ENHANCING ENGINEERING SCIENCES WITH UMA DISTRIBUTION: A PERFECT FIT AND VALUABLE CONTRIBUTIONS

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Abstract

In this study, we introduce a novel class of distributions called the length biased Uma distribution. This distribution is a specific instance of the broader weighted distribution family, known for its versatility in various applications. We explore the structural properties of the length biased Uma distribution and propose a robust parameter estimation technique based on maximum likelihood estimation. To assess its efficacy, we apply the newly introduced distribution to two real-world datasets, evaluating its flexibility and performance in comparison to existing models. The results obtained demonstrate the potential of the length biased Uma distribution as a valuable addition to the repertoire of statistical distributions, offering valuable insights for a wide range of practical applications.

Keywords: Length biased distribution, Uma distribution, Reliability analysis, Order statistics, Maximum likelihood estimation.

1. Introduction

In probability distribution theory, the concept of weighted probability models have attained a great importance for modeling different lifetime data sets occurring from various practical and applied fields like engineering, medical sciences, insurance, finance etc. There are several situations were classical distributions may not provide best fit to lifetime data. In such situation attempt has been made to generalize standard distribution by introducing an extra parameter to it. This extra parameter can be introduced through various techniques. One of such technique is of weighted technique. Fisher [7] introduced the concept of weighted distribution to model the ascertainment bias which was later formalized by Rao [16] in a unifying theory for problems were the observations fall in non-experimental, non-replicated and non-random categories. The weighted distributions are remarkable for efficient modeling of statistical data and prediction obviously when classical distributions are not suitable. The weighted distributions were formulated in such a situation to record the observation according to some weight function. The weighted distribution reduces to length biased distribution when the weight function considers only the length of units of interest. The probability of selecting an individual in a population is proportional to its magnitude is called length biased sampling. However, when observations are selected with probability proportional to their length, resulting distribution is called length biased distribution. The concept of length biased distribution was introduced by Cox [5] in renewal theory. Length biased sampling situation occurs were a proper sampling frame is absent. In such cases items are sampled at a rate proportional to their lengths so that the larger values could be sampled with higher probability.

A significant and remarkable contribution has been done by several authors to develop some important length biased probability models along with their applications in various fields. Oluwafemi and Olalekan [15] discussed on length and area biased exponentiated weibull distribution based on forest inventories. Ekhosuehil et al. [6] proposed the weibull length biased exponential distribution with statistical properties and applications. Atikankul et al. [2] discussed on the length biased weighted Lindley distribution with applications. Ratnaparkhi and Nimbalkar [18] presented the length biased lognormal distribution and its application in the analysis of data from oil field exploration studies. Mathew [14] proposed the reliability test plan for the Marshall Olkin length biased Lomax distribution. Mustafa and Khan [13] obtained the length biased power hazard rate distribution with its properties and applications. Chaito et al. [3] discussed on the length biased Gamma-Rayleigh distribution with applications. Al-omari and Alanzi [1] introduced the inverse length biased Maxwell distribution with statistical inference and application. Ghorbal [11] discussed on properties of length biased exponential model with applications. Ganaie and Rajagopalan [9] presented the weighted power Garima distribution with applications in blood cancer and relief times. Recently, Chaito and Khamkong [4] discussed on length-biased weibull-Rayleigh distribution for application to hydrological data.

Uma distribution is a recently introduced one parametric continuous lifetime distribution proposed by Shanker [19]. Its various statistical properties like moments, mean residual life function, hazard rate function, reverse hazard function, stochastic ordering, coefficient of variation, skewness, kurtosis and index of dispersion have been discussed. For estimating its parameter the method of maximum likelihood estimation has been discussed.

2. Length Biased Uma (LBU) Distribution

The probability density function of Uma distribution is given by

$$f(x;\theta)=\frac{\theta^4}{\theta^3+\theta^2+6}\left(1+x+x^3\right)e^{-\theta x};\ x>0,\ \theta>0 \tag{1}$$

and the cumulative distribution function of Uma distribution is given by

$$F(x;\theta)=1-\left(1+\frac{\theta x\left(\theta^2x^2+3\theta x+\theta^2+6\right)}{\theta^3+\theta^2+6}\right)e^{-\theta x};\quad x>0,\ \theta>0 \tag{2}$$

Consider X be the non-negative random variable with probability density function $f(x)$ .Let its non-negative weight function be $w(x)$, then the probability density function of weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)\, f(x)}{E(w(x))}, \quad x > 0.$$

Where $w(x)$ be the non-negative weight function and $E(w(x)) = \int w(x) f(x) dx < \infty.$

For various forms of weight function *w(x)* obviously when $w(x) = x^c$, resulting distribution is called weighted distribution. In this paper, we have considered the weight function at $w(x) = x$ to obtain the length biased version of Uma distribution and its probability density function is given by

$$f_l(x) = \frac{x\, f(x)}{E(x)} \tag{3}$$

Where $E(x) = \int_0^{\infty} x\, f(x,\theta) dx$

$$E(x) = \frac{\left(\theta^3 + 2\theta^2 + 24\right)}{\theta(\theta^3 + \theta^2 + 6)} \tag{4}$$

By using the equations (1) and (4) in equation (3), we will obtain the probability density function of length biased Uma distribution as

$$f_l(x) = \frac{x\theta^5}{(\theta^3 + 2\theta^2 + 24)}\left(1 + x + x^3\right) e^{-\theta x} \tag{5}$$

and the cumulative distribution function of length biased Uma distribution can be obtained as

$$F_l(x) = \int_0^x f_l(x) dx$$

$$F_l(x) = \int_0^x \frac{x\theta^5}{(\theta^3 + 2\theta^2 + 24)}\left(1 + x + x^3\right) e^{-\theta x} dx$$

$$F_l(x) = \frac{1}{(\theta^3 + 2\theta^2 + 24)} \int_0^x x\theta^5 \left(1 + x + x^3\right) e^{-\theta x} dx$$

$$F_l(x) = \frac{1}{(\theta^3 + 2\theta^2 + 24)}\left(\theta^5 \int_0^x x e^{-\theta x} dx \; + \; \theta^5 \int_0^x x^2 e^{-\theta x} dx \; + \; \theta^5 \int_0^x x^4 e^{-\theta x} dx\right) \tag{6}$$

Put $\theta x = t \;\Rightarrow \theta dx = dt \;\Rightarrow dx = \frac{dt}{\theta}$, Also $x = \frac{t}{\theta}$

When $x \to x,\ t \to \theta x$ and When $x \to 0,\ t \to 0$

After the simplification of equation (6), we will obtain the cumulative distribution function of length biased Uma distribution as

$$F_l(x) = \frac{1}{(\theta^3 + 2\theta^2 + 24)}\left(\theta^3 \gamma(2,\ \theta x) + \theta^2 \gamma(3,\ \theta x) + \ \gamma(5,\ \theta x)\right) \tag{7}$$

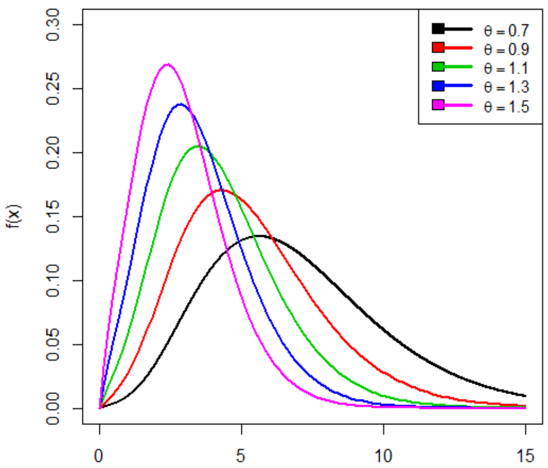

Fig. 1: *Pdf plot of LBU distribution*

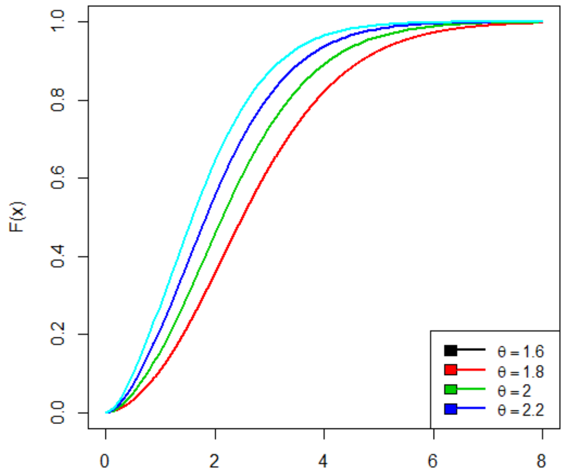

Fig. 2: *Cdf plot of LBU distribution*

3. Reliability Analysis

In this section, we will discuss about the reliability function, hazard rate function, reverse hazard rate function and Mills ratio of the proposed length biased Uma distribution.

3.1 Reliability function

The reliability function is termed as survival function and the reliability function of executed length biased Uma distribution can be determined as

$$R(x) = 1 - F_l(x)$$

$$R(x) = 1 - \frac{1}{(\theta^3 + 2\theta^2 + 24)}\left(\theta^3\gamma(2, \theta x) + \theta^2\gamma(3, \theta x) + \gamma(5, \theta x)\right) \tag{8}$$

3.2 Hazard function

The hazard function is also known as failure rate or force of mortality and is given by

$$h(x) = \frac{f_l(x)}{1 - F_l(x)}$$

$$h(x) = \frac{x\theta^5(1 + x + x^3)\, e^{-\theta x}}{(\theta^3 + 2\theta^2 + 24) - (\theta^3\gamma(2, \theta x) + \theta^2\gamma(3, \theta x) + \gamma(5, \theta x))} \tag{9}$$

3.3 Reverse hazard function

The reverse hazard rate function is given by

$$h_r(x) = \frac{f_l(x)}{F_l(x)}$$

$$h_r(x) = \frac{x\theta^5(1+x+x^3)\ e^{-\theta x}}{(\theta^3\gamma(2,\ \theta x)+\theta^2\gamma(3,\ \theta x)+\gamma(5,\ \theta x))} \tag{10}$$

3.4 Mills Ratio

The Mills Ratio of proposed model is given by

$$M.R = \frac{1}{h_r(x)} = \frac{(\theta^3\gamma(2,\ \theta x)+\theta^2\gamma(3,\ \theta x)+\gamma(5,\ \theta x))}{x\theta^5(1+x+x^3)\ e^{-\theta x}} \tag{11}$$

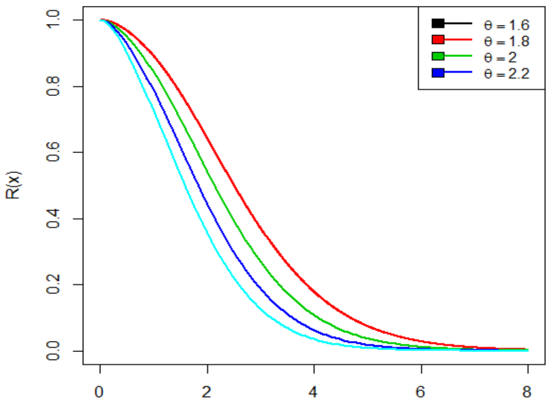

Fig. 3: *Reliability plot of LBU distribution*

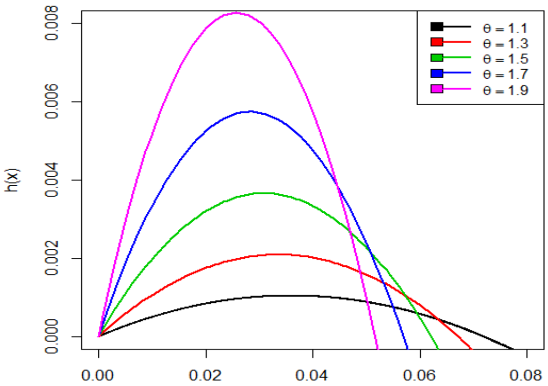

Fig. 4: *Hazard rate plot of LBU distribution*

4. Order Statistics

Suppose X(1), X(2),…, X(n) be the order statistics of a random sample X1, X2,…, Xn from a continuous population with cumulative distribution function FX(x) and probability density function fx (x) then the probability density function of rth order statistics X(r) is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\ f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \tag{12}$$

Now substituting the equations (5) and (7) in equation (12), we will obtain the probability density function of r^{th} order statistics $X_{(r)}$ of length biased Uma distribution as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\frac{x\theta^5}{(\theta^3+2\theta^2+24)}\left(1+x+x^3\right)e^{-\theta x}\right)\left(\frac{1}{(\theta^3+2\theta^2+24)}\left(\theta^3\gamma(2,\ \theta x)+\theta^2\gamma(3,\ \theta x)+\gamma(5,\theta x)\right)\right)^{r-1}$$

$$\times\left(1-\frac{1}{(\theta^3+2\theta^2+24)}\left(\theta^3\gamma(2,\ \theta x)\ +\ \theta^2\gamma(3,\ \theta x)+\gamma(5,\ \theta x)\right)\right)^{n-r}$$

Therefore, the probability density function of higher order statistic $X_{(n)}$ of length biased Uma distribution can be determined as

$$f_{x(n)}(x) = \frac{n\,x\,\theta^5}{(\theta^3 + 2\theta^2 + 24)}\left(1 + x + x^3\right)e^{-\theta x}\left(\frac{1}{(\theta^3 + 2\theta^2 + 24)}\left(\theta^3\gamma(2,\,\theta x) + \theta^2\gamma(3,\,\theta x) + \gamma(5,\,\theta x)\right)\right)^{n-1}$$

and probability density function of first order statistic $X_{(1)}$ of length biased Uma distribution can be determined as

$$f_{x(1)}(x) = \frac{n\,x\,\theta^5}{(\theta^3 + 2\theta^2 + 24)}\left(1 + x + x^3\right)e^{-\theta x}\left(1 - \frac{1}{(\theta^3 + 2\theta^2 + 24)}\left(\theta^3\gamma(2,\,\theta x) + \theta^2\gamma(3,\,\theta x) + \gamma(5,\,\theta x)\right)\right)^{n-1}$$

5. Test for Length biasedness of Length biased Uma distribution

Consider X1, X2,…., Xn be the random sample of size n from length biased Uma distribution. To analyse its flexibility consider the hypothesis.

$$H_o : f(x) = f(x;\theta) \quad \text{against} \quad H_1 : f(x) = f_l(x;\theta)$$

In order to determine, whether the random sample of size n comes from Uma distribution or length biased Uma distribution, the following test statistic rule is employed.

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n}\frac{f_l(x;\theta)}{f(x;\theta)} \tag{13}$$

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n}\left(\frac{x_i\,\theta(\theta^3 + \theta^2 + 6)}{(\theta^3 + 2\theta^2 + 24)}\right) \tag{14}$$

$$\Delta = \frac{L_1}{L_o} = \left(\frac{\theta(\theta^3 + \theta^2 + 6)}{(\theta^3 + 2\theta^2 + 24)}\right)^n \prod_{i=1}^{n} x_i \tag{15}$$

We should refuse to accept the null hypothesis, if

$$\Delta = \left(\frac{\theta(\theta^3 + \theta^2 + 6)}{(\theta^3 + 2\theta^2 + 24)}\right)^n \prod_{i=1}^{n} x_i > k \tag{16}$$

Equivalently, we should also refrain to retain the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k\left(\frac{(\theta^3 + 2\theta^2 + 24)}{\theta(\theta^3 + \theta^2 + 6)}\right)^n \tag{17}$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, \text{Where } k^* = k\left(\frac{(\theta^3 + 2\theta^2 + 24)}{\theta(\theta^3 + \theta^2 + 6)}\right)^n$$

Whether, the $2\log\Delta$ is distributed as chi-square distribution with one degree of freedom if the sample is large of size n and also p-value is determined by using chi-square distribution. Thus, we should refuse to accept the null hypothesis, if the probability value is given by

$p\left(\Delta^* > \gamma^*\right)$, Where $\gamma^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance and $\prod_{i=1}^{n} x_i$ is the experimental value of the statistic Δ^*.

6. Statistical properties

In this section, we will derive about the different structural properties of length biased Uma distribution those include moments, harmonic mean, moment generating function and characteristic function.

6.1 Moments

Let X be the random variable following length biased Uma distribution with parameter θ, then the r^{th} order moment $E(X^r)$ of proposed distribution can be obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_l(x)dx \tag{18}$$

$$E(X^r) = \mu_r' = \int_0^\infty x^r \frac{x\theta^5}{(\theta^3+2\theta^2+24)}\left(1+x+x^3\right) e^{-\theta x}dx \tag{19}$$

$$E(X^r) = \mu_r' = \int_0^\infty \frac{x^{r+1}\theta^5}{(\theta^3+2\theta^2+24)}\left(1+x+x^3\right) e^{-\theta x}dx \tag{20}$$

$$E(X^r) = \mu_r' = \frac{\theta^5}{(\theta^3+2\theta^2+24)}\int_0^\infty x^{r+1}\left(1+x+x^3\right) e^{-\theta x}dx \tag{21}$$

$$E(X^r) = \mu_r' = \frac{\theta^5}{(\theta^3+2\theta^2+24)}\left(\begin{array}{l}\int_0^\infty x^{(r+2)-1}e^{-\theta x}dx + \int_0^\infty x^{(r+3)-1}e^{-\theta x}dx \\ + \int_0^\infty x^{(r+5)-1}e^{-\theta x}dx\end{array}\right) \tag{22}$$

After simplification, we obtain from equation (22)

$$E(X^r) = \mu_r' = \frac{\theta^3\Gamma(r+2)+\theta^2\Gamma(r+3)+\Gamma(r+5)}{\theta^r(\theta^3+2\theta^2+24)} \tag{23}$$

Now substituting r = 1, 2, 3 and 4 in equation (23), we will obtain the first four moments of length biased Uma distribution as

$$E(X) = \mu_1' = \frac{2\theta^3+6\theta^2+120}{\theta(\theta^3+2\theta^2+24)} \tag{24}$$

$$E(X^2) = \mu_2' = \frac{6\theta^3+24\theta^2+720}{\theta^2(\theta^3+2\theta^2+24)} \tag{25}$$

$$E(X^3) = \mu_3' = \frac{24\theta^3+120\theta^2+5040}{\theta^3(\theta^3+2\theta^2+24)} \tag{26}$$

$$E(X^4) = \mu_4' = \frac{120\theta^3+720\theta^2+40320}{\theta^4(\theta^3+2\theta^2+24)} \tag{27}$$

$$\text{Variance} = \frac{6\theta^3+24\theta^2+720}{\theta^2(\theta^3+2\theta^2+24)} - \left(\frac{2\theta^3+6\theta^2+120}{\theta(\theta^3+2\theta^2+24)}\right)^2$$

$$S.D(\sigma) = \sqrt{\left(\frac{6\theta^3 + 24\theta^2 + 720}{\theta^2(\theta^3 + 2\theta^2 + 24)} - \left(\frac{2\theta^3 + 6\theta^2 + 120}{\theta(\theta^3 + 2\theta^2 + 24)}\right)^2\right)}$$

6.2 Harmonic mean

The harmonic mean for proposed length biased Uma distribution can be obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x)dx \tag{28}$$

$$H.M = \int_0^\infty \frac{1}{x} \frac{x\theta^5}{(\theta^3 + 2\theta^2 + 24)}\left(1 + x + x^3\right) e^{-\theta x} dx \tag{29}$$

$$H.M = \int_0^\infty \frac{\theta^5}{(\theta^3 + 2\theta^2 + 24)}\left(1 + x + x^3\right) e^{-\theta x} dx \tag{30}$$

$$H.M = \frac{\theta^5}{(\theta^3 + 2\theta^2 + 24)} \int_0^\infty \left(1 + x + x^3\right) e^{-\theta x} dx \tag{31}$$

$$H.M = \frac{\theta^5}{(\theta^3 + 2\theta^2 + 24)}\left(\int_0^\infty x^{(2)-2} e^{-\theta x} dx + \int_0^\infty x^{(2)-1} e^{-\theta x} dx + \int_0^\infty x^{(4)-1} e^{-\theta x} dx\right) \tag{32}$$

After simplification, we obtain

$$H.M = \frac{\theta(\theta^2 + \theta^2 + 6)}{(\theta^3 + 2\theta^2 + 24)} \tag{33}$$

6.3 Moment generating function and characteristic function

Let X be the random variable following length biased Uma distribution with parameter θ, then the moment generating function of executed distribution can be determined as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x)dx \tag{34}$$

Using Taylor's series, we obtain

$$M_X(t) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \ldots\right) f_l(x)dx$$

$$M_X(t) = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_l(x)dx$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j'$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\theta^3\Gamma(j+2) + \theta^2\Gamma(j+3) + \Gamma(j+5)}{\theta^j(\theta^3 + 2\theta^2 + 24)}\right) \tag{35}$$

$$M_X(t) = \frac{1}{(\theta^3 + 2\theta^2 + 24)} \sum_{j=0}^{\infty} \frac{t^j}{j!\theta^j} \left(\theta^3 \Gamma(j+2) + \theta^2 \Gamma(j+3) + \Gamma(j+5)\right) \tag{36}$$

Similarly, the characteristic function of length biased Uma distribution can be determined as

$$\varphi_x(t) = M_X(it)$$

$$M_X(it) = \frac{1}{(\theta^3 + 2\theta^2 + 24)} \sum_{j=0}^{\infty} \frac{it^j}{j!\theta^j} \left(\theta^3 \Gamma(j+2) + \theta^2 \Gamma(j+3) + \Gamma(j+5)\right) \tag{37}$$

7. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are also termed as income distribution or classical curves are mostly being used in economics to measure the distribution of inequality in income or poverty. The bonferroni and Lorenz curves can be written as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x\, f_l(x) dx \tag{38}$$

$$L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x\, f_l(x) dx \quad \text{and} \quad q = F^{-1}(p)$$

Where $\mu_1' = \dfrac{\left(2\theta^3 + 6\theta^2 + 120\right)}{\theta(\theta^3 + 2\theta^2 + 24)}$

$$B(p) = \frac{\theta(\theta^3 + 2\theta^2 + 24)}{p(2\theta^3 + 6\theta^2 + 120)} \int_0^q x \frac{x\theta^5}{\left(\theta^3 + 2\theta^2 + 24\right)} \left(1 + x + x^3\right) e^{-\theta x} dx \tag{39}$$

$$B(p) = \frac{\theta(\theta^3 + 2\theta^2 + 24)}{p(2\theta^3 + 6\theta^2 + 120)} \int_0^q \frac{x^2 \theta^5}{\left(\theta^3 + 2\theta^2 + 24\right)} \left(1 + x + x^3\right) e^{-\theta x} dx \tag{40}$$

$$B(p) = \frac{\theta^6}{p(2\theta^3 + 6\theta^2 + 120)} \int_0^q x^2 \left(1 + x + x^3\right) e^{-\theta x} dx \tag{41}$$

$$B(p) = \frac{\theta^6}{p(2\theta^3 + 6\theta^2 + 120)} \left(\int_0^q x^{(3)-1} e^{-\theta x} dx + \int_0^q x^{(4)-1} e^{-\theta x} dx + \int_0^q x^{(6)-1} e^{-\theta x} dx \right) \tag{42}$$

After simplification, we obtain

$$B(p) = \frac{\theta^6}{p(2\theta^3 + 6\theta^2 + 120)} \left(\gamma(3, \theta q) + \gamma(4, \theta q) + \gamma(6, \theta q)\right) \tag{43}$$

$$L(p) = \frac{\theta^6}{(2\theta^3 + 6\theta^2 + 120)} \left(\gamma(3, \theta q) + \gamma(4, \theta q) + \gamma(6, \theta q)\right) \tag{44}$$

8. Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the method of maximum likelihood estimation to estimate the parameters of length biased Uma distribution. Consider $X_1, X_2, \ldots, X_n$ be a random sample of size n from the length biased Uma distribution, then the likelihood function can be defined as

$$L(x) = \prod_{i=1}^{n} f_l(x)$$

$$L(x) = \prod_{i=1}^{n} \left(\frac{x_i \, \theta^5}{\left(\theta^3 + 2\theta^2 + 24\right)} \left(1 + x_i + x_i^3\right) e^{-\theta x_i} \right) \tag{45}$$

$$L(x) = \frac{\theta^{5n}}{(\theta^3 + 2\theta^2 + 24)^n} \prod_{i=1}^{n} \left(x_i \left(1 + x_i + x_i^3\right) e^{-\theta x_i} \right) \tag{46}$$

The log likelihood function is given by

$$\log L = 5n \log\theta - n\log(\theta^3 + 2\theta^2 + 24) + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log\left(1 + x_i + x_i^3\right) - \theta \sum_{i=1}^{n} x_i \tag{47}$$

By differentiating log likelihood equation (47) with respect to parameter θ and must satisfy the following normal equation

$$\frac{\partial \log L}{\partial\theta} = \frac{5n}{\theta} - n\left(\frac{(3\theta^2 + 4\theta)}{\left(\theta^3 + 2\theta^2 + 24)\right)} \right) - \sum_{i=1}^{n} x_i = 0 \tag{48}$$

The above likelihood equation is too complicated to solve it algebraically. Therefore, we use numerical technique like Newton-Raphson method for estimating required parameter of proposed distribution. To use the asymptotic normality results for determining the confidence interval. We have that if $(\hat{\lambda} = \hat{\theta})$ denotes the MLE $of(\lambda = \theta)$. We can determine the results as

$$\sqrt{n}(\hat{\lambda} - \lambda) \to N(0, I^{-1}(\lambda))$$

Where $I^{-1}(\lambda)$ is Fisher' s informatio n matrix.i.e .,

$$I(\lambda) = -\frac{1}{n}\left(E\left(\frac{\partial^2 \log L}{\partial\theta^2} \right) \right)$$

Here, we can define that

$$E\left(\frac{\partial^2 \log L}{\partial\theta^2} \right) = -\frac{5n}{\theta^2} - n\left(\frac{(\theta^3 + 2\theta^2 + 24)(6\theta + 4) - (3\theta^2 + 4\theta)^2}{(\theta^3 + 2\theta^2 + 24)^2} \right) \tag{49}$$

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidenc interval for θ.

9. Application

In this section, we have applied the two real data sets in length biased Uma distribution to determine its goodness of fit and then comparison has been developed in order to realize that the length biased Uma distribution provides quite satisfactory results over Uma, Shanker, Garima and Lindley distributions. The two real data sets are given below as

The following first real data set reported from Lawless [12] represents the failure times in hours of an accelerated life test of 59 conductors without any censored observation. and the data set is given below in table 1.

Table 1: *Data regarding the failure times in hours of an accelerated life test of 59 conductors by Lawless (2003)*

2.997	4.137	4.288	4.531	4.700	4.706	5.009	5.381	5.434	5.459	5.589	5.640
5.807	5.923	6.033	6.071	6.087	6.129	6.352	6.369	6.476	6.492	6.515	6.522
6.538	6.545	6.573	6.725	6.869	6.923	6.948	6.956	6.958	7.024	7.224	7.365
7.398	7.459	7.489	7.495	7.496	7.543	7.683	7.937	7.945	7.974	8.120	8.336
8.532	8.591	8.687	8.799	9.218	9.254	9.289	9.663	10.092	10.491	11.038	

The following data set obtained from Folks and Chhikara [8] and studied by Gadde et al. [10] represents runoff amounts at Jug Bridge, Maryland and the data set is given as under in table 2

Table 2: *Data regarding the runoff amounts at Jug Bridge, Maryland reported by Gadde et al. (2019)*

0.17	1.19	0.23	0.33	0.39	0.39	0.40	0.45	0.52	0.56
0.59	0.64	0.66	0.70	0.76	0.77	0.78	0.95	0.97	1.02
1.12	1.24	1.59	1.74	2.92					

To compute the model comparison criterions along with estimation of unknown parameters, the technique of R software is applied. In order to compare the performance of length biased Uma distribution over Uma, Shanker, Garima and Lindley distributions, we use criterion values like *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected), *CAIC* (Consistent Akaike Information Criterion), Shannon's entropy *H(X)* and *-2logL*. The distribution is better which shows corresponding criterions *AIC, BIC, AICC, CAIC, H(X)* and *-2logL* values smaller as compared with other distributions. For determining the criterion values *AIC, BIC, AICC, CAIC, H(X)* and *-2logL* given below following formulas are used.

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L, \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$CAIC = -2\log L + \frac{2kn}{n-k-1} \qquad and \qquad H(X) = -\frac{2\log L}{n}$$

Where n is the sample size, k is the number of parameters in statistical model and $-2\log L$ is the maximized value of log-likelihood function under the considered model.

Table 3: *Shows MLE and S. E Estimate for Data set 1 and Data set 2*

Distributions	MLE	S.E
Data set 1		
Length Biased Uma	$\hat{\theta} = 0.69944543$	$\hat{\theta} = 0.04014727$
Uma	$\hat{\theta} = 0.54873604$	$\hat{\theta} = 0.03478584$
Shanker	$\hat{\theta} = 0.27241572$	$\hat{\theta} = 0.02435319$
Garima	$\hat{\theta} = 0.21941841$	$\hat{\theta} = 0.02422882$
Lindley	$\hat{\theta} = 0.25722036$	$\hat{\theta} = 0.02393044$
Data set 2		
Length biased Uma	$\hat{\theta} = 3.5827918$	$\hat{\theta} = 0.3376072$
Uma	$\hat{\theta} = 2.3356047$	$\hat{\theta} = 0.2501545$

Shanker	$\hat{\theta} = 1.5164881$	$\hat{\theta} = 0.2188903$
Garima	$\hat{\theta} = 1.5342883$	$\hat{\theta} = 0.2647162$
Lindley	$\hat{\theta} = 1.6358862$	$\hat{\theta} = 0.2574658$

Table 4: *Shows Comparison and Performance of fitted distributions for Data 1*

Distributions	-2logL	AIC	BIC	AICC	CAIC	H(X)
LBU	261.0041	263.0041	265.0817	263.0742	263.0742	4.4237
Uma	274.1516	276.1516	278.2291	276.2217	276.2217	4.6466
Shanker	308.9524	310.9524	313.03	311.0225	311.0225	5.2364
Garima	335.3371	337.3371	.339.4147	337.4072	337.4072	5.6836
Lindley	316.7054	318.7054	320.783	318.7755	318.7755	5.3678

Table 5: *Shows Comparison and Performance of fitted distributions for Data 2*

Distributions	-2logL	AIC	BIC	AICC	CAIC	H(X)
LBU	32.62635	34.62635	35.84523	34.8002	34.8002	1.3050
Uma	42.32838	44.32838	45.54726	44.5022	44.5022	1.6931
Shanker	40.29075	42.29075	43.50963	42.4646	42.4646	1.6116
Garima	40.39727	42.39727	43.61614	42.5711	42.5711	1.6158
Lindley	39.60251	41.60251	42.82138	41.7764	41.7764	1.5841

From results given above in table 4 and table 5, it has been clearly revealed and observed that the length biased Uma distribution has lesser *AIC*, *BIC*, *AICC*, *CAIC*, *H(X)* and *-2logL* values as compared to the Uma, Shanker, Garima and Lindley distributions. Hence, it can be concluded that the length biased Uma distribution leads to a better fit over Uma, Shanker, Garima and Lindley distributions.

9. Conclusion

This article introduces a novel distribution known as the length-biased Uma distribution, which has been carefully developed and compared to its baseline distribution. Throughout the study, various essential characteristics of the length-biased Uma distribution, including moments, reliability function, hazard rate function, reverse hazard function, shapes of PDF, CDF, hazard, and reliability function, order statistics, Bonferroni, and Lorenz curves, have been thoroughly explored and presented.

Furthermore, the article employs the maximum likelihood estimation method to estimate the parameters of the length-biased Uma distribution, enhancing the robustness and applicability of the proposed model.

To assess its practical effectiveness, the newly introduced distribution has been tested with two real-world datasets. The results demonstrate the superiority of the length-biased Uma distribution over existing distributions such as Uma, Shanker, Garima, and Lindley distributions. This superiority is evidenced by the significantly improved performance and satisfactory outcomes obtained from the proposed length-biased Uma distribution.

In conclusion, the study provides valuable insights into the characteristics and applications of the length-biased Uma distribution, making it a promising addition to the field of statistical modeling and probability theory. The findings open up new avenues for researchers and practitioners to explore the potential of this distribution in various real-world scenarios and data analyses.

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