A NEW EXTENDED EXPONENTIATED DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

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Abstract

This manuscript focuses on the statistical properties and estimation methods of the exponentiated Suja distribution, which is characterized by two parameters: scale and shape. From a frequentist perspective, our primary emphasis is on estimation techniques. Additionally, we derive statistical and reliability characteristics for the model. We explore various estimation procedures, including order statistics, entropies, reliability analysis, and the maximum likelihood method. To assess the model's superiority and practical utility, we analyze real lifetime data sets. Overall, this study provides a comprehensive analysis of the exponentiated Suja distribution, offering insights into its statistical properties, estimation techniques, and real-life applications.

Keywords: Exponentiated technique, Suja distribution, Order statistics, Entropies, Reliability analysis, Maximum likelihood Estimation.

1. Introduction

A new theory of distributions was introduced by Gupta et al. [1], who discussed a new family of distributions namely the exponentiated exponential distribution. The family has two parameters scale and shape, which are similar to the weibull and gamma family. Later Gupta and Kundu [2] studied some properties of the distribution. They observed that many properties of the new family are similar to those of the weibull or gamma family. Hence the distribution can be used an alternative to a weibull or gamma distribution. The two parametric gamma and weibull are the most popular distributions for analyzing any lifetime data. The gamma distribution has a lot of applications in different fields other than lifetime distributions. the two parameters of gamma distribution represent the scale and the shape parameter and because of the scale and shape parameter, it has quite a bit of flexibility to analyze any positive real data. But one major disadvantage of the gamma distribution is that, if the shape parameter is not an integer, the distribution function or survival function cannot be expressed in a closed form. This makes gamma distribution little bit unpopular as compared to the weibull distribution, whose survival function and hazard function are simple and easy to study. Nowadays exponential distributions and their mathematical properties are widely studied for applied science experimental data sets. Rodrigues *et al.* [3] studied the exponentiated generalized Lindley distribution. Hassan et al. [4] discussed the exponentiated Lomax geometric distribution with its properties and applications. Nasiru *et al*. [5] obtained exponentiated generalized power series family of distributions. Rather and Subramanian [6] discussed the exponentiated Mukherjee-Islam distribution. Rather and Subramanian [7] obtained the exponentiated Ishita distribution with properties and applications. Maradesa Adeleke [10] discussed exponentiated exponential Lomax distribution and its Properties. Nasir *et al* [11] obtained the exponentiated Burr XII power series distribution with properties and its applications. Recently, Rather and Subramanian [8] discussed the exponentiated Garima distribution with applications in engineering sciences.

Suja distribution is a newly introduced one parametric lifetime model proposed by Shanker [9] for engineering sciences. The potentiality and usefulness of the proposed distribution in modeling lifetime data was greater as compared to other one parametric distributions namely Lindley, exponential, sujatha, Shanker and Aradhana. The different statistical properties of the proposed model have been derived and discussed such as order statistics, moments and associated measures, hazard and mean residual life function, stochastic ordering, Bonferroni and Lorenz curves and stress strength reliability. The parameters of the proposed distribution are estimated by employing the maximum likelihood estimation method. Finally, the goodness of fit of the proposed Suja distribution has been described by analyzing the real life data set and the fit has been found quite satisfactory over Lindley, exponential, Shanker, Aradhana, Sujatha and Amarendra distributions.

2. Exponentiated Suja Distribution (ESD)

The probability density function (pdf) of Suja distribution is given by

$$g(x)=\frac{\theta^5}{\theta^4+24}\left(1+x^4\right)e^{-\theta x};\ x>0,\theta>0 \tag{1}$$

and the cumulative distribution function (cdf) of the Suja distribution is given by

$$G(x)=1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\ ;x>0,\theta>0 \tag{2}$$

A random variable X is said to have an exponentiated distribution, if its cumulative distribution function is given by

$$F_\alpha(x)\ =\ (G(x))^\alpha\ \ ;x\in R^{'},\ \alpha>0 \tag{3}$$

Then X is said to have an exponentiated distribution.
The probability density function of X is given by

$$f_\alpha(x)=\alpha(G(x))^{\alpha-1}g(x) \tag{4}$$

on Substituting (2) in (3), we will get the cumulative distribution function of Exponentiated Suja distribution

$$F_\alpha(x)=\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha};\ x>0,\theta>0,\alpha>0 \tag{5}$$

and the probability density function of Exponentiated Suja distribution can be obtained as

$$f_\alpha(x)=\frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1} \tag{6}$$

The graphical representation of Pdf and Cdf are shown in Fig. 1 and Fig. 2.

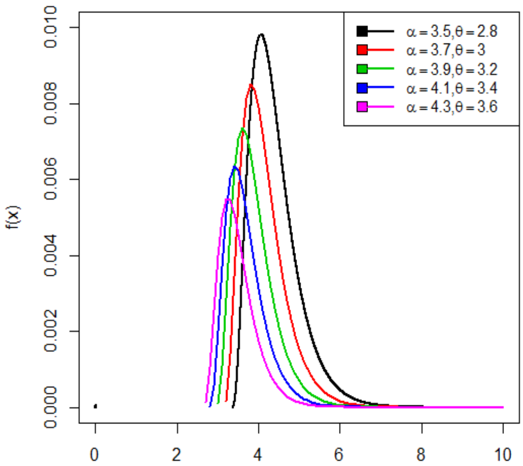

Fig. 1: *Pdf plot of exponentiated Suja distribution*

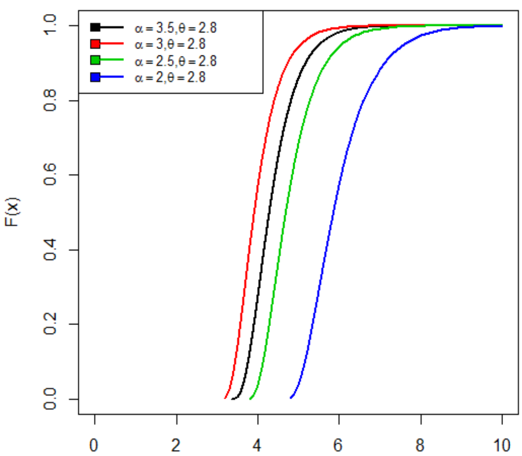

Fig. 2: *Cdf plot of exponentiated Suja distribution*

3. Reliability Analysis

In this section, we will discuss the survival function, hazard function and Reverse hazard rate function of the Exponentiated Suja distribution. Many researchers have discussed system reliability by using different techniques like Tillman et al [14], Wei et al. [15] and Wang [16].

The survival function of Exponentiated Suja distribution is given below and its graphical representation is in Fig. 3.

$$S(x)=1-\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha} \tag{7}$$

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality and is given by

$$h(x)=\left(\frac{\frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}}{1-\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}}\right) \tag{8}$$

The reverse hazard rate is given by

$$h_r(x) = \frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{(\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}}$$

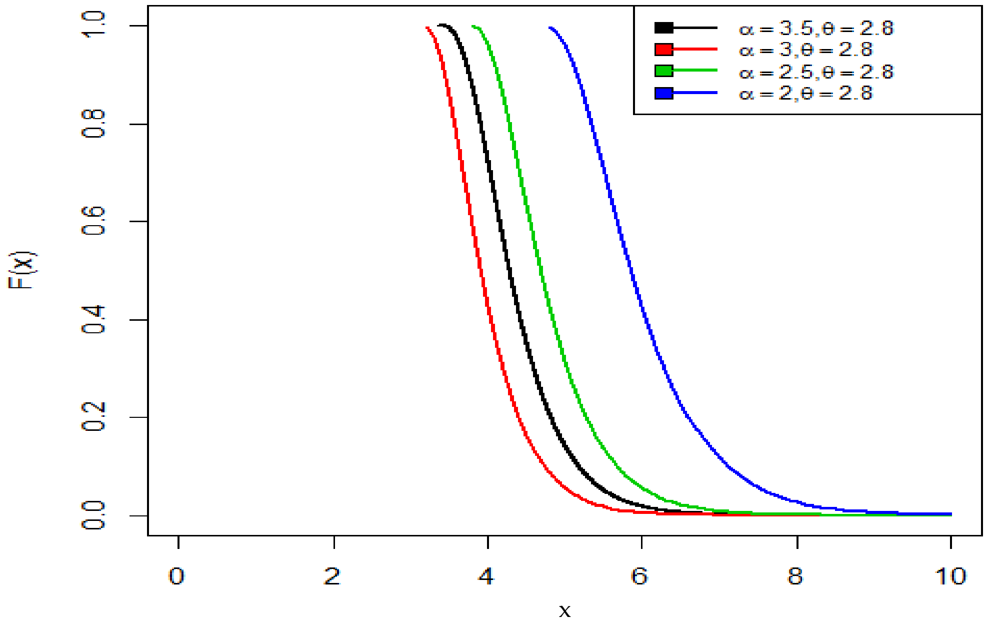

Fig. 3: *Survival function of exponentiated Suja distribution*

4. Statistical Properties

4.1 Moments

Suppose X is a random variable following exponentiated Suja distribution with parameters α and θ, then the rth order moment $E(X^r)$ for a given probability distribution is given by

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_\alpha(x)dx$$

$$E(X^r) = \int_0^\infty x^r \frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}dx \qquad (9)$$

$$E(X^r) = \frac{\alpha\theta^5}{\theta^4+24}\int_0^\infty x^r\left(1+x^4\right)e^{-\theta x}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}dx \qquad (10)$$

Using Binomial expansion of

$$\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1} = \sum_{i=0}^\infty \binom{\alpha-1}{i}\left\{\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right\}^i(-1)^i \qquad (11)$$

Equation (10) will become

$$E(X^r) = \frac{\alpha\theta^5}{\theta^4+24}\sum_{i=0}^\infty(-1)^i\binom{\alpha-1}{i}\int_0^\infty x^r\left(1+x^4\right)e^{-\theta x(1+i)}\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^i dx \qquad (12)$$

Again using Binomial expansion of

$$\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^i=\sum_{k=0}^{\infty}\binom{i}{k}\left(\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^k \tag{13}$$

Equation (12) becomes

$$E(X^r)=\frac{\alpha\theta^5}{\theta^4+24}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\left(\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^k\int_o^{\infty}x^r\left(1+x^4\right)e^{-\theta x(1+i)}dx \tag{14}$$

After simplification, we obtain

$$E(X^r)=\alpha\theta^5\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{\left(\theta^4+4\theta^3+12\theta^2+24\theta\right)^k}{(\theta^4+24)^{k+1}}\left(\frac{\theta(1+i)^4\Gamma(r+10k+1)+\Gamma(r+10k+5)}{\theta(1+i)^{r+10k+5}}\right) \tag{15}$$

Since equation (15) is a convergent series for all r ≥ 0, therefore all the moments exist. Therefore

$$E(X)=\alpha\theta^5\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{\left(\theta^4+4\theta^3+12\theta^2+24\theta\right)^k}{(\theta^4+24)^{k+1}}\left(\frac{\theta(1+i)^4\Gamma(10k+2)+\Gamma(10k+6)}{\theta(1+i)^{10k+6}}\right) \tag{16}$$

$$E(X^2)=\alpha\theta^5\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{\left(\theta^4+4\theta^3+12\theta^2+24\theta\right)^k}{(\theta^4+24)^{k+1}}\left(\frac{\theta(1+i)^4\Gamma(10k+3)+\Gamma(10k+7)}{\theta(1+i)^{10k+7}}\right) \tag{17}$$

Therefore, the Variance of X can be obtained as

$$V(X)=E(X^2)-(E(X))^2$$

4.2 Harmonic mean

The Harmonic mean for the proposed Exponentiated Suja distribution can be obtained as

$$H.M=E\left(\frac{1}{x}\right)=\int_0^{\infty}\frac{1}{x}f_{\alpha}(x)dx$$

$$=\int_0^{\infty}\frac{1}{x}\frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}dx \tag{18}$$

$$=\frac{\alpha\theta^5}{\theta^4+24}\int_0^{\infty}\frac{1}{x}\left(1+x^4\right)e^{-\theta x}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}dx \tag{19}$$

Using Binomial expansion in equation (19), we get

$$=\frac{\alpha\theta^5}{\theta^4+24}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\int_0^{\infty}\frac{1}{x}\left(1+x^4\right)e^{-\theta x(1+i)}\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^i dx \tag{20}$$

On using Binomial expansion in equation (20), we obtain

$$H.M=\frac{\alpha\theta^5}{\theta^4+24}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\left(\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^k\int_0^{\infty}\frac{1}{x}\left(1+x^4\right)e^{-\theta x(1+i)}dx \tag{21}$$

After the simplification of equation (21), we obtain

$$H.M=\alpha\theta^5\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\binom{i}{k}\frac{\left(\theta^4+4\theta^3+12\theta^2+24\theta\right)^k}{\left(\theta^4+24\right)^{k+1}}\left(\frac{\theta(1+i)^3\Gamma(10k+1)+\Gamma(10k+4)}{\theta(1+i)^{10k+4}}\right) \tag{22}$$

4.3 Moment Generating Function and Characteristics Function

Let X have an ESD, then the moment generating function of X is obtained as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_\alpha(x)dx$$

Using Taylor's series, we get

$$M_X(t) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + ...\right) f_\alpha(x)dx$$

$$M_X(t) = \alpha\theta^5 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i \binom{\alpha-1}{i}\binom{i}{k}\frac{t^j}{j!}\frac{\left(\theta^4+4\theta^3+12\theta^2+24\theta\right)^k}{(\theta^4+24)^{k+1}}\left(\frac{\theta(1+i)^4\Gamma(j+10k+1)+\Gamma(j+10k+5)}{\theta(1+i)^{j+10k+5}}\right) \quad (23)$$

Similarly, the Characteristics function of Exponentiated Suja distribution is given by

$$\varphi_X(t) = \alpha\theta^5 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i \binom{\alpha-1}{i}\binom{i}{k}\frac{mt^j}{j!}\frac{\left(\theta^4+4\theta^3+12\theta^2+24\theta\right)^k}{(\theta^4+24)^{k+1}}\left(\frac{\theta(1+i)^4\Gamma(j+10k+1)+\Gamma(j+10k+5)}{\theta(1+i)^{j+10k+5}}\right) \quad (24)$$

5. Order Statistics

Order statistics has wide field in reliability and life testing. There is also an extensive role of order statistics in several aspects of statistical inference. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots X_n$ drawn from the continuous population with probability density function *fx(x)* and cumulative distribution function Fx(x), then the pdf of rth order statistics $X_{(r)}$ can be written as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \quad (25)$$

Substitute the values of equation (5) and (6) in equation (25), we will obtain the pdf of rth order statistics X(r) for exponentiated Suja distribution and is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\frac{\alpha\theta^5(1+x^4)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}$$
$$\times\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha(r-1)} \quad (26)$$
$$\times\left(1-\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right)^{n-r}$$

The probability density function of higher order statistics $X_{(n)}$ can be obtained as

$$f_{x(n)}(x) = n\frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}$$
$$\times\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha(n-1)} \quad (27)$$

Similarly, the pdf of first order statistics $X_{(1)}$ can be obtained as

$$f_{x(1)}(x) = n\frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}$$
$$\times\left(1-\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right)^{n-1} \quad (28)$$

6. Maximum Likelihood Estimation

In this section, we will discuss the maximum likelihood estimators of the parameters of exponentiated Suja distribution. Let X1, X2,….Xn be the random sample of size n from the Exponentiated Suja distribution, then the likelihood function can be written as

$$L(\alpha,\theta) = \frac{\left(\alpha\theta^5\right)^n}{(\theta^4+24)^n}\prod_{i=1}^{n}\left((1+x^4)e^{-\theta x}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}\right) \quad (29)$$

The log likelihood function is given by

$$\log L(\alpha,\theta) = n\log\alpha + 5n\log\theta - n\log(\theta^4+24) + \sum_{i=1}^{n}\log\left(1+x_i{}^4\right) - \theta\sum_{i=1}^{n}x_i$$
$$+(\alpha-1)\sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right) \quad (30)$$

The maximum likelihood estimates of α, θ which maximizes (30), must satisfy the normal equations given by

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right) = 0 \quad (31)$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\log\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)} \quad (32)$$

$$\frac{\partial\log L}{\partial\theta} = \frac{5n}{\theta} - n\left(\frac{4\theta^3}{\theta^4+24}\right) - \sum_{i=1}^{n}x_i + (\alpha-1)\psi\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right) = 0 \quad (33)$$

Where ψ (.) is the digamma function.

At this point it is important to mention that the analytical solution of the above system of non-linear equation is unknown. Algebraically it is very difficult to solve the complicated form of likelihood system of nonlinear equations. Therefore, we use R and wolfram mathematics for estimating the required parameters.

7. Information Measures

7.1 Renyi Entropy

Entropies quantify the diversity, uncertainty, or randomness of a system. The Renyi entropy is named after Alfred Renyi in the context of fractual dimension estimation, the Renyi entropy forms the basis of the concept of generalized dimensions. The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$e(\beta)=\frac{1}{1-\beta}\log\left(\int_0^\infty f^{\beta}(x)dx\right)$$

Where, $\beta>0$ and $\beta\neq 1$

$$=\frac{1}{1-\beta}\log\left(\int_0^\infty\left\{\frac{\alpha\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}\right\}^{\beta}dx\right) \quad (34)$$

$$=\frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\beta}\int_0^\infty\left(1+x^4\right)^{\beta}e^{-\theta\beta x}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\beta(\alpha-1)}dx\right) \quad (35)$$

Using binomial expansion in (35), we get

$$=\frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\beta}\sum_{i=0}^{\infty}(-1)^i\binom{\beta(\alpha-1)}{i}\int_0^\infty\left(1+x^4\right)^{\beta}e^{-\theta x(\beta+i)}\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^{i}dx\right) \quad (36)$$

Again using binomial expansion in (16), we get

$$=\frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\beta}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\beta(\alpha-1)}{i}\binom{i}{k}\left(\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^{k}\int_0^\infty\left(1+x^4\right)^{\beta}e^{-\theta x(\beta+i)}dx\right) \quad (37)$$

After the simplification of (37) we obtain

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\beta(\alpha-1)}{i}\binom{i}{k}\binom{\beta}{j}\left(\frac{\theta^4+4\theta^3+12\theta^2+24\theta}{\theta^4+24}\right)^{k}\frac{\Gamma(4j+10k+1)}{\theta(\beta+i)^{4j+10k+1}}\right) \quad (38)$$

7.2 Tsallis Entropy

A generalization of Boltzmann-Gibbs (B-G) statistical mechanics initiated by Tsallis has gained a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy for a continuous random variable it is defined as

$$S_{\lambda}=\frac{1}{\lambda-1}\left(1-\int_0^\infty f^{\lambda}(x)dx\right)$$

$$= \frac{1}{\lambda-1}\left(1-\int_0^\infty\left\{\frac{\alpha\theta^5(1+x^4)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}\right\}^{\lambda}dx\right) \quad (39)$$

$$= \frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\lambda}\int_0^\infty(1+x^4)^{\lambda}e^{-\lambda\theta x}\left(1-\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)e^{-\theta x}\right)^{\lambda(\alpha-1)}dx\right) \quad (40)$$

Using binomial expansion in (40), we get

$$= \frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\lambda}\sum_{i=0}^{\infty}(-1)^i\binom{\lambda(\alpha-1)}{i}\int_0^\infty(1+x^4)^{\lambda}e^{-\theta x(\lambda+i)}\left(1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^{i}dx\right) \quad (41)$$

Again using binomial expansion in (41), we obtain

$$= \frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\lambda}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\lambda(\alpha-1)}{i}\binom{i}{k}\left(\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right)^{k}\int_0^\infty(1+x^4)^{\lambda}e^{-\theta x(\lambda+i)}dx\right) \quad (42)$$

After the simplification of (42), we get

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\left(\frac{\alpha\theta^5}{\theta^4+24}\right)^{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\binom{\lambda(\alpha-1)}{i}\binom{i}{k}\binom{\lambda}{j}\left(\frac{\theta^4+4\theta^3+12\theta^2+24\theta}{\theta^4+24}\right)^{k}\frac{\Gamma(4j+10k+1)}{\theta(\lambda+i)^{4j+10k+1}}\right) \quad (43)$$

8. Data Analysis

In this section, we use two real-life data sets in exponentiated Suja distribution and the model has been compared with Suja and exponential distributions

Data Set 1: The following data set of 40 patients suffering from blood cancer (leukemia) is reported by one of ministry of health hospitals in Saudi Arabia by Abouammah et al. [13]. The ordered lifetimes (in years) is provided below in table 1.

Table 1: *Data represents the blood cancer patients (leukemia)*

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.37	2.532	2.693	2.805	2.91	2.912	2.192	3.263	3.348
3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244
4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381

Data set 2: The second data set represents the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm which were originally reported by M. G. Bader and A. M. Priest [12]. The data set is provided below in table 2.

Table 2: *Data regarding the tensile strength (GPa) of 69 carbon fibers*

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.14	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554

2.566	2.57	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.88	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

In order to compare the exponentiated Suja distribution with Suja and exponential distribution, we consider the Criteria like BIC (Bayesian information criterion), AIC (Akaike information criterion), AICC (Corrected Akaike information criterion) and -2logL. The better distribution is which corresponds to lesser values of AIC, BIC, AICC and -2logL. For calculating AIC, BIC, AICC and -2logL can be evaluated by using the formulas as follows.

$$AIC = 2k - 2\log L \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1} \quad \text{and} \quad BIC = k\log n - 2\log L$$

Where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model.

Table 3: *Fitted distributions of the two data sets and criteria for comparison*

Data sets	Distribution	MLE	S.E	-2logL	AIC	BIC	AICC
1	Exponentiated Suja	$\hat{\alpha}=3.9557520$ $\hat{\theta}=1.9201488$	$\hat{\alpha}=1.2186387$ $\hat{\theta}=0.1531425$	120.792	124.792	128.169	125.117
	Suja	$\hat{\theta}=1.41132109$	$\hat{\theta}=0.08834353$	142.669	144.669	146.358	144.775
	Exponential	$\hat{\theta}=0.31839887$	$\hat{\theta}=0.05034278$	171.557	173.557	175.246	173.67
2	Exponentiated Suja	$\hat{\alpha}=91.3523215$ $\hat{\theta}=3.6448593$	$\hat{\alpha}=45.077905$ $\hat{\theta}=0.2307272$	77.5453	81.5453	85.6314	81.728
	Suja	$\hat{\theta}=1.6558639$	$\hat{\theta}=0.0831825$	164.252	166.252	168.295	166.312
	Exponential	$\hat{\theta}=0.40927188$	$\hat{\theta}=0.0542091$	215.846	217.846	219.889	217.905

From table 3, it can be easily seen that the exponentiated Suja distribution have the lesser AIC, BIC, AICC and -2logL values as compared to Suja and exponential distributions. Hence we can conclude that the exponentiated Suja distribution leads to a better fit than the Suja and exponential distributions.

9. Conclusion

In conclusion, this study has introduced a new generalization of the Suja distribution, known as the exponentiated Suja distribution, which incorporates two parameters: scale and shape. The distribution was generated using the exponentiated technique, and the parameters were estimated using the maximum likelihood estimator. Various statistical properties and reliability measures of the exponentiated Suja distribution were discussed.

Furthermore, the study demonstrated the practical applications of the new distribution in real-life time data. The results of two real lifetime data sets were compared with the Suja and exponential distributions, revealing that the exponentiated Suja distribution provides a better fit than both alternatives.

These findings highlight the potential of the exponentiated Suja distribution to enhance the modeling and analysis of data in various fields, such as clinical trials, epidemiological studies, and public health research. The improved fit observed in real-life applications suggests that the exponentiated Suja distribution can offer more accurate predictions and better capture the underlying characteristics of the data.

Overall, this study contributes to the field of biostatistics by introducing a novel distribution and demonstrating its advantages over existing models. Further research and applications of the exponentiated Suja distribution are encouraged to explore its full potential in various domains of statistical analysis and decision-making.

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