ANALYSIS OF AN $M/M/1/K$ FEEDBACK WORKING VACATION QUEUE WITH RENEGING

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Abstract

The analysis of an $M/M/1/N$ feedback working vacation queueing system with reneging is presented in this paper. Customers may become impatient and even disappointed when they see a long line. In the literature on queueing, customer dissatisfaction caused on by unsatisfactory service is referred as feedback. In the case of feedback, customers retry services after receiving unsatisfactory or incomplete. First, we create the equations for the steady-state probabilities using the Markov process method. The steady-state probabilities are then solved by the matrix method. We then provide some system performance measures. We create a cost model using performance analysis. Finally, we give some numerical examples to show how the various model parameters affect the system's behaviour.

Keywords: reneging, performance measure, steady-state probabilities, cost model

1. Introduction

Because of the wide range of applications in practical situations such as inventory systems, hospital emergency rooms dealing with critical patients, computer and communication systems, impatient telephone switchboard customers, and manufacturing and production systems, many researchers have been drawn to performance modelling of Markovian queues with different customer behaviour. In queueing literature, feedback is defined as consumers' dissatisfaction due to poor service quality.

Takacs [1] studied feedback queue with a single server and determined the queue size distribution. Dhosi [2] give an overview of some general decomposition results and the methodology used to obtain these results for two vacation models. The analysis of a $M/M/1$ queue with multiple and single working vacations was presented by Vijaya Laxmi et al. [3]. Wortman et al., [4] investigated $M/GI/1$ Bernoulli feedback queue, where the server goes on vacation according to Bernoulli schedules. Zhang et al. [5] presented an analysis for an $M/M/1/N$ queueing system with balking, reneging and server vacations.

Kumar et al. [6] studied optimization of an $M/M/1/N$ feedback queue with retention of reneged customer. Rakesh and sumeet [7] researched a multi-server Markovian feedback queueing model with finite capacity that includes balking, reneging, and keeping reneged customers. Jewkes and Buzacott, [8] examined queueing system of a K class $M/G/1$ with feedback. Vijaya Laxmi and Jyothsna [9] worked on the effect of balking and reneging in a single server queue under variant working vacation policy.Bouchentouf et al. [10] analyzed and dealt with variant multiple working vacations and impatience timers of Bernoulli feedback queueing system that is dependent on the server states.

Sundar et al. [11] talks about a single server queuing system where units come in batches of different sizes under a Poisson stream. In k stations, the server offers services. Each station's service times are distributed randomly. The system uses the feedback, vacation, and reneging principles. Van den Berg and Boxma, [12] studied feedback mechanism of an $M/M/1$ queue with a general probability. Bouchentouf et al. [13] studied the examination of a Bernoulli feedback, single vacation, waiting server, and impatient customer Markovian queueing system.

With Bernoulli feedback, synchronous multiple and single working vacations, balking and reneging during busy and working vacation periods, Yahiaoui et al. [14] investigated the cost optimization analysis of a discrete-time, finite capacity, multiserver queueing system. Choi et al., [15, 16] considered $M/M/c$ retrial queues with feedback, geometric loss, and multi-class customers and the Bernoulli feedback policy of an $M/G/1$ queue. A single server $M/M/1/N$ feedback queueing system with vacation, balking, reneging, and retention of reneged customers is the subject of a study by Bouchentouf et al. [17]. Kumar et al., [18, 19, 20] studied retrial queue with Bernoulli feedback, control retrial rate with balking and multi-server, as well as a generalized $M/G/1$ feedback queue in which customers are either "positive" or "negative".

Bouchentouf and Guendouzi [21] deal with the study of an $M^X/M/c$ Bernoulli feedback queueing system with two different policies of synchronous vacations and waiting servers. Ke and Chang [22] studied balking and Bernoulli feedback with a general retrial queue, where a modified vacation policy is operated by the server. Markovian queueing system with working vacation, Bernoulli schedule interruption, setup time with feedback, reneging of impatient customers, and retention of reneged customers are all discussed by Gupta [23]. Jeeva and Kumari [24] gave a mathematical method to generate the membership function with non-linear programming of the $M/G/1$ system, feedback, and bulk arrival queues with server vacation facility, in which departure probability, service time, arrival rate, vacation time, and batch size was all considered as fuzzy numbers. Jain [25] presented various schemes under a probability-based model for queue scheduling operating system. M/G/1 feedback queue is also studied by [26, 27, 28].

In this paper, a single server with a finite capacity queueing system is analyzed. Performance analysis of the Markovian feedback working vacation queue with and reneging is also discussed. Customer dissatisfaction as a result of poor service quality is referred to as feedback in queueing literature. Customers retries service after receiving partial or incomplete service in the case of feedback. The model assumptions are described in section 2. The steady-state equations and their matrix solution is calculated in section 3 and 4 respectively. The system's performance measures are discussed in section 5. Numerical results and cost analysis of the system are described in sections 6 and 7 and the further paper concludes in the next section.

2. Model Assumptions

We consider a $M/M/1/K$ queueing model to analyze its performance and effectiveness of feedback working vacation queue with balking and reneging. The basic assumptions underlying the model are as:

1. Customer's arrival follows a Poisson distribution with arrival rate λ.
2. When the server is unoccupied, then customer may join the system with probability q or leave with complementary probability $p = (1 - q)$. probability of not joining. If a customer receives service and is dissatisfied, they may leave the system or return with a probability of $p_1 = 1 - q_1$ depending on whether they were a feedback customer or not.
3. Every customer has to wait for a certain time interval T, before being served. If the service hasn't begun by then, the customer may become irritated and leave the line, the density function is given below with the random variable T is

$$d(t) = \alpha e^{-\alpha t}, \ \ t \geq 0, \alpha > 0.$$

with mean as $\frac{1}{\alpha}$.

4. Since the departure and arrival rate of the impatient customer without service is independent, therefore the average reneging rate $r(k)$ is defined by the function as

$$r(k) = (k-i)\alpha, \;\; i \le k \le K, i = 0,1$$

$$r(k) = 0, \;\; k > K$$

5. The server goes on vacation when the system is empty. After a vacation has ended, the server resumes regular service if there are still patrons in line; otherwise, he leaves for another vacation. The server remains operational and offers service to the arriving customers at a different service rate while on vacation. The assumption is that the vacation and service periods will follow a Poisson distribution with parameters ϕ and η, respectively. While on regular duty and while on working vacation, inter-arrival times, vacation times, and service times are all separate from one another.

$$v(t) = \eta e^{-\eta t}, \;\; t \ge 0, \eta > 0.$$

where η is the vacation rate of a server.

3. Steady-state Equation

In this section, we use the Markov process method to derive the steady-state probabilities. Let $\vartheta_{0,k}$ represent the probability that there are k customer in the system when the server is unavailable, When the server is online, $\vartheta_{1,k}$ represents the probability that k customers are present in the system. By using the Markov process theory, we are able to generate the following collection of steady-state equations.

$$0 = -\lambda\vartheta_{0,0} + (\eta p_1 + \alpha)\vartheta_{0,1} + \mu p_1 \vartheta_{1,1} \tag{1}$$
$$0 = \lambda\vartheta_{0,k-1} - (\eta p_1 + k\alpha + \lambda + \phi)\vartheta_{0,k} + (\eta p_1 + (k+1)\alpha)\vartheta_{0,k+1} \tag{2}$$
$$0 = \lambda\vartheta_{0,K-1} - (\eta p_1 + K\alpha + \phi)\vartheta_{0,K} \tag{3}$$
$$0 = -(\lambda + \eta p_1)\vartheta_{1,1} + (\mu p_1 + \alpha)\vartheta_{1,2} + \phi\vartheta_{0,1} \tag{4}$$
$$0 = \lambda\vartheta_{1,k-1} - (\lambda + \mu p_1 + (k-1)\alpha)\vartheta_{1,k} + (\mu p_1 + k\alpha)\vartheta_{1,k+1} + \phi\vartheta_{0,k} \tag{5}$$
$$0 = \lambda\vartheta_{1,K-1} - (\mu p_1 + (K-1)\alpha)\vartheta_{1,K} + \phi\vartheta_{0,K} \tag{6}$$

4. Solution

The steady-state probabilities $\vartheta_{j,k}, j = 0,1, j \le k \le K$, are acquired when a set of equations has been solved (1)-(6) using matrices. Let $\mathbf{Y} = (\mathbf{Y_0}, \mathbf{Y_1})$ be the steady-state probability vector, where $\mathbf{Y_0} = (\vartheta_{0,0}, \vartheta_{0,1}, \cdots, \vartheta_{0,K})$ and $\mathbf{Y_1} = (\vartheta_{1,1}, \vartheta_{1,2}, \cdots, \vartheta_{1,K})$. The equation (1)-(6) the matrix form of which reads as

$$\mathbf{Y\Theta} = \mathbf{0} \tag{7}$$
$$\mathbf{YE} = \mathbf{1} \tag{8}$$

where $\mathbf{\Theta}$ is the block-formed transition rate matrix for the Markov process and $\mathbf{E}$ is a column vector with each component equal to one

$$\mathbf{\Theta} = \begin{pmatrix} \mathbf{B_1} & \mathbf{B_2} \\ \mathbf{B_3} & \mathbf{B_4} \end{pmatrix}$$

The elements of the matrices $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$ and $\mathbf{B_4}$ are given by

$$\mathbf{B_1} = \begin{cases} -\lambda, & \text{if } k = j = 1, \\ \lambda, & \text{if } k = j-1, j \geq 2, \\ \eta p_1 + j\alpha, & \text{if } k = j+1, \\ u_j, & \text{if } k = j, j \geq 2, \\ 0, & \text{or else} \end{cases}$$

$$\mathbf{B_2} = \begin{cases} \phi, & \text{if } k = j+1, \\ 0, & \text{or else} \end{cases}$$

$$\mathbf{B_3} = \begin{cases} \mu p_1, & \text{if } k = j = 1, \\ 0, & \text{or else} \end{cases}$$

$$\mathbf{B_4} = \begin{cases} v_j, & \text{if } k = j, \\ \lambda, & \text{if } k = j-1, j \geq 2, \\ \mu p_1 + (j-1)\alpha, & \text{if } k = j+1, \\ 0, & \text{or else} \end{cases}$$

where for $1 \leq j \leq K, u_j = -(\eta p_1 + (j-1)\alpha + \lambda); v_j = -(\lambda + \mu p_1 + (j-1)\alpha)$. $\mathbf{B_1}$ is a square matrix of order $K+1, \mathbf{B_2}$ is a $(K+1) \times K, \mathbf{B_3}$ is a $K \times (K+1), \mathbf{B_4}$ and is a square matrix of order K.
Based on the partition, $\mathbf{Y} = (\mathbf{Y_0}, \mathbf{Y_1})$, equation (7) and (8) can be written as:

$$\mathbf{Y_0 B_1} + \mathbf{Y_1 B_3} = \mathbf{0} \tag{9}$$

$$\mathbf{Y_0 B_2} + \mathbf{Y_1 B_4} = \mathbf{0} \tag{10}$$

$$\mathbf{Y_0 E_0} + \mathbf{Y_1 E_1} = \mathbf{1} \tag{11}$$

where $\mathbf{E_0}, \mathbf{E_1}$ are, respectively, column vectors of orders $K+1$ and K, each with one element.
From equation (9), we have

$$\mathbf{Y_0} = -\mathbf{Y_1 B_3 B_1^{-1}} \tag{12}$$

Using equation (12) in (10) and (11), we get

$$\mathbf{Y_1}(\mathbf{I} - \mathbf{B_3 B_1^{-1} B_2 B_4^{-1}}) = \mathbf{0} \tag{13}$$

$$\mathbf{Y_1}(\mathbf{E_1} - \mathbf{B_3 B_1^{-1} E_0}) = \mathbf{1} \tag{14}$$

The matrices $\mathbf{B_2}$ and $\mathbf{B_3}$ can be written as

$$\mathbf{B_3} = \begin{pmatrix} v_1 & \mathbf{O_1} \\ \mathbf{O_2} & \mathbf{O_3} \end{pmatrix}_{K \times (K+1)}, \mathbf{B_2} = \phi \begin{pmatrix} \mathbf{O_1} \\ \mathbf{I}_{K \times K} \end{pmatrix}_{(K+1) \times K}$$

where $\mathbf{O_1}, \mathbf{O_2}$ and $\mathbf{O_3}$ are zero matrices of order $1 \times K, (K-1) \times 1$ and $(K-1) \times K$, respectively. Let $\mathbf{B_1}^{-1} = [b_{k,j}]_{(K+1)\times(K+1)}$ and $\mathbf{w} = (b_{1,1}, b_{1,2}, \cdots, b_{1,K+1})$ denotes the 1st row of $\mathbf{B_1}^{-1}$.

$$\mathbf{B_3 B_1^{-1}} = \begin{pmatrix} v_1 \mathbf{w} \\ \mathbf{O_4} \end{pmatrix}_{K \times (K+1)} \tag{15}$$

where $\mathbf{O_4}$ is a zero matrix of order $(K-1) \times (K+1)$
Kow,

$$\mathbf{B_2 B_4^{-1}} = \phi \begin{pmatrix} \mathbf{O_1} \\ \mathbf{B_4^{-1}} \end{pmatrix}_{(K+1) \times K} \tag{16}$$

From equation (15) and (16), we have

$$\mathbf{B_3 B_1^{-1} B_2 B_4^{-1}} = v_1 \phi \begin{pmatrix} \mathbf{w_0 B_4^{-1}} \\ \mathbf{O_3} \end{pmatrix}_{(K+1) \times K} \tag{17}$$

where $\mathbf{w_0} = (b_{1,2}, b_{1,3}, \cdots, b_{1,K+1})$
Let us partition $\mathbf{Y_1}$ as $[\vartheta_{1,1}, \tilde{\mathbf{Y_1}}]$ where $\tilde{\mathbf{Y_1}} = [\vartheta_{1,k}, 2 \le k \le K]_{1\times(K-1)}$. From equation (13) and (17), we have

$$[\vartheta_{1,1}, \tilde{\mathbf{Y_1}}] = [\vartheta_{1,1}, \tilde{\mathbf{Y_1}}] \begin{pmatrix} v_1\phi \mathbf{w_0}\mathbf{B_4^{-1}} \\ \mathbf{O_3} \end{pmatrix}$$

As a result, the probabilities of the system's length for the regular service period are given by

$$\vartheta_{1,k} the = \vartheta_{11} v_1 \phi \mathbf{w_0}\mathbf{B_4^{-1}}\epsilon_i, 1 \le k \le K,$$

where ϵ_i is a column vector whose k^{th} component is unity and the remaining components are zero. From equation(12) and (15), the system length probabilities of the server being on working vacation is given by

$$[\vartheta_{0,0}, \vartheta_{0,1}, \cdots \vartheta_{0,K}] = -[\vartheta_{1,1}, \tilde{\mathbf{Y_1}}] \begin{pmatrix} v_1\mathbf{w} \\ \mathbf{O_4} \end{pmatrix}$$

Hence,

$$\vartheta_{0,k} = -\vartheta_{1,1} v_1 \mathbf{w}\epsilon_{k+1}, 0 \le k \le K$$

By applying the normalization condition $\sum_{j=0}^{k}\sum_{k=j}^{K}\vartheta_{k,j} = 1$ the only unknown $\vartheta_{1,1}$ is obtained as

$$\vartheta_{1,1} = \left(v_1\phi \sum_{k=1}^{K} \mathbf{w_0}\mathbf{B_4^{-1}}\epsilon_i - v_1 \sum_{k=0}^{K} \mathbf{w}\epsilon_{k+1} \right)^{-1} \tag{18}$$

This completes the evaluation of steady-state probabilities.

5. Performance Measures

The steady-state probabilities can then be used to calculate several model performance metrics. The probability that the server is actively performing routine service P_B, the typical number of users in the system R_R, and the likelihood that the server is on vacation P_V are all given by

$$P_B = \sum_{k=1}^{K} \vartheta_{1,k}; \tag{19}$$

$$P_V = \sum_{k=0}^{K} \vartheta_{0,k} = 1 - P_B \tag{20}$$

$$L_Q = \sum_{j=0}^{1}\sum_{k=0}^{K} (k-j)\vartheta_{j,k} \tag{21}$$

$$L_S = \sum_{k=1}^{K} k(\vartheta_{0,k} + \vartheta_{1,k}); \tag{22}$$

$$R_R = \sum_{j=0}^{1}\sum_{k=0}^{K} (k-j)\alpha\vartheta_{j,k} \tag{23}$$

where $(k-j)\alpha$ is the instantaneous reneging rate.

6. Cost Model

In the following subsection, service rates are used as the decision variables to formulate the cost function of total expected cost function per unit of time. The best service rates with the lowest overall expected cost function are what we are trying to determine. The following cost parameters are presumptive:

Table 1: *Cost per unit time*

Cost per unit time	When
C_1	customer waiting for service
C_2	A customer reneges
C_3	busy server
C_4	server on working vacation
C_5	feedback customer during service period
C_6	feedback customer during working vacation period

The total expected cost function (T_{EC}) per unit of time may be defined as follows using the definitions of each of the cost components mentioned above:

$$T_{EC} = C_1 L_S + C_2 R_R + \mu(C_3 + q_1 C_5) + \eta(C_4 + q_1 C_6) \tag{24}$$

where L_S and R_R are given in equation (22) and (23) respectively.

7. Numerical Results

To illustrate how the various model parameters affect the ideal service rate μ, the ideal expected cost of the system $f(\mu)$, and other system performance measures, we present some numerical examples in this subsection. We fix the maximum number of customers in the system $K = 15$ and the cost elements $C_1 = 10$, $C_2 = 12$, $C_3 = 20$, $C_4 = 14$, $C_5 = 12$, $C_6 = 18$.

(i) **Effect of λ on performance measures and cost:** Here, we study the variation of the performance measures defined in equation (22),(23) and (24). Figure 1(a)-(c) display the sensitivity of performance measures with parameters λ for three different values of K, arrival rate λ, $q = 0.02$, $p_1 = 0.3, \mu = 5$, $\eta = 0.01$, $\alpha = 0.1$, $\phi = 0.1$ are considered. From (a) for the different number of customers K and the arrival rate λ of customer, the length of the system R_R increases. From (b) when the arrival rate λ of customer increases the average reneging rate R_R increases. Figure (c) when the arrival rate λ of customer increases the expected total cost T_{EC} increases lightly.

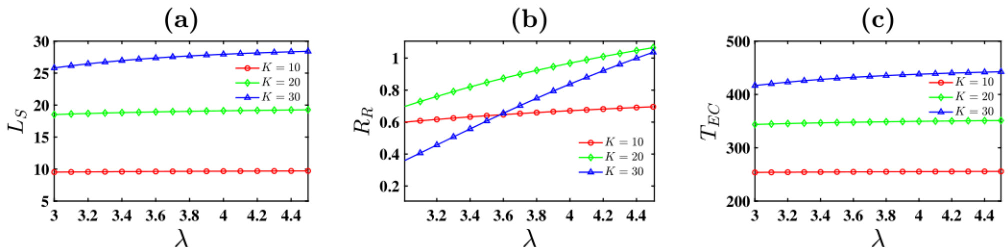

Figure 1: *Variation in performance measures with respect to λ of* ***(a)*** *Expected length of the system* ***(b)*** *Expected no. of customer loss* ***(c)*** *Total expected cost of the system, where $q = 0.02$, $p_1 = 0.3, \mu = 5$, $\eta = 0.01$, $\alpha = 0.1$, $\phi = 0.1$, K, and $\lambda = 3, 3.1, \cdots, 3.4$ respectively.*

(ii) **Effect of μ on performance measures and cost:** Figure 2(a)-(c) shows the sensitivity of performance measures with parameters μ for different value of K, arrival rate λ, $q = 0.02$, $p_1 = 0.3, \lambda = 3$, $\eta = 0.01$, $\alpha = 0.1$, $\phi = 0.1$ are considered. From (a) for the different number of customers K and the service rate μ of customer, the length of the system R_S decreases as obvious. From (b) when the service rate μ of customer increases the average customer loss R_R also decreases. Figure (c) when the service rate μ of customer increases the expected total cost T_{EC} increases when system capacity decreases.

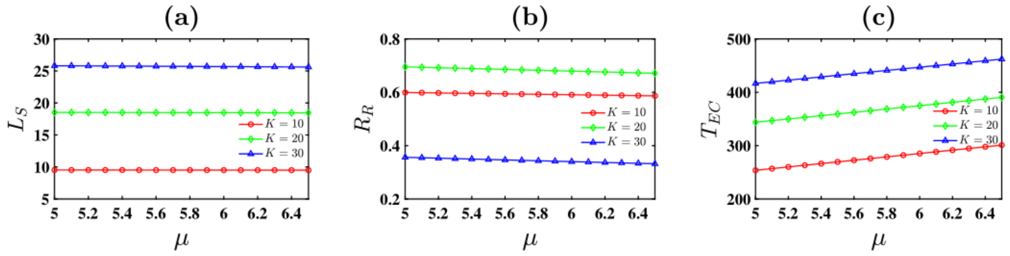

Figure 2: *Variation in performance measures with respect to μ of **(a)** Expected length of the system **(b)** Expected no. of customer loss **(c)** Total expected cost of the system, where $q = 0.02$, $p_1 = 0.3, \lambda = 5$, $\eta = 0.01$, $\alpha = 0.1$, $\phi = 0.1$, K, and $\mu = 5, 5.1, \cdots, 5.4$ respectively.*

(iii) **Effect of α on performance measures and cost:** The sensitivity of performance measures with parameters α of under three different value of K, arrival rate α, $q = 0.02$, $p_1 = 0.3, \lambda = 3$, $\mu = 5$, $\eta = 0.01$, $\phi = 0.1$ are considered can be viewed in figure 3(a)-(c). From (a) for the different number of customers K and the case parameter α of increases the length of the system R_R decreases. From (b) when the parameter α, the average customer loss L_R also decreases. Figure (c) when the parameter α increases the expected total cost T_{EC} decreases greatly.

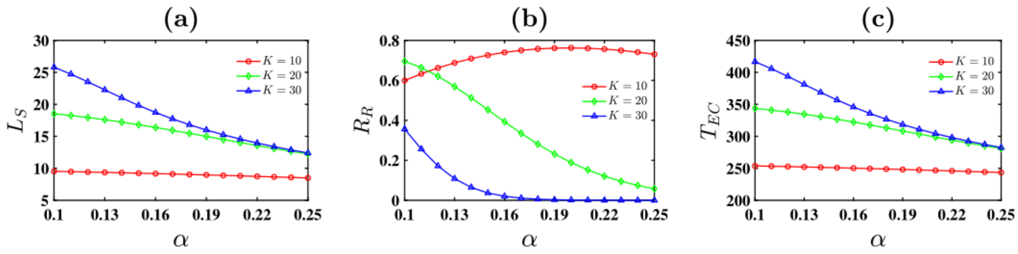

Figure 3: *Variation in performance measures with respect to α of **(a)** Expected length of the system **(b)** Expected no. of customer loss **(c)** Total expected cost of the system, where $q = 0.02$, $p_1 = 0.3$, $\lambda = 3, \mu = 5$, $\eta = 0.01$, $\phi = 0.1$, K, and $\alpha = 0.1, 0.101, \cdots, 0.114$ respectively.*

(iv) **Effect of η on performance measures and cost:** Figure 4(a)-(c) represents the sensitivity of performance measures with parameter η for three different value of K, arrival rate $q = 0.02$, $p_1 = 0.3$, $\lambda = 3$, $\mu = 5$, $\alpha = 0.1$, $\phi = 0.1$ are considered. From (a) for the different number of customer K and the parameter η increases the length of the system R_R decreases lightly. From (b) the average customer loss L_R decreases more when the system capacity is high. Figure (c) when the parameter η increases the expected total cost T_{EC} decreases lightly.

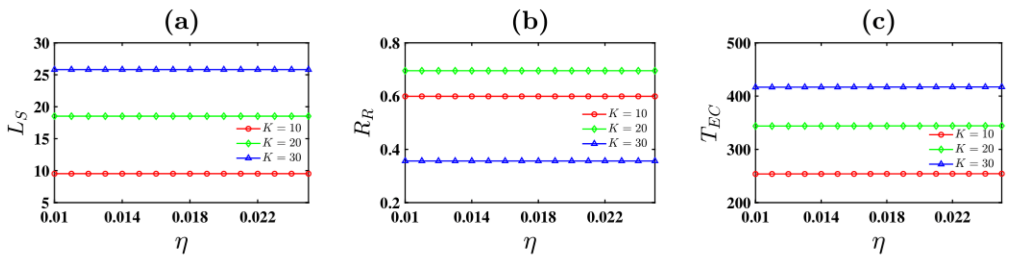

Figure 4: *Variation in performance measures with respect to η of **(a)** Expected length of the system **(b)** Expected no. of customer loss **(c)** Total expected cost of the system, where $q = 0.02$, $p_1 = 0.3, \lambda = 3$, $\mu = 5$, $\alpha = 0.1$, $\phi = 0.1$, K, and $\eta = 0.1, 0.11, \cdots, 0.25$ respectively.*

(v) **Effect on performance measures and cost with respect to k:** Figure 5(a)-(c) display the sensitivity of performance measures with parameters μ under three parameters for different value of K, arrival rate $\lambda = 3, 3.1, \cdots, 3.4$, $\mu = 5, 5.1, \cdots, 5.4$ and $\alpha = 0.1, 0.101, \cdots, 0.114\, q = 0.02$, $p_1 = 0.3$, $\eta = 0.01$, $\phi = 0.1$ are considered. From (a) when we make the variation in K, then R_R increases greatly w.r.t. λ, but increases very slowly w.r.t. μ and α. From (b) when we make the variation in the parameter α then the average customer loss is less. Figure (c) total cost of the system is low after making a variation in α and k.

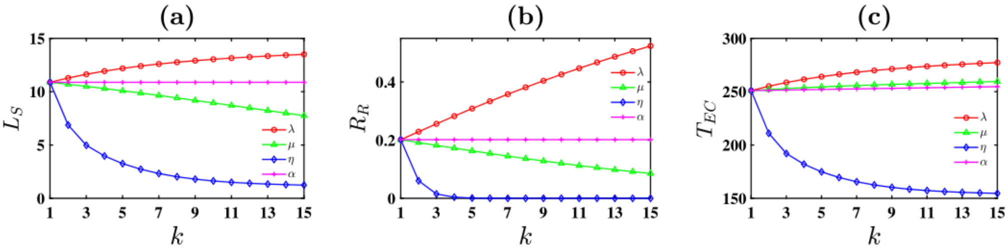

Figure 5: *Variation in performance measures with respect to λ, μ, α of **(a)** Expected length of the system **(b)** Expected no. of customer loss **(c)** Total expected cost of the system, where $q = 0.02$, $p_1 = 0.3$, $\eta = 0.01$, $\phi = 0.1$, and $k = 1, 2, \cdots, 15$, $\lambda = 3, 3.1, \cdots, 3.4$, $\mu = 5, 5.1, \cdots, 5.4$ and $\alpha = 0.1, 0.101, \cdots, 0.114$ respectively.*

(vi) **Effect on system lengh w.r.t. parameters in pair:** Figure 6(a)-(e) display the sensitivity of performance measures with parameters and system capacity $K = 15$ From (a)-(c) making variation in λ and other three parameters μ, α,η, R_R is increasing and decreasing as obvious. From (d) and (e) making variation in μ and other three parameters α and η, R_R is decreasing. From (f) when making variation in α and η, R_R is decreasing when α and η increasing.

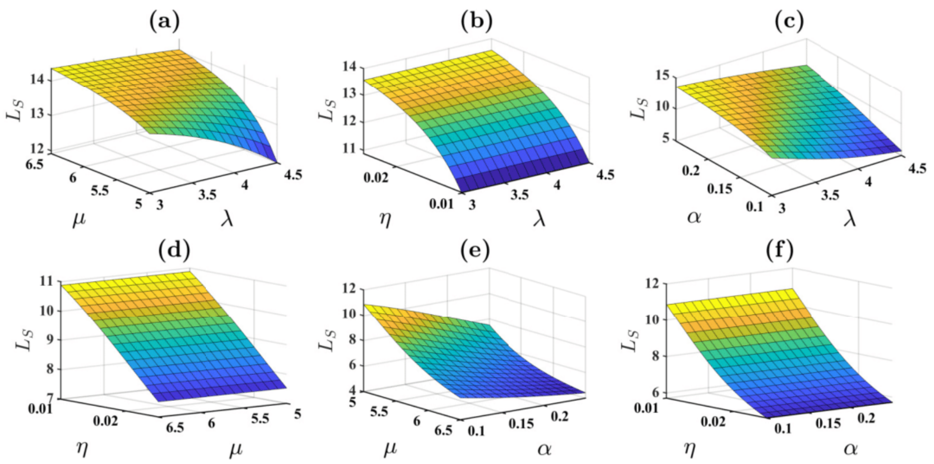

Figure 6: *Variation in length of the system with respect to **(a)** λ and μ **(b)** λ and α **(c)** λ and η **(d)** μ and α **(e)** μ and η **(f)** α and η,where $\lambda = 3, 3.1, \cdots, 6.5$, $\mu = 5, 5.1, \cdots, 8.5$,$\eta = \alpha = 0.1, 0.011, \cdots, 0.045$ else $q = 0.02$, $p_1 = 0.3$,$\lambda = 3$, $\mu = 5$, $\eta = 0.01$, $\alpha = 0.1$, $\phi = 0.1$, $K = 15$ respectively.*

(vii) **Effect on average reneging rate w.r.t. parameters in pair:** Figure 7(a)-(e) shows the sensitivity of performance measures with parameters and system capacity $K = 15$ From (a)-(c) making variation in λ and other three parameters μ, α, η, R_R is increasing and decreasing as obvious. From (d) and (e) making variation in μ and other three parameters α and η, L_R is decreasing. From (f) when making variation in α and η, L_R is decreasing when α and η increasing.

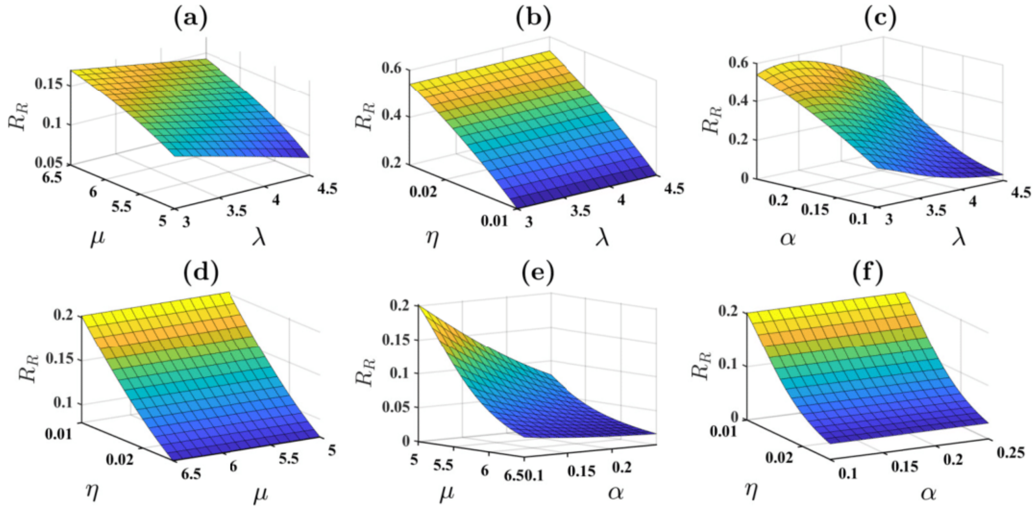

Figure 7: *Variation in average customer loss with respect to* ***(a)*** λ *and* μ ***(b)*** λ *and* α ***(c)*** λ *and* η ***(d)*** μ *and* α ***(e)*** μ *and* η ***(f)*** α *and* η*,where* $\lambda = 3, 3.1, \cdots, 6.5$, $\mu = 5, 5.1, \cdots, 8.5, \eta = \alpha = 0.1, 0.011, \cdots, 0.045$ *else* $q = 0.02$, $p_1 = 0.3, \lambda = 3$, $\mu = 5$, $\eta = 0.01$, $\alpha = 0.1$, $\phi = 0.1$, $K = 15$ *respectively.*

(viii) **Effect on total expected cost w.r.t. parameters in pair:** Figure 8(a)-(e) display the sensitivity of performance measures with parameters and system capacity $K = 15$ From (a)-(c) making variation in λ and other three parameters μ, α,η, T_{EC} is increasing and decreasing as obvious. From (d) and (e) making variation in μ and other three parameters α and η, T_{EC} is decreasing. From (f) when making variation in α and η, T_{EC} is decreasing when α and η increasing.

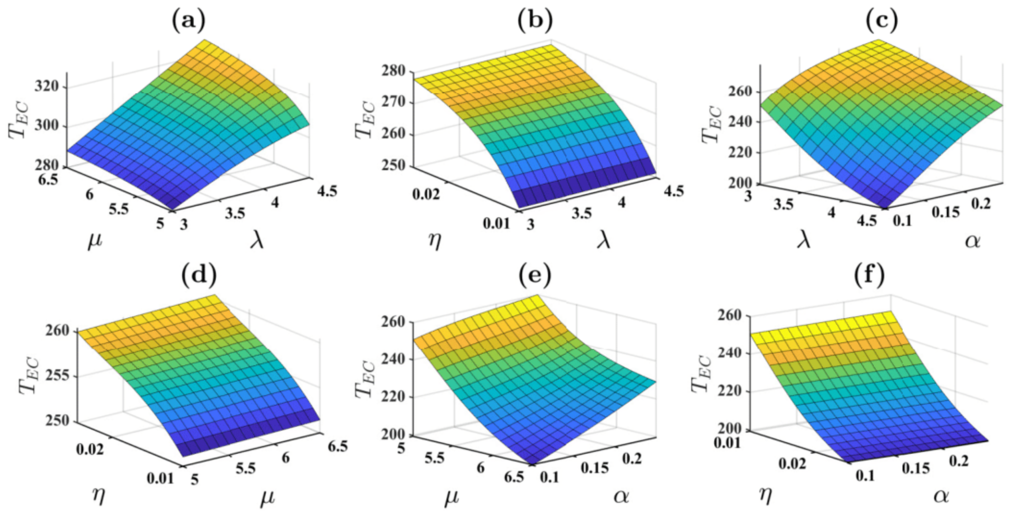

Figure 8: *Variation in total expected cost with respect to* ***(a)*** λ *and* μ ***(b)*** λ *and* α ***(c)*** λ *and* η ***(d)*** μ *and* α ***(e)*** μ *and* η ***(f)*** α *and* η*,where* $\lambda = 3, 3.1, \cdots, 6.5$, $\mu = 5, 5.1, \cdots, 8.5, \eta = \alpha = 0.1, 0.011, \cdots, 0.045$ *else* $q = 0.02$, $p_1 = 0.3, \lambda = 3$, $\mu = 5$, $\eta = 0.01$, $\alpha = 0.1$, $\phi = 0.1$, $K = 15$ *respectively.*

1

Conclusion

In this study, we investigated Markovian feedback queue with reneging and working vacations. We have obtained the steady-state probabilities and solve them using the matrix technique. The model results may be useful in modeling various production and service processes involving feedback and impatient customers. The analysis of the model is restricted to a fixed size. The model's unrestricted size case can also be investigated. Furthermore, to obtain time-dependent results model can be solved in a transient state.

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