ANALYZING LOAD SHARING SYSTEM RELIABILITY: A MODIFIED WEIBULL DISTRIBUTION APPROACH

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Abstract

Load sharing systems have the ability to distribute the workload among its components. For analyzing a two component parallel load sharing system, the accelerated failure time (AFT) based model with component lifetimes as linear failure rate distribution have been recently proposed in the literature. In the present study, the component lifetimes are assumed to follow a modified Weibull distribution, which is the generalization of many standard lifetime distributions such as exponential, Weibull, Rayleigh, and linear failure rate. The use of modified Weibull distribution leads to a new family of bivariate distributions for ordered random variables. We have also looked into the associated inference techniques for the proposed model. In order to evaluate the effectiveness of the suggested estimating approaches, we conducted a simulation study. In order to provide a practical application and better understanding, we carefully examine a dataset related to motors.

Keywords: accelerated failure time model, conditional distribution, load sharing, modified Weibull distribution, order statistics

1. Introduction

Load sharing systems are characterized by their ability to distribute the workload among multiple components, such that if one component fails, the remaining components bear the additional workload. This can either increase or decrease the load on each surviving component. Load sharing systems have been extensively investigated in various engineering domains, such as software and hardware reliability, power plants, computing workload analysis, and fiber composites (Wang *et al.* [1]).

Liu [2] presents various instances that illustrate the concept of load sharing systems. These include scenarios like electric generators distributing an electrical load within a power plant, CPUs operating in a multiprocessor computer system, cables supporting a suspension bridge, bolts fastening a wheel assembly onto a truck, and valves or pumps functioning in a hydraulic system. When any of these components fail, the remaining components must bear an additional load, which can elevate their failure rates. In an intriguing study by Drummond et al. [3] conducted on vertebrate species, it was observed that when a litter mate dies due to food shortage, the surviving offsprings receive a larger portion of the available food supply, leading to improved growth. This finding highlights how the failure of one individual can inadvertently benefit the surviving members. Furthermore, in the realm of software testing, the detection of a fault can uncover previously undetected critical faults. This demonstrates that a component's failure can facilitate the discovery of other hidden issues, thereby enhancing the overall reliability of the system. These examples collectively illustrate that when a component fails, it can actually enhance the remaining components' remaining lifespan, resulting in a higher growth rate for the surviving components.

Daniels [4] conducted the first study on the phenomenon of load sharing and load sharing systems. A thorough analysis of load sharing systems up till 2009 is present in Dewan and Naik-Nimbalkar [5]. Deshpande et al. [6], Park [7], Singh and Gupta [8], Park [9], Gurov and Utkin [10], Sutar and Naik-Nimbalkar [11]-[12], Krivtsov et al. [13], Wang et al. [1] and Sutar and Naik-Nimbalkar [14] have all published studies and modeled the load sharing phenomenon since then.

The study of load-sharing systems with a k-out-of-m configuration was suggested by Sutar and Naik-Nimbalkar [11] with modelling strategy based on the accelerated failure time (AFT) model. They concentrated on a particular configuration, a parallel load sharing system consisting of two components with baseline as the linear failure rate distribution. The associated inference techniques were also covered by the researchers. The distributions used there in for the ordered random variables are a subset of a larger family of distributions known as sequential order statistics. Kamps [15] first described this family of distributions and Cramer and Kamps [16] further developed them.

This study utilizes the load sharing model based on accelerated failure time (AFT), proposed by Sutar and Naik-Nimbalkar [11], to examine the load sharing phenomenon within a parallel system consisting of two components. We adopt a modified Weibull distribution (MWD) as the baseline distribution for the components in the system, which is characterized by three parameters: λ_1, λ_2, and λ_3. The introduction of this three-parameter MWD was done by Sarhan and Zaindin [17].

It is important to highlight that the three-parameter MWD provides a comprehensive representation of various distributions, including exponential, Weibull, Rayleigh and linear failure rate. Thus, the MWD serves as a versatile baseline distribution for the component lifetime in any load sharing system. The subsequent sections of this paper are organized as follows.

In Section 2, we address the AFT-based load-sharing model for a parallel load sharing system consisting of two components and with a modified Weibull distribution for component lifetimes. In Section 3, the inference procedures are thoroughly examined, while Section 4 focuses on the simulation study. Section 5 demonstrates an application using real data, and the last section presents the concluding remarks.

2. Proposed AFT based Load sharing model

We investigate a parallel system consisting of two components. The cumulative distribution function (c.d.f.) of the components follows a MWD characterized by three parameters: λ_1, λ_2, and λ_3. The probability density function (p.d.f.), survival function (s.f.), and hazard rate function of the MWD with parameters λ_1, λ_2, and λ_3 are provided as follows.

$$f(u) = \left(\lambda_1 + \lambda_2\lambda_3 u^{\lambda_3 - 1}\right) \exp\left\{-\left(\lambda_1 u + \lambda_2 u^{\lambda_3}\right)\right\},\ u > 0, \lambda_1, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_2 > 0 \quad (1)$$

$$\bar{F}(u) = \exp\left\{-\left(\lambda_1 u + \lambda_2 u^{\lambda_3}\right)\right\}, u > 0, \lambda_1, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_2 > 0$$
$$h(u) = \left(\lambda_1 + \lambda_2 \lambda_3 u^{\lambda_3 - 1}\right), u > 0, \lambda_1, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_2 > 0$$

The three-parameter modified Weibull distribution, denoted as MWD($\lambda_1, \lambda_2, \lambda_3$), serves as a generalization of the following distributions.

(a) It represents the Exponential distribution, ED(λ_1), when λ_2 is set to 0 and λ_3 is finite.

(b) It encompasses the Weibull distribution, WD(λ_2, λ_3), when λ_1 is set to 0.

(c) It corresponds to the Rayleigh distribution, RD(λ_2), when λ_3 is set to 2 and λ_1 is set to 0.

(d) It encompasses the Linear failure rate, LFR(λ_1, λ_2), when λ_3 is set to 2.

For more details on MWD($\lambda_1, \lambda_2, \lambda_3$), one can refer Sarhan and Zaindin [17] and references cited therein.

The load sharing behavior observed in a system comprising two components is captured by the AFT model, which was introduced by Sutar and Naik-Nimbalkar [11]. We denote the lifetimes of the two components in the system as V_1 and V_2. These lifetimes are considered independent and identically distributed random variables. The baseline densities of V_1 and V_2 are denoted as $f_1(\cdot)$ and $f_2(\cdot)$, respectively, while their corresponding baseline survival functions are denoted as $\bar{F}_1(\cdot)$ and $\bar{F}_2(\cdot)$. Let $X = \min(V_1, V_2)$ denote time of the first failure and $Y = \max(V_1, V_2)$ denote the time of the second failure or the system failure time. Consequently, the marginal density of the first failure can be expressed as follows.

$$g(x) = \left(2\lambda_1 + 2\lambda_2\lambda_3 x^{\lambda_3 - 1}\right) \exp\left\{-\left(2\lambda_1 x + 2\lambda_2 x^{\lambda_3}\right)\right\},\ x > 0, \lambda_1 > 0, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_2 > 0. \tag{2}$$

It is worth mentioning that the distribution of the first failure is identical to the baseline distribution, which is a modified Weibull distribution with parameters ($2\lambda_1, 2\lambda_2, \lambda_3$). Following the AFT load sharing model, the conditional density of variable Y given that $X = x$, as well as the joint density of the variables X and Y, can be expressed in the following manner.

$$g(y|x) = \left\{\frac{\lambda_1}{\beta} + \frac{\lambda_2\lambda_3 y^{\lambda_3 - 1}}{\beta^{\lambda_3}}\right\} \exp\left\{-\frac{\lambda_1}{\beta}(y - x) - \frac{\lambda_2}{\beta^{\lambda_3}}(y^{\lambda_3} - x^{\lambda_3})\right\}, \tag{3}$$
$$0 < x < y < \infty,\ \beta > 0,\ \lambda_1 > 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_1 + \lambda_2 > 0,$$

$$g(x,y) = 2\left(\lambda_1 + \lambda_2\lambda_3 x^{\lambda_3 - 1}\right)\left\{\frac{\lambda_1}{\beta} + \frac{\lambda_2\lambda_3 y^{\lambda_3 - 1}}{\beta^{\lambda_3}}\right\}$$
$$\times \exp\left\{-\frac{\lambda_1}{\beta}(y - x) - \frac{\lambda_2}{\beta^{\lambda_3}}(y^{\lambda_3} - x^{\lambda_3}) - \left(2\lambda_1 x + 2\lambda_2 x^{\lambda_3}\right)\right\}, \tag{4}$$
$$0 < x < y < \infty,\ \beta > 0,\ \lambda_1 > 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_1 + \lambda_2 > 0,$$

It is important to note that when the parameter β is equal to 1, the joint density described in equation (4) simplifies to the joint density of independent random variables X and Y. Essentially, when $\beta = 1$, it indicates no load sharing effect, and hence the occurrences of the two failures (first and second) are independent of each other. The parameter β is referred to as the load sharing parameter in this context. In the following section, we will delve into the inference procedures associated with this concept.

3. Inference procedures

In this section, we examine different methods for estimating the unknown parameters and introduce a testing procedure to assess the presence of the load sharing effect.

3.1. Direct estimation procedure

The complete data, denoted as $(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \{(x_i, y_i) : x_i \leq y_i;\ i = 1, 2, ..., n\}$, represents the set of observations. The log-likelihood function based on this complete data can be expressed as follows.

$$\log L = n \log 2 + \sum_{i=1}^{n} \log\left(\lambda_1 + \lambda_2\lambda_3 x_i^{\lambda_3 - 1}\right) + \sum_{i=1}^{n} \log\left(\lambda_1\beta^{\lambda_3} + \lambda_2\lambda_3\beta y_i^{\lambda_3 - 1}\right) - n(\lambda_3 + 1)\log\beta$$
$$- \frac{\lambda_1}{\beta}\sum_{i=1}^{n}(y_i - x_i) - \frac{\lambda_2}{\beta^{\lambda_3}}\sum_{i=1}^{n}(y_i^{\lambda_3} - x_i^{\lambda_3}) - 2\lambda_1\sum_{i=1}^{n} x_i - 2\lambda_2\sum_{i=1}^{n} x_i^{\lambda_3}.$$

We observe that, the log-likelihood equations $\frac{\partial \log L}{\partial \beta} = 0$, $\frac{\partial \log L}{\partial \lambda_1} = 0$, $\frac{\partial \log L}{\partial \lambda_2} = 0$ and $\frac{\partial \log L}{\partial \lambda_3} = 0$ do not have explicit solutions for the parameters λ_1, λ_2, λ_3, β. The mathematical expressions for the score functions, specifically $\frac{\partial \log L}{\partial \beta} = 0$, $\frac{\partial \log L}{\partial \lambda_1} = 0$, $\frac{\partial \log L}{\partial \lambda_2} = 0$, and $\frac{\partial \log L}{\partial \lambda_3} = 0$, can be found in Appendix (A).

In the following subsection, we outline a two-step approach for determining the values of the unknown parameters λ_1, λ_2, λ_3, and β.

3.2. Two-step parameter estimation procedure

The process of estimating the values of λ_1, λ_2, λ_3, and β has been conducted using a two-step methodology.
Step 1. We observe the first failure, X, and estimate baseline parameters, namely, λ_1, λ_2 and λ_3 by using the MCEM procedure proposed by Sutar [18].
Step 2. In order to estimate the load sharing parameter β, we utilize the conditional distribution of Y given $X = x$, as expressed in equation (3). The estimates of λ_1, λ_2, and λ_3 obtained in *Step 1* are then substituted into that equation to perform the estimation. We refer to this estimation process as a two-step estimation procedure, and the subsequent subsections outline these two steps in detail.

3.2.1 Estimation of λ_1, λ_2 and λ_3 (*Step 1*)

It is worth noting that the distribution of the first failure, X, is also a modified Weibull distribution (MWD) with parameters $2\lambda_1$, $2\lambda_2$, and λ_3. Let us denote $2\lambda_1 = \gamma_1$, $2\lambda_2 = \gamma_2$, thus distribution of X is MWD with parameters γ_1, γ_2 and λ_3. We use the MCEM algorithm proposed by Sutar [18] for finding the estimates of γ_1, γ_2 and λ_3. To implement the proposed MCEM algorithm, we take two independent random variables U_1 and U_2, which has exponential (γ_1) and Weibull (γ_2, λ_3) distributions, with their respective survival functions as $\exp(-\gamma_1 u_1)$ and $\exp(-\gamma_2 u_2^{\lambda_3})$. Let $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\lambda}_3$ be the MLEs of γ_1, γ_2 and λ_3 obtained through MCEM algorithm, then the MLEs of λ_1, λ_2 and λ_3 can be obtained as $\hat{\lambda}_1 = \frac{\hat{\gamma}_1}{2}$, $\hat{\lambda}_2 = \frac{\hat{\gamma}_2}{2}$ and $\hat{\lambda}_3$.

3.2.2 Estimation of β (*Step 2*)

To estimate the load sharing parameter β, we utilize the conditional distribution of Y given $X = x$ as described in equation (3). In this study, two methods are proposed for estimating β, which are discussed as follows.
Method I : It can be noted, the conditional distribution of Y given $X = x$ as truncated MWD with parameters $\frac{\lambda_1}{\beta} = \theta_1$(say), $\frac{\lambda_2}{\beta^{\lambda_3}} = \theta_2$ (say), $\lambda_3 = \theta_3$ (say) truncated at $X = x$. Furthermore, the conditional distribution mentioned is equivalent to the distribution of the minimum value between

two independent random variables, denoted as W_1 and W_2. Specifically, W_1 follows a truncated exponential distribution, which is truncated below x and has a parameter θ_1. On the other hand, W_2 follows a truncated Weibull distribution, also truncated below x, with parameters θ_2 and θ_3. The survival functions of W_1 and W_2 are $\exp\{-\theta_1(w_1 - x)\}$ and $\exp\{-\theta_2(w_2^{\theta_3} - x^{\theta_3})\}$, respectively. Let us consider the complete data for $i = 1, 2, ..., n$ as a set of $2n$ independent random variables denoted as (W_{1i}, W_{2i}). The random variables W_{1i} represent truncated exponential distributions, truncated below x_i, with a parameter θ_1, while W_{2i} represent truncated Weibull distributions, truncated at x_i, with parameters (θ_2, θ_3). Additionally, we define Z_{2i} as the minimum value between W_{1i} and W_{2i}. Consequently, Z_{2i} follows a truncated MWD (Minimum of Weibull and Exponential Distribution) distribution, characterized by a probability density function (p.d.f)

$$g(z_{2i}) = \left(\theta_1 + \theta_2\theta_3 z_{2i}^{\theta_3 - 1}\right) \exp\left\{-\theta_1(z_{2i} - x_i) - \theta_2(z_{2i}^{\theta_3} - x_i^{\theta_3})\right\},$$

$$0 < x_i < z_{2i} < \infty,\ \beta > 0, \theta_1 > 0, \theta_2 \geq 0, \theta_3 \geq 0, \theta_1 + \theta_2 > 0.$$

We can regard the observed values $\underline{y} \equiv (y_1, y_2, ..., y_n)$ as corresponding to the values of $\underline{Z}_2 \equiv (Z_{21}, Z_{22}, ..., Z_{2n})$.

The joint density of W_1 and W_2 given $\underline{X} = \underline{x}(\equiv (x_1, x_2, ..., x_n))$ can be written as

$$g(w_1, w_2|\mathbf{x}) = \{\theta_1, \theta_2, \theta_3\}^n \prod_{i=1}^{n} w_{2i}^{\theta_3 - 1} \exp\left\{-\theta_1(w_{1i} - x_i) - \theta_2(w_{2i}^{\theta_3} - x_i^{\theta_3})\right\}, \tag{5}$$

The log-likelihood can be expressed in the following manner.

$$\log L = n \log(\theta_1\theta_2\theta_3) + \theta_3 \sum_{i=1}^{n} \log(u_{2i}) - \theta_1 \sum_{i=1}^{n} u_{1i} - \theta_2 \sum_{i=1}^{n} u_{2i}^{\theta_3}.$$

In order to perform the E step, it is necessary to calculate the conditional expectation of $E_c[\log L|\underline{Z}_2]$. This can be represented as follows.

$$\begin{aligned} E_c[\log L|\underline{Z}_2] &= n \log(\theta_1^*\theta_2^*\theta_3^*) + \theta_3^* E_c\left[\sum_{i=1}^{n} \log(U_{2i})|\underline{Z}_2\right] \\ &- \theta_1^* E_c\left[\sum_{i=1}^{n} U_{1i}|\underline{Z}_2\right] - \theta_2^* E_c\left[\sum_{i=1}^{n} U_{2i}^{\theta_3^*}|\underline{Z}_2\right]. \end{aligned} \tag{6}$$

Remark 1. For i equal to 1, 2, and 3, the variables θ_i and θ_i^* represent the values of θ_i at the current iteration and the next iteration of the MCEM (Monte Carlo Expectation-Maximization) algorithm. Specifically, if $\theta_i = \theta_i^{(p)}$ represents the estimated value of θ_i at the p-th iteration, and $\theta_i^* = \theta_i^{(p+1)}$ represents the estimated value of θ_i at the $(p+1)$-th iteration, then θ_i and θ_i^* respectively denote the values of θ_i at the p-th and $(p+1)$-th iterations of the MCEM algorithm.

The conditional density of W_{11} given $X = x$ and $Z_{21} = z_{21}$ is a mixed probability density function (p.d.f.) and can be expressed as follows.

$$g(w_1|x, z_{21}) = \frac{\theta_1}{\left(\theta_1 + \theta_2\theta_3 z_{21}^{\theta_3 - 1}\right)} \left\{I_{(w_1 = z_{21})} + \theta_2\theta_3 z_{21}^{\theta_3 - 1} \exp\{-\theta_1(w_1 - z_{21})\} I_{(w_1 > z_{21})}\right\}.$$

where, $I_A(\cdot)$ is indicator function defined on set A. The details regarding the same are Appendix (B). Thus, the conditional expectation of W_{11} given X and Z_{21} can be obtained as

$$\begin{aligned} E(W_1|X, Z_{21}) &= \int w_1 g(w_1|x, z_{21}) dw_1 \\ &= \frac{\theta_1}{\left(\theta_1 + \theta_2\theta_3 z_{21}^{\theta_3 - 1}\right)} \left\{Z_{21} + \theta_2\theta_3 Z_{21}^{\theta_3 - 1} \exp\{\theta_1 Z_{21}\} \int_{Z_{21}}^{\infty} w_1 \exp\{-\theta_1 w_1\} dw_1\right\}. \end{aligned}$$

By using the result,

$$\int_{Z_{21}}^{\infty} w_1 \exp\{-\theta_1 w_1\} dw_1 = \frac{(\theta_1 Z_{21} + 1) \exp\{-\theta_1 Z_{21}\}}{\theta_1^2},$$

we get

$$E(W_1|X, Z_{21}) = \frac{\theta_1}{\left(\theta_1 + \theta_2\theta_3 z_{21}^{\theta_3-1}\right)} \left\{ Z_{21} + \theta_2\theta_3 Z_{21}^{\theta_3-1} \frac{\exp\{\theta_1 Z_{21}\}(\theta_1 Z_{21}+1)\exp\{-\theta_1 Z_{21}\}}{\theta_1^2} \right\}$$
$$= \frac{1}{\theta_1} + \left\{ Z_{21} - \left(\theta_1 + \theta_2\theta_3 Z_{21}^{\theta_3-1}\right)^{-1} \right\} = K(Z_{21})(\text{say}),$$

and hence

$$E\left(\sum_{i=1}^{n} W_{1i} | \underline{X}, \underline{Z}_2\right) = \frac{n}{\theta_1} + \sum_{i=1}^{n} Z_{2i} - \sum_{i=1}^{n} \frac{1}{\left(\theta_1 + \theta_2\theta_3 Z_{21}^{\theta_3-1}\right)}.$$

Thus, given $\{Z_{2i} = y_i, i = 1, 2, ..., n\}$ and $\{X_i = x_i, i = 1, 2, ..., n\}$, we get

$$E\left(\sum_{i=1}^{n} W_{1i} | \underline{x}, \underline{y}\right) = \frac{n}{\theta_1} + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \frac{1}{\left(\theta_1 + \theta_2\theta_3 y_i^{\theta_3-1}\right)}.$$

Likewise, the conditional density function of W_{21} given $X = x$, $Z_{21} = z_{21}$, and the corresponding conditional expectation can be expressed as follows.

$$g(w_2|x, z_{21}) = \frac{\theta_2\theta_3}{\left(\theta_1 + \theta_2\theta_3 z_{21}^{\theta_3-1}\right)} \left\{ z_{21}^{\theta_3-1} I_{(w_2=z_{21})} + \theta_1 w_2^{\theta_3-1} e^{\{-\theta_2(W_2^{\theta_3} - z_{21}^{\theta_3})\}} I_{(w_2>z_{21})} \right\}$$

and

$$E_c\left\{W_{21}^{\theta_3^*} | \underline{X}, Z_{21}\right\} = \frac{\theta_2\theta_3 Z_{21}^{\theta_3^*+\theta_3+1}}{\left(\theta_1 + \theta_2\theta_3 Z_{21}^{\theta_3-1}\right)} + \frac{\theta_1\theta_2\theta_3 e^{\theta_2 Z_{21}^{\theta_3}}}{\left(\theta_1 + \theta_2\theta_3 Z_{21}^{\theta_3-1}\right)} \int_{Z_1}^{\infty} W_2^{\theta_3^*+\theta_3-1} e^{-\theta_2 W_2^{\theta_3}} dW_2.$$

After simplification, we get,

$$E_c\left\{W_{21}^{\theta_3^*} | \underline{X}, Z_{21}\right\} = \frac{\theta_2\theta_3 Z_{21}^{\theta_3^*+\theta_3+1}}{\left(\theta_1 + \theta_2\theta_3 Z_{21}^{\theta_3-1}\right)} + \frac{\theta_1 \theta_2^{-\frac{\theta_3^*}{\theta_3}}}{\left(\theta_1 + \theta_2\theta_3 Z_{21}^{\theta_3-1}\right)} \int_0^{\infty} \left(u + \theta_2 Z_{21}^{\theta_3}\right)^{\frac{\theta_3^*}{\theta_3}} e^{-u} du$$

Let

$$T = \int_0^{\infty} \left(u + \theta_2 Z_{21}^{\theta_3}\right)^{\frac{\theta_3^*}{\theta_3}} e^{-u} du = E\left[V + \theta_2 Z_{21}^{\theta_3}\right]^{\frac{\theta_3^*}{\theta_3}},$$

where V has an exponential distribution with mean 1. By employing the Monte Carlo technique, we can replace T with a Monte Carlo sum, which is given as follows.

$$T = E\left[V + \theta_2 Z_{21}^{\theta_3}\right]^{\frac{\theta_3^*}{\theta_3}} = \frac{1}{m} \sum_{j=1}^{m} \left(v_j + \theta_2 Z_{21}^{\theta_3}\right)^{\frac{\theta_3^*}{\theta_3}},$$

where, $\{v_1, v_2, ..., v_m\}$ is random sample of size m (sufficiently large) from exponential distribution with mean 1. Thus, we get

$$E_c\left\{\sum_{i=1}^{n} W_{2i}^{\theta_3^*} | \underline{X}, \underline{Z}_2\right\} = \sum_{i=1}^{n} \left\{ \frac{\theta_2\theta_3 Z_{2i}^{\theta_3^*+\theta_3-1}}{\left(\theta_1 + \theta_2\theta_3 Z_{2i}^{\theta_3-1}\right)} + \frac{\theta_1 \theta_2^{-\frac{\theta_3^*}{\theta_3}}}{m\left(\theta_1 + \theta_2\theta_3 Z_{2i}^{\theta_3-1}\right)} \sum_{j=1}^{m} \left(v_j + \theta_2 Z_{2i}^{\theta_3}\right)^{\frac{\theta_3^*}{\theta_3}} \right\}$$

and hence $\{Z_{2i} = y_i, i = 1,2,...,n\}$, that is $\underline{Z}_2 = \underline{y}$, we get

$$E_c\left\{\sum_{i=1}^{n} W_{2i}^{\theta_3^*}|\underline{x},\underline{y}\right\} = \sum_{i=1}^{n}\left\{\frac{\theta_2\theta_3 y_i^{\theta_3^*+\theta_3-1}}{\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)} + \frac{\theta_1\theta_2^{-\frac{\theta_3^*}{\theta_3}}}{m\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}\sum_{j=1}^{m}\left(v_j+\theta_2 y_i^{\theta_3}\right)^{\frac{\theta_3^*}{\theta_3}}\right\}.$$

By applying similar arguments to calculate $E_c\left\{\sum_{i=1}^{n}\log\left(W_{2i}\right)|\underline{x},\underline{y}\right\}$, we obtain

$$E_c\left\{\sum_{i=1}^{n}\log\left(W_{2i}\right)|\underline{x},\underline{y}\right\} = \sum_{i=1}^{n}\left\{\frac{\theta_2\theta_3\log y_i}{\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}\right\}$$
$$+\sum_{i=1}^{n}\left\{\frac{\theta_1}{m\theta_2\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}\left(\sum_{j=1}^{m}\left(v_j+\theta_2 y_i^{\theta_3}\right)-\log\theta_2\right)\right\}.$$

To carry out the M-step, we obtain the following expression from equation (6).

$$\frac{\partial E_c\left[\log L|\underline{X},\underline{Z}_2\right]}{\partial\theta_1^*} = \frac{n}{\theta_1^*} - E_c\left[\sum_{i=1}^{n} W_{1i}|\underline{X},\underline{Z}_2\right], \tag{7}$$

$$\frac{\partial E_c\left[\log L|\underline{X},\underline{Z}_2\right]}{\partial\theta_2^*} = \frac{n}{\theta_2^*} - E_c\left[\sum_{i=1}^{n} W_{2i}^{\theta_3^*}|\underline{X},\underline{Z}_2\right], \tag{8}$$

$$\frac{\partial E_c\left[\log L|\underline{X},\underline{Z}_2\right]}{\partial\theta_3^*} = \frac{n}{\theta_3^*} - E_c\left[\sum_{i=1}^{n}\log(W_{2i})|\underline{X},\underline{Z}_2\right] - \theta_2^*\frac{\partial E_c\left[\sum_{i=1}^{n} W_{2i}^{\theta_3^*}|\underline{X},\underline{Z}_2\right]}{\partial\theta_3^*}. \tag{9}$$

From (7) and (8), we get

$$\frac{\partial E_c\left[\log L|\underline{X},\underline{Z}_2\right]}{\partial\theta_1^*} = 0 \Rightarrow \theta_1^* = \frac{n}{E_c\left[\sum_{i=1}^{n} W_{1i}|\underline{X},\underline{Z}_2\right]}, \tag{10}$$

$$\frac{\partial E_c\left[\log L|\underline{X},\underline{Z}_2\right]}{\partial\theta_2^*} = 0 \Rightarrow \theta_2^* = \frac{n}{E_c\left[\sum_{i=1}^{n} W_{2i}^{\theta_3^*}|\underline{X},\underline{Z}_2\right]}. \tag{11}$$

Based on the expression (5), it can be inferred that $t(W_1) = \sum_{i=1}^{n} W_{1i}$ serves as the sufficient statistic for θ_1. In order to carry out the M-step, we equate the sufficient statistic to its expectation, which takes the following form in our scenario.

$$E[W_1] = \frac{1}{\theta_1}. \tag{12}$$

The EM iterations alternate between expressions (12) and (10). Let $\theta_1^{(p)}$ represent the estimate of θ_1 at the p-th iteration step of the MCEM algorithm. The updated estimate $\theta_1^{(p+1)}$ is determined by the following equation.

$$\theta_1^{(p+1)} = \left\{\frac{1}{\theta_1^{(p)}} + \frac{1}{n}\left[\sum_{i=1}^{n}(y_i - x_i) - \sum_{i=1}^{n}\frac{1}{\left(\theta_1^{(p)}+\theta_2^{(p)}\theta_3^{(p)}y_i^{\theta_3^{(p)}-1}\right)}\right]\right\}^{-1}. \tag{13}$$

To estimate θ_3, we substitute equation (11) into equation (9), resulting in a nonlinear equation in terms of θ_3. This equation can be expressed as follows.

$$\frac{\partial E_c\left[\log L|\underline{x},\underline{y}\right]}{\partial\theta_3^*} = \frac{n}{\theta_3^*} - E_c\left[\sum_{i=1}^{n}\left(\log(W_{2i})\right)|\underline{x},\underline{y}\right] - n\frac{\left\{\frac{\partial E_c\left[\log W_{2i}^{\theta_3^*}|\underline{x},\underline{y}\right]}{\partial\theta_3^*}\right\}}{E_c\left[\sum_{i=1}^{n} W_{2i}^{\theta_3^*}|\underline{x},\underline{y}\right]}.$$

We employ the Newton-Raphson iterative method to estimate θ_3^*. Let $\theta_3^{(r)}$ denote the estimate of θ_3 at the r-th iteration of the Newton-Raphson method. The updated estimate $\theta_3^{(r+1)}$ is determined by the following equation.

$$\theta_3^{(r+1)} = \theta_3^{(r)} - \frac{\left\{ \frac{\partial E_c\left[\log L|\underline{x},\underline{y}\right]}{\partial \theta_3^{(r)}} \right\}}{\left\{ \frac{\partial^2 E_c\left[\log L|\underline{x},\underline{y}\right]}{\partial (\theta_3^{(r)})^2} \right\}}. \tag{14}$$

The detailed expressions used in equation (14) can be found in Appendix (C). The process should be iterated until the convergence criterion is satisfied. The value of θ_3 at the $(p+1)$-th iteration of MCEM, denoted as $\theta_3^* = \theta_3^{(p+1)}$, represents the stabilized value of θ_3 obtained through the Newton-Raphson method. Once we get the estimate of θ_3, we can obtain θ_2 at $(p+1)^{th}$ iteration by substituting (14) in (11). That is $\theta_2^{(p+1)}$ is given by

$$\theta_2^{(p+1)} = \frac{n}{E_c\left[\sum_{i=1}^n W_{2i}^{\theta_3^{(p+1)}} |\underline{x}, \underline{y}\right]}. \tag{15}$$

Thus, we get the estimates $\widehat{\theta}_1 = \widehat{(\lambda_1/\beta)}$, $\widehat{\theta}_2 = \widehat{(\lambda_2/\beta^{\lambda_3})}$ of (λ_1/β) and $\left(\lambda_2/\beta^{\lambda_3}\right)$ respectively. Now, we have the two estimates of β as $\widehat{\beta}_1 = (\widehat{\lambda}_1/\widehat{\theta}_1)$ and $\widehat{\beta}_2 = (\widehat{\lambda}_2/\widehat{\theta}_2)^{1/\widehat{\lambda}_3}$.Here, $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, and $\widehat{\lambda}_3$ represent the estimates of λ_1, λ_2, and λ_3, respectively, obtained in Step 1. The estimate of β is obtained by taking the average of $\widehat{\beta}_1$ and $\widehat{\beta}_2$. This estimation method is referred to as the 'average method' for estimating β.

Method II : The likelihood, based on the conditional density of Y given $X = x$, can be expressed as follows.

$$L = \prod_{i=1}^n \left\{ \frac{\lambda_1}{\beta} + \frac{\lambda_2\lambda_3 y_i^{\lambda_3 - 1}}{\beta^{\lambda_3}} \right\} \exp\left\{ -\frac{\lambda_1}{\beta}\sum_{i=1}^n (y_i - x_i) - \frac{\lambda_2}{\beta^{\lambda_3}}\sum_{i=1}^n (y_i^{\lambda_3} - x_i^{\lambda_3}) \right\}.$$

Then the log-likelihood can be written as

$$\log L = \sum_{i=1}^n \log\left\{ \frac{\lambda_1}{\beta} + \frac{\lambda_2\lambda_3 y_i^{\lambda_3 - 1}}{\beta^{\lambda_3}} \right\} - \frac{\lambda_1}{\beta}\sum_{i=1}^n (y_i - x_i) - \frac{\lambda_2}{\beta^{\lambda_3}}\sum_{i=1}^n (y_i^{\lambda_3} - x_i^{\lambda_3}).$$

Then we have,

$$\frac{\partial}{\partial \beta}\log L = -\sum_{i=1}^n \frac{\left\{ \frac{\lambda_1}{\beta^2} + \frac{\lambda_2\lambda_3^2 y_i^{\lambda_3-1}}{\beta^{\lambda_3+1}} \right\}}{\left\{ \frac{\lambda_1}{\beta} + \frac{\lambda_2\lambda_3 y_i^{\lambda_3-1}}{\beta^{\lambda_3}} \right\}} + \frac{\lambda_1}{\beta^2}\sum_{i=1}^n (y_i - x_i) - \frac{\lambda_2\lambda_3}{\beta^{\lambda_3+1}}\sum_{i=1}^n (y_i^{\lambda_3} - x_i^{\lambda_3}),$$

$$\frac{\partial^2}{\partial \beta^2}\log L = -\sum_{i=1}^n \frac{\left\{ \frac{\lambda_1}{\beta^2} + \frac{\lambda_2\lambda_3^2 y_i^{\lambda_3-1}}{\beta^{\lambda_3+1}} \right\}\left\{ \frac{2\lambda_1}{\beta^3} + \frac{\lambda_2\lambda_3^2(\lambda_3+1) y_i^{\lambda_3-1}}{\beta^{\lambda_3+2}} \right\} + \left\{ \frac{\lambda_1}{\beta^2} + \frac{\lambda_2\lambda_3^2 y_i^{\lambda_3-1}}{\beta^{\lambda_3+1}} \right\}^2}{\left\{ \frac{\lambda_1}{\beta} + \frac{\lambda_2\lambda_3 y_i^{\lambda_3-1}}{\beta^{\lambda_3}} \right\}^2}$$
$$-\frac{2\lambda_1}{\beta^3}\sum_{i=1}^n (y_i - x_i) - \frac{\lambda_2\lambda_3(\lambda_3+1)}{\beta^{\lambda_3+2}}\sum_{i=1}^n (y_i^{\lambda_3} - x_i^{\lambda_3}).$$

To estimate the load sharing parameter β, we substitute the maximum likelihood estimates (MLEs) $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_3$ of λ_1, λ_2, and λ_3, respectively, into the aforementioned expressions. The Newton-Raphson iteration method is then employed to obtain the estimate of β. Let $\beta^{(m)}$ be the estimate of β at m^{th} iteration. The estimate of β at $(m+1)^{th}$ iteration is given by

$$\beta^{(m+1)} = \beta^{(m)} - \frac{\left(\frac{\partial}{\partial \beta}\log L\right)|_{(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3)}}{\left(\frac{\partial^2}{\partial \beta^2}\log L\right)|_{(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3)}}.$$

We termed this procedure to estimate β as the 'iteration method'. The subsequent section presents the test procedure used to assess the presence of the load sharing effect.

3.3. Test Procedure

To test the load sharing effect, we set up the null hypothesis H_0 stating that the failure of a component does not affect the survival components, and the alternative hypothesis H_1 stating that there exists a load sharing phenomenon. Specifically, we express the null hypothesis as $H_0 : \beta = 1$, indicating no load sharing effect, and the alternative hypothesis as $H_1 : \beta \neq 1$, indicating the presence of a load sharing effect. To test these hypotheses, we employ a score type test, as used by Sutar and Naik-Nimbalkar [11]. The test statistic follows an asymptotic χ^2 distribution with 1 degree of freedom.

In the subsequent section, we present a simulation study to evaluate and compare the performance of two estimation methods: the average method and the iterative method. The simulation study aims to assess the accuracy and efficiency of these methods under various scenarios and conditions. By conducting simulations and analyzing the results, we can gain insights into the strengths and limitations of each method and make informed decisions about their suitability for practical applications.

4. Simulation study

In this section, we performed a simulation study to assess the performance of the proposed estimation procedure in estimating unknown parameters. We generated a total of 10,000 samples from the joint density described in equation (4) for different combinations of sample sizes (n) and parameter values. This allowed us to examine the behavior and accuracy of the estimation procedure under various scenarios and conditions.

For sample sizes of n = 20, 30, 50, and 100, we considered different parameter combinations, namely $(\lambda_1, \lambda_2, \lambda_3, \beta)$ as (1,2,0.5,0.5), (1,2,0.5,1), (1,2,0.5,1.5), (1,2,1,0.5), (1,2,1,1), (1,2,1,1.5), (1,2,2,0.5), (1,2,2,1), (1,2,2,1.5), (2,2,0.5,0.5), (2,2,0.5,1), (2,2,0.5,1.5), (2,2,1,0.5), (2,2,1,1), (2,2,1,1.5), (2,2,2,0.5), (2,2,2,1), and (2,2,2,1.5).

The average estimates of the unknown parameters $(\lambda_1, \lambda_2, \lambda_3, \beta)$ obtained through Method-I (Two-step Procedure), denoted as $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\beta})$, along with their corresponding standard errors (SE), i.e., SE($\hat{\lambda}_1$), SE($\hat{\lambda}_2$), SE($\hat{\lambda}_3$), and SE($\hat{\beta}$), are presented in Table 1 and Table 2. The simulation results reveal a clear pattern: as the sample size grows larger, the standard errors exhibit a decreasing trend. This indicates that larger sample sizes lead to enhanced precision in estimating the parameters, implying that the estimates become more accurate and reliable.

We also conducted a simulation study for the iterative method. We generated 10,000 samples with sizes $n = 30, 50$, and 100 from the joint density given in equation (4). We considered different parameter combinations as (1,2,0.5,0.5), (1,2,0.5,1), (1,2,0.5,1.5), (1,2,1,0.5), (1,2,1,1), (1,2,1,1.5), (1,2,2,0.5), (1,2,2,1), (1,2,2,1.5).

The estimates of the unknown parameters $(\lambda_1, \lambda_2, \lambda_3, \beta)$, where the estimate of β obtained through the Method-II (iterative method), along with their corresponding standard errors (SE($\hat{\lambda}_1$), SE($\hat{\lambda}_2$), SE($\hat{\lambda}_3$), and SE($\hat{\beta}$)), are reported in Table 3. We observed that compared to the estimates obtained by the average method, the estimates obtained by the iterative method had higher standard errors and tended to be overestimated.

Same phenomenon is observed for the simulation for study corresponding to the parameter combination $(\lambda_1, \lambda_2, \lambda_3, \beta)$ as (2,2,0.5,0.5), (2,2,0.5,1), (2,2,0.5,1.5), (2,2,1,0.5), (2,2,1,1), (2,2,1,1.5), (2,2,2,0.5), (2,2,2,1) and (2,2,2,1.5). Hence, we decided not to report the simulation results corresponding to these parameter combination.

Table 1: *The average estimates of ($\lambda_1, \lambda_2, \lambda_3, \beta$) obtained through the two-step procedure.*

n	λ_1	λ_2	λ_3	β	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$	$\widehat{\lambda}_3$	$\widehat{\beta}$	SE($\widehat{\lambda}_1$)	SE($\widehat{\lambda}_2$)	SE($\widehat{\lambda}_1$3)	SE($\widehat{\beta}$)
30	1	2	0.5	0.5	1.6347	3.2042	1.4232	0.7157	0.0798	0.0854	0.0693	0.0516
50	1	2	0.5	0.5	1.5741	2.5862	1.0873	0.6217	0.0759	0.0871	0.0669	0.0463
100	1	2	0.5	0.5	1.2432	2.3124	0.7554	0.5482	0.0748	0.0854	0.0675	0.0454
30	1	2	0.5	1	1.6865	3.256	1.8661	1.4245	0.0981	0.0847	0.0775	0.0611
50	1	2	0.5	1	1.4885	2.5789	1.0832	1.3029	0.0868	0.1001	0.0798	0.0579
100	1	2	0.5	1	1.1941	2.2879	0.7286	1.1229	0.0782	0.0861	0.0692	0.0461
30	1	2	0.5	1.5	1.8624	3.2677	1.9431	1.8289	0.1031	0.0962	0.0885	0.0721
50	1	2	0.5	1.5	1.5765	2.4989	0.9867	1.7110	0.0887	0.1042	0.0875	0.0598
100	1	2	0.5	1.5	1.2299	2.3093	0.7589	1.5603	0.0798	0.0954	0.0781	0.0476
30	1	2	1	0.5	1.8921	3.2776	2.4102	0.7372	0.0986	0.0865	0.0703	0.0627
50	1	2	1	0.5	1.5132	2.5867	1.5305	0.6134	0.0764	0.0967	0.0686	0.0511
100	1	2	1	0.5	1.2389	2.3123	1.2397	0.5623	0.0831	0.0876	0.0779	0.0467
30	1	2	1	1	1.7868	3.2682	2.3405	1.4321	0.1031	0.0881	0.0832	0.0684
50	1	2	1	1	1.5105	2.6105	1.5193	1.3139	0.0872	0.0989	0.0794	0.0673
100	1	2	1	1	1.1962	2.2961	1.2204	1.1283	0.0864	0.0872	0.0689	0.0551
30	1	2	1	1.5	1.8611	3.2692	2.3952	1.8382	0.1084	0.1051	0.0967	0.0685
50	1	2	1	1.5	1.5902	2.5156	1.4734	1.7102	0.0972	0.1098	0.0935	0.0637
100	1	2	1	1.5	1.2346	2.3203	1.2087	1.5589	0.0876	0.0971	0.0798	0.0472
30	1	2	2	0.5	1.9614	3.2658	3.4231	0.7267	0.0889	0.0847	0.0704	0.0542
50	1	2	2	0.5	1.5837	2.5991	2.5237	0.6079	0.0769	0.0885	0.0679	0.0476
100	1	2	2	0.5	1.2437	2.2984	2.2389	0.5472	0.0768	0.0853	0.0672	0.0462
30	1	2	2	1	1.8773	3.2674	3.4212	1.4298	0.0974	0.0869	0.0773	0.0616
50	1	2	2	1	1.4932	2.5813	2.5326	1.2998	0.0867	0.0983	0.0795	0.0568
100	1	2	2	1	1.2167	2.2916	2.2193	1.1183	0.0792	0.0851	0.0693	0.0463
30	1	2	2	1.5	1.8823	3.2593	3.4261	1.8672	0.1006	0.0975	0.0869	0.0578
50	1	2	2	1.5	1.5824	2.5139	2.4934	1.6672	0.0891	0.1092	0.0879	0.0573
100	1	2	2	1.5	1.2305	2.3027	2.1979	1.5723	0.07967	0.0945	0.0797	0.0493

Table 2: *The average estimates of ($\lambda_1, \lambda_2, \lambda_3, \beta$) obtained through the two-step procedure.*

n	λ_1	λ_2	λ_3	β	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$	$\widehat{\lambda}_3$	$\widehat{\beta}$	SE($\widehat{\lambda}_1$)	SE($\widehat{\lambda}_2$)	SE($\widehat{\lambda}_1 3$)	SE($\widehat{\beta}$)
30	2	2	0.5	0.5	2.9587	3.2681	1.9672	0.7361	0.0893	0.0842	0.0669	0.0578
50	2	2	0.5	0.5	2.5831	2.5951	1.0851	0.6138	0.0765	0.0879	0.0677	0.0456
100	2	2	0.5	0.5	2.2360	2.3013	0.7542	0.5479	0.0754	0.0851	0.0679	0.0442
30	2	2	0.5	1	2.8476	3.2821	1.8567	1.4310	0.0981	0.0843	0.0774	0.0621
50	2	2	0.5	1	2.4881	2.5813	1.0829	1.3021	0.0870	0.0995	0.0792	0.0582
100	2	2	0.5	1	2.1933	2.2929	0.7300	1.1232	0.0798	0.0856	0.0686	0.0450
30	2	2	0.5	1.5	2.8621	3.2713	1.9413	1.8348	0.1116	0.0961	0.0873	0.0635
50	2	2	0.5	1.5	2.5855	2.5018	0.9964	1.7027	0.0877	0.1030	0.0862	0.0604
100	2	2	0.5	1.5	2.2320	2.3058	0.7542	1.5590	0.0802	0.0934	0.0770	0.0482
30	2	2	1	0.5	2.9745	3.2799	2.3941	0.7416	0.0971	0.0845	0.0689	0.0618
50	2	2	1	0.5	2.5941	2.5979	1.5238	0.6156	0.0771	0.0962	0.0690	0.0504
100	2	2	1	0.5	2.2468	2.3015	1.2416	0.5601	0.0815	0.0860	0.0758	0.0485
30	2	2	1	1	2.8891	3.2801	2.3407	1.4392	0.1028	0.0871	0.0817	0.0671
50	2	2	1	1	2.4932	2.5918	1.5262	1.3124	0.0881	0.0998	0.0811	0.0656
100	2	2	1	1	2.2003	2.3056	1.2185	1.1273	0.0841	0.0887	0.0694	0.0529
30	2	2	1	1.5	2.8601	3.2713	2.4006	1.8425	0.1061	0.1027	0.0946	0.0684
50	2	2	1	1.5	2.5888	2.5139	1.4866	1.7073	0.0926	0.1115	0.0915	0.0630
100	2	2	1	1.5	2.2387	2.3117	1.2032	1.5622	0.0857	0.0965	0.0831	0.0490
30	2	2	2	0.5	2.9601	3.2589	3.4098	0.7244	0.0862	0.0818	0.0671	0.0512
50	2	2	2	0.5	2.5785	2.5853	2.5164	0.6021	0.0741	0.0861	0.0664	0.0447
100	2	2	2	0.5	2.2351	2.2937	2.2336	0.5423	0.0727	0.0815	0.0639	0.0413
30	2	2	2	1	2.8744	3.2701	3.4189	1.4251	0.0948	0.0841	0.0751	0.0591
50	2	2	2	1	2.4821	2.5784	2.5261	1.2982	0.0836	0.0971	0.0783	0.0555
100	2	2	2	1	2.1927	2.2891	2.2164	1.1153	0.0774	0.0816	0.0683	0.0436
30	2	2	2	1.5	2.8513	3.2587	3.4228	1.8294	0.0987	0.0939	0.0849	0.0611
50	2	2	2	1.5	2.5751	2.4992	2.4839	1.6982	0.0838	0.1018	0.0839	0.0572
100	2	2	2	1.5	2.2194	2.2943	2.1926	1.5468	0.0782	0.0896	0.0744	0.0451

Table 3: *The average estimates of $(\lambda_1, \lambda_2, \lambda_3, \beta)$ obtained through the iterative method.*

n	λ_1	λ_2	λ_3	β	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$	$\widehat{\lambda}_3$	$\widehat{\beta}$	SE($\widehat{\lambda}_1$)	SE($\widehat{\lambda}_2$)	SE($\widehat{\lambda}_1 3$)	SE($\widehat{\beta}$)
30	1	2	0.5	0.5	1.9887	3.3081	1.9873	0.8154	0.0893	0.0902	0.0678	0.0603
50	1	2	0.5	0.5	1.6912	2.6332	1.0923	0.7385	0.0782	0.0893	0.0691	0.0472
100	1	2	0.5	0.5	1.3497	2.3213	0.7743	0.6293	0.0772	0.0869	0.0695	0.0449
30	1	2	0.5	1	1.9534	3.3109	1.9576	1.4734	0.1023	0.0953	0.0867	0.0703
50	1	2	0.5	1	1.5934	2.6156	1.0921	1.3921	0.0899	0.1012	0.0823	0.0612
100	1	2	0.5	1	1.3173	2.4679	0.7591	1.1581	0.0823	0.0897	0.0728	0.0499
30	1	2	0.5	1.5	1.9727	3.3278	1.9825	1.9568	0.1792	0.1034	0.0957	0.0684
50	1	2	0.5	1.5	1.7455	2.5357	0.9764	1.8627	0.0893	0.1125	0.0897	0.0692
100	1	2	0.5	1.5	1.5671	2.3513	0.7934	1.6193	0.0934	0.1045	0.0842	0.0502
30	1	2	1	0.5	1.9834	3.2895	2.4231	0.8245	0.1034	0.0957	0.0725	0.0769
50	1	2	1	0.5	1.6761	2.6281	1.5482	0.7372	0.0821	0.0993	0.0723	0.0584
100	1	2	1	0.5	1.3182	2.3756	1.2949	0.6429	0.0902	0.0931	0.0784	0.0521
30	1	2	1	1	1.9233	3.2956	2.3756	1.5334	0.1342	0.0913	0.0882	0.0705
50	1	2	1	1	1.5421	2.6492	1.5849	1.4294	0.0917	0.1034	0.0942	0.0736
100	1	2	1	1	1.2951	2.3682	1.2735	1.2661	0.0879	0.0921	0.0704	0.0569
30	1	2	1	1.5	1.9349	3.2937	2.4623	1.9173	0.1236	0.1412	0.1034	0.0756
50	1	2	1	1.5	1.5923	2.5634	1.4954	1.8728	0.1054	0.1532	0.1031	0.0713
100	1	2	1	1.5	1.3681	2.3542	1.2348	1.6212	0.0886	0.0993	0.0902	0.0534
30	1	2	2	0.5	1.9789	3.2782	3.4267	0.8178	0.0921	0.0941	0.0714	0.0589
50	1	2	2	0.5	1.5845	2.5936	2.5372	0.6912	0.0797	0.0901	0.0686	0.0484
100	1	2	2	0.5	1.3383	2.3417	2.2756	0.6389	0.0810	0.0889	0.0725	0.0467
30	1	2	2	1	1.8972	3.2852	3.4462	1.5343	0.1034	0.0872	0.0792	0.0602
50	1	2	2	1	1.4939	2.5789	2.5319	1.4014	0.0913	0.0989	0.0810	0.0579
100	1	2	2	1	1.3214	2.2973	2.2682	1.2314	0.0824	0.0901	0.0713	0.0498
30	1	2	2	1.5	1.9344	3.2604	3.4610	1.9282	0.1083	0.0991	0.0892	0.0659
50	1	2	2	1.5	1.6952	2.5021	2.4934	1.8317	0.0884	0.1153	0.0897	0.0604
100	1	2	2	1.5	1.3156	2.3023	2.2103	1.7116	0.0816	0.0927	0.0829	0.0523

5. Illustration

In this section, we have applied the AFT based load sharing model and estimation procedures to motor data obtained from Reliability Edge Home [19]. The dataset consists of 18 systems, each consisting of two motors operating continuously in parallel. The failure times of both motors, along with their identification labels A and B, were recorded.

Our objectives were twofold. Firstly, we aimed to determine whether the modified Weibull distribution (MWD) is an appropriate baseline distribution for modeling the lifetimes of both components. Secondly, we aimed to test whether there exists a load sharing phenomenon, where the failure of one motor affects the working of the other.

To assess the appropriateness of the MWD as the baseline distribution, we conducted a

Kolmogorov-Smirnov type test, which confirmed its suitability. However, it should be noted that the test was conservative due to the estimation of unknown parameters. We utilized a two-step estimation procedure, with the estimation of β being conducted using the 'average method'. The estimated values of λ_1, λ_2, λ_3, and β were found to be 0.0028, 2.08168×10^{16}, 31.6118, and 1.9847, respectively.

To investigate the presence of load sharing among the motor failure times, we employed a score-type test proposed by Sutar and Naik-Nimbalkar [11]. The computed test statistic value was 19.564, which surpassed the critical values at both the 1% and 5% significance levels. Consequently, we can infer that the failure of one motor has a significant impact on the lifetime of the other. This finding supports the existence of a load sharing phenomenon, where the failures of individual components influence the performance of the remaining components in the system. This conclusion is further supported by the estimated value of $\hat{\beta}$ being 1.9847 (significatly different than 1), suggesting that these 18 parallel systems exhibit load sharing or a load sharing phenomenon among the component failures.

6. Conclusions

In our study, we focused on a two-component parallel load sharing system and utilized the accelerated failure time based load sharing model to capture the load sharing behavior observed in this system. We chose the modified Weibull distribution as the baseline distribution for the component lifetime. We proposed two procedures for estimating the model parameters within this framework and also discussed a test procedure for assessing the presence of load sharing in such systems. Furthermore, we conducted a simulation study to evaluate the performance of the proposed estimation procedures, which demonstrated satisfactory results. To illustrate the practical applicability of the load sharing system, we analyzed a specific dataset. It is worth mentioning that the modeling and analysis of load sharing phenomena can be extended to more complex systems, such as a k-out-of-m system.

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Appendix (A): Expressions involved in Score functions

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{\left(\lambda_1\lambda_3\beta^{\lambda_3-1} + \lambda_2\lambda_3 y_i^{\lambda_3-1}\right)}{\left(\lambda_1\beta^{\lambda_3} + \lambda_2\lambda_3\beta y_i^{\lambda_3-1}\right)} - \frac{n(\lambda_3+1)}{\beta} + \frac{\lambda_1}{\beta^2}\sum_{i=1}^{n}(y_i - x_i)$$
$$- \frac{\lambda_2\lambda_3}{\beta^{\lambda_3+1}}\sum_{i=1}^{n}(y_i^{\lambda_3} - x_i^{\lambda_3}) = 0, \tag{16}$$

$$\frac{\partial \log L}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{\beta^{\lambda_3-1}}{\left(\lambda_1\beta^{\lambda_3} + \lambda_2\lambda_3\beta y_i^{\lambda_3-1}\right)} + \sum_{i=1}^{n} \frac{1}{\left(\lambda_1 + \lambda_2\lambda_3 x_i^{\lambda_3-1}\right)}$$
$$- \frac{1}{\beta}\sum_{i=1}^{n}(y_i - x_i) - 2\sum_{i=1}^{n} x_i = 0,$$

$$\frac{\partial \log L}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{\beta\lambda_3 y_i^{\lambda_3-1}}{\left(\lambda_1\beta^{\lambda_3} + \lambda_2\lambda_3\beta y_i^{\lambda_3-1}\right)} + \sum_{i=1}^{n} \frac{\lambda_3 x_i^{\lambda_3-1}}{\left(\lambda_1 + \lambda_2\lambda_3 x_i^{\lambda_3-1}\right)}$$
$$- \frac{\lambda_2}{\beta^{\lambda_3}}\sum_{i=1}^{n}(y_i^{\lambda_3} - x_i^{\lambda_3}) - 2\lambda_2\sum_{i=1}^{n} x_i^{\lambda_3} = 0,$$

and

$$\frac{\partial \log L}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{\left(\lambda_1\beta^{\lambda_3}\log\beta + \lambda_2\beta y_i^{\lambda_3-1} + \lambda_2\lambda_3\beta y_i^{\lambda_3-1}\log y_i\right)}{\left(\lambda_1\beta^{\lambda_3} + \lambda_2\lambda_3\beta y_i^{\lambda_3-1}\right)}$$
$$+ \sum_{i=1}^{n} \frac{\lambda_2 x_i^{\lambda_3-1} + \lambda_2\lambda_3 x_i^{\lambda_3-1}\log(x_i)}{\left(\lambda_1 + \lambda_2\lambda_3 x_i^{\lambda_3-1}\right)} - n\log\beta - \lambda_2\sum_{i=1}^{n}\left(\frac{y_i}{\beta}\right)^{\lambda_3}\log\left(\frac{x_i}{\beta}\right)$$
$$- \lambda_2\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\lambda_3}\log\left(\frac{y_i}{\beta}\right) - 2\lambda_2\sum_{i=1}^{n} x_i^{\lambda_3}\log(x_i) = 0.$$

Appendix (B): The information pertaining to the conditional density of W_{11} given $X = x$ and $Z_{21} = z_{21}$

Consider

$$\begin{aligned}\bar{G}(w_1, z_{21}|X = x) &= P(Z_{21} > z_{21}, W_{11} > w_1|X = x)\\ &= \bar{F}_{W_1|X=x}(\max(z_{21}, w_1))\bar{F}_{W_2|X=x}(z_{21})\\ &= \exp\left\{-\theta_1[\max(z_{21}, w_1) - x] - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\right\}.\end{aligned}$$

Due to ordering in z_{21} and w_1, we have following three cases-

1. $z_{21} > w_1$ i.e. $\max(z_{21}, w_1) = z_{21}$.
2. $z_{21} < w_1$ i.e. $\max(z_{21}, w_1) = w_1$.
3. $z_{21} = w_1$ i.e. $\max(z_{21}, w_1) = z_{21}$ *or* w_1.

When, $z_{21} > w_1$ we have,

$$\bar{G}(w_1, z_{21}|X = x) = \exp\left\{-\theta_1(z_{21} - x) - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\right\}.$$

Thus, we have

$$g(w_1, z_{21}|X = x) = \frac{\partial^2}{\partial z_{21}\partial w_1}\bar{G}(z_{21}, w_1|x) = 0,\ z_{21} > w_1 > 0.$$

When, $z_{21} < w_1$ we have,

$$\bar{G}(w_1, z_{21}|X = x) = \exp\left\{-\theta_1(w_1 - x) - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\right\}\ , w_1 > z_{21} > 0,$$

and hence

$$g(w_1, z_{21}|X = x) = \frac{\partial^2}{\partial z_{21}\partial w_1}\bar{G}(z_{21}, w_1|x)\ , w_1 > z_{21} > 0.$$

That is

$$g(w_1, z_{21}|X = x) = \theta_1\theta_2\theta_3 z_{21}^{\theta_3 - 1}\exp\left\{-\theta_1(w_1 - x) - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\right\}\ , w_1 > z_{21} > 0.$$

When, $z_{21} = w_1$ we have,

$$\bar{G}(w_1, z_{21}|X = x) = \exp\{-\theta_1(z_{21} - x) - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\}\ , w_1 = z_{21} > 0.$$

Thus, we have

$$\begin{aligned}g(w_1, z_{21}|X = x) &= \frac{\partial^2}{\partial z_{21}\partial w_1}\bar{G}(z_{21}, w_1|x)\ , w_1 = z_{21} > 0\\ &= \theta_1\exp\left\{-\theta_1(z_{21} - x) - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\}\right\}, z_{21} = w_1\ , w_1 = z_{21} > 0.\end{aligned}$$

Thus, by combining all the above cases, we can write the joint density as

$$\begin{aligned}g(w_1, z_{21}|X = x) = &\ \theta_1\theta_2\theta_3 z_{21}^{\theta_3 - 1}\exp\{-\theta_1(w_1 - x) - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\}I(z_{21} = w_1)\\ &+ \theta_1\exp\{-\theta_1(z_{21} - x) - \theta_2(z_{21}^{\theta_3} - x^{\theta_3})\}I(z_{21} < w_1).\end{aligned}$$

Hence, the conditional density of W_{11} given $X = x, Z_{21} = z_{21}$ can be obtained as

$$\begin{aligned}g(w_1|x, z_{21}) &= \frac{g(w_1, z_{21}|X = x)}{g(z_{21}|X = x)}\\ &= \frac{\theta_1}{\left(\theta_1 + \theta_2\theta_3 z_{21}^{\theta_3 - 1}\right)}\left\{I_{(w_1 = z_{21})} + \theta_2\theta_3 z_{21}^{\theta_3 - 1}\exp\{-\theta_1(w_1 - z_{21})\}\, I_{(w_1 > z_{21})}\right\}.\end{aligned}$$

Appendix (C): Expressions involved in (14)

$$\frac{\partial^2 E_c\left[\log L|\underline{x},\underline{y}\right]}{\partial(\theta_3^{(r)})^2} = -\frac{n}{(\theta_3^{(r)})^2} - n\frac{\left\{E_c\left[W_{2i}^{\theta_3^{(r)}}|\underline{x},\underline{y}\right]\frac{\partial^2 E_c\left[W_{2i}^{\theta_3^{(r)}}|\underline{x},\underline{y}\right]}{\partial(\theta_3^{(r)})^2}\right\} - \left\{\frac{\partial E_c\left[W_{2i}^{\theta_3^{(r)}}|\underline{x},\underline{y}\right]}{\partial(\theta_3^{(r)})^2}\right\}}{\left\{E_c\left[W_{2i}^{\theta_3^{(r)}}|\underline{x},\underline{y}\right]\right\}^2},$$

with

$$\frac{\partial E_c\left\{\sum_{i=1}^n W_{2i}^{\theta_3^{(r+1)}}|\underline{x},\underline{y}\right\}}{\partial\theta_3^{(r)}} = \sum_{i=1}^n \frac{\theta_2\theta_3 y_i^{\theta_3^{(r+1)}+\theta_3-1}\log(y_i)}{\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}$$

$$-\sum_{i=1}^n\sum_{j=1}^m \frac{\theta_1^{(r)}\theta_2^{(r)^{-\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}}\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)^{\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}}{m\theta_3^{(r)}\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}$$

$$+\sum_{i=1}^n\sum_{j=1}^m \frac{\theta_1^{(r)}\theta_2^{(r)^{-\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}}\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)^{\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}\log\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)}{m\theta_3^{(r)}\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}$$

and

$$\frac{\partial^2 E_c\left\{\sum_{i=1}^n W_{2i}^{\theta_3^{(r+1)}}|\underline{x},\underline{y}\right\}}{\partial(\theta_3^{(r)})^2} = \sum_{i=1}^n \frac{\theta_2\theta_3 y_i^{\theta_3^{(r+1)}+\theta_3-1}(\log(y_i))^2}{\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}$$

$$+\sum_{i=1}^n\sum_{j=1}^m \frac{\theta_1^{(r)}\theta_2^{(r)^{-\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}}\left(\log\theta_2^{(r)}\right)^2\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)^{\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}}{m\left(\theta_3^{(r)}\right)^2\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}$$

$$-\sum_{i=1}^n\sum_{j=1}^m \frac{\theta_1^{(r)}\theta_2^{(r)^{-\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}}\left(\log(\theta_2^{(r)})\right)\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)^{\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}\log\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)}{m\left(\theta_3^{(r)}\right)^2\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}$$

$$-\sum_{i=1}^n\sum_{j=1}^m \frac{\theta_1^{(r)}\theta_2^{(r)^{\frac{-\theta_3^{(r+1)}}{\theta_3^{(r)}}}}\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)^{\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}\log\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)}{m\left(\theta_3^{(r)}\right)^2\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}$$

$$+\sum_{i=1}^n\sum_{j=1}^m \frac{\theta_1^{(r)}\theta_2^{(r)^{\frac{-\theta_3^{(r+1)}}{\theta_3^{(r)}}}}\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)^{\frac{\theta_3^{(r+1)}}{\theta_3^{(r)}}}\left(\log\left(v_j+\theta_2^{(r)}y_i^{\theta_3}\right)\right)^2}{m\left(\theta_3^{(r)}\right)^2\left(\theta_1+\theta_2\theta_3 y_i^{\theta_3-1}\right)}.$$