A New Class of Sin-G Family of Distributions with Applications to Medical Data

LAXMI PRASAD SAPKOTA, PANKAJ KUMAR AND VIJAY KUMAR

•

Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, UP, India.
lpsapkota75@gmail.com, pankajagadish@gmail.com, vkgkp@rediffmail.com

Abstract

This article is dedicated to the study of the new class of distributions and one of its particular members. Based on the ratio of CDF $G(x)$ and $1+G(x)$ of the baseline distribution, we have developed the new trigonometric family of distributions by transforming the sine function, and we named it the new class sin-G (NCS-G) family of distributions. The general properties of the suggested family of distributions are provided. Using the inverted Weibull distribution as a baseline distribution, we have introduced a member of the suggested family having a reverse-j or increasing, or inverted bathtub-shaped hazard function. Some statistical properties of this NCS-IW distribution are explored. The associated parameters of the new distribution are estimated through the MLE method. To assess the estimation procedure, we conducted a Monte Carlo simulation and found that even for small samples, biases and mean square errors decreased as the size of the sample increased. Two real medical data sets are considered for the application of the NCS-IW distribution. Using some criteria for model selection and goodness of fit test statistics, we empirically proved that the suggested model performs better than six other existing models (most of which have more parameters).

Keywords: Sine-G distribution, Inverse Weibull distribution, Maximum Likelihood Estimation, Entropy, Quantile Function

1. INTRODUCTION

Statistical distributions are frequently used to investigate real-world phenomena. The theory of statistical distributions is extensively studied, as are new developments in their application. Several families of distributions have been developed to describe various real-world phenomena. In reality, this new development in distribution theory is a continuing practice. Many probability distributions proposed in the literature have a large number of parameters to make the model more versatile. However, obtaining estimates for these parameters can be challenging using numerical resources, as per some authors Marshall and Olkin [17]. Hence, it is better to create models with fewer parameters and greater flexibility for modeling actual data. To achieve this objective, a group of researchers searched for new distributions employing trigonometric functions. In the last few years, researchers have been attracted to trigonometric models due to their flexibility and mathematical tractability. Among the various trigonometric G-family members, Kumar et al. [15] have defined a new class of distribution using the sine trigonometric function and defined the sin-exponential model as its member. The cumulative distribution function (CDF) of this family is given by

$$F(x;\chi) = \sin\left\{\frac{\pi}{2}K(x;\chi)\right\} \quad x \in R, \tag{1}$$

where $K(x;\chi)$ is the CDF of any base continuous distribution. Instantaneously, Souza [24] introduced another trigonometric model based on the sine function and Gomez-Deniz and Caldern-Ojeda [9] define the arc-tan-G family of distributions using the arctangent function. Gomez-Deniz and Caldern-Ojeda [9] demonstrated the new distribution family that was used to characterize Norwegian fire insurance data.

This distribution family was introduced for an underlying Pareto distribution and a new model named the Pareto arctan distribution, and it was discovered that when compared to other well-known distributions, this distribution offers an excellent fit. Similarly, the hyperbolic cosine-F families of distributions were defined using a hyperbolic trigonometric function by Kharazmi and Saadatinik [14], and the hyperbolic cosine Rayleigh distribution was defined by Sakthivel and Rajkumar [22]. Using a similar technique as used in sin-G, the Cos-G family of distributions was introduced by Souza et al.[25] who also introduce the Cos-Weibull distribution as a member of Cos-G class. Similarly, Souza et al. [26] have introduced another sin-G class as defined by Kumar et al. [15] with bathtub-shaped or reverse-j, or increasing failure rate function, and studied the Sine inverse Weibull distribution as a particular member. The CDF of the Sin-G class of distribution is

$$F(x;\omega)=\int_{0}^{\frac{\pi}{2}K(x;\omega)}\cos(t)dt=\sin\left[\frac{\pi}{2}K(x;\omega)\right];x\in R \tag{2}$$

where $K(x;\omega)$ is the CDF of any parent distribution and $\omega>0$ is the vector of parameters of the parent distribution. Also, Mahmood et al. [16] have developed the new sin-G family and analyzed the sin-inverse Weibull model in particular. Chesneau and Jamal [6] have defined the sine Kumaraswamy-G family of distributions as having two extra parameters to this family. Muhammad et al. [19] have defined the exponentiated sine-G family and analyzed the particular distribution as an exponentiated sine-Weibull distribution. Another trigonometric function-related probability model introduced by Chaudhary et al.[3] is called Arctan generalized exponential distribution. Using the sine-G family of distributions, Isa et al. [12] have developed a new two-parameter model called the sine Burr XII distribution. Hence, we have noticed that the simple functions are associated with trigonometric distributions and are mathematically tractable (see [15], [26]). Further, the sine transformation can remarkably enhance the flexibility of $G(x)$ without any additional parameters Chesneau and Jamal [6]. Due to these pleasant features, we are motivated towards the sine transformation family. In this study, we have developed a new family of trigonometric models using the sine function, and we called it the "new class of sine-G family" (NCS-G) of distributions. The other parts of this study are organized as follows: Section 2 introduces the model development methodology as well as some key functions of the distribution family. Some general properties and parameter estimation of the NCS-G family are presented in Sections 3 and 4 respectively. In Section 5, a particular member of the NCS-G family is introduced. A detailed study and application of this model are also presented in this section. Finally, we present the conclusion in Section 6.

2. The NCS-G Family of Distribution (NCS-G FD)

Using the T-X approach proposed by Alzaatreh et al. [1], this study proposes a new family of distributions known as the NCS-G family of distributions. Let $G(x;\xi)$ be a baseline CDF of a continuous random variable X and $\xi>0$ be a vector of associated parameters, and then the CDF $F(x;\xi)$ of the NCS-G FD is defined as

$$F(x;\xi)=\int_{0}^{\pi\left(\frac{G(x;\xi)}{1+G(x;\xi)}\right)}\cos(t)dt=\sin\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right];x\in R. \tag{3}$$

Differentiating the CDF defined in Equation (3), the PDF $f(x;\xi)$ of the family is expressed as

$$f(x;\xi)=\pi\cos\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right]\frac{g(x;\xi)}{(1+G(x;\xi))^{2}};x\in R. \tag{4}$$

2.1. Reliability Function

The Reliability function of NCS-G FD is given by

$$R(x;\xi)=1-\sin\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right];x\in R. \tag{5}$$

2.2. Hazard Function

The Hazard function of NCS-G FD is given as

$$H(x;\xi) = \pi \cos\left[\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right] \frac{g(x;\xi)}{(1+G(x;\xi))^2}\left[1-\sin\left(\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^{-1}; x \in R \tag{6}$$

2.3. The Quantile Function (QF)

The p^{th} quantile can be calculated by solving, $Q(p) = F^{-1}(p)$. Now the QF of NCS-G FD is given by

$$Q_X(p;\xi) = G^{-1}\left[\frac{\sin^{-1}(p)}{\pi - \sin^{-1}(p)}\right], \tag{7}$$

where p has $U(0,1)$ distribution.

2.4. Random Deviate Generation

Random deviate for the NCS-G FD can be generated

$$x = G^{-1}\left[\frac{\sin^{-1} u}{\pi - \sin^{-1} u}\right] \tag{8}$$

where $u \in U(0,1)$ distribution.

2.5. Skewness and Kurtosis

Bowley's measure of skewness was defined by Kenney and Keeping [13] as,

$$S_k(B) = \frac{Q(3/4;\xi) + Q(1/4;\xi) - 2Q(1/2;\xi)}{Q(3/4;\xi) - Q(1/4;\xi)} \tag{9}$$

and the coefficient of Moor's kurtosis defined by Moors [18] is given by

$$K_u(M) = \frac{Q(0.875;\xi) - Q(0.625;\xi) + Q(0.375;\xi) - Q(0.125;\xi)}{Q(3/4;\xi) - Q(1/4;\xi)}. \tag{10}$$

3. General Properties of NCS-G FD

3.1. Linear form

Using the following Taylor series expansions, we can express the density function of NCS-G FD in a linear form as

$$\cos x = \sum_{n=0}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots; -\infty < x < \infty. \tag{11}$$

$$(1+x)^c = \sum_{n=0}^{\infty} \binom{c}{n} x^n = 1 + \frac{c}{1!}x + \frac{c(c-1)}{2!}x^2 + \frac{c(c-1)(c-2)}{3!}x^3 + \cdots; |x| < 1. \tag{12}$$

The PDF of NCS-G FD is

$$f(x;\xi) = g(x;\xi) \sum_{i=0}^{\infty} \frac{\pi^{2i+1}(-1)^{2i}}{(2i)!} (1+G(x;\xi))^{2(i-1)} (G(x;\xi))^{2i}. \tag{13}$$

Further expanding Equation (13) using generalized binomial series expansion. The expression for $f(x;\xi)$ becomes

$$f(x;\xi) = g(x;\xi) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} T_{ij} \{G(x;\xi)\}^{2i+j}; x \in R \tag{14}$$

here

$$T_{ij} = \frac{\pi^{2i+1}(-1)^{2i}}{(2i)!} \binom{2(i-1)}{j}. \tag{15}$$

3.2. Moments

The r^{th} order non-central moment (μ_r') for the NCS-G FD is

$$\begin{aligned}\mu_r' = E(X^r) &= \int_{-\infty}^{\infty} x^r f(x)dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij}\int_{-\infty}^{\infty} x^r \left(G(x;\xi)\right)^{2i+j} g(x;\xi)dx\end{aligned} \tag{16}$$

Further moments can also be calculated using the quantile function for more detail (see Balakrishnan and Cohen [2]). Let $G(x;\xi) = p \Rightarrow g(x;\xi)dx = dp; 0 \leqslant p \leqslant 1$, then r^{th} moment can be computed using

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij}\int_{0}^{1} p^{2i+j} Q_G^r(p)dp;\ \ 0 < p < 1. \tag{17}$$

where $Q_G(p)$ is the QF of any distribution.

3.3. Moment Generating Function

The MGF ($M_X(t)$) for the NCS-G FD is

$$\begin{aligned}M_X(t) &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\mu_r' \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^k}{k!}T_{ij}\int_{-\infty}^{\infty} x^r g(x;\xi)\left(G(x;\xi)\right)^{2i+j}dx.\end{aligned} \tag{18}$$

Using the quantile function, MGF can be expressed as

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^k}{k!}T_{ij}\int_{0}^{1} p^{2i+j} Q_G^r(p)\,dp, \quad 0 < p < 1. \tag{19}$$

where $Q_G(p)$ is the QF of any distribution.

3.4. Incomplete Moment

The incomplete moment can be defined as $M_r(y) = \int_0^y x^r f(x)dx$. Therefore incomplete moment for NCS-G FD is given by

$$M_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{-\infty}^{y} T_{ij}x^r g(x;\xi)\left\{G(x;\xi)\right\}^{2i+j}dx \tag{20}$$

Alternately, $M_r(y)$ may be expressed in terms of QF as

$$M_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij}\int_{0}^{G(y)} p^{2i+j} Q_G^r(p)dp;\ \ 0 < p < 1 \tag{21}$$

3.5. Mean Residual Life (MRL)

The MRL of the random variable X is defined as

$$\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \int_{-\infty}^{y} xf(x)dx\right] - y. \tag{22}$$

Therefore, MRL for NCS-G FD is given by

$$\bar{M}(y)=\frac{1}{F(y)}\left[\mu-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{ij}\int_{-\infty}^{y}xg(x;\xi)\left\{G(x;\xi)\right\}^{2i+j}dx\right]-y. \tag{23}$$

Alternately, $\bar{M}(y)$ can be expressed in terms of QF as

$$\bar{M}(y)=\frac{1}{F(y)}\left[\mu-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{ij}\int_{0}^{G(y)}p^{2i+j}Q_G(p)dp\right]-y. \tag{24}$$

3.6. Inequality Measure

In several fields, including insurance, econometrics, and reliability, we can employ Lorenz and Bonferroni curves to measures such as income, poverty, etc.
i) Lorenz Curve
The function of the Lorenz curve is written as hence Lorenz curve for NCS-G FD is given by

$$L_{F(y)}=\frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{ij}\int_{-\infty}^{y}xg(x;\xi)\left(G(x;\xi)\right)^{2i+j}dx. \tag{25}$$

Alternatively, it can be written in terms of QF as

$$L_{F(y)}=\frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{ij}\int_{-\infty}^{G(y)}p^{2i+j}Q_G(p)dp. \tag{26}$$

ii) Boneferroni Curve
The Boneferroni curve can be calculated using $B_{F(y)}=\frac{L_{F(y)}}{F(y)}$. From Equation (25), the Boneferroni curve for the NCS-G FD is calculated as

$$B_{F(y)}=\frac{1}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{ij}\int_{-\infty}^{y}xg(x;\xi)\left(G(x;\xi)\right)^{2i+j}dx \tag{27}$$

3.7. Entropy

Entropy quantifies the uncertainty or variation of a random variable. Its application spans numerous disciplines, including econometrics, probability theory, engineering, and life sciences in general. There are several types of entropy, some of which are as follows:
i) Renyi's Entropy
Entropy is used as a measure of uncertainty or variation in a random variable in many disciplines, including engineering, econometrics, insurance, etc. Renyi [20] introduced entropy measures, which can be used to calculate the variability of uncertainty.

$$R_\rho(X)=\frac{1}{1-\rho}\log\int_{-\infty}^{\infty}\left\{f(x)\right\}^{\rho}dx \tag{28}$$

and $\rho\neq 1$. The PDF of NCS-G FD $[f(x,\xi)]^\rho$ can be defined in the form of

$$[f(x;\xi)]^\rho=\pi^\rho\left(g(x;\xi)\right)^\rho\left[\cos\left(\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^\rho\left(1+G(x;\xi)\right)^{-2\rho} \tag{29}$$

By considering the Taylor series of the function

$$\left[\cos\left(\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^\rho \tag{30}$$

at the point s=1/4, we can write

$$[\cos(\pi s)]^{\rho} = \sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k \binom{k}{r} (-1)^{k-r} \left(\frac{1}{4}\right)^{k-r} s^r \tag{31}$$

where $a_k = \frac{1}{k!}\left[\{\cos(\pi s)\}^{\rho}\right]^{(k)}\Big|_{s=\frac{1}{4}}$ using this relation Equation (29) becomes

$$[f(x;\xi)]^{\rho} = \pi^{\rho}\left(g(x;\xi)\right)^{\rho} \sum_{k=0}^{\infty}\sum_{r=0}^{k} a_k \binom{k}{r} (-1)^{k-r} \left(\frac{1}{4}\right)^{k-r} \left(G(x;\xi)\right)^{r} \left(1+G(x;\xi)\right)^{-(2\rho+r)} \tag{32}$$

Further expanding Equation (32) using generalized binomial series expansion. The expression for $[f(x;\xi)]^{\rho}$ becomes

$$[f(x;\xi)]^{\rho} = \pi^{\rho} \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} (-1)^{m+k-r} a_k \binom{k}{r} \left(\frac{1}{4}\right)^{k-r} \binom{(2\rho+r)+m-1}{m} \left(G(x;\xi)\right)^{r+m} \left(g(x;\xi)\right)^{\rho} \tag{33}$$

Substituting $[f(x,\xi)]^{\rho}$ into the expression defining Equation (28), Renyi's entropy for NCS-G FD is given by

$$R_{\rho}(X) = \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} Z_{krm} \int_{-\infty}^{\infty} \left(g(x;\xi)\right)^{\rho} \left(G(x;\xi)\right)^{r+m} dx\right] \tag{34}$$

where $Z_{krm} = (-1)^{m+k-r}\pi^{\rho} a_k \binom{k}{r} \left(\frac{1}{4}\right)^{k-r} \binom{(2\rho+r)+m-1}{m}$.

ii) q-Entropy

The q-entropy is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\int_{-\infty}^{\infty} \{f(x)\}^{\rho} dx\right]. \tag{35}$$

$\rho > 0$ and $\rho \neq 1$. Substituting $[f(x;\xi)]^{\rho}$ from Equation (32) into the expression for $H(\rho)$, the q-Entropy for NCS-G FD is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} Z_{krm} \int_{-\infty}^{\infty} \left(g(x;\xi)\right)^{\rho} \left(G(x,\xi)\right)^{r+m} dx\right] \tag{36}$$

where $\rho > 0$ and $\rho \neq 1$.

iii) Shannon's Entropy

When $\rho \uparrow 1$, Shannon's entropy for a random variable X with PDF $f(x)$ is a particular case of Renyi's entropy. Shannon entropy is defined as $\eta_X = E(-\log f(x))$. For the NCS-G FD is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} g(x;\xi)\left(G(x;\xi)\right)^{2i+j}\right\}\right]. \tag{37}$$

4. Estimation Method NCS-G FD

4.1. Maximum Likelihood Estimation (MLE)

In this section, the parameters of the NCS-G FD are estimated using the MLE method. Given a random sample $x_1,...,x_n$ of size n with parameters vector ξ from the NCS-G FD, we can compute the MLEs. Let $u = \xi^T$ be $(p \times 1)$ parameter vectors, the log density and total log-likelihood function, respectively, are given by

$$L(x;\xi) = \log\pi + \log\left[\cos\left\{\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right\}\right] - 2\log\left(1+G(x;\xi)\right) + \log g(x;\xi), \tag{38}$$

and

$$l(\underline{x},\xi) = n\log\pi + \sum_{i=1}^{n}\log\left[\cos\left\{\pi\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\right] - 2\sum_{i=1}^{n}\log\left(1+G(x_i;\xi)\right) + \sum_{i=1}^{n}\log g(x_i;\xi). \tag{39}$$

Partially differentiating the Equation (39) with respect to ξ gives the score function's components of the $V(u) = \left(\frac{\partial l}{\partial \xi}\right)^T$ as follows

$$\frac{\partial l}{\partial \xi} = -\pi\sum_{i=1}^{n}\tan\left\{\pi\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\frac{G_k'(x_i;\xi)}{\left(1+G(x_i;\xi)\right)^2} - 2\sum_{i=1}^{n}\frac{G_k'(x_i;\xi)}{\left(1+G(x_i;\xi)\right)} + \sum_{i=1}^{n}\frac{g_k'(x_i;\xi)}{g(x_i;\xi)},$$

where $g_k'(x_i;\xi) = \frac{dg(x_i;\xi)}{d\xi}$, $g_k'(x_i;\xi) = \frac{d^2g(x_i;\xi)}{d^2\xi}$, $G_k'(x_i;\xi) = \frac{dG(x_i;\xi)}{d\xi}$ and $G_k'(x_i;\xi) = \frac{d^2G(x_i;\xi)}{d^2\xi}$.

4.2. Method of Least Square Estimation (LSE)

Another method of estimation was introduced by Swain et al. [27] named the ordinary LSE and weighted LSE to estimate the distribution parameters. Consider $x_{(1)},...,x_{(n)}$ be order statistics of the random sample of size n from $F(x,\xi)$. The LSE for the NCS-G FD can be obtained by minimizing

$$K(X;\xi) = \sum_{i=1}^{n}\left[F(x_{(i)};\xi) - \frac{i}{n+1}\right]^2. \tag{40}$$

with respect to ξ. The least-square estimates for the NCS-G FD also become

$$K(X;\xi) = \sum_{i=1}^{n}\left[\sin\left[\pi\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right] - \frac{i}{n+1}\right]^2. \tag{41}$$

Now differentiating Equation (41) with respect to ξ we get

$$\frac{\partial K}{\partial \xi} = 2\pi\sum_{i=1}^{n}\left[\sin\left[\pi\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right] - \frac{i}{n+1}\right]\cos\left[\pi\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right]\frac{G_k'(x_{(i)};\xi)}{\left(1+G(x_{(i)};\xi)\right)^2}. \tag{42}$$

where $G_k'(x_i;\xi) = \frac{dG(x_{(i)};\xi)}{d\xi}$. By solving $\frac{dK}{d\xi} = 0$, we will get the LSEs.

4.3. Cramer-von Mises Minimum Distance Estimator (CVME)

Cramer-von Mises estimators (CVMEs) are specific types of statistical estimators that minimize the difference between the estimated and the empirical CDF. These estimators are considered to have a lower bias compared to other minimum distance estimators. In the context of estimating parameters for the NCS-G FD distribution, CVMEs can be used to obtain more accurate estimates by minimizing

$$C(X;\xi) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{(i)};\xi) - \frac{2i-1}{2n}\right]^2. \tag{43}$$

with respect to ξ . The CVMEs for the NCS-G FD also become

$$C(X;\xi) = \sum_{i=1}^{n}\left[\sin\left[\pi\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right] - \frac{2i-1}{2n}\right]^2 \tag{44}$$

Now differentiating Equation (44) with respect to ξ we get

$$\frac{\partial C}{\partial \xi} = 2\pi\sum_{i=1}^{n}\left[\sin\left[\pi\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right] - \frac{2i-1}{2n}\right]\cos\left[\pi\frac{G(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}\right]\frac{G_k'(x_{(i)};\xi)}{\left(1+G(x_{(i)};\xi)\right)^2}, \tag{45}$$

where $G_k'(x_{(i)};\xi) = \frac{dG(x_{(i)};\xi)}{d\xi}$. By solving $\frac{dC}{d\xi} = 0$, we will get the CVMEs.

5. Special Member of NCS-G FD

Generalization of several distributions can be made using the NCS-G FD. Here we have considered the inverse Weibull (IW) distribution as a parent distribution to introduce a special member.

5.1. A New Class Sin Inverse Weibull (NCS-IW) Distribution

The CDF and PDF of the IW distribution are respectively given by

$$G(x;\delta,\theta) = exp(-\theta x^{-\delta}); x>0, \delta>0, \theta>0$$

and

$$g(x;\delta,\theta) = \delta\theta x^{-(\delta+1)} exp(-\theta x^{-\delta}).$$

The CDF and PDF of the NCS-IW distribution are given by

$$F(x;\theta,\delta) = \sin\left[\pi \frac{\exp(-\theta x^{-\delta})}{1+\exp(-\theta x^{-\delta})}\right]; x>0. \tag{46}$$

$$f(x;\theta,\delta) = \pi\theta\delta x^{-(\delta+1)} \cos\left[\pi \frac{\exp(-\theta x^{-\delta})}{1+\exp(-\theta x^{-\delta})}\right] \frac{\exp(-\theta x^{-\delta})}{\left(1+\exp(-\theta x^{-\delta})\right)^2}; x>0. \tag{47}$$

The reliability and hazard functions, respectively, are given by

$$R(x;\theta,\delta) = 1-\sin\left[\pi \frac{\exp(-\theta x^{-\delta})}{1+\exp(-\theta x^{-\delta})}\right]; x>0. \tag{48}$$

and

$$h(x;\theta,\delta) = \pi\theta\delta x^{-(\delta+1)} \frac{\exp(-\theta x^{-\delta})}{\left(1+\exp(-\theta x^{-\delta})\right)^2} \cos\left[\pi \frac{\exp(-\theta x^{-\delta})}{1+\exp(-\theta x^{-\delta})}\right] \left[1-\sin\left(\pi \frac{\exp(-\theta x^{-\delta})}{1+\exp(-\theta x^{-\delta})}\right)\right]^{-1}; x>0. \tag{49}$$

The possible shapes of PDF and HRF of NCS-IW distribution are shown in Figure (1) and it is observed that HRF can have reverse-j, or inverted bathtub or increasing hazard function. The quantile function and random deviate generation for the NCS-IW distribution, respectively, are given by

$$Q_X(p) = \left[-\frac{1}{\theta}\log\left(\frac{\sin^{-1} p}{\pi-\sin^{-1} p}\right)\right]^{-\frac{1}{\delta}}. \tag{50}$$

and

$$x = \left[-\frac{1}{\theta}\log\left(\frac{\sin^{-1} u}{\pi-\sin^{-1} u}\right)\right]^{-\frac{1}{\delta}}. \tag{51}$$

5.2. Linear Expansion

Using Equation (14), Equation (47) can be expressed in linear form as

$$f(x;\xi) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{ij} x^{-(\delta+1)} \exp\left\{-(2i+j+1)\theta x^{-\delta}\right\} \tag{52}$$

where $B_{ij} = \frac{(-1)^{2i}\theta\delta\pi^{2i+1}}{(2i)!}\binom{2(i-1)}{j}$.

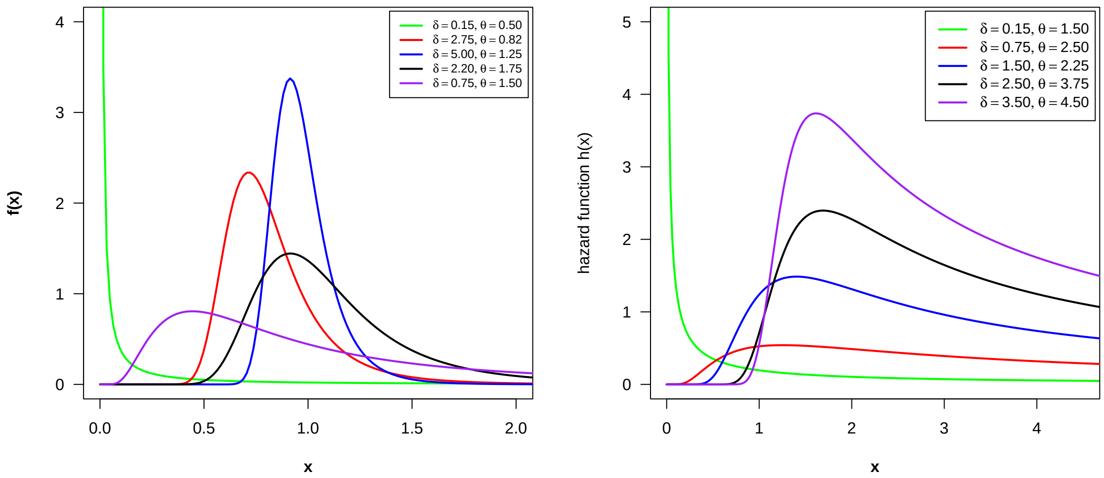

Figure 1: *Shapes of PDF and HRF of NCS-IW distribution*

5.3. Moments

Using the PDF defined in Equation (52), the r^{th} order non-central moment ($\mu_r^{'}$) for the NCS-IW distribution can be presented as

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{ij}^{*} \frac{\Gamma\left(\frac{\delta-r}{\delta}\right)}{\left[\theta\{(2i+j)+1\}\right]^{\frac{\delta-r}{\delta}}}; \quad \forall \delta > r, \tag{53}$$

where $B_{ij}^{*} = \frac{(-1)^{2i}\theta\pi^{2i+1}}{(2i)!}\binom{2(i-1)}{j}$ and $\Gamma(.)$ is the gamma function.

5.4. Moment Generating Function (MGF)

The MGF ($M_X(t)$) for the NCS-IW distribution is

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^k B_{ij}^{*}}{k!} \frac{\Gamma\left(\frac{\delta-r}{\delta}\right)}{\left[\theta\{(2i+j)+1\}\right]^{\frac{\delta-r}{\delta}}}; \quad \forall \delta > r. \tag{54}$$

5.5. Incomplete moment

The incomplete moment for NCS-IW distribution is presented as

$$\begin{aligned} M_r(y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{ij} \int_0^y x^{r-(\delta+1)} \exp\left\{-(2i+j+1)\theta x^{-\delta}\right\} dx \\ &= \frac{1}{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{ij} \frac{\gamma\left(\frac{\delta-r}{\delta}, (2i+j+1)\theta y^{-\delta}\right)}{\{(2i+j+1)\theta\}^{\frac{\delta-r}{\delta}}}, \end{aligned} \tag{55}$$

where $\gamma(.)$ incomplete gamma function.

5.6. Mean Residual Life

The MRL for NCS-IW distribution is given by

$$\begin{aligned}\bar{M}(y) &= \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{ij}\int_{0}^{y} x^{-\delta}\exp\left\{-(2i+j+1)\theta x^{-\delta}\right\}dx\right] - y \\ &= \frac{1}{F(y)}\left[\mu - \frac{1}{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{ij}\frac{\gamma\left(\frac{\delta-1}{\delta},(2i+j+1)\theta y^{-\delta}\right)}{\{(2i+j+1)\theta\}^{\frac{\delta-1}{\delta}}}\right] - y,\end{aligned} \tag{56}$$

where $\gamma(.)$ is the incomplete gamma function. Using the Equations (9 and 10) for NCS-IW distribution, we have plotted the graphs of skewness and kurtosis in Figure (2) for different values of the parameters δ and θ.

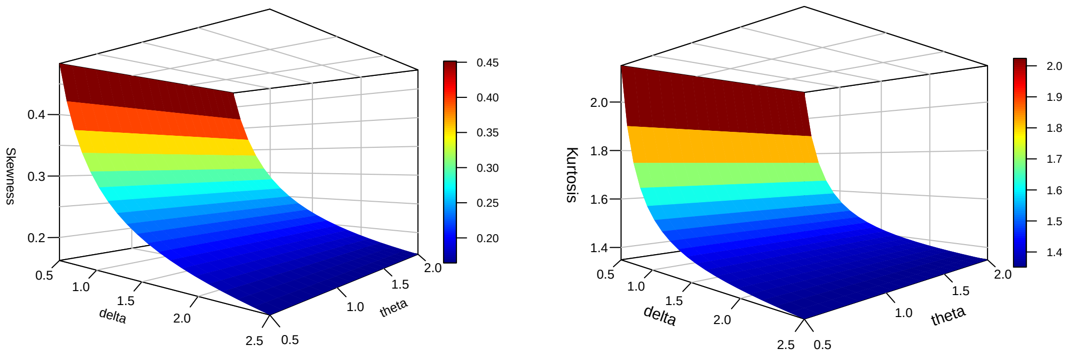

Figure 2: *Skewness and Kurtosis plots of NCS-IW distribution.*

5.7. Entropy

i) Renyi's Entropy
Renyi's entropy for NCS-IW distribution is given by

$$\begin{aligned}R_\rho(X) &= \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} Z_{krm}(\delta\theta)^{\rho}\int_{0}^{\infty} x^{-\rho(\delta+1)}\exp(-(r+m+\rho)\theta x^{-\delta})dx\right] \\ &= \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty} Z_{krm}\frac{(\delta\theta)^{\rho}}{\delta}\frac{\Gamma\left\{\frac{(\rho-1)(\delta+1)}{\delta}+1\right\}}{\{(r+m+\rho)\theta\}^{\frac{(\rho-1)(\delta+1)}{\delta}+1}}\right],\end{aligned} \tag{57}$$

where $Z_{krm} = (-1)^{m+k-r}\pi^{\rho}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

ii) q-Entropy

The q-Entropy for NCS-IW distribution is given by

$$\begin{aligned} H(\rho) &= \frac{1}{1-\rho}\log\left[1 - Z_{krm}(\delta\theta)^{\rho}\int_0^{\infty} x^{-\rho(\delta+1)}\exp(-(r+m+\rho)\theta x^{-\delta})dx\right] \\ &= \frac{1}{1-\rho}\log\left[1 - Z_{krm}\frac{(\delta\theta)^{\rho}}{\delta}\frac{\Gamma\left\{\frac{(\rho-1)(\delta+1)}{\delta}+1\right\}}{\{(r+m+\rho)\theta\}^{\frac{(\rho-1)(\delta+1)}{\delta}+1}}\right] \end{aligned} \tag{58}$$

where $\rho > 0$ and $\rho \neq 1$. where $Z_{krm} = (-1)^{m+k-r}\pi^{\rho}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

iii) Shannon's Entropy

The Shannon entropy for the NCS-IW distribution is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\pi^{2i+1}(-1)^{2i}}{(2i)!}\binom{2(i-1)}{j}x^{-(\delta+1)}\exp(-\theta(2i+j+1)x^{-\delta})\right\}\right]. \tag{59}$$

5.8. Inequality Measure

i) Lorentz Curve

The Lorenz curve for NCS-IW distribution is given by

$$\begin{aligned} L_{F(y)} &= \frac{\delta\theta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}B_{ij}\int_0^{y}x^{-\delta}\exp(-\theta(2i+j+1)x^{-\delta})dx \\ &= \frac{\theta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}B_{ij}\frac{\gamma\left(\frac{\delta-1}{\delta},(2i+j+1)\theta y^{-\delta}\right)}{\{(2i+j+1)\theta\}^{\frac{\delta-1}{\delta}}}. \end{aligned} \tag{60}$$

where $\gamma(.)$ is the incomplete gamma function.

ii) Boneferroni Curve

The Boneferroni curve for the NCS-IW distribution is given by

$$\begin{aligned} B_{F(y)} &= \frac{1}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}B_{ij}\int_0^{y}x^{-\delta}\exp(-\theta(2i+j+1)x^{-\delta})dx \\ &= \frac{1}{\delta\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}B_{ij}\frac{\gamma\left(\frac{\delta-1}{\delta},(2i+j+1)\theta y^{-\delta}\right)}{\{(2i+j+1)\theta\}^{\frac{\delta-1}{\delta}}}. \end{aligned} \tag{61}$$

where $\gamma(.)$ is the incomplete gamma function.

5.9. Estimation MLE for NCS-IW distribution

We now investigate the MLE for estimating the parameters of the NCS-IW model. As a result, we intend to compute MLEs for the parameters δ and θ. Let $X = (x_1, ..., x_n)^T$ be a vector of size n of independent random variables from the NCS-IW distribution. Then, the log-likelihood is given by

$$l(x;\delta,\theta) = n\log(\pi\theta\delta) - (\delta+1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\cos\left[\pi\frac{\exp(-\theta x_i^{-\delta})}{1+\exp(-\theta x_i^{-\delta})}\right] - 2\sum_{i=1}^{n}\log\left(1+\exp(-\theta x_i^{-\delta})\right) - \theta\sum_{i=1}^{n}x_i^{-\delta} \tag{62}$$

Partially differentiating the Equation (62) with respect to δ and θ gives the score function's components of $V(u) = \left(\frac{\partial l}{\partial \delta}, \frac{\partial l}{\partial \theta}\right)^T$ as,

$$\begin{aligned}\frac{\partial l}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} \log x_i + \pi\theta \sum_{i=1}^{n} \frac{x_i^{-\delta} \log(x_i) \exp(-\theta x_i^{-\delta})}{\left(1+\exp(-\theta x_i^{-\delta})\right)^2} \tan\left[\pi \frac{\exp(-\theta x_i^{-\delta})}{1+\exp(-\theta x_i^{-\delta})}\right] \\ + 2\theta \sum_{i=1}^{n} \frac{x_i^{-\delta} \log(x_i) \exp(-\theta x_i^{-\delta})}{\left(1+\exp(-\theta x_i^{-\delta})\right)} + \theta \sum_{i=1}^{n} x_i^{-\delta} \log(x_i)\end{aligned} \tag{63}$$

and

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \pi \sum_{i=1}^{n} \frac{x_i^{-\delta} exp(-\theta x_i^{-\delta})}{\left(1+\exp(-\theta x_i^{-\delta})\right)^2} \tan\left[\pi \frac{\exp(-\theta x_i^{-\delta})}{1+\exp(-\theta x_i^{-\delta})}\right] - 2\sum_{i=1}^{n} \frac{x_i^{-\delta} exp(-\theta x_i^{-\delta})}{\left(1+\exp(-\theta x_i^{-\delta})\right)} - \sum_{i=1}^{n} x_i^{-\delta} \tag{64}$$

The MLEs of δ and θ are obtained by maximizing $l(x;\delta,\theta)$ in δ and θ , which can be done by solving simultaneously the equations $\frac{\partial l}{\partial \delta} = 0$ and $\frac{\partial l}{\partial \theta} = 0$.

5.10. Simulation

Using the maxLik R package introduced by Henningsen and Toomet [10], we generated samples from the quantile function defined in Equation (50) for various parameter combinations of the NCS-IW distribution and calculated the MLEs for each sample using the `maxLik()` function with the BFGS algorithm. This allows us to test parameter estimation problems such as the sharpness or flatness of the likelihood function, as well as estimate the size and direction (underestimate or overestimate) of the MLEs bias. Sample sizes of 20, 30, 40, 50, and 75 are used in the simulation. The procedure is repeated 10,000 times, and the average estimate value, bias, and mean square error (MSE) are calculated. The experiment is summarized in Table 1, which shows the average estimate, bias, and MSEs for each parameter. As can be seen, the MLE method consistently overestimates the parameter δ and underestimates the parameter θ, but as sample size increases, MLEs gradually approach the actual values of δ and θ.

Table 1: *The estimated values, Biases, and MSEs based on 10000 simulations of NCS-IW distribution.*

	Actual values		MLEs		Bias		MSEs	
n	delta	theta	$\hat{\delta}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\theta}$
	0.25	0.50	0.268	0.4796	0.018	-0.0204	0.0029	0.0219
20	0.50	0.75	0.5372	0.7263	0.0372	-0.0237	0.0115	0.0318
	0.75	1.00	0.805	0.9816	0.055	-0.0184	0.026	0.0415
	0.25	0.50	0.2621	0.4869	0.0121	-0.0131	0.0016	0.0142
30	0.50	0.75	0.5241	0.7343	0.0241	-0.0157	0.0066	0.0215
	0.75	1.00	0.7874	0.9848	0.0374	-0.0152	0.0154	0.0277
	0.25	0.50	0.2593	0.4889	0.0093	-0.0111	0.0012	0.0109
40	0.50	0.75	0.5175	0.7377	0.0175	-0.0123	0.0046	0.0157
	0.75	11.00	0.7768	0.9911	0.0268	-0.0089	0.0103	0.0201
	0.25	0.50	0.257	0.4919	0.007	-0.0081	0.0009	0.0087
50	0.50	0.75	0.5146	0.7398	0.0146	-0.0102	0.0037	0.0129
	0.75	1.00	0.7696	0.992	0.0196	-0.008	0.0078	0.0159
	0.25	0.50	0.2546	0.4943	0.0046	-0.0057	0.0006	0.0059
75	0.50	0.75	0.5089	0.7444	0.0089	-0.0056	0.0022	0.0084
	0.75	1.00	0.7646	0.994	0.0146	-0.006	0.0052	0.0105

5.11. Application

Employing two real data sets, we exhibit the application of the NCS-IW distribution in this section. The data sets employed for the application of the suggested distribution are given as follows
i) Data set
Data set 1 (cancer data):
The data set contains information on the survival times of 44 patients. These patients who received radiotherapy have head and neck cancer, and this data set was reported by Efron [8].
"12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776".
Data set 2 (relief time data):
The real data set is considered from Clark and Gross [7], which provides the relief times of 20 patients receiving an analgesic. The data are:
"1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.0".
ii) Model Analysis
We calculate some well-known goodness-of-fit statistics to analyze data sets 1 and 2 and the fitted models are evaluated using the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (AD), Kolmogrov-Smirnov (KS) with p-values, and Cram'er-von Mises (CVM) for more detail (see Chen and Balakrishnan [5]). All the essential computations are carried out in R-software. For the comparison of fitting capability, we have selected some models such as inverse Weibull (IW), transformed sine Weibull (TSW) Sakthivel and Rajkumar [23], arctan generalized exponential (AGE) Chaudhary et al. [3], arctan Lomax (ALomx) Chaudhary and Kumar [4], arcsine exponential (ASE) Rahman [21], and arcsine exponentiated Weibull (ASEW) He et al. [11]. The PDFs of candidate models are as follows

$$f_{IW}\left(x;\delta,\theta\right)=\delta\theta x^{-\delta-1}e^{-\theta x^{-\delta}},x,\delta,\theta>0.$$

$$f_{TSW}\left(x;\alpha,\beta,\lambda\right)=\frac{\pi}{2}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left[\frac{\pi\lambda}{2}\left(1-e^{-\alpha x^{\beta}}\right)\cos\left(\frac{\pi}{2}e^{-\alpha x^{\beta}}\right)-(1-\lambda)\sin\left(\frac{\pi}{2}e^{-\alpha x^{\beta}}\right)\right],x,\alpha,\beta,\lambda>0.$$

$$f_{AGE}\left(x;\alpha,\beta,\lambda\right)=\frac{\alpha\beta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\beta-1}}{arc\tan\left(\alpha\right)\left[1+\left\{\alpha\left(1-\left(1-e^{-\lambda x}\right)^{\beta}\right)\right\}^{2}\right]};x,\alpha,\beta,\lambda>0.$$

$$f_{ALomx}\left(x;\alpha,\beta,\lambda\right)=\frac{\alpha\beta\lambda}{arc\tan\left(\alpha\right)}\frac{\left(1+\beta x\right)^{-\lambda-1}}{\left[1+\left\{\alpha\left(1+\beta x\right)^{-\lambda}\right\}^{2}\right]};x,\alpha,\beta,\lambda>0.$$

$$f_{ASE}\left(x;\alpha\right)=\frac{\sqrt{e^{-x/\alpha}}}{\pi\alpha\sqrt{1-e^{-x/\alpha}}};x,\alpha>0.$$

$$f_{ASEW}\left(x;\alpha,\beta,\lambda\right)=\frac{2}{\pi}\alpha\beta\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}}\frac{\left(1-e^{-\lambda x^{\alpha}}\right)^{\beta-1}}{\sqrt{1-\left(1-e^{-\lambda x^{\alpha}}\right)^{2\beta}}};x,\alpha,\beta,\lambda>0.$$

In Tables 2 and 3, we have presented the estimated values of the parameters and their associated standard error (SE in parentheses) of the models under study using the MLE method for cancer and relief time data. Similarly, in Tables 4 and 5, we have presented the model selection and goodness of fit statistics like log-likelihood, HQIC, AIC, KS, AD, and CVM for both data sets. It has been observed that the suggested model has the least statistics as compared to IW, AGE, ALomx, ASE, ASEW, and TSW. Hence NCS-IW is more flexible (even four trigonometric distributions having three parameters) and provides a good fit. Also, we have displayed the graphical illustrations of the fitted models in Figures 5 and 6. These figures also verified that the NCS-IW model can perform well as compared to candidate models

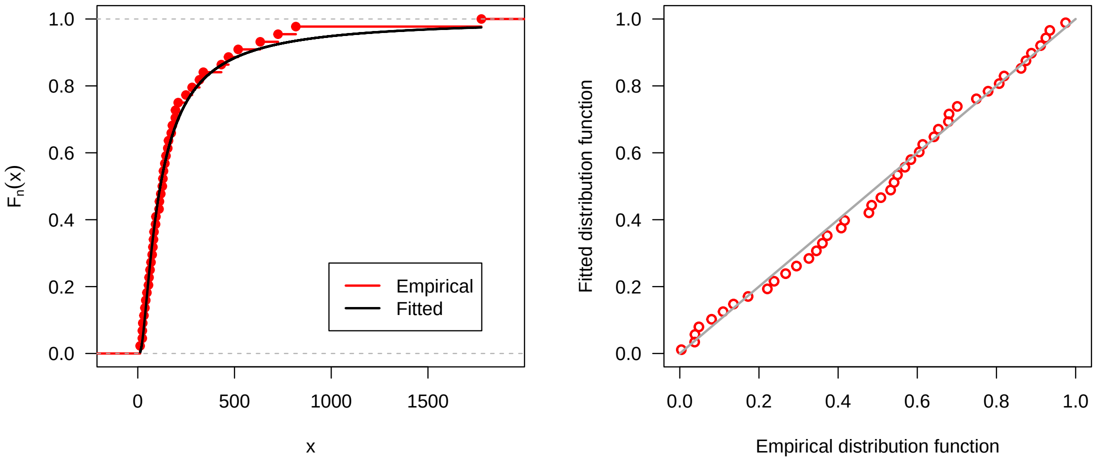

Figure 3: *KS and P-P plots (data-I).*

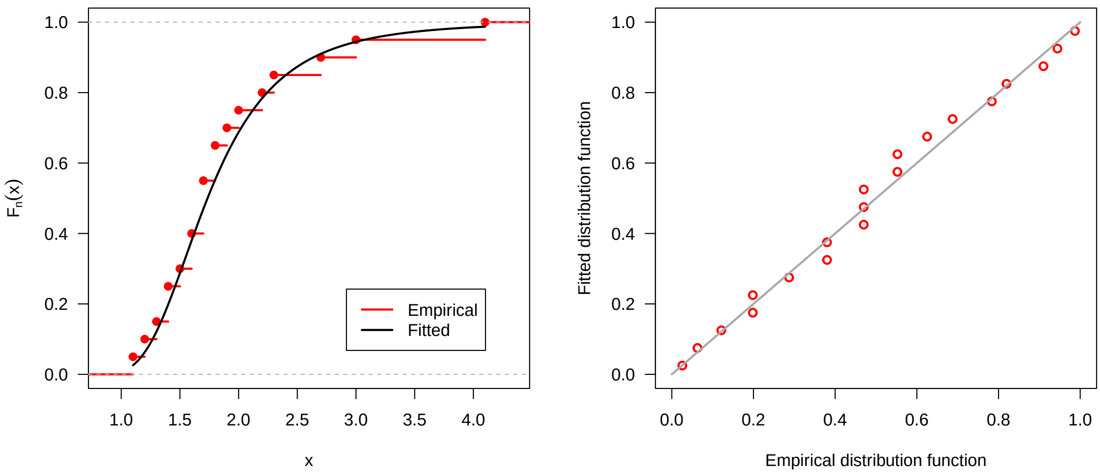

Figure 4: *KS and P-P plots (data-II).*

Table 2: *MLEs with SE (in parentheses) (data-I).*

Distribution	Parameter(SE)	Parameter(SE)	Parameter(SE)
NCS-IW(δ, θ)	0.6317(0.0508)	32.4048(6.9827)	–
IW(δ, θ)	0.9985(0.0393)	75.557(4.4651)	–
AGE(α, β, λ)	0.0179(0.5939)	1.0688(0.2216)	0.0047(9.00E-04)
ALomx(α, β, λ)	27.525(5.8997)	0.0640(0.0335)	1.5273(0.2833)
ASE(α)	341.8104(4.1943)	–	–
ASEW(α, β, λ)	0.4578(0.0133)	13.1876(4.5521)	0.4031(0.0919)
TSW(α, β, λ)	0.0039(0.0025)	0.9742(0.1073)	0.1327(0.1312)

Table 3: *MLEs with SE (in parentheses) (data-II).*

Distribution	Parameter(SE)	Parameter(SE)	Parameter(SE)
NCS-IW(δ,θ)	2.3934(0.4249)	6.0185(1.3910)	–
IW(δ,θ)	4.0175(0.706)	6.0224(2.0083)	–
AGE(α,β,λ)	29.0366(6.6483)	2.9010(3.1180)	2.5293(0.567)
ALomx(α,β,λ)	187.9197(5.1477)	0.2891(0.3043)	12.8568(11.0058)
ASE(α)	127.8946(4.8432)	–	–
ASEW(α,β,λ)	1.0488(0.1284)	104.561(19.0921)	3.1656(0.1303)
TSW(α,β,λ)	0.0811(0.0398)	2.9331(0.4532)	0.1297(0.1388)

Table 4: *Some selection criteria and goodness-of-fit statistics (data-I).*

Distribution	-2logL	AIC	HQIC	KS(p-value)	CVM(p-value)	AD(p-value)
NCS-IW	556.1646	560.1646	561.4879	0.0706(0.9698)	0.0302(0.9768)	0.2106(0.9873)
IW	559.1617	563.1617	564.4851	0.0916(0.8218)	0.0806(0.6906)	0.5084(0.7373)
AGE	563.9111	569.9111	571.8961	0.1496(0.2518)	0.1963(0.2753)	1.0219(0.3455)
ALomx	556.8248	562.8248	564.8098	0.0532(0.9990)	0.0142(0.9998)	0.1482(0.9988)
ASE	587.1019	589.1019	589.7636	0.2771(0.0018)	1.0569(0.0017)	5.3298(0.0020)
ASEW	554.7389	560.7389	562.7239	0.0634(0.9896)	0.0214(0.9959)	0.1351(0.9994)
TSW	561.8178	567.8178	569.8028	0.1126(0.5930)	0.1014(0.5799)	0.6674(0.5857)

Table 5: *Some selection criteria and goodness-of-fit statistics (data-I).*

Distribution	-2logL	AIC	HQIC	KS(p-value)	CVM(p-value)	AD(p-value)
NCS-IW	31.0171	35.0171	35.4059	0.0975(0.9913)	0.0254(0.9906)	0.1594(0.9979)
IW	30.8174	34.8174	35.2062	0.1020(0.9854)	0.0266(0.988)	0.1545(0.9984)
AGE	36.8149	42.8149	43.398	0.1193(0.9385)	0.0577(0.8338)	0.5597(0.6847)
ALomx	35.4117	41.4117	41.9949	0.1136(0.9587)	0.0565(0.8416)	0.4783(0.7670)
ASE	154.7472	156.7472	156.9416	0.8863(0.0000)	5.1247(0.0000)	31.4397(0.0000)
ASEW	31.1885	37.1885	37.7716	0.1170(0.9470)	0.0363(0.9551)	0.2096(0.9877)
TSW	39.7066	45.7066	46.2898	0.1694(0.6147)	0.1415(0.4194)	0.8932(0.4170)

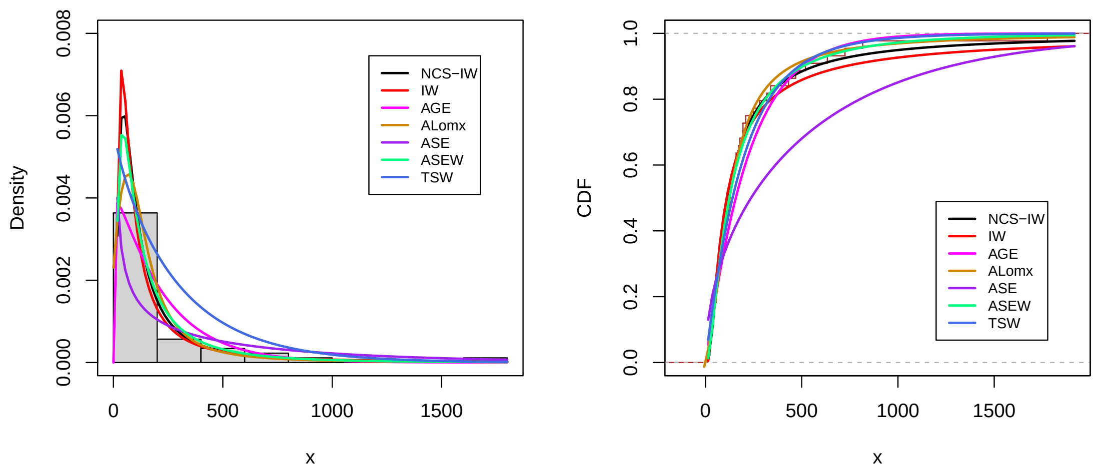

Figure 5: *Estimated PDF (left) and empirical vs estimated CDF (right) (data-I).*

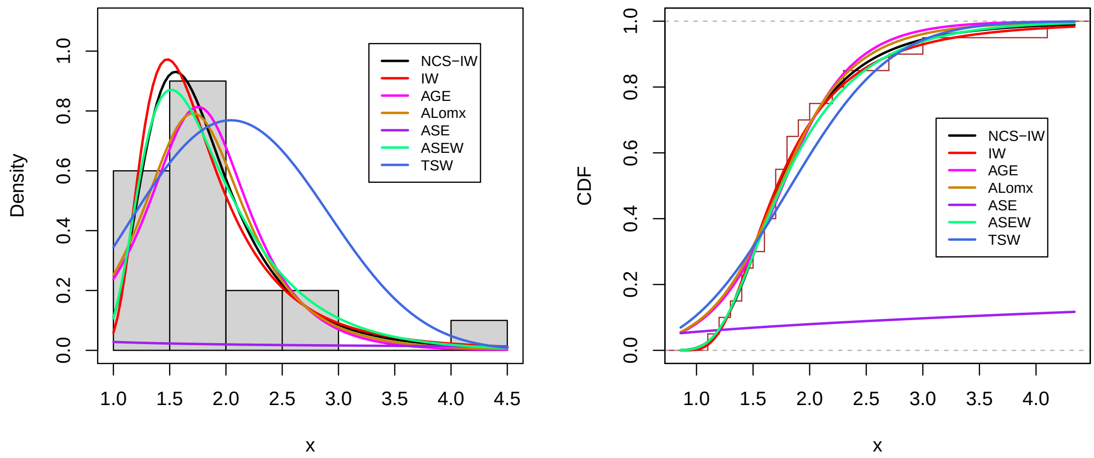

Figure 6: *Estimated PDF (left) and empirical vs estimated CDF (right) (data-II).*

6. Conclusion

Based on the ratio of CDF $G(x)$ and $1+G(x)$ of baseline distribution, we developed the new trigonometric family of distributions by transforming the sine function and we named it the new class sin-G family of distributions. General properties of the suggested family of distributions are provided. Using Inverse Weibull distribution as a baseline distribution, we have introduced a member of the suggested family having reverse-j or increasing or inverted bathtub-shaped hazard function. Some statistical characteristics of this NCS-IW distribution are explored. The associated parameters of the new distribution are estimated through the MLE method. To assess the estimation procedure, we conducted a Monte Carlo simulation and found that even for small samples, biases and mean square errors decreased as the size of the sample increased. Two real medical data sets are considered for the application of the NCS-IW distribution. Using some model selection criteria and goodness of fit test statistics, we empirically proved that the suggested model performs better than six other existing models (most of which have more parameters). Hence, we expect that the suggested family and its member distribution can be used in broader areas like medical science, reliability engineering, survival analysis, etc., and one can generate a new model using this family of distributions in the future.

References

[1] Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, 71(1), 63-79.

[2] Balakrishnan, N., & Cohen, A. C. (1991). *Order statistics & inference: estimation methods*. Academic Press, London.

[3] Chaudhary, A. K., Sapkota, L. P. & Kumar, V. (2021). Some properties and applications of arctan generalized exponential distribution. *International Journal of Innovative Research in Science, Engineering and Technology (IJIRSET)*, 10(1), 456-468.

[4] Chaudhary, A. K., & Kumar, V. (2021). The ArcTan Lomax distribution with properties and applications. International Journal of Scientific Research in Science, Engineering and Technology, 4099, 117-125.

[5] Chen, G., & Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, 27(2), 154-161.

[6] Chesneau, C., & Jamal, F. (2020). The sine Kumaraswamy-G family of distributions. *Journal of Mathematical Extension*, 15.

[7] Clark, V. A., & Gross, A. J. (1975). *Survival distributions: reliability applications in the biomedical sciences*. New York, John Wiley Sons.

[8] Efron, B. (1988). Logistic regression, survival analysis, and the Kaplan-Meier curve. *Journal of the American Statistical Association*, 83(402), 414-425.

[9] Gomez-Deniz, E., & Calderin-Ojeda, E. (2015). Modelling insurance data with the Pareto ArcTan distribution. *ASTIN Bulletin: The Journal of the IAA*, 45(3), 639-660.

[10] Henningsen, A., & Toomet, O. (2011). maxLik: A package for maximum likelihood estimation in R. Computational Statistics, 26, 443-458.

[11] He, W., Ahmad, Z., Afify, A. Z., & Goual, H. (2020). The arcsine exponentiated-X family: validation and insurance application. *Complexity*, 1-18.

[12] Isa, A. M., Ali, B. A., & Zannah, U. (2022). Sine Burr XII Distribution: Properties and Application to Real Data Sets. *AJBAR*, 1(3), 48-58.

[13] Kenney, J. F. & Keeping, E. S. (1962). *Mathematics of Statistics*, 3 edn, Chapman and Hall Ltd, New Jersey.

[14] Kharazmi, O. & Saadatinik, A. (2016). Hyperbolic cosine-F families of distributions with an application to exponential distribution. *Gazi University Journal of Science*, 29(4), 811-829.

[15] Kumar, D., Singh, U., & Singh, S. K. (2015). A new distribution using sine function-its application to bladder cancer patients data. *Journal of Statistics Applications & Probability*, 4(3), 417.

[16] Mahmood, Z., Chesneau, C., & Tahir, M. H. (2019). A new sine-G family of distributions: properties and applications. Bull. *Comput. Appl. Math.*, 7(1), 53-81.

[17] Marshall, A. W., & Olkin, I. (2007). *Life distributions* (Vol. 13). Springer, New York.

[18] Moors, J. J. A. (1988). A quantile alternative for kurtosis. Journal of the Royal Statistical Society: *Series D (The Statistician)*, 37(1), 25-32.

[19] Muhammad, M., Alshanbari, H. M., Alanzi, A. R., Liu, L., Sami, W., Chesneau, C., & Jamal, F. (2021). A new generator of probability models: the exponentiated sine-G family for lifetime studies. *Entropy*, 23(11), 1394.

[20] Renyi, A. (1961). *On measures of entropy and information*. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics (Vol. 4, pp. 547-562). University of California Press.

[21] Rahman, M. M. (2021). Arcsine-G Family of Distributions. *J. Stat. Appl. Pro. Lett.* 8(3), 169-179.

[22] Sakthivel, K. M. and Rajkumar, J. (2020). Hyperbolic cosine Rayleigh distribution and its application to breaking stress of carbon fibers. *Journal of Indian Society and Probability Statistics*, 21(2), 471-485.

[23] Sakthivel, K. M., & Rajkumar, J. (2021). Transmuted sine-G family of distributions: theory and applications. *Statistics and Applications*,(Accepted: 10 August 2021).

[24] Souza, L. (2015). *New trigonometric classes of probabilistic distributions (Doctoral dissertation, Thesis*, Universidade Federal Rural de Pernambuco).

[25] Souza, L., Junior, W. R. D. O., de Brito, C. C. R., Ferreira, T. A., & Soares, L. G. (2019a). General properties for the Cos-G class of distributions with applications. *Eurasian Bulletin of Mathematics* (ISSN: 2687-5632), 63-79.

[26] Souza, L., Junior, W., De Brito, C., Chesneau, C., Ferreira, T., & Soares, L. (2019b). On the Sin-G class of distributions: theory, model and application. *Journal of Mathematical Modeling*, 7(3), 357-379.

[27] Swain, J. J., Venkatraman, S., & Wilson, J. R. (1988). Least-squares estimation of distribution functions in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29(4), 271-297.