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EXPLORING THE LENGTH BIASED TORNUMONKPE DISTRIBUTION: PROPERTIES, ESTIMATIONS AND PRACTICAL APPLICATIONS

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Abstract

In this study, we introduce a novel extension of the Tornumonkpe distribution, known as the length biased Tornumonkpe distribution. This distribution holds particular significance as it belongs to the family of weighted distributions, specifically the length biased variant. Through an in-depth analysis, we explore the mathematical and statistical properties of this distribution, shedding light on its unique characteristics. To estimate the model parameters of this new distribution, we employ the well-established technique of maximum likelihood estimation. This allows us to accurately determine the parameters and enhance our understanding of the distribution's behavior. To demonstrate the practical applicability and advantages of the length biased Tornumonkpe distribution, we showcase its performance using a real-life time data set. Through this empirical examination, we investigate the distribution's superiority and flexibility, providing valuable insights into its potential use in various domains.

Keywords: Length biased distribution, Tornumonkpe distribution, order statistics, maximum likelihood estimation

1. Introduction

Weighted distributions have emerged as a unifying and powerful tool to address biases in unequally weighted sample data, providing a comprehensive approach for modeling and representing statistical information. This concept was initially suggested by Fisher [2], exploring the influence of ascertainment methods on the distribution of recorded observations. Subsequently, Rao [5] further developed and unified the theory, particularly in situations where standard distributions were inadequate for capturing observations with equal probabilities. The theory of weighted distributions also provides an integrative conceptualization for model stipulation and data representation problems. The weighted distributions are used as a tool in selection of appropriate models for observed data especially when samples are drawn without a proper frame. The weighted distribution reduces to length biased distribution when the weight function considers only the length of units of interest. The concept of length biased sampling was introduced by Cox [1] and Zelen [12]. The application of length biased distributions has found widespread use in various biomedical areas, including family history analysis, survival analysis, clinical trials, intermediate events, reliability theory, and population studies. In situations where a proper sampling frame is absent, length biased distributions offer an elegant solution by sampling items at a rate proportional to their lengths, thereby granting larger values a higher probability of being sampled. Many studies on length biased distribution has been published, for example; Rather and Subramanian [6], Rather and Subramanian [7], Rather and Ozel [8], Rather and Subramanian [9], Rather et al. [10].

Tornumonkpe distribution is a recently executed one parametric continuous probability distribution proposed by Nwikpe [4]. Its various mathematical and statistical properties such as order statistics, crude and raw moments, moment generating function, hazard rate function, graphs of pdf, cdf and hazard function and Renyi entropy have been discussed. Its parameter has also been estimated by using the maximum likelihood estimation.

2. Length Biased Tornumonkpe (LBT) Distribution

The probability density function of Tornumonkpe distribution is given by

$$f(x;\theta)=\frac{\theta^3}{(\theta^2+2)}\left(x^2+x\theta\right)e^{-\theta x};\ x>0,\ \theta>0 \tag{1}$$

and the cumulative distribution function of Tornumonkpe distribution is given by

$$F(x;\theta)=\left(1-\left(1+\frac{\theta^2x^2+\theta x(\theta^2+2)}{(\theta^2+2)}\right)e^{-\theta x}\right);\quad x>0,\ \theta>0 \tag{2}$$

Suppose the random variable X following non-negative condition with probability density function $f(x)$ .Let $w(x)$ be its non-negative weight function, then the probability density function of weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0.$$

Where $w(x)$ be the non-negative weight function and $E(w(x))=\int w(x)f(x)dx<\infty.$

Depending upon the various choices of weighted function $w(x)$ obviously when $w(x) = x^c$, resulting distribution is known as weighted distribution. In this paper, we have to study the length biased version of Tornumonkpe distribution called as length biased Tornumonkpe distribution. So, the weight function considered at $w(x) = x$, resulting distribution is called length biased distribution with its probability density function given by

$$f_l(x)=\frac{x\,f(x)}{E(x)} \tag{3}$$

Where $E(x)=\int_0^\infty x\,f(x,\theta)dx$

$$E(x)=\frac{(6+2\theta^2)}{\theta(\theta^2+2)} \tag{4}$$

By substituting equations (1) and (4) in equation (3), we will obtain the probability density function of length biased Tornumonkpe distribution as

$$f_l(x)=\frac{x\,\theta^4}{\left(6+2\theta^2\right)}\left(x^2+x\theta\right)e^{-\theta x} \tag{5}$$

and the cumulative distribution function of length biased Tornumonkpe distribution can be obtained as

$$F_l(x) = \int_0^x f_l(x)dx$$

$$F_l(x) = \frac{1}{\left(6+2\theta^2\right)}\left(\theta^4\int_0^x x^3 e^{-\theta x}dx \; + \; \theta^5\int_0^x x^2 e^{-\theta x}dx\right) \tag{6}$$

Put $\theta x = t \quad \Rightarrow \theta dx = dt \quad \Rightarrow dx = \frac{dt}{\theta}$, Also $x = \frac{t}{\theta}$

When $x \to x,\ t \to \theta x$ and When $x \to 0,\ t \to 0$

After the simplification of equation (6), we will obtain the cumulative distribution function of length biased Tornumonkpe distribution as

$$F_l(x) = \frac{1}{\left(6+2\theta^2\right)}\left(\gamma(4,\ \theta x) + \theta^2\gamma(3,\ \theta x)\right) \tag{7}$$

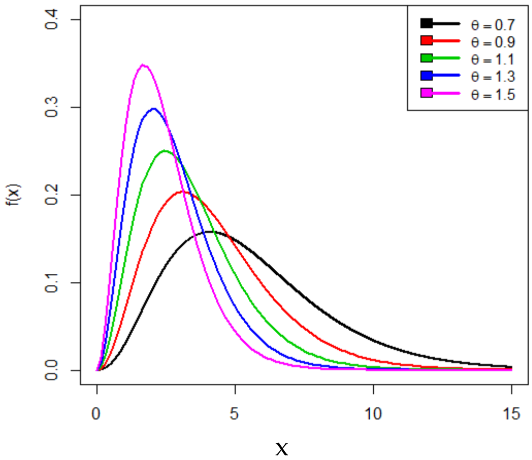

Figure 1: *Pdf plot of LBT distribution*

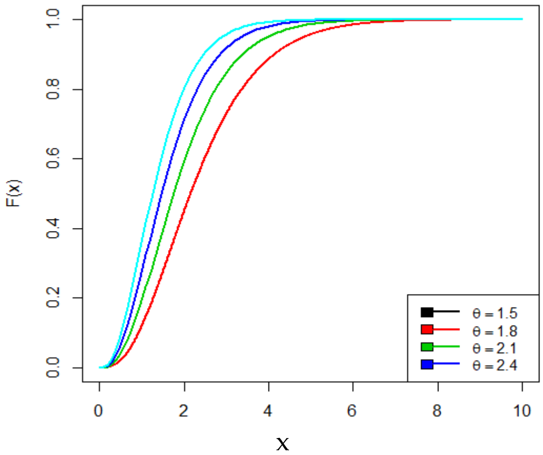

Figure 2: *Cdf plot of LBT distribution*

3. Survival Analysis

In this section, we will discuss about the survival function, hazard rate function, reverse hazard rate function and Mills ratio of proposed length biased Tornumonkpe distribution.

3.1 Survival function

The survival or reliability function of the length biased Tornumonkpe distribution can be obtained as

$$S(x) = 1 - F_l(x)$$

$$S(x) = 1 - \frac{1}{\left(6+2\theta^2\right)}\left(\gamma(4,\ \theta x) + \ \theta^2\gamma(3,\ \theta x)\right) \tag{8}$$

3.2 Hazard function

The hazard function is also known as hazard rate or failure rate or force of mortality and is given by

$$h(x) = \frac{f_l(x)}{1 - F_l(x)}$$

$$h(x) = \frac{x\theta^4 (x^2 + x\theta)\, e^{-\theta x}}{(6 + 2\theta^2) - (\gamma(4, \theta x) + \theta^2 \gamma(3, \theta x))} \tag{9}$$

3.3 Reverse hazard function

The reverse hazard rate function is given by

$$h_r(x) = \frac{f_l(x)}{F_l(x)}$$

$$h_r(x) = \frac{x\theta^4 (x^2 + x\theta)\, e^{-\theta x}}{(\gamma(4, \theta x) + \theta^2 \gamma(3, \theta x))} \tag{10}$$

3.4 Mills Ratio

$$M.R = \frac{1}{h_r(x)} = \frac{(\gamma(4, \theta x) + \theta^2 \gamma(3, \theta x))}{x\theta^4 (x^2 + x\theta)\, e^{-\theta x}} \tag{11}$$

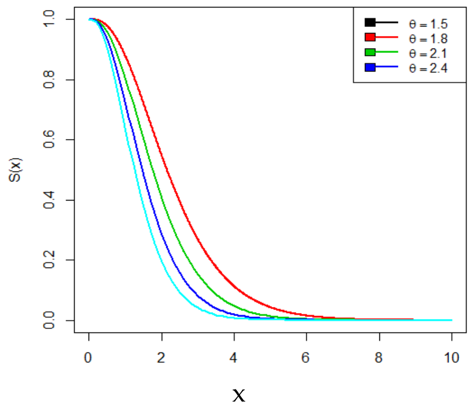

Fig. 3: Survival plot of LBT distribution

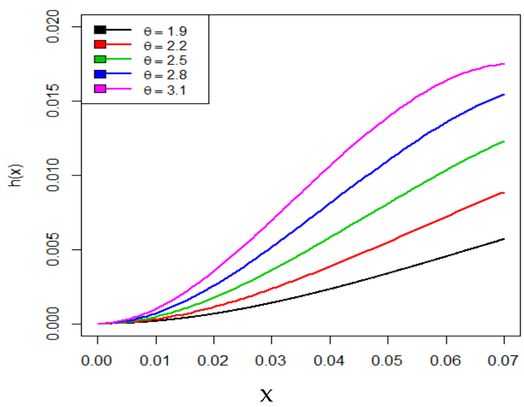

Fig. 4: Hazard plot of LBT distribution

4. Statistical Properties

In this section, we will discuss various statistical properties of length biased Tornumonkpe distribution those include moments, harmonic mean, moment generating function and characteristic function.

4.1 Moments

Let X be the random variable following length biased Tornumonkpe distribution with parameter θ, then the r^{th} order moment $E(X^r)$ of proposed distribution can be obtained as

$$E(X^r)=\mu_r'=\int_0^\infty x^r f_l(x)dx$$

$$E(X^r)=\mu_r'=\int_0^\infty x^r \frac{x\theta^4}{\left(6+2\theta^2\right)}\left(x^2+x\theta\right)e^{-\theta x}dx \tag{12}$$

$$E(X^r)=\mu_r'=\int_0^\infty \frac{x^{r+1}\theta^4}{\left(6+2\theta^2\right)}\left(x^2+x\theta\right)e^{-\theta x}dx \tag{13}$$

$$E(X^r)=\mu_r'=\frac{\theta^4}{\left(6+2\theta^2\right)}\int_0^\infty x^{r+1}\left(x^2+x\theta\right)e^{-\theta x}dx \tag{14}$$

$$E(X^r)=\mu_r'=\frac{\theta^4}{\left(6+2\theta^2\right)}\left(\int_0^\infty x^{(r+4)-1}e^{-\theta x}dx+\theta\int_0^\infty x^{(r+3)-1}e^{-\theta x}dx\right) \tag{15}$$

After the simplification of equation (15), we obtain

$$E(X^r)=\mu_r'=\frac{\Gamma(r+4)+\theta^2\Gamma(r+3)}{\theta^r(6+2\theta^2)} \tag{16}$$

Now putting r = 1, 2, 3 and 4 in equation (16), we will obtain the first four moments of length biased Tornumonkpe distribution as

$$E(X)=\mu_1'=\frac{24+6\theta^2}{\theta(6+2\theta^2)} \tag{17}$$

$$E(X^2)=\mu_2'=\frac{120+24\theta^2}{\theta^2(6+2\theta^2)} \tag{18}$$

$$E(X^3)=\mu_3'=\frac{720+120\theta^2}{\theta^3(6+2\theta^2)} \tag{19}$$

$$E(X^4)=\mu_4'=\frac{5040+720\theta^2}{\theta^4(6+2\theta^2)} \tag{20}$$

$$\text{Variance}=\frac{120+24\theta^2}{\theta^2(6+2\theta^2)}-\left(\frac{24+6\theta^2}{\theta(6+2\theta^2)}\right)^2 \tag{21}$$

$$S.D(\sigma)=\sqrt{\left(\frac{120+24\theta^2}{\theta^2(6+2\theta^2)}-\left(\frac{24+6\theta^2}{\theta(6+2\theta^2)}\right)^2\right)} \tag{22}$$

4.2 Harmonic mean

The harmonic mean for proposed length biased Tornumonkpe distribution can be obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x)dx \tag{23}$$

$$H.M = \int_0^\infty \frac{1}{x} \frac{x\theta^4}{\left(6+2\theta^2\right)}\left(x^2 + x\theta\right)e^{-\theta x}dx \tag{24}$$

$$H.M = \int_0^\infty \frac{\theta^4}{\left(6+2\theta^2\right)}\left(x^2 + x\theta\right)e^{-\theta x}dx \tag{25}$$

$$H.M = \frac{\theta^4}{\left(6+2\theta^2\right)}\int_0^\infty \left(x^2 + x\theta\right)e^{-\theta x}dx \tag{26}$$

$$H.M = \frac{\theta^4}{\left(6+2\theta^2\right)}\left(\int_0^\infty x^{(3)-1} e^{-\theta x}dx \ + \ \theta\int_0^\infty x^{(2)-1}e^{-\theta x}dx\right) \tag{27}$$

After the simplification of equation (27), we obtain

$$H.M = \frac{\theta(2+\theta^2)}{\left(6+2\theta^2\right)} \tag{28}$$

4.3 Moment generating function and characteristic function

Let X be the random variable following length biased Tornumonkpe distribution with parameter θ, then the moment generating function of proposed distribution can be obtained as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x)dx \tag{29}$$

Using Taylor's series, we obtain

$$M_X(t) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \dots\right) f_l(x)dx \tag{30}$$

$$M_X(t) = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_l(x)dx \tag{31}$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j' \tag{32}$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\Gamma(j+4) + \theta^2\Gamma(j+3)}{\theta^j(6+2\theta^2)}\right) \tag{33}$$

$$M_X(t) = \frac{1}{(6+2\theta^2)} \sum_{j=0}^\infty \frac{t^j}{j!\theta^j}\left(\Gamma(j+4) + \theta^2\Gamma(j+3)\right) \tag{34}$$

Similarly, the characteristic function of length biased Tornumonkpe distribution can be obtained as

$$\varphi_x(t) = M_X(it)$$

$$M_X(it) = \frac{1}{(6+2\theta^2)} \sum_{j=0}^{\infty} \frac{it^j}{j!\theta^j} \left(\Gamma(j+4) + \theta^2 \Gamma(j+3) \right) \tag{35}$$

5. Order Statistics

Consider X(1), X(2),…, X(n) be the order statistics of a random sample X1, X2,…, Xn from a continuous population with probability density function fx (x) and cumulative distribution function FX(x), then the probability density function of rth order statistics X(r) is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) \left(F_X(x)\right)^{r-1} \left(1 - F_X(x)\right)^{n-r} \tag{36}$$

By using the equations (5) and (7) in equation (36), we will obtain the probability density function of rth order statistics of length biased Tornumonkpe distribution as

$$\begin{aligned} f_{x(r)}(x) = {} & \frac{n!}{(r-1)!(n-r)!} \left(\frac{x\theta^4}{\left(6+2\theta^2\right)} \left(x^2 + x\theta\right) e^{-\theta x} \right) \\ & \times \left(\frac{1}{\left(6+2\theta^2\right)} \left(\gamma(4, \theta x) + \theta^2 \gamma(3, \theta x) \right) \right)^{r-1} \times \left(1 - \frac{1}{\left(6+2\theta^2\right)} \left(\gamma(4, \theta x) + \theta^2 \gamma(3, \theta x) \right) \right)^{n-r} \end{aligned} \tag{37}$$

Therefore, the probability density function of higher order statistic X(n) of length biased Tornumonkpe distribution can be obtained as

$$f_{x(n)}(x) = \frac{n x\theta^4}{\left(6+2\theta^2\right)} \left(x^2 + x\theta\right) e^{-\theta x} \left(\frac{1}{\left(6+2\theta^2\right)} \left(\gamma(4, \theta x) + \theta^2 \gamma(3, \theta x) \right) \right)^{n-1} \tag{38}$$

and the probability density function of first order statistic X(1) of length biased Tornumonkpe distribution can be obtained as

$$\begin{aligned} f_{x(1)}(x) = {} & \frac{n x\theta^4}{\left(6+2\theta^2\right)} \left(x^2 + x\theta\right) e^{-\theta x} \\ & \times \left(1 - \frac{1}{\left(6+2\theta^2\right)} \left(\gamma(4, \theta x) + \theta^2 \gamma(3, \theta x) \right) \right)^{n-1} \end{aligned} \tag{39}$$

6. Likelihood Ratio Test

Let X1, X2,…., Xn be the random sample of size n from length biased Tornumonkpe distribution. To examine its significance, we use the hypothesis.

$$H_o : f(x) = f(x;\theta) \quad \text{against} \quad H_1 : f(x) = f_l(x;\theta)$$

In order to investigate, whether the random sample of size n comes from the Tornumonkpe distribution or length biased Tornumonkpe distribution, the following test statistic is used.

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \frac{f_l(x;\theta)}{f(x;\theta)}$$

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \left(\frac{x_i\, \theta(\theta^2+2)}{\left(6+2\theta^2\right)} \right) \tag{40}$$

$$\Delta = \frac{L_1}{L_o} = \left(\frac{\theta(\theta^2+2)}{\left(6+2\theta^2\right)} \right)^n \prod_{i=1}^{n} x_i \tag{41}$$

We should refuse to accept the null hypothesis, if

$$\Delta = \left(\frac{\theta(\theta^2+2)}{\left(6+2\theta^2\right)} \right)^n \prod_{i=1}^{n} x_i > k \tag{42}$$

Equivalently we should also refuse to retain the null hypothesis, where

$$\Delta^* = \prod_{i=1}^{n} x_i > k \left(\frac{\left(6+2\theta^2\right)}{\theta(\theta^2+2)} \right)^n \tag{43}$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, \text{Where } k^* = k \left(\frac{\left(6+2\theta^2\right)}{\theta(\theta^2+2)} \right)^n \tag{44}$$

For large sample of size *n*, *2log Δ* is distributed as chi-square distribution with one degree of freedom and also *p*-value is obtained by using the chi square distribution. Thus, we should refuse to accept the null hypothesis, when the probability value is given by

$p\left(\Delta^* > \alpha^*\right)$, Where $\alpha^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance and $\prod_{i=1}^{n} x_i$ is the observed value of the statistic Δ^*.

7. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are known as income distribution or classical curves are mostly being used to measure the distribution of inequality in income or poverty. The bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x\, f_l(x)dx$$

$$L(p) = pB(p) = \frac{1}{\mu_1'}\int_0^q x\, f_l(x)dx \quad \text{and} \quad q = F^{-1}(p)$$

Where $\mu_1' = \frac{\left(24+6\theta^2\right)}{\theta(6+2\theta^2)}$

$$B(p) = \frac{\theta(6+2\theta^2)}{p(24+6\theta^2)}\int_0^q \frac{x^2\theta^4}{\left(6+2\theta^2\right)}\left(x^2+x\theta\right)e^{-\theta x}dx \tag{45}$$

$$B(p) = \frac{\theta^5}{p(24+6\theta^2)}\int_0^q x^2\left(x^2+x\theta\right)e^{-\theta x}dx \tag{46}$$

$$B(p) = \frac{\theta^5}{p(24+6\theta^2)}\left(\int_0^q x^{(5)-1}e^{-\theta x}dx + \theta\int_0^q x^{(4)-1}e^{-\theta x}dx\right) \tag{47}$$

After simplification, we obtain

$$B(p) = \frac{\theta^5}{p(24+6\theta^2)}\left(\gamma(5,\,\theta q)+\theta\,\gamma(4,\,\theta q)\right) \tag{48}$$

$$L(p) = \frac{\theta^5}{(24+6\theta^2)}\left(\gamma(5,\,\theta q)+\theta\,\gamma(4,\,\theta q)\right) \tag{49}$$

8. Parameter Estimation

In this section, we will discuss the method of maximum likelihood estimation to estimate the parameters of length biased Tornumonkpe distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from length biased Tornumonkpe distribution, then the likelihood function can be defined as

$$L(x) = \prod_{i=1}^n f_l(x)$$

$$L(x) = \prod_{i=1}^n\left(\frac{x_i\,\theta^4}{\left(6+2\theta^2\right)}\left(x_i^2+x_i\,\theta\right)e^{-\theta x_i}\right) \tag{50}$$

$$L(x) = \frac{\theta^{4n}}{(6+2\theta^2)^n}\prod_{i=1}^n\left(x_i\left(x_i^2+x_i\,\theta\right)e^{-\theta x_i}\right) \tag{51}$$

The log likelihood function is given by

$$\log L = 4n\log\theta - n\log(6+2\theta^2) + \sum_{i=1}^n \log x_i + \sum_{i=1}^n \log\left(x_i^2+x_i\,\theta\right) - \theta\sum_{i=1}^n x_i \tag{52}$$

Now differentiating the log likelihood equation (52) with respect to parameter θ. We must satisfy the following normal equation

$$\frac{\partial \log L}{\partial\theta} = \frac{4n}{\theta} - n\left(\frac{4\theta}{6+2\theta^2}\right) + \sum_{i=1}^n\left(\frac{x_i}{\left(x_i^2+x_i\,\theta\right)}\right) - \sum_{i=1}^n x_i = 0 \tag{53}$$

The above likelihood equation is too complicated to solve it algebraically. Therefore, we use numerical technique like Newton-Raphson method for estimating the required parameter of proposed distribution.

In order to use the asymptotic normality results for obtaining the confidence interval. We have that if $(\hat{\beta}=\hat{\theta})$ denotes the MLE of $(\beta=\theta)$. We can state the results as

$$\sqrt{n}(\hat{\beta}-\beta)\rightarrow N(0,\, I^{-1}(\beta))$$

Where $I^{-1}(\beta)$ is Fisher's information matrix i.e.,

$$I(\beta)=-\frac{1}{n}\left(E\left(\frac{\partial^2 \log L}{\partial\theta^2}\right)\right) \tag{54}$$

Here, we define

$$E\left(\frac{\partial^2 \log L}{\partial\theta^2}\right)=-\frac{4n}{\theta^2}-n\left(\frac{(4)(6+2\theta^2)-(16\theta^2)}{(6+2\theta^2)^2}\right)-\sum_{i=1}^{n}\left(\frac{E(x_i)^2}{(x_i^{\,2}+x_i\,\theta)^2}\right) \tag{55}$$

Since β being unknown, we estimate $I^{-1}(\beta)$ by $I^{-1}(\hat{\beta})$ and this can be used to obtain asymptotic confidenc interval for θ.

9. Applications

In this section, we have fitted a real life data set in length biased Tornumonkpe distribution in order to show that length biased Tornumonkpe distribution fits better over Tornumonkpe, exponential and Lindley distributions. The real life data set is given below as

The following data set represents the failure times in hours of an accelerated life test of 59 conductors without any censored observation is obtained by Lawless [3] and the data set is given below in table 1.

Table 1: *Data regarding the failure times in hours*

2.997	4.137	4.288	4.531	4.700	4.706	5.009	5.381	5.434	5.459	5.589	5.640
5.807	5.923	6.033	6.071	6.087	6.129	6.352	6.369	6.476	6.492	6.515	6.522
6.538	6.545	6.573	6.725	6.869	6.923	6.948	6.956	6.958	7.024	7.224	7.365
7.398	7.459	7.489	7.495	7.496	7.543	7.683	7.937	7.945	7.974	8.120	8.336
8.532	8.591	8.687	8.799	9.218	9.254	9.289	9.663	10.092	10.491	11.038	

To compute the model comparison criterion values along with the estimation of unknown parameters, the technique of R software is used. In order to compare the performance of length biased Tornumonkpe distribution over Tornumonkpe, exponential and Lindley distributions, we use the criterion values such as *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected) and *-2logL*. The better distribution is which corresponds to the lesser values of *AIC*, *BIC*, *AICC* and *-2logL*. For calculating the criterion values like *AIC, BIC, AICC* and *-2logL* following formulas are used.

$$AIC=2k-2\log L,\qquad BIC=k\log n-2\log L\qquad \text{and}\qquad AICC=AIC+\frac{2k(k+1)}{n-k-1}$$

Where n is the sample size, k is the number of parameters in statistical model and $-2\log L$ is the maximized value of log-likelihood function under the considered model.

Table 2: *Shows Comparison and Performance of fitted distributions*

Distributions	MLE	S.E	-2logL	AIC	BIC	AICC
Length Biased Tornumonkpe	$\hat{\theta} = 0.55730745$	$\hat{\theta} = 0.03557811$	270.4884	272.4884	274.5659	272.5585
Tornumonkpe	$\hat{\theta} = 0.41550194$	$\hat{\theta} = 0.03053944$	285.505	287.505	289.5825	287.5751
Exponential	$\hat{\theta} = 0.14326745$	$\hat{\theta} = 0.01865093$	347.2809	349.2809	351.3585	349.3510
Lindley	$\hat{\theta} = 0.25722036$	$\hat{\theta} = 0.02393044$	316.7054	318.7054	320.783	318.7755

From results given above in table 2, it has been clearly realized and observed that the length biased Tornumonkpe distribution has lesser *AIC*, *BIC*, *AICC* and *-2logL* values as compared to the Tornumonkpe, exponential and Lindley distributions. Hence, it can be revealed that length biased Tornumonkpe distribution leads to a better fit over Tornumonkpe, exponential and Lindley distributions.

10. Conclusion

In this article, we have introduced the length biased Tornumonkpe distribution as a novel approach within the field. By applying the length biased technique to its baseline distribution, we have generated a new distribution that exhibits unique properties and characteristics. Throughout our analysis, we have explored and derived various structural properties of the length biased Tornumonkpe distribution. These properties include moments, the shape of the probability density function (PDF) and cumulative distribution function (CDF), mean and variance, harmonic mean, survival function, hazard rate function, moment generating function, reverse hazard rate function, order statistics, Bonferroni and Lorenz curves. Through a comprehensive study of these properties, we have gained valuable insights into the behavior and statistical properties of the length biased Tornumonkpe distribution. To estimate the parameters of the length biased Tornumonkpe distribution, we employed the widely used technique of maximum likelihood estimation. This allowed us to obtain reliable estimates and enhance our understanding of the distribution's characteristics.

Furthermore, we have validated the practical applicability and superiority of the length biased Tornumonkpe distribution through its examination with real-life data sets. By comparing its fit with other well-known distributions such as the Tornumonkpe, exponential, and Lindley distributions, we have demonstrated that the length biased Tornumonkpe distribution outperforms these alternatives in terms of goodness-of-fit.

Finally, the length biased Tornumonkpe distribution represents a significant advancement in statistical modeling. Its distinctive properties, derived through rigorous analysis, and its superior fit with real-life data sets emphasize its potential for various applications. Researchers and practitioners can benefit from incorporating the length biased Tornumonkpe distribution into their analyses, enabling more accurate and robust modeling in diverse fields.

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