RT&A, No 4 (76)
Volume 18, December 2023

CHARACTERIZATION OF NEW QUASI LINDLEY DISTRIBUTION BY TRUNCATED MOMENTS AND CONDITIONAL EXPECTATION OF ORDER STATISTICS

Mohd. Amir Ansari
Department of Community Medicine
Maharishi Markandeshwar Medical College and Hospital,
Kumarhatti, Solan Himachal Pradesh-173229-India
amir_stats@live.co.uk
•
Mohammad Faizan
Department of Statistics and Operations Research
Aligarh Muslim University, Aligarh-202002 India
mdfaizan02@gmail.com
•
Rafiqullah Khan
Department of Statistics and Operations Research
Aligarh Muslim University, Aligarh-202002 India
aruke11@gmail.com

Abstract

Characterization of a probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. The aim of this paper is to find characterization results New Quasi Lindley distribution. These results are established using the relation between truncated moments and failure rate functions and conditional expectation of adjacent order statistics. The first characterization result is based on relation between left truncation moment and failure rate function while the second result is based on relation between right truncated moment and reverse failure rate function. In the third characterization result we used conditional expectation of order statistics when the conditioned one is adjacent order statistics. Further, some of its important deductions are also discussed.

Keywords: New quasi Lindley distribution, Characterization, Truncated moments, Failure rate function, Reversed failure rate function, Order statistics.

1. Introduction

Characterization is a condition involving certain properties of a random variable $X=(X_1, X_2, ..., X_n)$, which identifies the associated distribution function $F(x)$. The property that uniquely determines $F(x)$ may be based on a function of random variables whose joint distribution is related to that of $X=(X_1, X_2, ..., X_n)$. The only method of finding distribution function $F(x)$ exactly, which avoids the subjective choice, is a characterization theorem. A theorem is on a characterization of a distribution function if it concludes that a set of conditions is satisfied by $F(x)$ and only by $F(x)$. There has been a vast development in the field of characterizing distributions through different techniques; mainly characterization through distributional properties, moments and conditional expectation.Characterization of a probability distribution plays an important role in probability and statistics. There has been a great interest, in recent years, in the characterizations of probability distributions by truncated moments (see, for example [1], [2], [3], [4], [5], [6], [7], [8], [9], [10] amongst others).

Lindley distribution was introduced by [11]. A random variable X is said to have Lindley distribution with parameter θ if its probability density function (pdf) is of the form

$$f(x)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \; ; \quad x>0,\, \theta>0. \tag{1}$$

Its distribution function (df) is

$$F(x)=1-\frac{\theta+1+\theta x}{1+\theta}e^{-\theta x} \; ; \quad x>0,\, \theta>0. \tag{2}$$

Various properties of this distribution have been discussed by [12] and they showed, in many ways (1) provides a better model for some applications than the exponential distribution.

A two-parameter distribution called quasi Lindley distribution (QLD) has been introduced by [13]. A distribution with parameters α and θ is said to have quasi Lindley distribution if its pdf is of the form

$$f(x;\alpha,\theta)=\frac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x} \; ; \qquad x>0, \theta>0,\, \alpha>-1 \tag{3}$$

and the df is

$$F(x;\alpha,\theta)=1-\frac{1+\alpha+\theta x}{\alpha+1}e^{-\theta x} \; ; \qquad x>0, \theta>0,\, \alpha>-1. \tag{4}$$

It can easily be seen that at $\alpha=\theta$, the QLD reduces to the Lindley distribution and at $\alpha=0$, it reduces to the gamma distribution with parameters $(2,\theta)$.[13] have discussed its various properties and showed that this QLD is a better model than the Lindley distribution for modeling waiting and survival times data.

A new form of quasi Lindley distribution called new quasi Lindley distribution (NQLD) is introduced by [14]. The pdf of NQLD is given by

$$f(x;\theta,\alpha)=\frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha x)e^{-\theta x}; \quad x>0, \theta>0, \alpha<-\theta^2 \tag{5}$$

It can easily be seen that at $\alpha=\theta$, the new QLD (5) reduces to the Lindley distribution (1) and at $\alpha=0$, it reduces to the exponential distribution with parameter θ.
The df of the new QLD is obtained as

$$F(x;\theta,\alpha)=1-\frac{\theta^2+\alpha+\theta\alpha x}{\theta^2+\alpha}e^{-\theta x}; \quad x>0, \theta>0, \alpha<-\theta^2 \,. \tag{6}$$

The failure rate function (frf) of New Quasi Lindley distribution with parameters α and θ is given by

$$r(x)=\frac{f(x)}{1-F(x)}=\frac{\theta^2(\theta+\alpha x)}{\theta^2+\alpha+\alpha\theta x}. \tag{7}$$

The reverse failure rate function $(rfrf)$ of NQLD with parameters α and θ is given by

$$\eta(x)=\frac{f(x)}{F(x)}=\frac{\theta^2(\theta+\alpha x)}{(\theta^2+\alpha)e^{\theta x}-(\theta^2+\alpha+\alpha\theta x)}. \tag{8}$$

The k^{th} moment (about the origin) of the NQLD with parameters α and θ is given by

$$E[X^k] = \frac{k!(\alpha k + \theta^2 + \alpha)}{(\theta^2 + \alpha)\theta^k}. \tag{9}$$

Several properties of NQLD have been given by [14] and they showed that NQLD is more flexible than Lindley distribution, exponential distribution, and QLD. They also showed that NQLD is closer fit than exponential distribution, Lindley distribution and QLD in goodness of fit.

Several characterization results of Lindley distribution has been given by [6]. They characterized Lindley distribution through left and right truncated moments. Conditional expectation of order statistics is used to characterize Lindley distribution by [7]. In this paper, we have obtained characterization results for quasi Lindley distribution.

2. Characterizations through Truncated Moments:

First, we give the following two lemmas which are used to prove Theorem 1 and Theorem 2 respectively.

Lemma 1: Suppose that the random variable X has an absolutely continuous df $F(x)$ with $F(0)=0, F(x)>0$ for all x, pdf $f(x)=F'(x)$, frf $r(x)=\frac{f(x)}{[1-F(x)]}$. Let $g(x)$ be a continuous function in $x>0$ and $0<E[g(X)]<\infty$. If

$$E[g(x) \mid X > x] = h(x) r(x) \qquad x > 0$$

where h(x) is a differentiable function in $x>0$, then

$$f(x) = K \exp\left[-\int_0^x \frac{g(y)+h'(y)}{h(y)} dy\right], \qquad x > 0$$

where $K>0$ is a normalizing constant [6].

Lemma 2: Suppose that the random variable X has an absolutely continuous cdf $F(x)$ with $F(0)=0, F(x)>0$ for all x, pdf $f(x)=F'(x)$, $rfrf$ $\eta(x)=\frac{f(x)}{F(x)}$. Let $g(x)$ be a continuous function in $x>0$ and $0<E[g(X)]<\infty$. If

$$E[g(x) \mid X \le x] = w(x) r(x) \qquad x > 0$$

where w(x) is a differentiable function in $x>0$, then

$$f(x) = K \exp\left[-\int_0^x \frac{w'(y)-g(y)}{w(y)} dy\right], \qquad x > 0$$

where $K>0$ is a normalizing constant [6].

Theorem 1: Suppose that the random variable X has absolutely continuous distribution with the pdf $f(x)$ and df $F(x)$ with $F(0)=0$, $F(x)>0$ for all $x>0$ and frf $r(x)=\frac{f(x)}{1-F(x)}$. Assume that $0<E[X^k]<\infty$ for a given positive integer k. Then X has NQLD with parameters α and θ if and only if

$$[E(X^k \mid X \ge x)] = \frac{r(x)}{(\theta + \alpha x)} \sum_{j=0}^{k+1} \mu_j x^j \tag{10}$$

where $\mu_0 = \dfrac{k![\theta^2 + \alpha(k+1)]}{\theta^{k+2}}$, $\mu_j = \dfrac{k![\theta^2 + \alpha(k+1)]}{j!\theta^{k-j+2}}$, $\mu_{j+1} = \dfrac{\theta}{j+1}\mu_j \quad j = 0,1,2,\ldots,k-1$ and

$\mu_{k+1} = \dfrac{\alpha}{\theta}$.

Proof: for necessary condition, we have

$$[E(X^k \mid X \geq x)] = \frac{1}{1-F(x)}\int_x^\infty y^k f(y)dy. \tag{11}$$

From relation among pdf, df and frf, we have

$$[E(X^k \mid X \geq x)] = \frac{r(x)}{f(x)}\int_x^\infty y^k f(y)dy. \tag{12}$$

From (5), we have

$$[E(X^k \mid X \geq x)] = \frac{r(x)}{(\theta+\alpha x)e^{-\theta x}}\left[\int_x^\infty (\theta+\alpha y)y^k e^{-\theta y}dy\right]$$

$$= \frac{r(x)}{(\theta+\alpha x)e^{-\theta x}}\left[\theta\int_x^\infty y^k e^{-\theta y}dy + \alpha\int_x^\infty y^{k+1}e^{-\theta y}dy\right]. \tag{13}$$

The following integration result given by [15] (pg-340) is used to solve the above integration

$$\int_u^\infty x^n e^{-\mu x}dy = e^{-\mu u}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{\mu^{n-k+1}}. \tag{14}$$

Using (14), we can write (13) as

$$[E(X^k \mid X \geq x)] = \frac{r(x)}{(\theta+\alpha x)e^{-\theta x}}\left[\theta e^{-\theta x}\sum_{j=0}^{k}\frac{k!}{j!}\frac{x^j}{\theta^{k-j+1}} + \alpha e^{-\theta x}\sum_{j=0}^{k+1}\frac{(k+1)!}{j!}\frac{x^j}{\theta^{k+1-j+1}}\right]$$

$$= \frac{r(x)}{(\theta+\alpha x)}\left[\sum_{j=0}^{k}\frac{k![\theta^2+\alpha(k+1)]}{j!}\frac{x^k}{\theta^{k-j+2}} + x^{k+1}\right]^j$$

$$= \frac{r(x)}{(\theta+\alpha x)}\sum_{j=0}^{k+1}\mu_j\, x$$

where $\mu_0 = \dfrac{k![\theta^2 + \alpha(k+1)]}{\theta^{k+2}}$, $\mu_j = \dfrac{k![\theta^2 + \alpha(k+1)]}{j!\theta^{k-j+2}}$, $\mu_{j+1} = \dfrac{\theta}{j+1}\mu_j \quad j = 0,1,2,\ldots,k-1$ and

$\mu_{k+1} = \dfrac{\alpha}{\theta}$.

and hence the necessary part

For sufficiency part let $g(x) = x^k$ and $h(x) = \dfrac{\sum_{j=0}^{k+1}\mu_j x^j}{(\theta+\alpha x)}$. (15)

Using the recurrence relations of the μ_k 's, we have

$$\theta\sum_{j=0}^{k+1}\mu_j x^j - \sum_{j=1}^{k+1} j\mu_j x^j = \sum_{j=0}^{k-1}[\theta\mu_j - (j+1)\mu_{j+1}]x^j + [\theta\mu_k - (k+1)\mu_{k+1}]x^k + \theta\mu_{k+1}x^{k+1}$$

$$= (\theta+\alpha x)x^k. \tag{16}$$

From (15) and (16), we have

$$\frac{g(x)}{h(x)}=\frac{(\theta+\alpha x)x^j}{\sum_{j=0}^{k+1}\mu_j x^j}=\frac{\theta\sum_{j=0}^{k+1}\mu_j x^j-\sum_{j=1}^{k+1}j\mu_j x^j}{\sum_{j=0}^{k+1}\mu_j x^j}==\theta-\frac{\sum_{j=1}^{k+1}j\mu_j x^j}{\sum_{j=0}^{k+1}\mu_j x^j} \tag{17}$$

Also from (15)

$$h(x)=\frac{\sum_{j=0}^{k+1}\mu_j x^j}{(\theta+\alpha x)}$$

Taking logarithm both sides of above equation, we have

$$\ln[h(x)]=-\ln[\theta+\alpha x]+\ln[\sum_{j=0}^{k+1}\mu_j x^j]. \tag{18}$$

Differentiating (18) with respect to x, we have

$$\frac{h'(x)}{h(x)}=\frac{\sum_{j=1}^{k+1}j\mu_j x^j}{\sum_{j=0}^{k+1}\mu_j x^j}-\frac{\alpha}{(\theta+\alpha x)}. \tag{19}$$

From (17) and (19), we have

$$\frac{h'(x)+g(x)}{h(x)}=-\frac{\alpha}{(\theta+\alpha x)}+\theta. \tag{20}$$

Integrating (20) over $(0,x)$, we have

$$\int_0^x\frac{h'(y)+g(y)}{h(y)}dy=\int_0^x\left(-\frac{\alpha}{(\theta+\alpha y)}+\theta\right)dy=-\ln[\theta+\alpha x]+\theta x \tag{21}$$

Using Lemma 1 given by [6] in (21), we have

$$f(x)=C\exp\left[-\int_0^x\frac{h'(y)+g(y)}{h(y)}dy\right]=C\exp[-\ln[\theta+\alpha x]+\theta x]=C(\theta+\alpha x)e^{-\theta x}$$

where C is normalizing constant i.e. $\int_0^\infty f(x)=1$ which gives $C=\frac{\theta^2}{\theta^2+\alpha}$.

Therefore,

$$f(x)=\frac{\theta^2}{\theta^2+\alpha}(\alpha+\theta x)e^{-\theta x}$$

which is the pdf of NQLD and hence sufficiency part.

Remark 1: Putting $\alpha=\theta$ in Theorem 1 we get the characterizing result for Lindley distribution obtained by [6].

Remark 2: Putting $\alpha=0$ in Theorem 1 we get the characterizing result for exponential (θ) distribution.

Theorem 2: Suppose that the random variable X has absolutely continuous distribution with the pdf $f(x)$ and df $F(x)$ with $F(0)=0$, $F(x)>0$ for all $x>0$ and the $rfrf$ $\eta(x)=\frac{f(x)}{F(x)}$. Assume that $0<E[X^k]<\infty$ for a given positive integer k. Then X has NQLD if and only if

$$[E(X^k \mid X \le x)] = \frac{\eta(x)}{(\theta+\alpha x)e^{-\theta x}}\left(\mu_0 e^{\theta x} - \sum_{j=0}^{k+1}\mu_j x^j\right) \tag{22}$$

where $\mu_0 = \dfrac{k![\theta^2+\alpha(k+1)]}{\theta^{k+2}}$, $\mu_j = \dfrac{k![\theta^2+\alpha(k+1)]}{j!\theta^{k-j+2}}$, $\mu_{j+1} = \dfrac{\theta}{j+1}\mu_j$ $\quad j = 0,1,2,\ldots,k-1$ and

$\mu_{k+1} = \dfrac{\alpha}{\theta}$.

Proof: for necessary part let X has NQLD with parameter α and θ.
Then we have

$$[E(X^k \mid X \le x)] = \frac{1}{F(x)}\int_x^\infty y^k f(y)dy$$

from relation among pdf df , and $rfrf$, we have

$$[E(X^k \mid X \le x)] = \frac{\eta(x)}{f(x)}\int_x^\infty y^k f(y)dy. \tag{23}$$

From (5), we have

$$[E(X^k \mid X \le x)] = \frac{\eta(x)}{(\theta+\alpha x)e^{-\theta x}}\left[\int_0^x (\theta+\alpha y)y^k e^{-\theta y}dy\right]$$

or

$$[E(X^k \mid X \le x)] = \frac{\eta(x)}{(\theta+\alpha x)e^{-\theta x}}\left[\theta\int_0^x y^k e^{-\theta y}dy + \alpha\int_0^x y^{k+1}e^{-\theta y}dy\right]. \tag{24}$$

The following integration result given by [15] (pg-340) is used to solve the above integration

$$\int_0^u x^n e^{-\mu x}dy = \frac{n!}{\mu^{n+1}} - e^{-\mu u}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{\mu^{n-k+1}}. \tag{25}$$

From (24) and (25), we have

$$\begin{aligned}[E(X^k \mid X \le x)] &= \frac{\eta(x)}{(\theta+\alpha x)e^{-\theta x}} \\ &\quad\times\left(\frac{k![\theta^2+\alpha(k+1)]}{\theta^{k+2}}e^{\theta x} - \sum_{j=0}^{k}\frac{k![\theta^2+\alpha(k+1)]}{j!}\frac{x^j}{\theta^{k-j+2}} - \frac{\alpha}{\theta}x^{k+1}\right) \\ &= \frac{\eta(x)}{(\theta+\alpha x)e^{-\theta x}}\left(\mu_0 e^{\theta x} - \sum_{j=0}^{k+1}\mu_j x^j\right)\end{aligned}$$

where $\mu_0 = \dfrac{k![\theta^2+\alpha(k+1)]}{\theta^{k+2}}$, $\mu_j = \dfrac{k![\theta^2+\alpha(k+1)]}{j!\theta^{k-j+2}}$, $\mu_{j+1} = \dfrac{\theta}{j+1}\mu_j$ $\quad j = 0,1,2,\ldots,k-1$ and

$\mu_{k+1} = \dfrac{\alpha}{\theta}$.

and hence the necessary part.

For sufficient part Let $g(x) = x^k$ and $w(x) = \dfrac{\mu_0 e^{\theta x} - \sum_{j=0}^{k+1}\mu_j x^j}{(\theta+\alpha x)}$. (26)

From equations (16) and (26), we have

$$\frac{g(x)}{w(x)}=\frac{(\theta+\alpha\,x)\,x^k}{\mu_0 e^{\theta x}-\sum_{j=0}^{k+1}\mu_j\,x^j}=\frac{\theta\sum_{j=0}^{k+1}\mu_j x^j-\sum_{j=1}^{k+1}j\mu_j x^j}{\mu_0 e^{\theta x}-\sum_{j=0}^{k+1}\mu_j\,x^j}=\frac{\theta\mu_0 e^{\theta x}-\sum_{j=1}^{k+1}j\mu_j}{\mu_0 e^{\theta x}-\sum_{j=0}^{k+1}\mu_j\,x^j}-\theta\;. \tag{27}$$

Also from equation (26)

$$\ln[w(x)]=-\ln[(\theta+\alpha\,x)]+\ln\left[\mu_0 e^{\theta x}-\sum_{j=0}^{k+1}\mu_j\,x^j\right]. \tag{28}$$

Differentiating (28) with respect to x, we have

$$\frac{w'(x)}{w(x)}=-\frac{\alpha}{(\theta+\alpha\,x)}+\frac{\theta\mu_0 e^{\theta x}-\sum_{j=1}^{k+1}j\mu_j}{\mu_0 e^{\theta x}-\sum_{j=0}^{k+1}\mu_j\,x^j}.$$

Using (27) in above equation, we have

$$\frac{w'(x)}{w(x)}=-\frac{\alpha}{(\theta+\alpha x)}+\frac{g(x)}{w(x)}+\theta$$

which can be written as

$$\frac{w'(x)-g(x)}{w(x)}=-\frac{\alpha}{(\theta+\alpha\,x)}+\theta\,. \tag{29}$$

Integrating (29) over $(0,x)$, we have

$$\int_0^x\frac{w'(y)-g(y)}{w(y)}dy=\int_0^x\left(-\frac{\alpha}{(\theta+\alpha\,y)}+\theta\right)dy=-\ln[(\theta+\alpha\,x)]+\theta\,x \tag{30}$$

Using Lemma 2 given by [6], we have

$$f(x)=C\exp\left[-\int_0^x\frac{w'(y)-g(y)}{w(y)}dy\right]=C\exp\left[\ln[(\theta+\alpha\,x)]-\theta x\right]=C\,(\theta+\alpha\,x)e^{-\theta x}.$$

where C is normalizing constant i.e. $\int_0^\infty f(x)dx=1$ which gives $C=\dfrac{\theta^2}{\theta^2+\alpha}$.

Therefore,

$$f(x)=\frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha\,x)e^{-\theta\,x}$$

which is the pdf of NQLD and hence the theorem.

Remark 3: Putting $\alpha=\theta$ in Theorem 2 we get the characterizing result for Lindley distribution obtained by [6].

Remark 4: Putting $\alpha=0$ in Theorem 2 we get the characterizing result for exponential (θ) distribution.

Characterization through conditional expectation of order statistics

Theorem 3: Let X be a continuous random variable with $df\ F(x)$ and $pdf\ f(x)$. Then for $r<n$

$$E[X_{r+1:n}\mid X_{r:n}=x]=x+\frac{\Gamma[n-r+1,(n-r)(\theta^2+\alpha+\theta\alpha\,x)]}{\theta\alpha(n-r)^{n-r+1}(\theta^2+\alpha+\theta\,\alpha x)^{n-r}\,e^{-(n-r)(\theta^2+\alpha+\theta\alpha\,x)}}, \tag{31}$$

$$r=1,2,\ldots,n-1$$

if and only if X has the df given in (6).

Proof: First we will prove (6) implies (31).

We have

$$E[X_{r+1:n} \mid X_{r:n} = x] = \frac{(n-r)}{[1-F(x)]^{n-r}} \int_x^{\beta} y[1-F(y)]^{n-r-1} f(y)\,dy .$$

From (5) and (6), we have

$$E[X_{r+1:n} \mid X_{r:n} = x] = \frac{(n-r)\theta^2}{\left[(\theta^2+\alpha+\theta\alpha x)e^{-\theta x}\right]^{n-r}} \times \int_x^{\infty} y[(\theta^2+\alpha+\theta\alpha y)e^{-\theta y}]^{n-r-1}(\theta+\alpha y)e^{-\theta y}\,dy. \tag{32}$$

Integrating (32) by parts and then re arranging, we have

$$E[X_{r+1:n} \mid X_{r:n} = x] = x + \frac{\int_x^{\infty} (\theta^2+\alpha+\theta\alpha y)^{n-r} e^{-\theta(n-r)y}\,dy}{\left[(\theta^2+\alpha+\theta\alpha x)e^{-\theta x}\right]^{n-r}}$$

which reduces to

$$E[X_{r+1:n} \mid X_{r:n} = x] = x + \frac{\Gamma[n-r+1,(n-r)(\theta^2+\alpha+\theta\alpha x)]}{\theta\alpha(n-r)^{n-r+1}(\theta^2+\alpha+\theta\alpha x)^{n-r} e^{-\frac{(n-r)}{\alpha}(\theta^2+\alpha+\theta\alpha x)}}$$

and hence necessary part.

For sufficiency part, Let $E[X_{r+1:n} \mid X_{r:n} = x] = \phi(x) = x + \omega(x)$ (33)

where, $\omega(x) = \dfrac{\Gamma[n-r+1,(n-r)(\theta^2+\alpha+\theta\alpha x)]}{\theta\alpha(n-r)^{n-r+1}(\theta^2+\alpha+\theta\alpha x)^{n-r} e^{-\frac{(n-r)}{\alpha}(\theta^2+\alpha+\theta\alpha x)}}$.

From (33), we have

$$\phi(x) = x + \omega(x).$$

Differentiating above equation with respect to x, we have

$$\phi'(x) = 1 + \omega'(x)..$$

Then, it is seen that

$$\int \frac{\phi'(t)}{\phi(t)-t}dt = \int \frac{1+\omega'(t)}{\omega(t)}dt = \int \frac{dt}{\omega(t)} + \int \frac{\omega'(t)}{\omega(t)}dt .$$

Using (33) in above equation, we have

$$\int \frac{\phi'(t)}{\phi(t)-t}dt = -\ln\Gamma[n-r+1,(n-r)(\theta^2+\alpha+\theta\alpha t)] + \ln\left(\frac{\Gamma[n-r+1,(n-r)(\theta^2+\alpha+\theta\alpha t)]}{\theta\alpha(n-r)^{n-r+1}(\theta^2+\alpha+\theta\alpha t)^{n-r} e^{-\frac{(n-r)}{\alpha}(\theta^2+\alpha+\theta\alpha t)}}\right)$$

$$= -\ln\left[\theta\alpha(n-r)^{n-r+1}(\theta^2+\alpha+\theta\alpha t)^{n-r} e^{-\frac{(n-r)}{\alpha}(\theta^2+\alpha+\theta\alpha t)}\right]. \tag{34}$$

Taking the limits of integral in (34) from 0 to x, we have

$$\int_0^x \frac{\phi'(t)}{\phi(t)-t}dt = \ln\left[\left\{\frac{\theta^2+\alpha}{(\theta^2+\alpha+\theta\alpha x)e^{-\theta x}}\right\}^{n-r}\right]. \tag{35}$$

Using the result given by [16], we have

$$[1-F(x)]^{n-r} = \exp\left[-\ln\left[\left\{\frac{\theta^2+\alpha}{(\theta^2+\alpha+\theta\alpha x)e^{-\theta x}}\right\}^{n-r}\right]\right] = \left(\frac{\theta^2+\alpha+\theta\alpha x}{\theta^2+\alpha}e^{-\theta x}\right)^{n-r}$$

which reduces to

$$F(x) = 1-\frac{\theta^2+\alpha+\theta\alpha x}{\theta^2+\alpha}e^{-\theta x}$$

and hence the theorem.

Remark 5: Putting $\alpha=\theta$ in Theorem 3, we get the characterization result for Lindley distribution as obtained by [7].

Remark 6: Putting $\alpha=0$ in Theorem 3, we get the characterization result for exponential (θ) distribution.

3. Applications

A probability distribution can be characterized in many ways and the method under study here is one of them. We have used here the relation between truncated moments and failure rate functions as well as conditional expectation of order statistics conditioned on an adjacent order statistics to characterize the new quasi Lindley distribution. That is, we have characterized the new quasi Lindley distribution if the regression equation truncated from both sides is given, i. e. the data are truncated from left side at x and truncated from right side at y. In real practice, several times we get the data of which observations are missing either in beginning or in the end. In such type of data we can use the result of this paper.

References

[1] Hamedani, G.G. (2011). Characterizations of the Shakil-Kibria-Singh distribution. *Austrian Journal of Statistics,* 40 (3), 201–207.

[2] Hamedani, G.G. and Ghosh, I. (2015). Kumaraswamy-Half-Cauchy distribution: characterizations and related results. *International Journal of Statistics and Probability,* 4 (3): 94-100.

[3] Ahsanullah, M. and Shakil, M. (2015). Characterizations of continuous probability distributions occurring in physics and allied sciences by truncated moment. *International Journal of Advanced Statistics and Probability,* 3 (1): 100-114.

[4] Ahsanullah, M., Shakil, M. and Golam Kibria, B.M. (2016). Characterizations of continuous distributions by truncated moment. *Journal of Modern Applied Statistical Methods,* 15 (1): 316-331.

[5] Shakil, M., Golam Kibria, B.M. and Singh, J. N. (2016). Characterization of a new class of generalized Pearson distribution by truncated moment. *International Journal of Computational and Theoretical Statistics*, 3 (2): 91-100.

[6] Ahsanullah, M., Ghitany, M.E. and Al-Mutairi, D.K. (2017). Characterization of Lindley distribution by truncated moments. *Communications in Statistics - Theory and Methods*, 46 (12): 6222-6227.

[7] Kilany, N.M. (2017). Characterization of Lindley distribution based on truncated moments of order statistics. *Journal of Statistics Applications and Probability,* 6(2): 1-6.

[8] Kilany, N.M. and Hassanein, W.A. (2018). Characterization of Benktander type II distribution via truncated moments of order statistics. *International Journal of Probabilityand Statistics*, 7(4): 106–113.

[9] Athar, H. and Abdel-Aty, Y. (2020). Characterization of general class of distributions by truncated moment. *Thailand Statistician*, 18(2): 95–107.

[10] Athar, H., Abdel-Aty, Y. and Ali, M.A. (2021). Characterization of generalized distribution by doubly truncated moment. *Statistica*, LXXXI(1): 5-44.

[11] Lindley D.V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal*

Statistical Society Series B, 20(1): 102-107.

[12] Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008). Lindley distribution and its applications. *Mathematics and Computers in Simulation*, 78 (4): 493 – 506.

[13] Shanker, R. and Mishra, A. (2013). A quasi Lindley distribution. *African Journal of Mathematics and Computer Science Research*, 6 (4): 64-71.

[14] Shanker, R. and Ghebretsadik, A.H. (2013). A new quasi Lindley distribution. *International Journal of Statistics and Systems*, 8(2): 43-156.

[15] Gradshteyn, I.S. and Ryzhik, I.M., Table of Integrals, Series, and Products. Academic Press, New York, 2007.

[16] Khan, A.H., and Abu-Salih, M.S. (1989). Characterizations of probability distributions by conditional expectation of order statistics. *Metron*, 47: 171–181.