THEORY AND APPLICATIONS OF THE ALPHA POWER TYPE II TOPP-LEONE- GENERATED FAMILY OF DISTRIBUTIONS

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Abstract

This paper introduces a composition of two single parameter generalized family of distributions: the alpha power transform and type II Topp-Leone-G families of distributions. Some basic mathematical treatments of the family of distributions are studied. The parameter estimates of the proposed family of distributions are derived via maximum likelihood estimation method and a Monte Carlo simulation study was conducted to examine the asymptotic behaviour of the parameter estimates of sub-model belonging to the proposed family of distributions. To illustrate the applicability of the proposed family of distributions in real world data fittings, two data sets consisting of the daily recovery and mortality rates of Covid-19 patients in Nigeria, from May 1 to June 30, 2020, was employed. The APTIITLK distribution arising from the proposed family of distributions, alongside with some bounded non-nested distributions was used to fit the two data sets and results obtained from the analysis clearly revealed that the APTIITLK distribution outperformed all the non-nested distributions used in fitting the two data sets. Some informative graphical plots for goodness of fit test were investigated to further validate the flexibility of the APTIITLK distribution over the competing distributions.

Keywords: Alpha Power Transformation; Type II Topp-Leone Generated; Quantile; Simulation Study

1. INTRODUCTION

The theory of statistical analysis has received a reasonable attention in the area of developing lifetime distributions. several lifetime distributions have been proposed to analyze real world phenomena in literature. Its utility has found tremendous applications in research fields such as engineering, biological sciences, machine learning, actuarial sciences, demography, agricultural sciences, etc. Regardless the numerous lifetime distributions in literature, an insatiable quest to develop more flexible and tractable models have evolved among researchers in the field of statistical distribution theory. It is noteworthy that many existing lifetime distributions have failed in providing good fit for certain complex datasets, thus, the drive to develop new ones. Several novel methodologies have been introduced to expand the utility of existing lifetime distributions. Thanks to [1] who developed the exponentiated Weibull family of distributions, [2] introduced the Marshall-Olkin extended family, [3] studied the beta-*G* class of distributions, [4] proposed the transmuted-G method, [5] used the idea of [3] to introduce the Kumaraswamy-*G* method, [6] proposed the transformed-transformer (*T*-*X*) method, and [7] developed the Weibull-*G* method.

Recently, [8] have suggested a new method of adding extra parameter to an existing lifetime distribution which they called "alpha-power transformation method". Let $G(t)$ denote the cdf of any continuous random variable *T*, [8] defined the alpha-power transformation of $G(t)$ as

$$F_{APT}(t,\alpha)=\begin{cases}\dfrac{\alpha^{G(t)}-1}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq1\\[2ex] G(t), & if \quad \alpha=1\end{cases} \tag{1}$$

The pdf associated to (1) is defined as

$$f_{APT}(t,\alpha)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}\,g(t)\,\alpha^{G(t)}, & if \quad \alpha>0,\ \alpha\neq1\\[2ex] g(t), & if \quad \alpha=1\end{cases} \tag{2}$$

The methodology defined in (1) and (2) has been adopted by researchers to generalize existing lifetime distributions. Such generalizations include the alpha-power Raleigh distribution by [9], alpha-power transformed Lindley distribution by [10], alpha-power transformed power Lindley distribution by [11], alpha-power inverse Lomax distribution by [12], alpha-power Tessier distribution by [13], alpha-power Topp-Leone distribution by [14], etc.

Another tractable method of generalization is the type II Topp-Leone-G family of distributions proposed by [15]. They adopted the idea of [16] to generalize the Topp-Leone distribution with the cdf defined by

$$\begin{aligned}G_{TIITL-G}(t,\gamma,\xi)&=1-2\gamma\int_0^{1-F(t,\xi)}t^{\gamma-1}\left(1-t\right)\left(2-t\right)^{\gamma-1}dt,\\ &=1-\left(1-F^2\left(t,\xi\right)\right)^{\gamma},\end{aligned} \tag{3}$$

and pdf obtained as

$$g_{TIITL-G}(t,\gamma,\xi)\;=\;2\gamma\, f\left(t,\xi\right)F\left(t,\xi\right)\left[1-F^2\left(t,\xi\right)\right]^{\gamma-1},\quad t>0,\gamma>0. \tag{4}$$

The one-parameter special case of the Topp-Leone distribution developed by [17] happens to be the simplest (single parameter) distribution with a bathtub hazard rate property and this unique feature has also motivated researchers to study different modification of the distribution to enhance its flexibility in data fitting. [18] developed the Topp-Leone inverse Weibull distribution, [19] proposed the Topp-Leone Weibull distribution, [20] discussed the Topp-Leone generated Weibull distribution, [21] studied the Topp-Leone power Lindley distribution, [22] developed the transmuted version of the Marshall-Olkin Topp-Leone distribution studied in [23], etc.

Inspired by the idea of [24], we construct a novel and more suitable two-parameter generalized class of distributions by considering the cdf defined in (3) as the new baseline distribution in (1). The cdf of the new two-parameter generalized class of distributions is thus, defined as

$$F_{APTIITL-G}(t,\alpha,\gamma,\xi)=\begin{cases}\dfrac{\alpha^{\left[1-\left(1-F^2(t,\xi)\right)^{\gamma}\right]}-1}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] 1-\left(1-F^2\left(t,\xi\right)\right)^{\gamma}, & if \quad \alpha=1\end{cases}, \tag{5}$$

the density function associated to (5) is obtained as

$$f_{APTIITL-G}(t,\alpha,\gamma,\xi)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}2\gamma f(t,\xi)F(t,\xi)\left(1-F^2\left(t,\xi\right)\right)^{\gamma-1}\alpha^{\left[1-\left(1-F^2(t,\xi)\right)^{\gamma}\right]}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] 2\gamma f(t,\xi)F(t,\xi)\left(1-F^2\left(t,\xi\right)\right)^{\gamma-1}, & if \quad \alpha=1\end{cases}. \tag{6}$$

The random variable *T* in (5) and (6) is said to follow the alpha power type II Topp-Leone-G family of distributions (APTIITL-G for short). The survival function (sf) and hazard rate function (hrf) of the APTIITL-G family are, respectively, defined as

$$S_{APTIITL-G}(t,\alpha,\gamma,\xi)=\begin{cases}\dfrac{\alpha\left(1-\alpha^{-\left(1-F^2(t,\xi)\right)^{\gamma}}\right)}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] \left(1-F^2\left(t,\xi\right)\right)^{\gamma}, & if \quad \alpha=1\end{cases}, \tag{7}$$

and

$$h_{APTIITL-G}(t,\alpha,\gamma,\xi)=\begin{cases}\dfrac{\log(\alpha)2\gamma f(t,\xi)F(t,\xi)\left(1-F^2\left(t,\xi\right)\right)^{\gamma-1}\alpha^{-\left(1-F^2(t,\xi)\right)^{\gamma}}}{\left(1-\alpha^{-\left(1-F^2(t,\xi)\right)^{\gamma}}\right)}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] \dfrac{2\gamma f(t,\xi)F(t,\xi)}{1-F^2\left(t,\xi\right)}, & if \quad \alpha=1\end{cases}. \tag{8}$$

The basic objectives for developing the APTIITL-G family in practice are:

(i) to capture distributions with exponentially decreasing (reversed-J), negatively-skewed, positively-skewed, symmetric shaped property;

(ii) to construct distributions that span various forms of hazard rate property;

(iii) to produce distributions with consistently better fits than existing nested and non-nested distributions.

The rest of this paper is structured into the following sections. In Section 2 presents the materials and method. In detail, we derive the linear representation of the APTIITL-G density function, introduce some special sub-models generated from the APTIITL-G family. Some statistical properties of the APTIITL-G family are studied and the parameter estimation of the APTIITL-G family are obtained via the maximum likelihood method. A simulation study is conducted to investigate the asymptotic behavior of the parameter estimates. In Section 3, two data sets are used to illustrate the potential of sub-model from the APTIITL-G family. Section 4 concludes the paper.

2. MATERIALS AND METHOD

2.1 The density function of APTIITL-G family: linear representation

Most generalized distributions lack closed form expression for some of their statistical properties, thus limiting their utility in data analysis. Statistical properties such as moments, moment generating function, probability weighted moments, etc., are derived from the density function of the distribution. Hence, there is a clear need to obtain the series representation of the density function. To obtain the series representation of the density function of APTIITL-G family, we consider the following useful expansions.

$$\alpha^t = \sum_{k=0}^{\infty} \frac{(\log(\alpha))^k \, t^k}{k!}, \tag{9}$$

$$(1-t)^n = \sum_{q=0}^{\infty} \binom{n}{q} (-1)^q \, t^q. \tag{10}$$

(See [25], pg. 26, 2007).

Using (9) and (10) in (6), we have

$$\alpha^{\left[1-\left(1-F^2(t,\xi)\right)^{\gamma}\right]} = \sum_{j=0}^{\infty} \frac{(\log(\alpha))^j}{j!} \left[1-\left(1-F^2(t,\xi)\right)^{\gamma}\right]^j,$$

$$\left[1-\left(1-F^2(t,\xi)\right)^{\gamma}\right]^j = \sum_{k=0}^{j} \binom{j}{k} (-1)^k \left(1-F^2(t,\xi)\right)^{\gamma k},$$

$$\left(1-F^2(t,\xi)\right)^{\gamma(k+1)-1} = \sum_{m=0}^{\gamma(k+1)-1} \binom{\gamma(k+1)-1}{m} (-1)^m F(t,\xi)^{2m},$$

By inserting into (6), we have

$$f_{APTIITL-G}(t,\alpha,\gamma,\xi) = \sum_{j,=0}^{\infty} \sum_{k=0}^{j} \sum_{m=0}^{\gamma(k+1)-1} \psi_{j,k,m} \, \pi_{2(m+1)}(t,\alpha,\gamma,\xi) \tag{11}$$

where,

$$\psi_{j,k,m} = \frac{\gamma(\log(\alpha))^{j+1}}{j!(m+1)(\alpha-1)} \binom{j}{k} \binom{\gamma(k+1)-1}{m} (-1)^{k+m}$$

and

$$\pi_{2(m+1)}(t,\alpha,\gamma,\xi) = 2(m+1)\, f(t,\xi)\left[F(t,\xi)\right]^{2(m+1)-1}.$$

The pdf of APTIITL-G family defined in (11) is expressed as an infinite linear combination of exp-G densities with power parameter $2(m+1)$. Whereas, the cdf of APTIITL-G family is expressed as a linear combination of the exp-G cdfs as

$$F_{APTIITL-G}(t,\alpha,\gamma,\xi)=\sum_{j,=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+1)-1}\psi_{j,k,m}\,\Pi_{2(m+1)}(t,\alpha,\gamma,\xi). \tag{12}$$

Where $\Pi_{2(m+1)}(t,\alpha,\gamma,\xi)$ is the exp-G cdf with power parameter $2(m+1)$.

2.2 Some special sub-models of APTTIITL-G family

In this section, the authors introduced five special sub-models from APTIITL-G family by allowing the baseline distribution in (5) to follow Kumaraswamy, Weibull, log-logistic, Lindley and Bur XII distributions.

2.2.1 The alpha power type II Topp-Leone Kumaraswamy (APTIITLK) distribution

The Kumaraswamy distribution is a bounded lifetime distribution developed by [26], with cdf and pdf, respectively, defined by

$$F(t,\beta,\lambda)=1-\left(1-t^{\beta}\right)^{\lambda},\qquad \beta,\lambda>0,\ \ 0<t<1, \tag{13}$$

and

$$f(t,\beta,\lambda)=\lambda\beta t^{\beta-1}\left(1-t^{\beta}\right)^{\lambda-1},\qquad \beta,\lambda>0,\ \ 0<t<1. \tag{14}$$

By inserting (13) into (5), the authors defined the cdf of alpha power type II Topp-Leone Kumaraswamy (APTIITLK) distribution by

$$F_{APTIITLK}(t)=\begin{cases}\dfrac{\alpha^{\left[1-\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma}\right]}-1}{\alpha-1}, & if\quad \alpha>0,\ \alpha\neq 1\\ 1-\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma}, & if\quad \alpha=1\end{cases}, \tag{15}$$

and the associated pdf defined as

$$f_{APTIITLK}(t)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}2\gamma\lambda\beta t^{\beta-1}\left(1-t^{\beta}\right)^{\lambda-1}\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma-1}\alpha^{\left[1-\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma}\right]}, & if\quad \alpha>0,\ \alpha\neq 1\\ 2\gamma\lambda\beta t^{\beta-1}\left(1-t^{\beta}\right)^{\lambda-1}\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma-1}, & if\quad \alpha=1\end{cases} \tag{16}$$

The sf and hrf of APTIITLK distribution are obtained, respectively, as

$$S_{APTIITLK}(t)=\begin{cases}\dfrac{\alpha\left(1-\alpha^{-\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma}}\right)}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\ \left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma}, & if \quad \alpha=1\end{cases} \tag{17}$$

and

$$h_{APTIITLK}(t)=\begin{cases}\dfrac{\log(\alpha)2\gamma\lambda\beta t^{\beta-1}\left(1-t^{\beta}\right)^{\lambda-1}\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma-1}}{\alpha^{\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma}}\left(1-\alpha^{-\left(1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}\right)^{\gamma}}\right)}, & if \quad \alpha>0,\ \alpha\neq 1\\ \dfrac{2\gamma\lambda\beta t^{\beta-1}\left(1-t^{\beta}\right)^{\lambda-1}\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]}{1-\left[1-\left(1-t^{\beta}\right)^{\lambda}\right]^{2}}, & if \quad \alpha=1\end{cases}. \tag{18}$$

The plots of the pdf and hrf of APTIITLK distribution are shown in Figure 1.

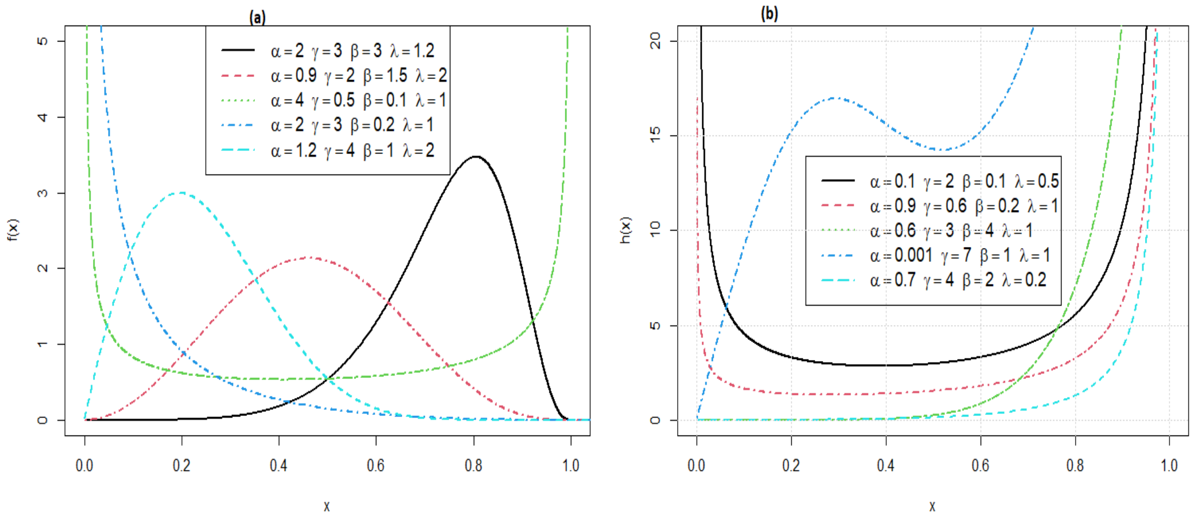

Figure 1: The pdf plot (a) and hrf plot (b) of APTIITLK distribution for different parameter value.

Figure 1 reveals that the pdf of APTIITLK distribution exhibits a decreasing (reserved J-shape), negatively-skewed, positively-skewed, symmetric and bathtub shapes, whereas, the hrf plots indicate an increasing, bathtub and inverted bathtub hazard properties.

2.2.2 The alpha power type II Topp-Leone Weibull (APTIITLW) distribution

Suppose the baseline distribution in (5) follow the Weibull distribution with $F(t,\lambda)=1-e^{-t^{\lambda}}$ and $f(t,\lambda)=\lambda t^{\lambda-1}e^{-t^{\lambda}}$, where $\lambda>0$ is the shape parameter, the authors defined the cdf of alpha power type II Topp-Leone Weibull (APTIITLW) distribution by

$$F_{APTIITLW}(t)=\begin{cases}\dfrac{\alpha^{\left[1-\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma}\right]}-1}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] 1-\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma}, & if \quad \alpha=1\end{cases}, \tag{19}$$

and the pdf of APTIITLW distribution is obtained as

$$f_{APTIITLW}(t)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}2\gamma\lambda t^{\lambda-1}e^{-t^{\lambda}}\left[1-e^{-t^{\lambda}}\right]\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma-1}\alpha^{\left[1-\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma}\right]}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] 2\gamma\lambda t^{\lambda-1}e^{-t^{\lambda}}\left[1-e^{-t^{\lambda}}\right]\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma-1}, & if \quad \alpha=1\end{cases}. \tag{20}$$

The sf and hrf of the APTIITLW distribution are obtained, respectively, as

$$S_{APTIITLW}(t)=\begin{cases}\dfrac{\alpha\left(1-\alpha^{-\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma}}\right)}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] \left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma}, & if \quad \alpha=1\end{cases}, \tag{21}$$

and

$$h_{APTIITLW}(t)=\begin{cases}\dfrac{\log(\alpha)2\gamma\lambda t^{\lambda-1}e^{-t^{\lambda}}\left[1-e^{-t^{\lambda}}\right]\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma-1}}{\alpha^{\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma}}\left(1-\alpha^{-\left(1-\left[1-e^{-t^{\lambda}}\right]^{2}\right)^{\gamma}}\right)}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] \dfrac{2\gamma\lambda t^{\lambda-1}e^{-t^{\lambda}}\left[1-e^{-t^{\lambda}}\right]}{1-\left[1-e^{-t^{\lambda}}\right]^{2}}, & if \quad \alpha=1\end{cases}. \tag{22}$$

Figure 2 presents the pdf and hrf plots of the APTIITLW distribution for some selected values of the parameters.

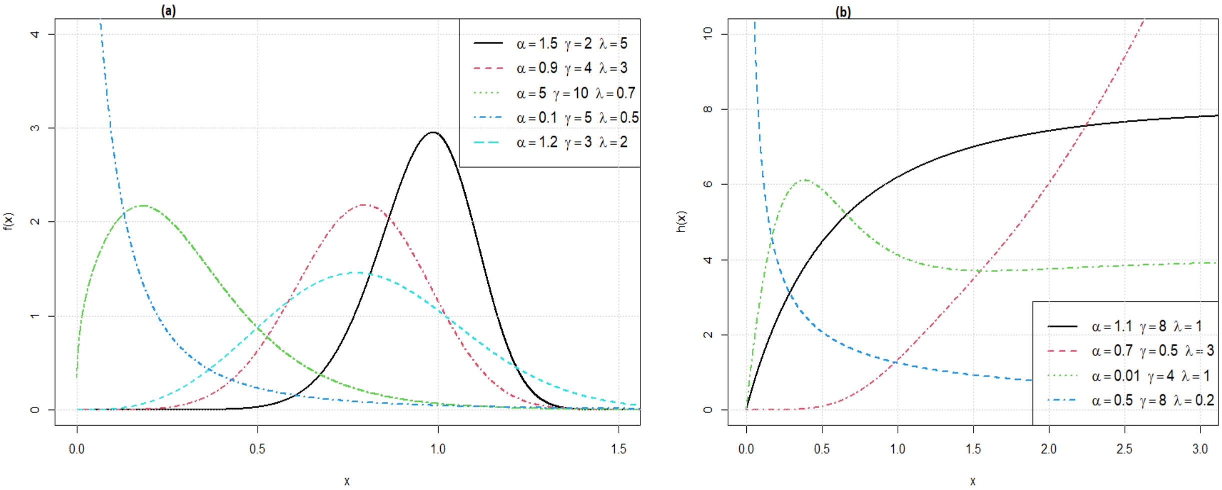

Figure 2: The pdf plot (a) and hrf plot (b) of the APTIITLW distribution for varying choices of parameter.

Clearly, the pdf plot in Figure 2 indicates a decreasing (reserved J-shape), negatively-skewed, positively-skewed, and symmetric shapes, whereas, the hrf plot indicate a decreasing, increasing, and inverted bathtub hazard properties.

2.2.3 The alpha power type II Topp-Leone log-logistic (APTIITL³) distribution

Let T be a random variable having the log-logistic cumulative distribution function (cdf), $F(t,\lambda)=1-\left(1+t^{\lambda}\right)^{-1}$ and density function (pdf), $f(t,\lambda)=\lambda t^{\lambda-1}\left(1+t^{\lambda}\right)^{-2}$. It is easy to define the cdf and pdf of a new distribution from (5) and (6), respectively, as

$$F_{APTIITL^3}(t)=\begin{cases}\dfrac{\alpha^{\left[1-\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma}\right]}-1}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\ 1-\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma}, & if \quad \alpha=1\end{cases}, \tag{23}$$

and

$$f_{APTIITL^3}(t)=\begin{cases}\dfrac{\log(\alpha)2\gamma\lambda t^{\lambda-1}}{(\alpha-1)\left(1+t^{\lambda}\right)^{2}}\left[1-\left(1+t^{\lambda}\right)^{-1}\right]\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma-1}\alpha^{\left[1-\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma}\right]}, & if \quad \alpha>0,\ \alpha\neq 1\\ \dfrac{2\gamma\lambda t^{\lambda-1}}{\left(1+t^{\lambda}\right)^{2}}\left[1-\left(1+t^{\lambda}\right)^{-1}\right]\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma-1}, & if \quad \alpha=1\end{cases}. \tag{24}$$

The cdf and pdf of the APTIITL³ distribution are readily defined by (23) and (24). The sf and hrf associated to (23) and (24) are obtained, respectively, as

$$S_{APTIITL^3}(t)=\begin{cases}\dfrac{\alpha\left(1-\alpha^{-\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma}}\right)}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\ \left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma}, & if \quad \alpha=1\end{cases},\tag{25}$$

and

$$h_{APTIITL^3}(t)=\begin{cases}\dfrac{\log(\alpha)2\gamma\lambda t^{\lambda-1}\left(1+t^{\lambda}\right)^{-2}\left[1-\left(1+t^{\lambda}\right)^{-1}\right]\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma-1}}{\alpha^{\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma}}\left(1-\alpha^{-\left(1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}\right)^{\gamma}}\right)}, & if\ \alpha>0,\ \alpha\neq 1\\ \dfrac{2\gamma\lambda t^{\lambda-1}\left(1+t^{\lambda}\right)^{-2}\left[1-\left(1+t^{\lambda}\right)^{-1}\right]}{1-\left[1-\left(1+t^{\lambda}\right)^{-1}\right]^{2}}, & if \quad \alpha=1\end{cases}.\tag{26}$$

The pdf and hrf plots of APTIITL³ distribution for selected values of the parameters are displayed in Figure 3.

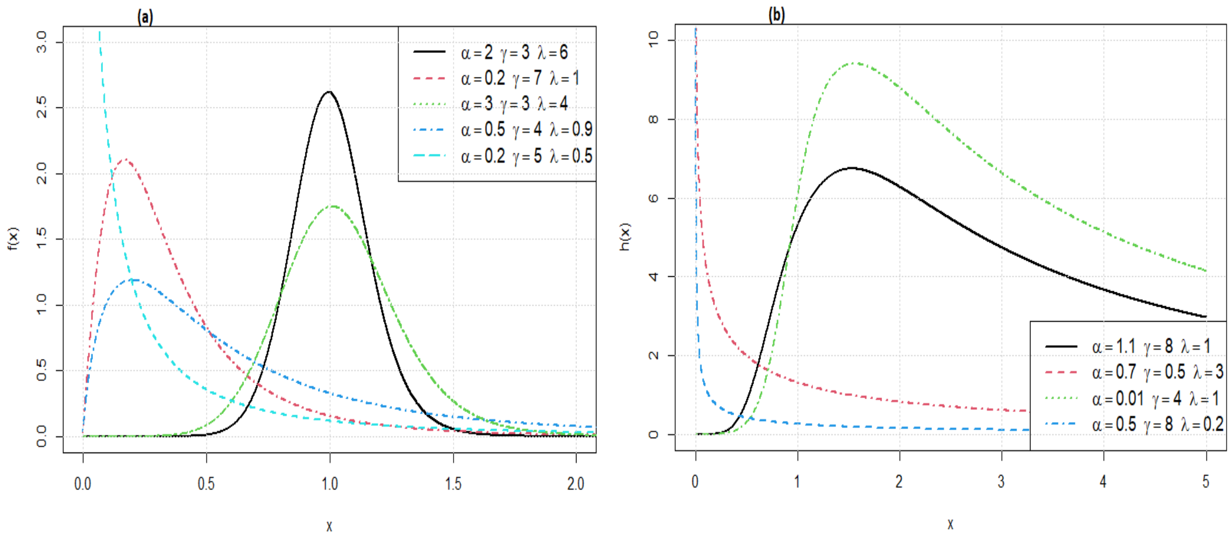

Figure 3: The pdf plot (a) and hrf plot (b) of APTIITL³ distribution for varying choices of parameter.

The pdf plots in Figure 3 indicates a decreasing (reserved J-shape), positively-skewed, and symmetric shapes, whereas, the hrf plots indicate a decreasing and inverted bathtub hazard properties.

2.2.4 The alpha power type II Topp-Leone Lindley (APTIITLL) distribution

The one-parameter Lindley distribution proposed by [27] is defined by the cdf and pdf, respectively, as

$$F(t,b) = 1-\left(1+\frac{bt}{1+b}\right)e^{-bt}, \quad b>0, \quad t>0, \tag{27}$$

and

$$f(t,b) = \frac{b^2}{1+b}(1+t)e^{-bt}, \quad b>0, \ t>0. \tag{28}$$

By inserting (27) and (28) into (5) and (6), the authors obtained the cdf and pdf of alpha power type II Topp-Leone Lindley (APTIITLL) distribution, respectively, as

$$F_{APTIITLL}(t) = \begin{cases} \dfrac{\alpha^{\left[1-\left(1-\left[1-\left(1+\frac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma}\right]}-1}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1 \\ 1-\left(1-\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma}, & if \quad \alpha=1 \end{cases}, \tag{29}$$

and

$$f_{APTIITLL}(t) = \begin{cases} \dfrac{\log(\alpha)2\gamma b^2}{(\alpha-1)(1+b)}(1+t)e^{-bt}\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]\left(1-\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma-1}\alpha^{\left[1-\left(1-\left[1-\left(1+\frac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma}\right]}, & if \quad \alpha>0,\ \alpha\neq 1 \\ \dfrac{2\gamma b^2}{(1+b)}(1+t)e^{-bt}\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]\left(1-\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma-1}, & if \quad \alpha=1 \end{cases} \tag{30}$$

The sf and hrf of APTIITLL distribution are obtained, respectively, as

$$S_{APTIITLL}(t) = \begin{cases} \dfrac{\alpha\left(1-\alpha^{-\left(1-\left[1-\left(1+\frac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma}}\right)}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1 \\ \left(1-\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma}, & if \quad \alpha=1 \end{cases}, \tag{31}$$

and

$$h_{APTIITLL}(t)=\begin{cases}\dfrac{\dfrac{\log(\alpha)2\gamma b^2}{(1+b)}(1+t)e^{-bt}\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]\left(1-\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma-1}}{\alpha^{\left(1-\left[1-\left(1+\frac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma}}\left(1-\alpha^{-\left(1-\left[1-\left(1+\frac{bt}{1+b}\right)e^{-bt}\right]^2\right)^{\gamma}}\right)}, & if \quad \alpha>0,\ \alpha\neq1 \\[2ex] \dfrac{\dfrac{2\gamma b^2}{(1+b)}(1+t)e^{-bt}\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]}{1-\left[1-\left(1+\dfrac{bt}{1+b}\right)e^{-bt}\right]^2}, & if \quad \alpha=1\end{cases}. \tag{32}$$

Figure 4 displays the pdf and hrf plots of the APTIITLL distribution for selected parameter values.

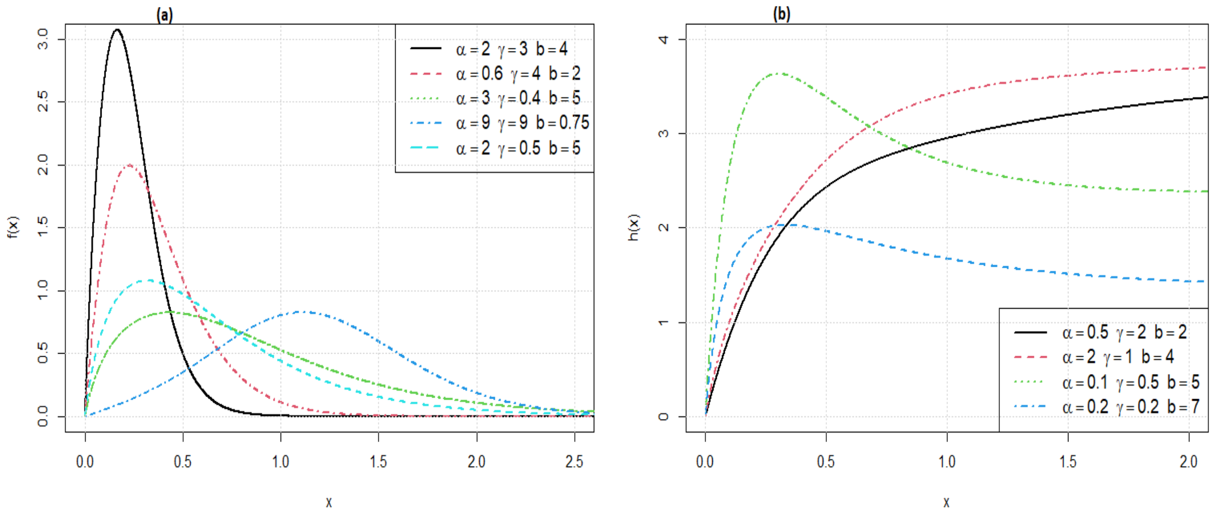

Figure 4: The pdf plot (a) and hrf plot (b) of APTIITLL distribution for varying choices of parameter.

From Figure 4, we observe that the pdf plots of APTIITLL distribution accommodates a positively-skewed and symmetric shapes, whereas, the hrf plots exhibit an increasing and inverted bathtub hazard properties.

2.2.5 The alpha power type II Topp-Leone Burr XII (APTIITLBXII) distribution

The Burr XII distribution is one of the most commonly used models among the twelve (12) special models introduced by [28]. The cdf and pdf of Burr XII distribution are defined, respectively, as

$$F(t,a,b)=1-\left(1+t^a\right)^{-b}, \quad a,b>0, \quad t>0, \tag{33}$$

and

$$f(t,a,b)=abt^{a-1}\left(1+t^a\right)^{-(b+1)}, \quad a,b>0, \quad t>0. \tag{34}$$

Utilizing the cdf defined in (33) as the baseline distribution in (5), the authors obtained the cdf of alpha power type II Topp-Leone Burr XII (APTIITLBXII) distribution by

$$F_{APTIITLBXII}(t)=\begin{cases}\dfrac{\alpha^{\left[1-\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma}\right]}-1}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] 1-\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma}, & if \quad \alpha=1\end{cases}, \tag{35}$$

and

$$f_{APTIITLBXII}(t)=\begin{cases}\dfrac{\log(\alpha)2\gamma abt^{a-1}}{(\alpha-1)\left(1+t^{a}\right)^{(b+1)}}\left[1-\left(1+t^{a}\right)^{-b}\right]\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma-1}\alpha^{\left[1-\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma}\right]}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] \dfrac{2\gamma abt^{a-1}}{\left(1+t^{a}\right)^{(b+1)}}\left[1-\left(1+t^{a}\right)^{-b}\right]\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma-1}, & if \quad \alpha=1\end{cases}. \tag{36}$$

The sf and hrf of APTIITLBXII distribution are obtained, respectively, as

$$S_{APTIITLBXII}(t)=\begin{cases}\dfrac{\alpha\left(1-\alpha^{-\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma}}\right)}{\alpha-1}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] \left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma}, & if \quad \alpha=1\end{cases}, \tag{37}$$

and

$$h_{APTIITLBXII}(t)=\begin{cases}\dfrac{\dfrac{\log(\alpha)2\gamma abt^{a-1}}{\left(1+t^{a}\right)^{(b+1)}}\left[1-\left(1+t^{a}\right)^{-b}\right]\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma-1}}{\alpha^{\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma}}\left(1-\alpha^{-\left(1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}\right)^{\gamma}}\right)}, & if \quad \alpha>0,\ \alpha\neq 1\\[2ex] \dfrac{\dfrac{2\gamma abt^{a-1}}{\left(1+t^{a}\right)^{(b+1)}}\left[1-\left(1+t^{a}\right)^{-b}\right]}{1-\left[1-\left(1+t^{a}\right)^{-b}\right]^{2}}, & if \quad \alpha=1\end{cases}, \tag{38}$$

Some useful pdf and hrf plots of the APTIITLBXII distribution are displayed in Figure 5.

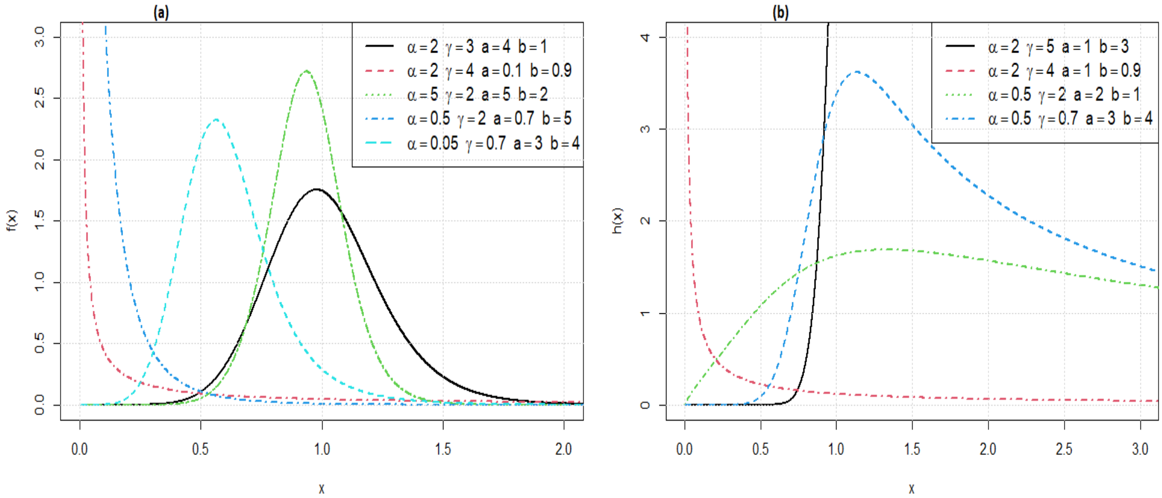

Figure 5: The pdf plot (a) and hrf plot (b) of APTIITLBXII distribution for different values of the parameters.

The density plots of the APTIITLBXII distribution displayed in Figure 5, shows a decreasing, positively-skewed and symmetric shapes, whereas, the hrf plots exhibit a decreasing, increasing and inverted bathtub hazard properties.

2.3 Statistical Properties

This section is devoted to derivation of some statistical properties of APTIITL-G family. In particular, the quantiles, r^{th}-moments, moment generating function, probability weighted moments (PWMs), Renyi entropy and order statistics are derived.

2.3.1 Quantile Function

The quantile function of APTIITL-G family of distributions is obtained as

$$Q_T(u) = F^{-1}\left\{\sqrt{1-\left[1-\frac{\log\left(u(\alpha-1)+1\right)}{\log(\alpha)}\right]^{1/\gamma}}\right\}, \qquad u\in(0,1). \tag{39}$$

By inserting $u = 0.5$ in (39), we obtain the median of APTIITL-G family as

$$Q_T(0.5) = F^{-1}\left\{\sqrt{1-\left[1-\frac{\log\left(\alpha+1\right)-\log\left(2\right)}{\log(\alpha)}\right]^{1/\gamma}}\right\}. \tag{40}$$

The utility of (39) is most essential in generating random sample from the distribution.

2.3.2 Moments and Incomplete Moments

Let T be a random variable having the density function of the APTIITL-G family, then from (11), the r^{th} moments of T is defined by

$$E(T^r) = \mu_r^{'} = \sum_{j,=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+1)-1} \psi_{j,k,m} \int_{-\infty}^{\infty} t^r \pi_{2(m+1)}\left(t,\alpha,\gamma,\xi\right) dt, \quad r = 1,2,3,4,... \tag{41}$$

The integral part of (41) can be expressed as $E\left[Y_{2(m+1)}^r\right]$, which is the r^{th} moments of the exp-G family with power parameter $2(m+1)$.

The mean $\left(\mu_1^{'}\right)$ of the APTIITL-G family is obtained from (41) when $r=1$. The variance $\left(\sigma^2\right)$, skewness $\left(S_k\right)$ and kurtosis $\left(K_s\right)$ are obtained as

$$\text{variance}\left(\sigma^2\right) = \mu_2^{'} - \left(\mu_1^{'}\right)^2, \qquad \text{skewness}\left(S_k\right) = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\left(\mu_1^{'}\right)^3}{\left(\mu_2^{'} - \left(\mu_1^{'}\right)^2\right)^{\frac{3}{2}}},$$

$$\text{kurtosis}\left(K_s\right) = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_3^{'}\left(\mu_1^{'}\right)^2 - 3\left(\mu_1^{'}\right)^4}{\left(\mu_2^{'} - \left(\mu_1^{'}\right)^2\right)^2}.$$

furthermore, we deduce the r^{th} lower incomplete moment of APTIITL-G family from (31) as

$$\varphi_r(q) = \sum_{j,=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+1)-1} \psi_{j,k,m} \int_{-\infty}^{q} t^r \pi_{2(m+1)}\left(t,\alpha,\gamma,\xi\right) dt. \tag{42}$$

Table 1 holds numerical values of the mean $\left(\mu_1^{'}\right)$, variance $\left(\sigma^2\right)$, measures of skewness $\left(S_k\right)$ and kurtosis $\left(K_s\right)$ of alpha power type II Topp-Leone Kumaraswamy (APTIITLK) distribution for some selected values of the parameters. Observations from the table reveal that APTIITLK distribution is negatively-skewed, positively-skewed, symmetric, platykurtic, leptokurtic as well as exhibiting a mesokurtic properties.

Table 1: *The r^{th}-moments of APTIITLK distribution for* $(\beta = 2,\ \gamma = 3)$

α	λ	$\mu_1^{'}$	σ^2	S	K
0.2	2	0.4423	0.0200	0.1322	3.0658
	4	0.3248	0.0123	0.3264	3.3290
	6	0.2686	0.0089	0.4609	3.1114
1.5	2	0.5236	0.0208	-0.2296	2.9934
	4	0.3896	0.0137	0.0230	2.7440
	6	0.3238	0.0101	0.0834	2.6767
3.0	2	0.5513	0.0198	-0.3418	3.0006
	4	0.4121	0.0134	-0.1413	2.7469
	6	0.3431	0.0099	-0.0618	2.5109

2.3.3 Moment generating function (mgf) and probability weighted moments (PWMs)

The mgf of APTIITL-G family is obtained as

$$\begin{aligned} M_T(q) &= E\left[e^{qT}\right] = \int_{-\infty}^{\infty} e^{qt} f(t)dt, \\ &= \sum_{j,n=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+1)-1} \psi_{j,k,m,n}^{*} E\left[Y_{2(m+1)}^{n}\right], \end{aligned} \tag{43}$$

where,

$$\psi^{*}_{j,k,m,n} = \frac{\gamma q^{n}\left(\log(\alpha)\right)^{j+1}}{j!n!(m+1)(\alpha-1)}\binom{j}{k}\binom{\gamma(k+1)-1}{m}(-1)^{k+m},$$

and $E\left[Y^{n}_{2(m+1)}\right]$ is the n^{th} moment of the exp-G family with power parameter $2(m+1)$.

The PWMs of a random variable T as defined in [29] is given as

$$\rho_{q,r} = E\left[T^{r}F^{q}(t)\right] = \int_{-\infty}^{\infty} t^{r} f(t)F^{q}(t)dt. \tag{44}$$

By inserting (5) and (6) into (44), the authors obtained the $(q,r)^{th}$ PWMs of APTIITL-G family as

$$f(t,\alpha,\gamma,\xi)F^{q}(t,\alpha,\gamma,\xi) = \sum_{l=0}^{\infty}(-1)^{q-l}\binom{q}{l}\frac{\log\alpha}{(\alpha-1)^{q+1}}2\gamma f(t,\xi)F(t,\xi)\left(1-F^{2}(t,\xi)\right)^{\gamma-1}\alpha^{(1+l)\left[1-\left(1-F^{2}(t,\xi)\right)^{\gamma}\right]}. \tag{45}$$

Further simplification of (45) and substituting into (44), yields

$$\rho_{q,r} = \sum_{l,j=0}^{\infty}\psi^{**}_{k,m}E\left[Y^{r}_{2(m+1)}\right], \tag{46}$$

where,

$$\psi^{**}_{k,m} = \sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+1)-1}\frac{\gamma(1+l)^{j}\left(\log(\alpha)\right)^{j+1}}{j!(m+1)(\alpha-1)^{q+1}}\binom{q}{l}\binom{j}{k}\binom{\gamma(k+1)-1}{m}(-1)^{q-l+k+m}.$$

2.3.4 Renyi Entropy

The Renyi entropy of a random variable T with a known pdf, $f(t)$ is given by

$$\tau_{R}(\omega) = \frac{1}{1-\omega}\log\int_{-\infty}^{\infty} f^{\omega}(t)dt, \qquad \omega>0, \omega\neq 1. \tag{47}$$

Applying (6) in (47), the Renyi entropy of APTIITL-G family is defined as follows.

$$\tau_{R}(\omega) = \frac{1}{1-\omega}\log\left[\left[\frac{\log\alpha}{\alpha-1}\right]^{\omega}(2\gamma)^{\omega}\int_{-\infty}^{\infty} f^{\omega}(t,\xi)F^{\omega}(t,\xi)\left(1-F^{2}(t,\xi)\right)^{\omega(\gamma-1)}\alpha^{\omega\left[1-\left(1-F^{2}(t,\xi)\right)^{\gamma}\right]}dt\right]. \tag{48}$$

Employing (11) and (12) into (48), yields

$$\tau_{R}(\omega) = \frac{1}{1-\omega}\log\left[\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+\omega)-\omega}\frac{(2\gamma)^{\omega}\left(\log(\alpha)\right)^{j+\omega}\omega^{j}}{j!(\alpha-1)^{\omega}}\binom{j}{k}\binom{\gamma(k+\omega)-\omega}{m}(-1)^{k+m}\int_{-\infty}^{\infty} f^{\omega}(t,\xi)F^{2m+\omega}(t,\xi)dt\right]. \tag{49}$$

2.3.5 Order Statistics

Suppose that $T_1, T_2, ..., T_n$ are random samples generated from a known probability distribution. Let $T_{r:n}$ denote the r^{th} order statistic, then the pdf of $T_{r:n}$ is defined as

$$f_{r:n}(t) = \frac{1}{B(r,n-r+1)}\sum_{p=0}^{n-r}\binom{n-r}{p}(-1)^{p} f(t)F(t)^{r+p-1}, \tag{50}$$

By inserting (5) and (6) into (50), the authors obtained the pdf of APTIITL-G r^{th} order statistics as follows.

$$f(t)F(t)^{r+p-1} = \sum_{l=0}^{\infty}(-1)^{r+p-l-1}\binom{r+p-1}{l}\frac{\log\alpha}{(\alpha-1)^{r+p}}2\gamma f(t,\xi)F(t,\xi)\left(1-F^{2}(t,\xi)\right)^{\gamma-1}\alpha^{(1+l)\left[1-\left(1-F^{2}(t,\xi)\right)^{\gamma}\right]} \tag{51}$$

Employing similar approach in (45), (51) is further simplified as

$$f(t)F(t)^{r+p-1}=\sum_{l,j=0}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+1)-1}\frac{2\gamma(1+l)^{j}(\log(\alpha))^{j+1}}{j!(\alpha-1)^{r+p}}\binom{r+p-1}{l}\binom{j}{k}\binom{\gamma(k+1)-1}{m}(-1)^{r+p+k+m-l-1}f(t,\xi)F^{2m+1}(t,\xi), \tag{52}$$

so that (50) now becomes

$$f_{r:n}(t)=\frac{1}{B(r,n-r+1)}\sum_{l,j=0}^{\infty}\varpi_{p,k,m}\,\pi_{2(m+1)}(t), \tag{53}$$

where,

$$\varpi_{p,k,m}=\sum_{p=0}^{n-r}\sum_{k=0}^{j}\sum_{m=0}^{\gamma(k+1)-1}\frac{\gamma(1+l)^{j}(\log(\alpha))^{j+1}}{j!(m+1)(\alpha-1)^{r+p}}\binom{r+p-1}{l}\binom{n-r}{p}\binom{j}{k}\binom{\gamma(k+1)-1}{m}(-1)^{r+2p+k+m-l-1}.$$

Whereas, the s^{th} moment of APTIITL-G r^{th} order statistic can be expressed as

$$E\left(T_r^s\right)=\frac{1}{B(r,n-r+1)}\sum_{l,j=0}^{\infty}\varpi_{p,k,m}\,E\left[Y_{2(m+1)}^{s}\right], \tag{54}$$

where $E\left[Y_{2(m+1)}^{s}\right]$ is the s^{th} moment of exp-G family with power parameter $2(m+1)$.

2.4 Parameter Estimation and Simulation Study

2.4.1 Maximum Likelihood Estimation

The method of maximum likelihood estimation is adopted to estimate the unknown parameters of APTIITL-G family of distributions. Suppose $(t_1,t_2,...,t_n)$ are random samples generated from APTIITL-G family, then the log-likelihood function is given as

$$\begin{aligned}\ell(t_i,\psi)&=n\ln(\ln\alpha)-n\ln(\alpha-1)+n\ln(2\gamma)+\sum_{i=1}^{n}\ln\left(f(t_i,\xi)\right)+\sum_{i=1}^{n}\ln\left(F(t_i,\xi)\right)\\&+(\gamma-1)\sum_{i=1}^{n}\ln\left(1-F^2(t_i,\xi)\right)+\ln\alpha\sum_{i=1}^{n}\left(1-\left(1-F^2(t_i,\xi)\right)^{\gamma}\right),\qquad \psi=(\alpha,\gamma,\xi)^T\end{aligned} \tag{55}$$

The score function $U(t_i,\psi)=\left[\frac{\partial\ell(t_i,\psi)}{\partial\alpha},\frac{\partial\ell(t_i,\psi)}{\partial\gamma},\frac{\partial\ell(t_i,\psi)}{\partial\xi}\right]^T$ associated with the log-likelihood function in (55) is obtained by taking the first derivative of (55) with respect to the parameters. These are expressed as

$$\frac{\partial\ell(t_i,\psi)}{\partial\alpha}=\frac{n}{\alpha\ln\alpha}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\left(1-F^2(t_i,\xi)\right)^{\gamma}\right),$$

$$\frac{\partial\ell(t_i,\psi)}{\partial\gamma}=\frac{n}{\gamma}+\sum_{i=1}^{n}\ln\left(1-F^2(t_i,\xi)\right)+\ln\alpha\sum_{i=1}^{n}\ln\left(1-F^2(t_i,\xi)\right)\left(1-F^2(t_i,\xi)\right)^{\gamma},$$

$$\frac{\partial\ell(t_i,\psi)}{\partial\xi_j}=\sum_{i=1}^{n}\frac{f'(t_i,\xi)}{f(t_i,\xi)}+\sum_{i=1}^{n}\frac{f(t_i,\xi)}{F(t_i,\xi)}-2(\gamma-1)\sum_{i=1}^{n}\frac{f(t_i,\xi)}{1-F^2(t_i,\xi)}+2\gamma\ln\alpha\sum_{i=1}^{n}\left(1-F^2(t_i,\xi)\right)^{\gamma-1}F(t_i,\xi)f(t_i,\xi),$$

where $f'(t_i,\xi)=\frac{\partial f(t_i,\xi)}{\partial\xi_j}$, and $\partial\xi_j$ is the j^{th} element of the vector of parameter ξ.

The maximum likelihood estimates (MLEs) of ψ say $\hat{\psi}=\left(\hat{\alpha},\hat{\lambda},\hat{\xi}\right)$, are obtained by solving the system of nonlinear equation $U(t_i,\psi)=0$. Statistical packages such as ***bbmle*** and ***optim*** in R software can be used to numerically compute the parameter estimates

2.4.2 Simulation Study

Again, taking the Kumaraswamy distribution as the generator, the study investigates the performance of the parameter estimates of the APTIITLK distribution via a Monte Carlo simulation study. Random samples of size $n=(100,200,500,800,1000)$ are generated from the APTIITLK distribution at two distinct sets of parameter values $(\alpha=0.2,\beta=0.8,\gamma=3,\lambda=2)$ and $(\alpha=0.2,\beta=0.8,\gamma=3,\lambda=2)$. At each case, the simulation is repeated 3000 times and the following quantities are computed:

i) mean estimate $(\bar{\psi})=\frac{1}{N}\sum_{i=1}^{N}\hat{\psi}_i$,

ii) average bias $=\frac{1}{N}\sum_{i=1}^{N}(\hat{\psi}_i-\bar{\psi})$,

iii) root mean square error (RMSE) $=\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\psi}_i-\bar{\psi})^2}$.

Tables 2 and 3 display the mean estimate, average bias and root mean square errors of the estimates of APTIITLK distribution.

Table 2: *Simulation results of APTIITLK distribution for* $(\alpha=0.7,\beta=0.8,\gamma=0.5,\lambda=2)$

Parameters	*N*	Mean	Bias	RMSE
	100	0.6672	0.6648	1.0672
	200	0.6708	0.5707	0.9121
α	500	0.6827	0.4825	0.3039
	800	0.6902	0.2401	0.0751
	1000	0.7121	0.1361	0.0022
	100	0.7646	0.1846	0.0721
	200	0.7920	0.0920	0.0352
β	500	0.8014	0.0614	0.0191
	800	0.8022	0.0355	0.0075
	1000	0.8155	0.0252	0.0013
	100	0.4635	0.0233	2.0535
	200	0.4669	0.0151	1.8012
γ	500	0.4702	-0.5750	0.9453
	800	0.4847	-1.2353	0.0413
	1000	0.5177	-1.3023	0.0085
	100	1.8647	0.4106	1.8287
	200	1.8707	0.2651	0.9518
λ	500	1.9237	0.1291	0.4737
	800	2.1101	0.0120	0.2042
	1000	2.1261	0.0014	0.0134

Table 3: *Simulation results of APTIITLK distribution for* $(\alpha = 0.2, \beta = 2, \gamma = 0.2, \lambda = 0.5)$

Parameters	n	Mean	Bias	RMSE
	100	0.1789	0.7149	2.0312
	200	0.1820	0.4812	1.3635
α	500	0.2002	0.2102	0.2738
	800	0.2016	0.1650	0.0225
	1000	0.2106	0.0567	0.0061
	100	1.8981	0.3981	0.3534
	200	1.9017	0.1517	0.2197
β	500	1.9157	0.8573	0.0536
	800	2.1195	0.0995	0.0129
	1000	2.2014	0.0418	0.0031
	100	0.1845	0.4459	0.5416
	200	0.1886	0.1325	0.3162
γ	500	0.2051	0.0640	0.1916
	800	0.2073	-0.9782	0.0177
	1000	0.2105	-1.1086	0.0042
	100	0.4749	0.8951	0.6725
	200	0.4850	0.6232	0.2524
λ	500	0.5002	0.3589	0.1626
	800	0.5160	0.1242	0.0884
	1000	0.5167	0.0253	0.0152

From the results in Tables 2 and 3, the following remarks were observed:

(i) the mean estimate for the parameters approaches the true parameter value as n increases;

(ii) parameter estimates α, β and λ are positively biased, while parameter estimate γ can both be positively and negatively biased;

(iii) the bias and root mean square error for all the parameters decrease as n increases.

These remarks are consistent with the properties of a good estimator.

3. DATA ANALYSIS, RESULTS AND DISCUSSIONS

In this Section, we illustrate the potential of alpha power type II Topp-Leone Kumaraswamy distribution (APTIITLKD) belonging to the APTIITL-G family of distributions using two real data sets. The data sets are concerned with the recovery and mortality rates of Covid-19 patients in Nigeria, covering a duration of two (2) months (May 1 to June 30, 2020).

The flexibility of APTIITLK distribution in data fittings is investigated by comparing its fit with the ones obtained from existing bounded non-nested models. The density function of these competitor distributions is defined as follows:

1. Odd log-logistic Kumaraswamy distribution (OLLKD) studied by [30];

$$f(x,a,b,\alpha) = \frac{ab\alpha x^{a-1}\left(1-x^{a}\right)^{b\alpha-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}}{\left[\left(1-\left(1-x^{a}\right)^{b}\right)^{\alpha}+\left(1-x^{a}\right)^{b\alpha}\right]^{2}};$$

2. Unit-Burr XII distribution (UBXIID) developed by [31];

$$f(x,\alpha,\beta) = \alpha\beta x^{-1}\left(-\log x\right)^{\beta-1}\left(1+\left(-\log x\right)^{\beta}\right)^{-(\alpha+1)};$$

3. Unit-Burr III distribution (UBIIID) proposed by [32];

$$f(x,\lambda,\beta) = \lambda\beta x^{-2}\left(x^{-1}-1\right)^{\beta-1}\left(1+\left(x^{-1}-1\right)^{\beta}\right)^{-(\lambda+1)};$$

4. Beta distribution reported in [33];

$$f(x,a,b) = \frac{x^{a-1}\left(1-x\right)^{b-1}}{B\left(a,b\right)}, \quad B\left(a,b\right)=\frac{\Gamma\left(a\right)\Gamma\left(b\right)}{\Gamma\left(a+b\right)};$$

5. Kumaraswamy distribution (KwD) developed by [26];

$$f(x,a,b) = abx^{a-1}\left(1-x^{a}\right)^{b-1}.$$

Data set 1: This data set comprises of the daily recovery rate of Covid-19 patients in Nigeria within the period of 2 months (May 1 to June 30, 2020). The data set is obtained from the ratio of total daily recovery and the total confirmed cases. The data is presented as follows:
0.1617512, 0.1469849, 0.1563722, 0.1488223, 0.1630508, 0.1697933, 0.1704481, 0.1735685, 0.1794748, 0.1768584, 0.1943547, 0.2003342, 0.2152484, 0.2285936, 0.2422018, 0.2618751, 0.2674945, 0.2662348, 0.2708952, 0.2755729, 0.2718073, 0.2764082, 0.2888653, 0.2886848, 0.2864403, 0.2858341, 0.2863850, 0.2907459, 0.2899376, 0.2898021, 0.2959063, 0.2951409, 0.2994731, 0.2981372, 0.3069642, 0.3120567, 0.3127606, 0.3170751, 0.3156003, 0.3123886, 0.3136308, 0.3087811, 0.3221790, 0.3252774, 0.3245260, 0.3211070, 0.3279100, 0.3364533, 0.3412879, 0.3437092, 0.3391559, 0.3398044, 0.3398346, 0.3433625, 0.3457312, 0.3458919, 0.3542364, 0.3582257, 0.3666300, 0.3740898, 0.3793103

Data set 2: This data set holds the daily records of mortality rate of Covid-19 patients in Nigeria within the same time frame in the first data set. It is computed from the ratio of daily death cases and the total confirmed cases. The data is given as follows: 0.003225806, 0.004187605, 0.005863956, 0.0007137759, 0.002033898, 0.001589825, 0.001418037, 0.001022495, 0.002409058, 0.002500568, 0.003232062, 0.001462294, 0.001609334, 0.001162340, 0.0005504587, 0.000711617, 0.0008390670, 0.0009716599, 0.001406030, 0.0001497679, 0.001425314, 0.001239499, 0.0009301090, 0.0003827019, 0.0006197323, 0.0008389262, 0.001832131, 0.0005608525, 0.0005375188, 0.0002029427, 0.001180870, 0.001323502, 0.001109160, 0.001343364, 0.00008683571, 0.0006754475, 0.0008174610, 0.0007208073, 0.0009374268, 0.0005199049, 0.0002883298, 0.001168064, 0.0003293591, 0.0002550695, 0.0009325459, 0.0008404370, 0.0002332634, 0.001747956, 0.0007575758, 0.0003133650, 0.0006058158, 0.0009385497, 0.0005736412, 0.0003275467, 0.0003633061, 0.0003979836, 0.0003004550, 0.0002076671, 0.0001628200, 0.0002785183, 0.0003113567.
Model selection criteria such as the maximized log-likelihood (Log-Lik), Akaike information criterion (AIC), and some goodness of fit test statistics including the Komolgorov-Smirnov (*K-S*) and Crammer von Mises (W^{*}) test statistics with their corresponding *p*-value are employed for model comparison. Tables 4 and 5 present the summary statistics of the recovery and mortality rate of Covid-19 patients in Nigeria, respectively.

Table 4: *Summary Statistics of the Covid-19 Recovery Data set*

Models	Estimates	Log-Lik	AIC	*K-S* *(p-value)*	W^* *(p-value)*
APTIITLKD	$\alpha = 3.5166$ $\beta = 2.7647$ $\gamma = 11.4660$ $\lambda = 10.7149$	85.3897	-165.7794	0.1223 **(0.2963)**	0.2019 **(0.2641)**
OLLKD	$a = 0.5458$ $b = 0.9851$ $\alpha = 6.7200$	78.4359	-150.8717	0.1366 **(0.1867)**	0.2883 **(0.1458)**
UBXIID	$\alpha = 0.0802$ $\beta = 50.7525$	84.7801	-165.5602	0.1370 **(0.1848)**	0.2756 **(0.1586)**
UBIIID	$\lambda = 0.0496$ $\beta = 20.8434$	40.7476	-77.4952	0.4037 **($1.711e^{-9}$)**	2.6963 **($2.644e^{-7}$)**
Beta	$a = 12.4278$ $b = 31.8699$	79.0322	-154.0643	0.1897 **(0.0214)**	0.4963 **(0.0404)**
Kumaraswamy	$a = 5.6206$ $b = 785.6500$	84.5196	-165.0392	0.1375 **(0.1815)**	0.2452 **(0.1948)**

Table 5: *Summary Statistics of the Covid-19 Mortality Data set*

Models	Estimates	Log-Lik	AIC	*K-S* *(p-value)*	W^* *(p-value)*
APTIITLKD	$\alpha = 68.5111$ $\beta = 0.4399$ $\gamma = 8.7893$ $\lambda = 12.7477$	369.706	-721.412	0.0549 **(0.9880)**	0.0284 **(0.9819)**
OLLKD	$a = 0.2645$ $b = 4.2375$ $\alpha = 5.0911$	360.7049	-715.4097	0.0726 **(0.8811)**	0.0349 **(0.9586)**
UBXIID	$\alpha = 0.0651$ $\beta = 7.7576$	216.3316	-428.6632	0.5653 **($1.665e^{-15}$)**	5.3930 **($2.2e^{-16}$)**
UBIIID	$\lambda = 0.0636$ $\beta = 2.1849$	257.1764	-510.3529	0.5169 **($1.887e^{-15}$)**	4.7144 **($2.2e^{-16}$)**
Beta	$a = 1.5592$ $b = 1446.418$	359.0777	-714.1554	0.0725 **(0.8821)**	0.0639 **(0.7913)**
Kumaraswamy	$a = 1.2125$ $b = 3639.133$	357.7849	-711.5698	0.0843 **(0.7469)**	0.0844 **(0.6686)**

3.1 Discussion of Results

By way of discussing the results in Tables 4 and 5, it is well known that the most appropriate model in fitting any real data set, corresponds to the one having the maximum value of log-likelihood and the minimum value in respect to AIC, *K-S* and W^* with the highest *p*-value. Clearly, from these tables we observed that the APTIITLK distribution satisfying the conditions, outperformed the rest competitor distributions. Thus, becoming the appropriate model in fitting the two data sets considered. Furthermore, we illustrate the flexibility of the APTIITLK distribution over the competitor distribution through graphical plots such as the density and distribution fits of the distributions for each data set as shown in Figures 6 and 7, respectively.

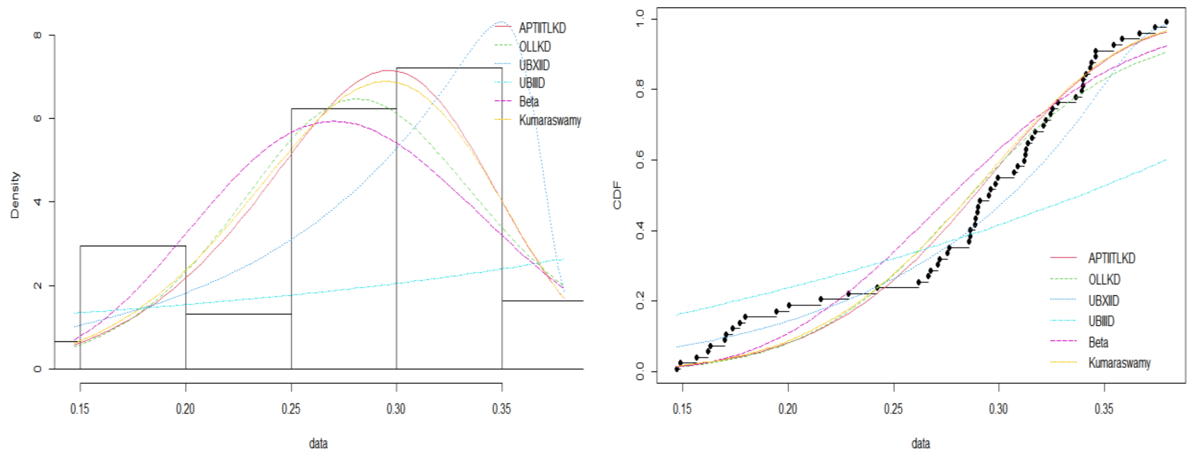

Figure 6: The fitted pdf and cdf of the distributions for Covid-19 recovery data set

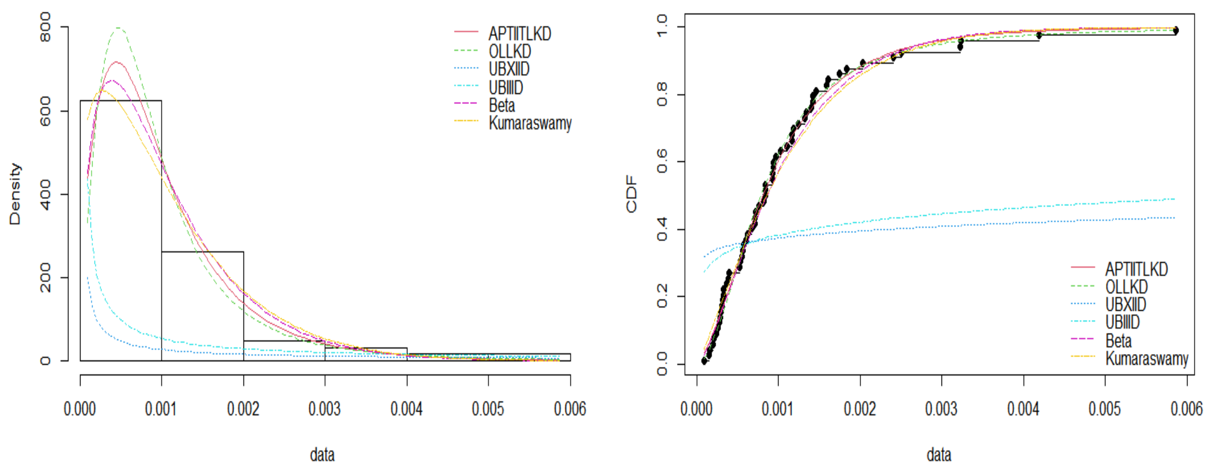

Figure 7: The fitted pdf and cdf of the distributions for Covid-19 mortality data set

4. CONCLUSION

In this paper, we have developed a new family of distributions called “Alpha Power Type II Topp-Leone-generated family of distributions” and some of its mathematical properties were derived. The maximum likelihood estimation method was adopted to obtain the parameter estimate of the family of distributions. A Monte Carlo simulation study was conducted in other to investigated the performance of the parameter estimates of sub-model belonging to the proposed family of distributions. Two data sets comprising of the daily recovery and mortality rates of Covid-19 patients in Nigeria, from May 1 to June 30, 2020, was employed to illustrate the potential of the proposed family in real world data fittings. Results obtained from the analysis clearly revealed that the APTIITLK distribution from the proposed family performed reasonably better than the compared non-nested distributions in analyzing the two Covid-19 datasets under study.

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