inspection at instant t $\mid S_0 = i$ within warranty period 'k'}.
$BM_i^k(t)(BU_i^k(t))$	P{manufacturer repairman is occupied with repair/replacement when the associated cost are to be paid by the manufacturer (user) at instant t $\mid S_0 = i$ within warranty period 'k'}.
$RM_i^k(t)(RU_i^k(t))$	expected number of replacement in (0,t], when expenses are met by manufacturer(user) $\mid S_0 = i$ during warranty period 'k'.

where k denotes normal(n), extended(et), expired(ex).

4. Stochastic Model

The probable states of described system are:

State 0: (O_{nw}); State 1: (F_{in}); State 2: ($F_{rn}^{(m)}$);
State 3: ($F_{rpn}^{(m)}$); State 4: ($F_{rn}^{(u)}$); State 5: ($F_{rpn}^{(u)}$);
State 6: (F_{iet}); State 7: ($F_{ret}^{(u)}$); State 8: ($F_{rpet}^{(u)}$);
State 9: ($F_{ret}^{(m)}$); State 10: ($F_{rpet}^{(m)}$); State 11: (O_{etw});
State 12: (F_{iex}); State 13: ($F_{rex}^{(u)}$); State 14: ($F_{rpex}^{(u)}$);
State 15: (O_{exw});

where,

O_{kw}	operative system in warranty period 'k'.
F_{ik}	failed system under inspection in warranty period 'k'.
$F_{rn}^{(m)}(F_{rpn}^{(m)})/F_{ret}^{(m)}(F_{rpet}^{(m)})$	failed system under repair(replacement) in normal/extended warranty period, for which expenses are to be borne by manufacturer.
$F_{rk}^{(u)}/F_{rpk}^{(u)}$	failed system under repair/ replacement in warranty period 'k', for which charges are to be borne by user itself.

Here, k denotes normal(n), extended(et), expired(ex).
By employing markov and regenerative process, the transition between various states is represented by Figure 2. The state space constitutes the set of regenerative states i.e., S={0, 1, 2, ..., 15}, where O={0, 11, 15} is operative, and F={1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14} is failed state space respectively.
The transition densities from state i to state j ($Q_{ij}(t)$) are:

$Q_{01}(t) = \int_0^t dF(u)$, $Q_{12}(t) = pr_1 \int_0^t dI_m(u)$, $Q_{13}(t) = pr_2 \int_0^t dI_m(u)$,
$Q_{14}(t) = qr_1 \int_0^t dI_m(u)$, $Q_{15}(t) = qr_2 \int_0^t dI_m(u)$, $Q_{20}(t) = \int_0^t dG_n(u)$,
$Q_{30}(t) = \int_0^t dH_n(u)$, $Q_{40}(t) = \int_0^t dG_n(u)$, $Q_{50}(t) = \int_0^t dH_n(u)$,
$Q_{06}(t) = \int_0^t dF(u)$, $Q_{67}(t) = qr_1 \int_0^t dI_m(u)$, $Q_{68}(t) = qr_2 \int_0^t dI_m(u)$,

$Q_{69}(t) = pr_1 \int_0^t dI_m(u)$, $Q_{6,10}(t) = pr_2 \int_0^t dI_m(u)$, $Q_{7,11}(t) = \int_0^t dG_{et}(u)$,
$Q_{8,11}(t) = \int_0^t dH_{et}(u)$, $Q_{9,11}(t) = \int_0^t dG_{et}(u)$, $Q_{10,11}(t) = \int_0^t dH_{et}(u)$,
$Q_{0,12}(t) = \int_0^t dF(u)$, $Q_{11,12}(t) = \int_0^t dF(u)$, $Q_{11,6}(t) = \int_0^t dF(u)$,
$Q_{12,13}(t) = r_1 \int_0^t dI_u(u)$, $Q_{12,14}(t) = r_2 \int_0^t dI_u(u)$, $Q_{13,15}(t) = \int_0^t dG_{ex}(u)$,
$Q_{14,15}(t) = \int_0^t dH_{ex}(u)$, $Q_{15,12}(t) = \int_0^t dF(u)$.

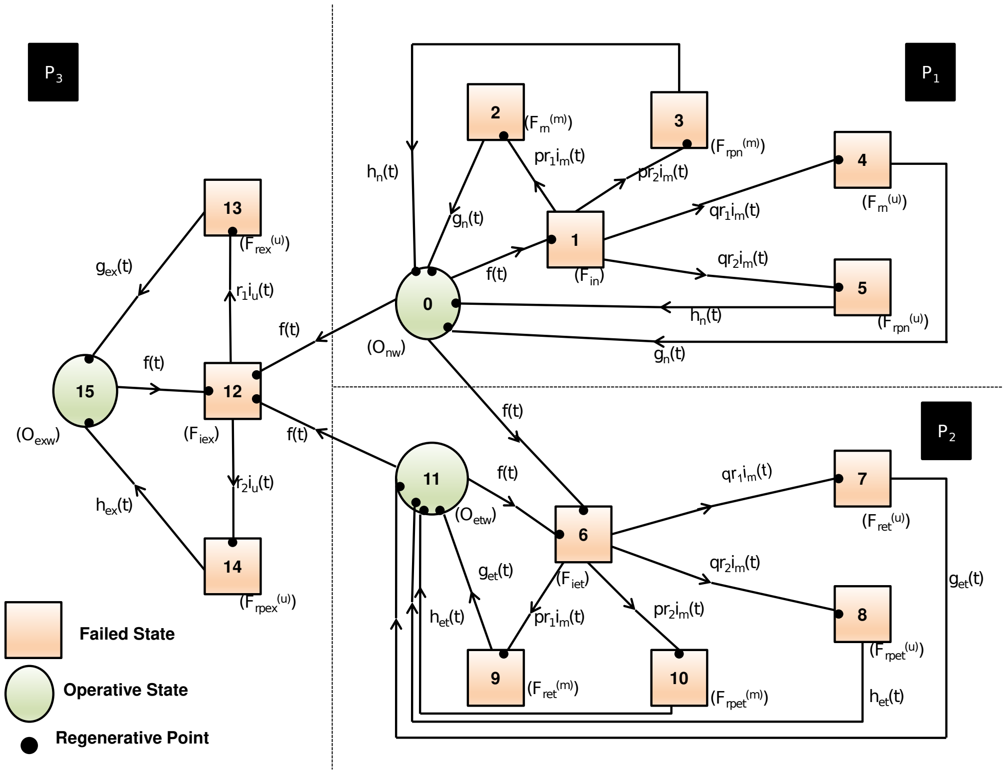

Figure 2: *State transition diagram*

Thus, transition probabilities from state i to j are:

$$p_{ij} = \lim_{s \to 0} q_{ij}^*(s), \quad where \quad q_{ij}(t) = \frac{dQ_{ij}(t)}{dt}.$$

These probabilities follow the property of transition probability matrix for each measure of system effectiveness.
Mean sojourn time (μ_i) in state i are:

$$\mu_0 = \int_0^\infty t f(t) dt = \int_0^\infty \overline{F}(t) dt.$$

Similarly,

$$\mu_1 = \int_0^\infty \overline{I}_m(t)dt, \quad \mu_2 = \int_0^\infty \overline{G}_n(t)dt, \quad \mu_3 = \int_0^\infty \overline{H}_n(t)dt, \quad \mu_4 = \int_0^\infty \overline{G}_n(t)dt,$$
$$\mu_5 = \int_0^\infty \overline{H}_n(t)dt, \quad \mu_6 = \int_0^\infty \overline{I}_m(t)dt, \quad \mu_7 = \int_0^\infty \overline{G}_{et}(t)dt, \quad \mu_8 = \int_0^\infty \overline{H}_{et}(t)dt,$$
$$\mu_9 = \int_0^\infty \overline{G}_{et}(t)dt, \quad \mu_{10} = \int_0^\infty \overline{H}_{et}(t)dt, \quad \mu_{11} = \int_0^\infty \overline{F}(t)dt, \quad \mu_{12} = \int_0^\infty \overline{I}_u(t)dt,$$
$$\mu_{13} = \int_0^\infty \overline{G}_{ex}(t)dt, \quad \mu_{14} = \int_0^\infty \overline{H}_{ex}(t)dt, \quad \mu_{15} = \int_0^\infty \overline{F}(t)dt.$$

Defining m_{ij}= $E(q_{ij}(t))$= $\int_0^\infty tq_{ij}(t)dt$, we have

$$m_{01} = \int_0^\infty tq_{01}(t)dt = \int_0^\infty tf(t)dt = \mu_0.$$

Similarly, we get the following relations

$m_{06} = m_{0,12} = \mu_0$, $m_{20} = \mu_2$, $m_{30} = \mu_3$, $m_{40} = \mu_4$,
$m_{50} = \mu_5$, $m_{7,11} = \mu_7$, $m_{8,11} = \mu_8$, $m_{9,11} = \mu_9$,
$m_{10,11} = \mu_{10}$, $m_{11,6} = m_{11,12} = \mu_{11}$, $m_{12,13} + m_{12,14} = \mu_{12}$, $m_{13,15} = \mu_{13}$,
$m_{14,15} = \mu_{14}$, $m_{15,12} = \mu_{15}$, $m_{12} + m_{13} + m_{14} + m_{15} = \mu_1$,
$m_{67} + m_{68} + m_{69} + m_{6,10} = \mu_6$.

The reliability and performance-indicating characteristics of the system are determined in the following section.

5. Reliability (R(t)) and Mean Time to Failure (MTTF)

Theorem 1. R(t) and MTTF are given as

$$R(t) = \overline{F(t)}, \qquad MTTF = \mu_0.$$

Proof. Let $\psi_0(t)$=P[system is operative until time t | $S_0 = 0$], then using probabilistic arguments, it can be seen from transition diagram, that

$$\psi_0(t) = p_1Q_{01}(t) + p_2Q_{06}(t) + p_3Q_{0,12}(t). \tag{1}$$

The expression on R.H.S. of equation (1) shows the system transit from state 0 to failed state 1 or 6 or 12, with probability $p_1Q_{01}(t)$, $p_2Q_{06}(t)$ and $p_3Q_{0,12}(t)$ respectively in time t. Taking Laplace-Stieltjes transformation of the above equation, we get

$$\psi_0^{**}(s) = p_1Q_{01}^{**}(s) + p_2Q_{06}^{**}(s) + p_3Q_{0,12}^{**}(s). \tag{2}$$

Thus,

$$\begin{aligned}
R(t) &= L^{-1}[\frac{1-\psi_0^{**}(s)}{s}] \\
&= L^{-1}[\frac{1-(p_1Q_{01}^{**}(s) + p_2Q_{06}^{**}(s) + p_3Q_{0,12}^{**}(s))}{s}] \qquad [Substituting(2)] \\
&= L^{-1}[\frac{1}{s}] - p_1L^{-1}[\frac{Q_{01}^{**}(s)}{s}] - p_2L^{-1}[\frac{Q_{06}^{**}(s)}{s}] - p_3L^{-1}[\frac{Q_{0,12}^{**}(s)}{s}] \\
&= 1 - p_1\int_0^t dF(u) - p_2\int_0^t dF(u) - p_3\int_0^t dF(u) \quad [Using\ transition\ densities] \qquad (3) \\
&= 1 - (p_1+p_2+p_3)\int_0^t dF(u) \\
&= 1 - \int_0^t dF(u) \quad (\because p_1+p_2+p_3 = 1) \\
&= 1 - \int_0^t f(u)du = 1 - F(t) = \overline{F(t)}.
\end{aligned}$$

and

$$\begin{aligned}
MTTF &= \lim_{s\to 0}[\frac{1-\psi_0^{**}(s)}{s}] \quad [\frac{0}{0}] \\
&= -\psi_0^{**'}(0).
\end{aligned} \tag{4}$$

Differentiating eq^n (2) both sides w.r.t. s and then taking $\lim s \to 0$, we get

$$\begin{aligned} \psi_0^{**'}(0) &= p_1 Q_{01}^{**'}(0) + p_2 Q_{06}^{**'}(0) + p_3 Q_{0,12}^{**'}(0) \\ &= -\mu_0. \end{aligned} \tag{5}$$

From eq^n (4) and (5),

$$\text{MTTF}=\mu_0.$$

■

6. System Availability

Theorem 2. The Laplace transformation of point-wise availability during **extended warranty period** is given by

$$A_0^{et*}(s) = \frac{N_1^{et*}(s)}{D_1^{et*}(s)},$$

where
$N_1^{et*}(s)$=

$$\begin{vmatrix} M_0^*(s) & -q_{06}^*(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -q_{67}^*(s) & -q_{68}^*(s) & -q_{69}^*(s) & -q_{6,10}^*(s) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -q_{7,11}^*(s) \\ 0 & 0 & 0 & 1 & 0 & 0 & -q_{8,11}^*(s) \\ 0 & 0 & 0 & 0 & 1 & 0 & -q_{9,11}^*(s) \\ 0 & 0 & 0 & 0 & 0 & 1 & -q_{10,11}^*(s) \\ M_{11}^*(s) & -q_{11,6}^*(s) & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$M_0^*(s) = M_{11}^*(s) = \frac{1-f^*(s)}{s}$,

$D_1^{et*}(s)$=

$$\begin{vmatrix} 1 & -q_{06}^*(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -q_{67}^*(s) & -q_{68}^*(s) & -q_{69}^*(s) & -q_{6,10}^*(s) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -q_{7,11}^*(s) \\ 0 & 0 & 0 & 1 & 0 & 0 & -q_{8,11}^*(s) \\ 0 & 0 & 0 & 0 & 1 & 0 & -q_{9,11}^*(s) \\ 0 & 0 & 0 & 0 & 0 & 1 & -q_{10,11}^*(s) \\ 0 & -q_{11,6}^*(s) & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

The system steady-state availability is given as

$$A_0^{et} = \frac{N_1^{et}}{D_1^{et}},$$

where

$$N_1^{et} = \mu_{11},$$

and

$$D_1^{et} = \mu_{11} + \mu_6 + \mu_7 q r_1 + \mu_8 q r_2 + \mu_9 p r_1 + \mu_{10} p r_2.$$

Proof. Considering $A_0^{et}(t)$ = P(system is operative at time t in extended warranty period | $S_0 = 0$), then from transition diagram, we have

$$\begin{aligned} A_0^{et}(t) &= \overline{F(t)} + \int_0^t q_{06}(u) A_6^{et}(t-u)du \\ &= M_0(t) + q_{06}(t) \odot A_6^{et}(t) \end{aligned} \quad (6)$$

where $M_0(t)$ represents that the system remains operative in state 0 instead of moving to any other state. The term $q_{06}(t)$ denotes the system transition probability from state 0 to state 6 in time u<t and thereafter remains operative from state 6 onwards for t-u time.
Similarly,

$$\begin{cases} A_6^{et}(t) = q_{67}(t) \odot A_7^{et}(t) + q_{68}(t) \odot A_8^{et}(t) + q_{69}(t) \odot A_9^{et}(t) + q_{6,10}(t) \odot A_{10}^{et}(t), \\ A_7^{et}(t) = q_{7,11}(t) \odot A_{11}^{et}(t), \\ A_8^{et}(t) = q_{8,11}(t) \odot A_{11}^{et}(t), \\ A_9^{et}(t) = q_{9,11}(t) \odot A_{11}^{et}(t), \\ A_{10}^{et}(t) = q_{10,11}(t) \odot A_{11}^{et}(t), \\ A_{11}^{et}(t) = M_{11}(t) + q_{11,6}(t) \odot A_6^{et}(t). \end{cases} \quad (7)$$

Taking Laplace Transformation of eq^n(6)-(7) and solving them for $A_0^{et*}(s)$ by method of determinants, we get

$$A_0^{et*}(s) = \frac{L_1(s)}{M_1(s)},$$

$$\begin{aligned} L_1(s) &= M_o^*(s) + M_{11}^*(s)q_{06}^*(s)q_{67}^*(s)q_{7,11}^*(s) + M_{11}^*(s)q_{06}^*(s)q_{68}^*(s)q_{8,11}^*(s) \\ &\quad + M_{11}^*(s)q_{06}^*(s)q_{69}^*(s)q_{9,11}^*(s) + M_{11}^*(s)q_{06}^*(s)q_{6,10}^*(s)q_{10,11}^*(s) \\ &\quad - M_o^*(s)q_{67}^*(s)q_{11,6}^*(s)q_{7,11}^*(s) - M_o^*(s)q_{68}^*(s)q_{11,6}^*(s)q_{8,11}^*(s) \\ &\quad - M_o^*(s)q_{69}^*(s)q_{11,6}^*(s)q_{9,11}^*(s) - M_o^*(s)q_{11,6}^*(s)q_{6,10}^*(s)q_{10,11}^*(s) \\ &= N_1^{et*}(s) \end{aligned} \quad (8)$$

$$\begin{aligned} M_1(s) &= 1 - q_{68}^*(s)q_{11,6}^*(s)q_{8,11}^*(s) - q_{69}^*(s)q_{11,6}^*(s)q_{9,11}^*(s) - q_{11,6}^*(s)q_{6,10}^*(s)q_{10,11}^*(s) \\ &\quad - q_{67}^*(s)q_{11,6}^*(s)q_{7,11}^*(s) \\ &= D_1^{et*}(s) \end{aligned} \quad (9)$$

Using Abel's lemma, the system's steady state availability is

$$A_0^{et} = \lim_{s \to 0} sA_0^{et*}(s) = \frac{N_1^{et*}(0)}{D_1^{et*\prime}(0)} = \frac{N_1^{et}}{D_1^{et}}, \quad (10)$$

Differentiating eq^n (9) w.r.t. s,

$$\begin{aligned} D_1^{et*\prime}(s) &= q_{11,6}^{*}{}'(s)(-q_{67}^*(s)q_{7,11}^*(s) - q_{68}^*(s)q_{8,11}^*(s) - q_{69}^*(s)q_{9,11}^*(s) - q_{6,10}^*(s)q_{10,11}^*(s)) \\ &\quad - q_{67}^{*}{}'(s)q_{7,11}^*(s)q_{11,6}^*(s) - q_{68}^{*}{}'(s)q_{8,11}^*(s)q_{11,6}^*(s) - q_{69}^{*}{}'(s)q_{9,11}^*(s)q_{11,6}^*(s) \\ &\quad - q_{6,10}^{*}{}'(s)q_{10,11}^*(s)q_{11,6}^*(s) - q_{7,11}^{*}{}'(s)q_{11,6}^*(s)q_{6,7}^*(s) \\ &\quad - q_{8,11}^{*}{}'(s)q_{11,6}^*(s)q_{6,8}^*(s) - q_{9,11}^{*}{}'(s)q_{11,6}^*(s)q_{6,9}^*(s) \\ &\quad - q_{10,11}^{*}{}'(s)q_{11,6}^*(s)q_{6,11}^*(s). \end{aligned} \quad (11)$$

Setting lim s $\to$ 0 in eq^n (8) and (11), we obtain

$$N_1^{et} = \mu_{11}, \quad (12)$$

$$\begin{aligned} D_1^{et} &= \mu_{11}(pr_1 + pr_2 + qr_1 + qr_2) + pr_1\mu_9 + m_{69} + pr_2\mu_{10} + m_{6,10} + qr_1\mu_7 \\ &\quad + m_{67} + m_{68} + \mu_8 qr_2 \\ &= \mu_{11} + \mu_6 + \mu_9 pr_1 + \mu_{10} pr_2 + \mu_7 qr_1 + \mu_8 qr_2. \end{aligned} \tag{13}$$

■

Similarly, availabilities during normal and expired warranty period are given as

$$A_0^n = \frac{N_1^n}{D_1^n}, \qquad A_0^{ex} = \frac{N_1^{ex}}{D_1^{ex}}. \tag{14}$$

where

$$\begin{aligned} N_1^n &= \mu_0, D_1^n = \mu_0 + \mu_1 + \mu_2 pr_1 + \mu_3 pr_2 + \mu_4 qr_1 + \mu_5 qr_2, \\ N_1^{ex} &= \mu_{15}, D_1^{ex} = \mu_{15} + \mu_{12} + \mu_{13} r_1 + \mu_{14} r_2. \end{aligned} \tag{15}$$

7. Expected Busy Period and Number of Replacements

Employing the definitions of BU_i^k, BM_i^k and I_i^k, i ∈ S (defined in Section 3) and follow the same probabilistic arguments as discussed in preceding Section 6, the expected time for which repairman remain involved in repair/replacement/inspection of a failed system in different warranty zones, in steady-state given as:

$BU_0^k = \frac{N_2^k}{D_1^k}$; $\quad BM_0^k = \frac{N_3^k}{D_1^k}$; $\quad I_0^k = \frac{N_4^k}{D_1^k}$; $\quad$ k=n,et,ex.

$N_2^n = q(\mu_4 r_1 + \mu_5 r_2)$; $\quad N_2^{et} = q(\mu_7 r_1 + \mu_8 r_2)$; $\quad N_2^{ex} = \mu_{13} r_1 + \mu_{14} r_2$;
$N_3^n = p(\mu_2 r_1 + \mu_3 r_2)$; $\quad N_3^{et} = p(\mu_9 r_1 + \mu_{10} r_2)$; $\quad N_4^n = \mu_1$;
$N_4^{et} = \mu_6$; $\quad N_4^{ex} = \mu_{12}$;

Further, by definitions of RU_i^k and RM_i^k,i ∈ S (defined in Section 3), in steady state, the expected number of replacements during different warranty time zones are:

$RU_0^k = \frac{N_5^k}{D_1^k}$; $\quad RM_0^k = \frac{N_6^k}{D_1^k}$; $\quad$ k=n,et,ex
$N_5^n = qr_2$; $\quad N_5^{et} = qr_2$; $\quad N_5^{ex} = r_2$;
$N_6^n = pr_2$; $\quad N_6^{et} = pr_2$;

D_1^n, D_1^{et} and D_1^{ex} are mentioned in eq^n (13) and (15).
Focusing on economic viability of defined system, cost-benefit analysis is performed in the succeeding section. Profitability analysis assists both the manufacturer and the user in categorizing the parameters which may cause long-term loss.

8. Cost-Benefit Analysis

To carry out a cost-benefit analysis, the profit function is defined for the manufacturer and user. Mathematically, the profit function for a system is the difference between total revenue generated and total expenditure incurred in a given period. So, in steady-state, the profit function for the manufacturer and user is formulated as follows:
Manufacturer Profit

$$\begin{aligned} P^m &= CP + EC - MC - C_1^m(p_1 I_0^n + p_2 I_0^{et}) - C_2^m(p_1 BM_0^n + p_2 BM_0^{et}) \\ &\quad - C_3^m(p_1 RM_0^n + p_2 RM_0^{et}). \end{aligned} \tag{16}$$

User Profit

$$P^u = C_0(p_1A_0^n + p_2A_0^{et} + p_3A_0^{ex}) - C_1^u(p_3I_0^{ex}) - C_2^u(p_1BU_0^n + p_2BU_0^{et} + p_3BU_0^{ex}) - C_3^u(p_1RU_0^n + p_2RU_0^{et} + p_3RU_0^{ex}) - CP - EC. \quad (17)$$

where,
CP=Cost price of the system
MC=Manufacturing cost of the system
EC=Cost for extended the warranty period.
C_0=Revenue generated by system.
$C_1^m(C_1^u)$= Cost of engaging the repairman by the manufacturer(user) for inspection.
$C_2^m(C_2^u)$=Cost of engaging the repairman by the manufacturer(user) for repair/replacement.
$C_3^m(C_3^u)$=Cost per replacement of system borne by manufacturer(user).
The above-mentioned costs are considered per unit of time.

9. Sensitivity Analysis

Sensitivity analysis is a concept that determines as to which parameter (independent variable) the obtained measures (dependent variable) are highly or least affected. Relative sensitivity analysis is used to assess the impact of different parameters because the numerical values for various parameters differ significantly. A normalised form of a sensitivity function is known as relative sensitivity function. Using the eq^ns (10), (14), (16) and (17), the sensitivity (D_y^k, Z_y^s) and relative sensitivity functions (d_y^k, z_y^s) for availabilities (A_0^n, A_0^{et}, A_0^{ex}) and profit functions (P^m,P^u) respectively are defined as

$$D_y^k = \frac{\partial(A_0^k)}{\partial y}; \quad d_y^k = \frac{D_y^k y}{A_0^k}; \quad k = n, et, ex. \quad (18)$$

and

$$Z_y^s = \frac{\partial(P^s)}{\partial y}; \quad z_y^s = \frac{Z_y^s y}{P^s}; \quad s = u, m. \quad (19)$$

10. Results and Discussions

In this section, measures obtained in Sections 5-9 respectively are illustrated via numerical examples.
Expressions for various measures have been derived using general probability time distributions. The system user may choose specific values for the parameters involved to illustrate the model based on records of failures, repairs, costs, and available probabilities. The particular distribution can be identified by applying the appropriate test to such data. Because real data on failures, repairs, costs, etc., could not be collected in our study, we used exponential, Weibull and Erlang distributions and assumed values for parameters involved to illustrate the model numerically.

10.1. Example 1:All the Time Distributions follows Exponential Distribution

Assuming failure/inspection/replacement/repair times are exponentially distributed with their p.d.f. given as:

$$\begin{cases} f(t) = \lambda_0 e^{-\lambda_0 t}, \quad i_m(t) = \gamma_m e^{-\gamma_m t}, \quad i_u(t) = \gamma_u e^{-\gamma_u t}, \quad h_n(t) = \beta_n e^{-\beta_n t}, \\ h_{et}(t) = \beta_{et} e^{-\beta_{et} t}, \quad g_n(t) = \alpha_n e^{-\alpha_n t}, \quad g_{et}(t) = \alpha_{et} e^{-\alpha_{et} t}, \quad g_{ex}(t) = \alpha_{ex} e^{-\alpha_{ex} t}, \\ h_{ex}(t) = \beta_{ex} e^{-\beta_{ex} t}. \end{cases} \quad (20)$$

Consider the value of parameters as

$$p = 0.7, q = 0.3, p_1 = 0.2, p_2 = 0.3, p_3 = 0.5, r_1 = 0.7, r_2 = 0.3, \lambda_0 = 0.0005,$$
$$\gamma_m = 1.5, \gamma_u = 1.2, \alpha_n = 0.5, \alpha_{et} = 0.4, \beta_n = 0.02, \beta_{et} = 0.02, \alpha_{ex} = 0.25,$$
$$\beta_{ex} = 0.01, CP = 150, EC = 15, MC = 120, C_0 = 500, C_1^m = 80, C_2^m = 100,$$
$$C_3^m = 15,000, C_1^u = 100, C_2^u = 120, C_3^u = 15,000. \quad (21)$$

10.1.1 Effect of Time (t) and Failure Rate (λ_0) on Reliability Measures

Taking the values of other parameters constant as mentioned in eq^n (21), the effect of parameters (t, λ_0) on reliability function (R(t)) is shown in Figure 3.
As both the parameters t and λ_0 increase, R(t) decreases. Further, Table 1 represents the values of availabilities (A_0^n, A_0^{et},A_0^{ex}) in three different time zones for varied λ_0. All three availabilities decrease with an increase in λ_0. Also for any particular λ_0, the availabilities satisfy the relation $A_0^n > A_0^{et} > A_0^{ex}$. In other words, during the expired warranty period, the system's availability is much influenced by λ_0 as compared to the other time zones.

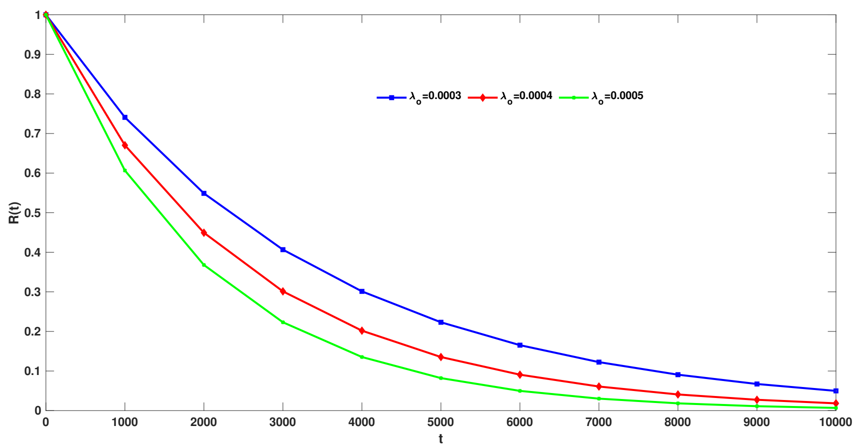

Figure 3: *Reliability w.r.t varied t and λ_0*

Table 1: *Value of Availability for varied λ_0*

	Availability		
λ_0	A_0^n	A_0^{et}	A_0^{ex}
0.001	0.9832	0.9829	0.9675
0.002	0.9670	0.9663	0.9370
0.003	0.9513	0.9503	0.9083
0.004	0.9361	0.9349	0.8814
0.005	0.9214	0.9199	0.8560
0.006	0.9071	0.9054	0.8321
0.007	0.8933	0.8913	0.8094
0.008	0.8799	0.8777	0.7880
0.009	0.8669	0.8645	0.7676
0.010	0.8542	0.8517	0.7483

10.1.2 Effect of Various Rates / Costs on Profit Functions

The outcomes of profit functions, P^m for varied (EC, MC), (CP, p) and P^u for varied (C_0, C_3^u), (λ_0, r_1) are studied. The other parameters are kept fixed, and their values are taken as in eq^n(21). The results obtained are represented by Figure 4, 5, 6, 7 and Table 2 respectively and summarised as follows:

1. P^m goes down as MC and p increases but hike in its value is observed when EC and CP increases.

2. As the parameter C_0 increases, P^u increases. Moreover, the rise in the values of λ_0, r_1, C_3^u respectively, results in the decreasing P^u.

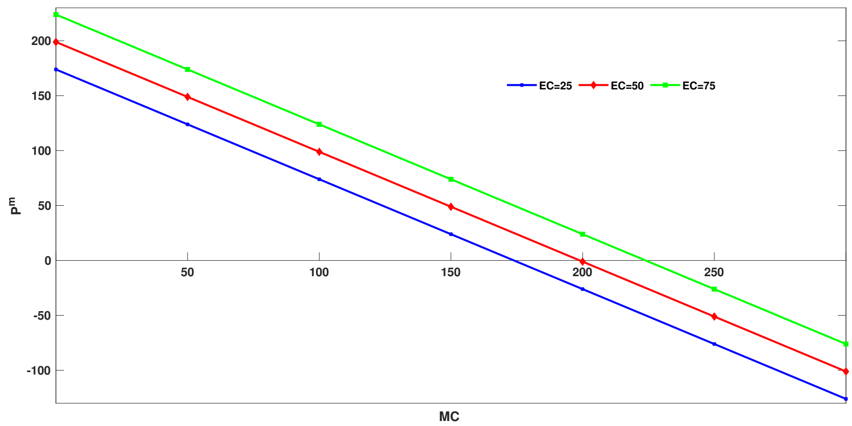

Figure 4: *Variation in P^m for varied MC and EC*

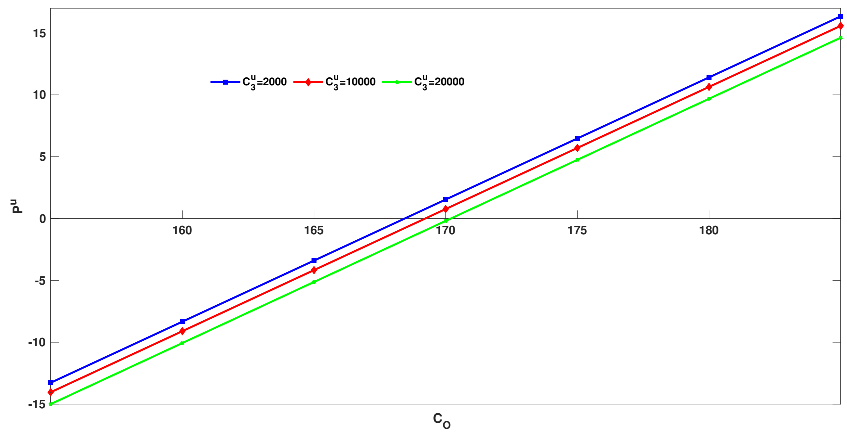

Figure 5: *Variation in P^u for varied C_0 and C_3^u*

3. The system should remain profitable for user as well as manufacturer.

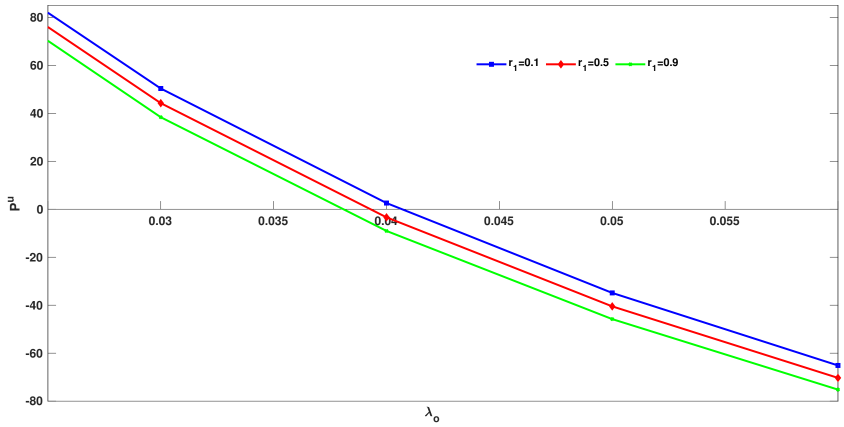

Figure 6: *Variation in P^u for varied λ_0 and r_1*

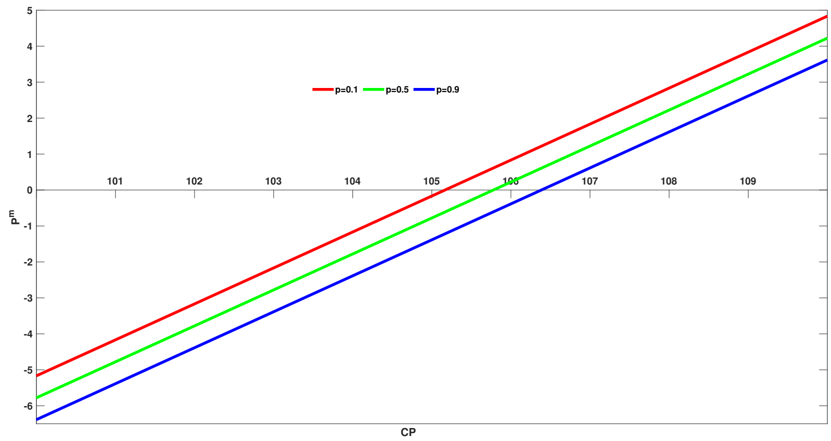

Figure 7: *Variation in P^m for varied CP and p*

Keeping that in mind, the bounds for some of the parameters are evaluated as:

(a) For EC=25, $P^m > 0$ iff MC < 175

(b) For C_3^u = 2000, $P^u > 0$ iff $C_0 > 168$

(c) For r_1=0.1, $P^u > 0$ iff $\lambda_0 < 0.038$

(d) For p=0.1, $P^m > 0$ iff $CP > 105.112$

Bounds for other values of EC, C_3^u and r_1 are also mentioned in Table 2.

Table 2: *Bounds for Revenue/Cost/Rate*

Cost/Revenue/Rate	Varied Parameter	Bounds For Profitability(P^u/P^m>0)
MC	EC=25	MC<175
	EC=50	MC<200
	EC=75	MC< 220
C_0	C_3^u=2000	C_0>168
	C_3^u=10000	C_0>169
	C_3^u=20000	C_0>170
λ_0	$r_1 = 0.1$	λ_0< 0.038
	$r_1 = 0.5$	λ_0< 0.039
	$r_1 = 0.9$	λ_0< 0.040
CP	$p = 0.1$	CP>105.112
	$p = 0.5$	CP>105.784
	$p = 0.9$	CP>106.396

10.2. Example 2: Failure Time follows Weibull Distribution

Assuming the failure time follows Weibull distribution with p.d.f.

$$f(t) = \frac{\delta}{\eta}.(\frac{t}{n})^{\delta-1}.e^{-(\frac{t}{n})^{\delta}} \tag{22}$$

where δ and η are shape and scale parameters respectively.
All the other time distributions follows exponential distribution with same p.d.f. and numerical value as taken in Section 10.1. Figure 8 reveals that the reliability decreases with time and for t<200, it is on the higher side for higher values of η and δ. However, the reverse trend of its values for η and δ is noticed for t>200.
Taking δ=0.3 and η=150, the behaviour of A_0^n, A_0^{et} w.r.t γ_m and A_0^{ex} w.r.t γ_u is shown in Table 3. The increasing trend of availabilities with an increase in γ_u and γ_m respectively are observed.However,the system availability (A_0^{ex}) during expired warranty period is lesser as compared to A_0^n, A_0^{et} for any particular value of γ_u(=γ_m).

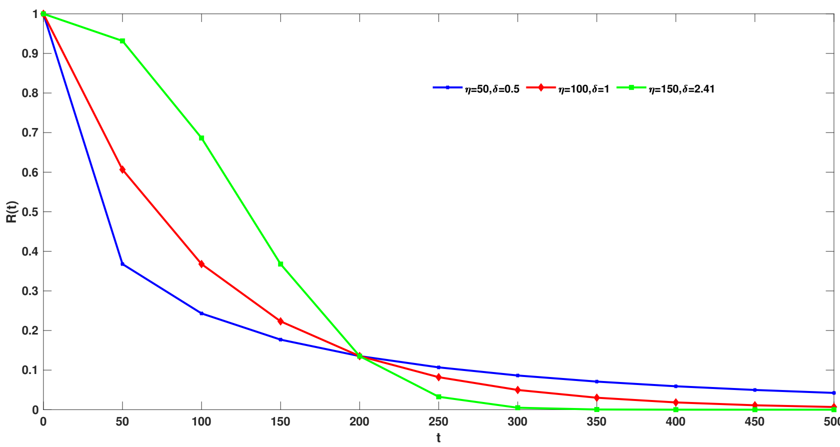

Figure 8: *Reliability w.r.t. varied t,η and δ*

Table 3: *Value of Availabilities for varied γ_m/γ_u*

	Availability		
γ_m/γ_u	A_0^n	A_0^{et}	A_0^{ex}
0.1	0.9813	0.9811	0.9701
0.2	0.9848	0.9846	0.9735
0.3	0.9860	0.9857	0.9746
0.4	0.9866	0.9863	0.9752
0.5	0.9869	0.9867	0.9756
0.6	0.9872	0.9869	0.9758
0.7	0.9873	0.9871	0.9760
0.8	0.9875	0.9872	0.9761
0.9	0.9876	0.9873	0.9762
1.0	0.9876	0.9874	0.9762

10.3. Example 3: Repair Time follows Erlang Distribution

Considering the repair time in different warranty time follows Erlang distribution with p.d.f.

$$\begin{cases} g_n(x) = \xi_n.\frac{x^{k_n-1}.e^{-\xi_n x}}{(k_n-1)!} \\ g_{et}(x) = \xi_{et}.\frac{x^{k_{et}-1}.e^{-\xi_{et} x}}{(k_{et}-1)!} \\ g_{ex}(x) = \xi_{ex}.\frac{x^{k_{ex}-1}.e^{-\xi_{ex} x}}{(k_{ex}-1)!} \end{cases} \tag{23}$$

where
$k_n/k_{et}/k_{ex}$ and $\xi_n/\xi_{et}/\xi_{ex}$ are shape and scale parameter during normal/extended/expired warranty period,
All of the other time distributions have the same p.d.f. ,i.e., exponential distribution and numerical values as in Section 10.1.

Table 4: *Value of Availabilities for varied λ_o*

	Availability		
λ_o	A_o^n	A_o^{et}	A_o^{ex}
0.0001	0.9907	0.9396	0.9383
0.0002	0.9817	0.8861	0.8837
0.0003	0.9727	0.8383	0.8351
0.0004	0.9640	0.7955	0.7916
0.0005	0.9554	0.7568	0.7524
0.0006	0.9469	0.7216	0.7169
0.0007	0.9386	0.6897	0.6846
0.0008	0.9304	0.6604	0.6551
0.0009	0.9224	0.6335	0.6280
0.001	0.9145	0.6087	0.6031

Taking ξ_n=ξ_{et}=ξ_{ex}=0.5 and k_n=k_{et}=k_{ex}=7, the behaviour of A_0^n, A_0^{et} and A_0^{ex} w.r.t λ_o is shown in Table 4. The decreasing trend of availabilities with an increase in λ_o is observed.

10.4. Numerical Calculations for Sensitivity Analysis

Considering all the p.d.f. involved as exponential as taken in eq^n (20) and assuming the values of parameters as mentioned in eq^n (21), the sensitivity analysis is performed. Using eq^n (18) and (19), the outcomes for the sensitivity and relative sensitivity functions of availability and profit functions are summarized in Table 5, 6 and 7 respectively. The magnitude of these functions is taken into account while drawing inferences about parameters and the order in which they influence the different measures.

Table 5: *Sensitivity and relative sensitivity analysis of Availabilities*

Parameter (y)	Sensitivity Function $D_y = \frac{\partial(A_0)}{\partial y}$	Relative Sensitivity Function $d_y = \frac{D_y y}{A_0}$
	Normal Warranty Period	
λ_0	-16.7791	-0.0085
γ_m	2.1848×10^{-4}	3.3052×10^{-4}
α_n	0.0014	7.0597×10^{-4}
β_n	0.3687	0.0074
p	-0.0081	-0.0057
q	-0.0081	-0.0025
r_1	-9.8315×10^{-4}	-6.9408×10^{-4}
r_2	-0.0246	-0.0073
	Extended Warranty Period	
λ_0	-16.1172	-0.0086
γ_m	2.1840×10^{-4}	3.3045×10^{-4}
α_{et}	0.0021	8.4732×10^{-4}
β_{et}	0.3686	0.0074
p	-0.0082	-0.0058
q	-0.0082	-0.0025
r_1	-0.0012	-8.4732×10^{-4}
r_2	-0.0246	-0.0074
	Expired Warranty Period	
λ_0	-32.54	-0.0166
α_{ex}	0.0053	0.0015
β_{ex}	1.4507	0.0147
γ_u	3.3582×10^{-4}	4.0978×10^{-4}
r_1	-0.0019	-0.0014
r_2	-0.0484	-0.0148

It has been determined that

1. Availabilities (A_0^n, A_0^{et}, A_0^{ex}) in three different periods are highly influenced by λ_0. Though these are least affected by variation in γ_u and γ_m respectively.
2. P^m and P^u both the profit functions are extremely sensitive to λ_0.
3. Variation in C_3^u and C_3^m results in a nominal change in P^m and P^u respectively.

Table 6: *Sensitivity and relative sensitivity analysis of User Profit*

Parameter (y)	Sensitivity Function $Z_y^u = \frac{\partial(P^u)}{\partial y}$	Relative Sensitivity Function $z_y^u = \frac{Z_y^u y}{P^u}$
λ_0	-1.7455×10^4	-0.0268
α_{ex}	1.6730	0.0013
β_{ex}	448.1202	0.0137
γ_u	0.1003	3.6900×10^{-4}
γ_m	0.0545	2.5063×10^{-4}
α_n	0.1474	2.2597×10^{-4}
β_n	39.4737	0.0024
α_{et}	0.3453	4.2349×10^{-4}
β_{et}	59.1900	0.0036
C_0	0.9874	1.5137
C_1^u	$-2.0489 \times 10-4$	-6.2816×10^{-5}
C_2^u	-0.0093	-0.0034
C_3^u	-9.6069×10^{-5}	-0.0044
CP	-1	-0.4599
EC	-1	-0.0460
p	-2.0369	-0.0044
q	-3.6463	-0.0034
r_1	-0.9001	-0.0019
r_2	-26.3183	-0.0242
p_1	494.8074	0.3034
p_2	494.7154	0.4551
p_3	487.5415	0.7474

Moreover, the order or sequence in which different parameters influence the availabilities (A_0^n, A_0^{et}, A_0^{ex}) and profit functions (P^m,P^u) are

- Availability(A_0^n): $\lambda_0 > \beta_n > r_2 > p > q > \alpha_n > r_1 > \gamma_m$.
- Availability(A_0^{et}): $\lambda_0 > \beta_{et} > r_2 > \alpha_{et} > p > q > r_1 > \gamma_m$.
- Availability(A_0^{ex}): $\lambda_0 > \beta_{ex} > r_2 > \alpha_{ex} > r_1 > \gamma_u$.
- Profit Function(P^u):$C_0 > p_3 > CP p_2 > p_1 >> EC > \lambda_0 > r_2 > \beta_{ex} > C_3^u p >> \beta_{et} > C_2^u > q > \beta_n > r_1 > \alpha_{ex} > \alpha_{et} > \gamma_u > \gamma_m > \alpha_n > C_1^u$.
- Profit Function(P^m):$CP > MC > EC > \lambda_0 > p > r_2 > C_3^m > p_2 > p_1 > C_2^m > \beta_{et} > C_1^m > \beta_n > r_1 > \alpha_{et} > \alpha_n > \gamma_m > q$.

Table 7: *Sensitivity and relative sensitivity analysis of Manufacturer Profit*

Parameter (y)	Sensitivity Function $Z_y^m = \frac{\partial(P^m)}{\partial y}$	Relative Sensitivity Function $z_y^m = \frac{Z_y^m y}{P^m}$
λ_0	-2.1458×10^3	-0.0244
γ_m	0.0086	2.9373×10^{-4}
α_n	0.0188	2.1404×10^{-4}
β_n	5.0452	0.0023
α_{et}	0.0441	4.0166×10^{-4}
β_{et}	7.5651	0.0034
C_1^m	$-1.6524 \times 10-4$	-3.0100×10^{-4}
C_2^m	-0.0029	-0.0066
C_3^m	-5.2050×10^{-5}	-0.0178
CP	1	3.4155
EC	1	0.3415
MC	-1	-2.7324
p	-1.5181	-0.0242
q	-0.0089	6.0795×10^{-5}
r_1	-0.0387	-6.1683×10^{-4}
r_2	-3.4432	-0.0235
p_1	-2.1573	-0.0098
p_2	-2.1690	-0.0148

11. Conclusion

A stochastic model of system functioning in normal, extended and expiry warranty conditions is developed in this paper. Markov and regenerative processes are employed to derive various reliability characteristics and profit functions for the manufacturer as well as the user of the system. The derived measures are further illustrated by discussing numerical examples for exponential, Weibull and Erlang cases. System is found available for a longer period in normal as compared to extended or expiry warranty periods. Upper/ lower bounds are obtained for involved rates/ costs, which can affect the system's profitability. Availabilities and profit functions are observed to be most sensitive to the failure rate. Further, for cost consideration, manufacturer and user profit functions are influenced most by the cost price of the system (CP) and revenue generated (C_o), respectively. Since the results for a described system are obtained using general probability time distribution, the finding of the study is lucrative from the standpoints of both the manufacturer and the user if they have real data on failures, repairs, costs, and so on.

Funding

The first author delightedly acknowledges the University Grants Commission (UGC), New Delhi, India for providing financial support.

Disclosure statement

The authors declare that they have no conflict of interest.

References

[1] Ives, B., and Vitale, M. (1988). After the sale: Leveraging maintenance with information technology. *MIS Quarterly*, 12(1): 7–21.

[2] Ritchken, P. H.,Chandramohan, J., and Tapiero, C. S. (1989). Servicing, quality design and control. *IIE Transactions*, 21(3): 213–220.

[3] Singpurwalla, N. D., and Wilson, S. (1993). The warranty problem: its statistical and game-theoretic aspects. *SIAM review*, 35(1): 17–42.

[4] Blischke, W. R., and Murthy, D. N. P. (1992). Product warranty management - I: A taxonomy for warranty policies. *European journal of operational research*, 62(2): 127–148.

[5] Jain, M., and Maheshwari, S. (2006). Discounted costs for repairable units under hybrid warranty. *Applied Mathematics and Computation*, 173(2): 887–901.

[6] Bai, J., and Pham, H. (2006). Cost analysis on renewable full-service warranties for multi-component systems. *European Journal of Operational Research*, 168(2): 492–508.

[7] Huang, H. Z., Liu, Z. J., and Murthy, D. (2007). Optimal reliability, warranty and price for new products. *IIE Transactions*, 39(8): 819–827.

[8] Kadyan, M., and Ramnivas (2013). Cost benefit analysis of a single-unit system with warranty for repair. *Applied Mathematics and Computation*, 223: 346–353.

[9] Rahman, A., and Chattopadhyay, G. (2015). Long term warranty and after sales service. *In Long Term Warranty and After Sales Service*, 1–15.

[10] Pham, H., and Bai, J. (2023). Warranty Policies: Analysis and Perspectives. *In Springer Handbook of Engineering Statistics*, 185–195.

[11] Taneja, G. (2010). Reliability Model for a System with Conditional Warranty and various types of Repair/Replacement. *Advances in Information Theory and Operations Research*, 214–220.

[12] Lei, Y., Liu, Q., and Shum, S. (2017). Warranty pricing with consumer learning. *European Journal of Operational Research*, 263(2): 596–610.

[13] Solkhe, M.,and Taneja, G. (2019). Reliability and Availability Analysis of a system with Periods Before and After Expiry of Conditional Warranty. *International Journal of Agricultural and Statistical Sciences* 15: 281–288.

[14] Niwas, R. (2023). Reliability and profit analysis of a markov model having cost-free warranty with waiting repair facility. *In Engineering Reliability and Risk Assessment*, 145–160.

[15] Hooti, F., Ahmadi, J., and Longobardi, M. (2020). Optimal extended warranty length with limited number of repairs in the warranty period. *Reliability Engineering & System Safety*, 203: 107111.

[16] Lorson, E., and Suiter, M. (2021). Do you want an extended warranty with that? *Page One Economics Newsletter.*

[17] Huang, Y. S., Fang, c. C., Lu, C. M., and Tseng, T. L. B. (2022). Optimal warranty policy for consumer electronics with dependent competing failure processes. *Reliability Engineering & System Safety*, 222: 108418.

[18] Salmasnia, A.,and Hatami, A. (2019). A three-level servicing contract with an integrated maintenance plan, warranty policy, technology level, and pricing model that considers the time value of money. *Scientia Iranica* 30(2): 674–690.