DIFFERENT ESTIMATION METHODS FOR THE PARAMETER OF XGAMMA DISTRIBUTION AND THEIR COMPARISON

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Abstract

The xgamma distribution is vital in reliability/survival analysis and biomedical research. In this article, different estimation methods are proposed for the parameter of this distribution. The distribution is a unique finite mixture of exponential distribution and gamma distribution. Some further properties of the distribution that are not available in the earlier literature are studied. We consider the maximum likelihood estimator, least squares estimator, weighted least squares estimator, percentile estimator, the maximum product spacing estimator, the minimum spacing absolute distance estimator, the minimum spacing absolute log-distance estimator, Cramér von Mises estimator, Anderson Darling estimator, right-tailed Anderson Darling estimator, and compare them using a comprehensive simulation study. For comparison purposes, the estimators' bias and mean squared error are considered. A real data example is also a part of this work. Some model selection techniques are used to choose the best fitting of the distribution to the data.

Keywords: Bootstrap confidence intervals; classical methods of estimation; entropy; mixture distribution; stress-strength reliability.

1. Introduction

The xgamma distribution was introduced by Sen et al. [14]. It is a mixture distribution of $F_1(x) \sim Exp(\theta)$ and $F_2(x) \sim Gamma(3, \theta)$ with their mixing proportions $\pi_1 = \theta/(1+\theta)$ and $\pi_2 = 1 - \pi_1$ respectively. The probability density function (pdf) and cumulative distribution function (cdf) of the xgamma distribution are, respectively, given by

$$f(x;\theta) = \frac{\theta^2}{(1+\theta)}\left(1 + \frac{\theta}{2}x^2\right)e^{-\theta x}, \qquad x > 0, \theta > 0 \tag{1}$$

and

$$F(x;\theta) = 1 - \frac{\left(1 + \theta + \theta x + \frac{\theta^2 x^2}{2}\right)}{(1+\theta)}e^{-\theta x}, \qquad x > 0, \theta > 0. \tag{2}$$

The distribution has gained widespread popularity using reliability, survival analysis and biomedical research. The distribution does not belong to the regular exponential family of distributions; hence, the statistical inferential aspects are not used for the exponential family. The present study aims to estimate the parameter of the xgamma distribution with seven different methods. From the literature survey, there is little attempt made in this direction, and this article is an effort to fill the gap. For this reason, the maximum likelihood estimator (MLE), least squares estimator (LSE), weighted least squares estimator (WLSE), Crmer"von Mises estimator (CvME), Maximum product of spacings estimator (MPSE), Anderson-Darling estimator (ADE), and Right-tail Anderson-Darling estimator (RADE) have been considered for estimation.
The article is organized as follows. Section 2 introduces some further properties of the xgamma distribution. In Section 3, we introduce seven different methods of estimation. A comprehensive Monte Carlo simulation study is presented to evaluate the performances of these estimators concerning bias and mean squared error (MSE) criteria in Section 4. In Section 5, we consider a real data illustration. The concluding remarks are made in Section 6.

2. New Properties

This section discusses some new statistical properties that have yet to be available in earlier literature.

2.1. Incomplete Moments, Mean Deviations, and Lorenz and Benferroni Curves

The r^{th} incomplete moment, say, $m_r^I(t)$, of the xgamma distribution is given by

$$\begin{aligned} m_r^I(t) &= \int_0^t x^r f(x)dx \\ &= \frac{\gamma(r+1,\theta t)}{(1+\theta)\theta^{r-1}} + \frac{\gamma(r+3,\theta t)}{2(1+\theta)\theta^r}. \end{aligned}$$

Apart from range and standard deviation, mean deviation about the mean, δ_1 and median, δ_2 are used as measures of spread in a population. Incomplete moments are used to define $\delta_1 = 2\mu_1^{'}F(\mu_1^{'}) - 2m_1^I(\mu_1^{'})$ and $\delta_2 = \mu_1^{'} - 2m_1^I(\mu_e)$,respectively. Here, $\mu_1^{'} = E(X)$ is to be obtained from r^{th} moment of xgamma distribution with $r = 1$, $F(\mu_1^{'})$ is to be calculated from (2), $m_1^I(\mu_1^{'})$ is the first incomplete function obtained from the above equation with $r = 1$.
The Lorenz and Benferroni curves are defined by $L(p) = m_1^I(x_p)/\mu_1^{'}$ and $B(p) = \frac{m_1^I(x_p)}{(p\mu_1^{'})}$, respectively, where $x_p = F^{-1}(p)$ can be computed numerically by the quantile function with $u = p$. These curves are significantly used in economics, reliability, demography, insurance, and medicine. We refer to Pundir, Arora, and Jain[28] and the references cited therein for details on this aspect.

2.2. Entropies

The entropy measures the variation of the uncertainty of X, a random variable. A popular entropy measure is Renyi entropy [13]. If X has the pdf, f(x), then Renyi entropy is defined by

$$H_R(\beta) = \frac{1}{1-\beta} \ln \left\{ \int_0^\infty f^\beta(x)dx \right\} \tag{3}$$

where $\beta > 0$ and $\beta \neq 1$. Suppose X has the pdf in (2). Then, the Renyi entropy of xgamma distribution is

$$H_R(\beta) = \frac{1}{1-\beta} \ln \left\{ \sum_{i=0}^{\beta} \binom{\beta}{i} \frac{\theta^{2\beta-i-1}}{2^i(1+\theta)^\beta} \frac{\Gamma(2i+1)}{\beta^{2i+1}} \right\}$$

Shannon measure of entropy is defined as

$$\begin{aligned} H(f) &= E[-\ln f(X)] \\ &= \left\{\frac{\theta+3}{\theta+1}\right\} - ln\left\{\frac{\theta^2}{1+\theta}\right\} - \sum_{i=1}^{\infty}(-1)^i\frac{\Gamma(2i+1)}{2^i(1+\theta)\theta^{i-1}} - \sum_{i=1}^{\infty}(-1)^i\frac{\Gamma(2i+3)}{2^{i+1}(1+\theta)\theta^i} \end{aligned}$$

2.3. Stress-Strength Reliability

The Stress-Strength model is the life of a component with a random strength X subjected to a random stress Y. When a component experiences stress greater than its capacity to withstand, i.e., strength, it breaks and works well when $X > Y$. So, Stress-Strength Reliability is $R = P(Y < X)$. Let $X \sim xgamma(\theta_1)$ and $Y \sim xgamma(\theta_2)$ be independent random variables. Then Stress-Strength Reliability

$$\begin{aligned} R &= P(Y < X) \\ &= \int_0^{\infty} G_y(x)f(x)dx \\ &= 1 - \frac{\theta_2^2}{(\theta_1+\theta_2)(1+\theta_1)(1+\theta_2)}. \\ & \left[(1+\theta_1) + \frac{\theta_1}{(\theta_1+\theta_2)} + \frac{\theta_1^2+\theta_1\theta_2+\theta_2}{(\theta_1+\theta_2)^2} + \frac{3\theta_1\theta_2}{(\theta_1+\theta_2)^3} + \frac{6\theta_1^2\theta_2}{(\theta_1+\theta_2)^4}\right]. \end{aligned}$$

Also if $\theta_1 = \theta_2 = \theta$, then

$$R = \frac{1-\theta}{2}.$$

2.4. Moments of the residual life

The residual life function is essential in reliability/survival analysis, social studies, bio-medical sciences, economics, population study, the insurance industry, maintenance and product quality control, and product technology. Let X denote the lifetime of a unit at age t, then $X_t = X - t \mid X > t$ is the remaining lifetime beyond that age t.

The cdf $F(x)$ is uniquely determined by the r^{th} moment of the residual life of X (for $r = 1, 2, ...$) [Navarro, Franco, and Ruiz [10], and it is given by

$$\begin{aligned} m_r(t) &= \frac{1}{\bar{F}(t)}\int_t^{\infty}(x-t)^r dF(x) \\ &= \frac{(1+\theta)e^{\theta x}}{(1+\theta+\theta x+\frac{\theta^2x^2}{2})}\left[\sum_{i=0}^{r}(-1)^i\binom{r}{i}t^i\frac{\Gamma(r+1-i,\theta t)}{(1+\theta)\theta^{r-i-1}} + \sum_{i=0}^{r}(-1)^i\binom{r}{i}t^i\frac{\Gamma(r+3-i,\theta t)}{2(1+\theta)\theta^{r-i}}\right]. \end{aligned}$$

In particular, if $r = 1$, then $m_1(t)$ represents an important function called the mean residual life (MRL) function, representing the average life length for a unit alive at age t.

2.5. Moments of the reversed residual life

In some real-life situations, uncertainty is not only related to the future but can also refer to the past. Consider a system whose state is observed only at a specific preassigned inspection time t. If the system is inspected for the first time and found to be 'down', failure relies on the past, i.e. on which instant in $(0, t)$ it has failed. So, the study of a dual notion of the residual life that deals with the past time seems worthwhile [see Di Crescenzo and Longobardi [6]]. If X, a random variable denotes the lifetime of a unit is down at age t, then $\bar{X}_t = t - X \mid X < t$ indicates the idle time or inactivity time or reversed residual life of the unit at age t.

In the case of forensic science, people may be interested in estimating $\bar{X}_t$ to ascertain a person's exact time of death. In the Insurance industry, it represents the period that remained unpaid by a policyholder due to death. The r^{th} moment of $\bar{X}_t$ (for $r = 1,\ 2,\ ...$) is given by

$$\begin{aligned}
\bar{m}_r(t) &= \frac{1}{F(t)} \int_0^t (t-x)^r dF(x) \\
&= \frac{1}{\left\{1 - \frac{(1+\theta+\theta x+\frac{\theta^2 x^2}{2})e^{-\theta x}}{(1+\theta)}\right\}} \left[\sum_{i=0}^{r} (-1)^i \binom{r}{i} t^{r-i} \frac{\gamma(i+1,\theta t)}{(1+\theta)\theta^{i-1}} + \sum_{i=0}^{r} (-1)^i \binom{r}{i} t^{r-i} \frac{\gamma(i+3,\theta t)}{2(1+\theta)\theta^{i}}\right]
\end{aligned}$$

In particular, if $r = 1$, then $\bar{m}_1(t)$ represents a function called the mean idle time or inactivity time (MIT) or reversed residual life (MRRL) function that indicates the mean inactive life length for a unit which is first observed down at age t. The properties of the MIT function have been explored by Ahmad, Kayid, and Pellerey [1] and Kayid and Ahmad [7].

3. Estimation on distribution parameter

In this section, we describe seven estimation methods, namely, MLE, LSE, WLSE, CvME, MPSE, ADE and RADE, to obtain the estimators of the parameter θ of the xgamma distribution.

3.1. Maximum likelihood estimator

Let $(X_1, X_2, \ldots, X_n)$ is a random sample from the distribution in (1). Then, the log-likelihood function is given by

$$\ell(\theta) = 2n \log\theta - n\log(1+\theta) + \sum_{i=1}^{n} \log(1 + \frac{\theta}{2} X_i^2) - \theta \sum_{i=1}^{n} X_i. \tag{4}$$

The derivative of the log-likelihood function is

$$\frac{d\ell(\theta)}{d\theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \frac{\frac{X_i^2}{2}}{1+\frac{\theta}{2}X_i^2}. \tag{5}$$

Equating this to zero does not yield a closed-form solution for the MLE; thus, a numerical method, like Newton Raphson, is used to solve this equation.

3.2. Ordinary and weighted least squares estimator

The ordinary least squares and weighted least squares estimators were proposed by Swain et al. [16] to estimate the parameters of Beta distributions. Suppose $F(X_{i:n}|\theta)$ denotes the cumulative distribution function of the ordered random variables $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ of size n from a distribution function $F(\cdot|\theta)$. Therefore, in this case, the **least square** estimator of θ, say, $\hat{\theta}_{LSE}$ can be obtained by minimizing the function

$$S(\theta) = \sum_{i=1}^{n} \left[F\left(X_{i:n}|\theta\right) - \frac{i}{n+1}\right]^2$$

with respect to θ, where $F(\cdot|\theta)$ is the cdf, given in Eqn. 2. Equivalently, this can be obtained by solving:

$$\sum_{i=1}^{n} \left[F\left(X_{i:n} \mid \theta\right) - \frac{i}{n+1}\right] \eta_1\left(X_{i:n} \mid \theta\right) = 0,$$

where,

$$\eta_1\left(X_{i:n} \mid \theta\right)=\frac{1}{1+\theta}\left[\theta+(\theta^2-1)(1+X_{i:n})+\theta\left(\frac{\theta^2}{2}-1\right)X_{i:n}^2\right]e^{-\theta X_{i:n}}. \tag{6}$$

The **weighted least squares** estimator of θ, say, $\hat{\theta}_{WLSE}$, can be obtained by minimizing

$$W(\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(X_{i:n} \mid \theta\right)-\frac{i}{n+1}\right]^2.$$

This estimator can also be obtained by solving the following:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(X_{i:n} \mid \theta\right)-\frac{i}{n+1}\right]\eta_1\left(X_{i:n} \mid \theta\right)=0$$

where $\eta_1(\cdot \mid \theta)$ is given in Equation (6).

3.3. Cramèr-von-Mises estimator

To motivate our choice of Cramèr-von-Mises type minimum distance estimators, MacDonald [8] provided empirical evidence that the estimator's bias is smaller than the other minimum distance estimators. Thus, the Cramèr-von-Mises estimator of θ, say $\hat{\theta}_{CvME}$ can be obtained by minimizing

$$C(\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(X_{i:n}|\theta\right)-\frac{2i-1}{2n}\right]^2$$

with respect to θ. This estimator can also be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(X_{i:n} \mid \theta\right)-\frac{2i-1}{2n}\right]\eta_1\left(X_{i:n} \mid \theta\right)=0$$

where $\eta_1(\cdot \mid \theta)$ is given in Equation (6).

3.4. Maximum product of spacings estimator

The maximum product spacing method was introduced by Cheng and Amin [4] as an alternative to MLE to estimate the unknown parameters of continuous univariate distributions. The maximum product spacing method was also derived independently by Ranneby [12] as an approximation to the Kullback-Leibler measure of information. To motivate our choice, Cheng and Amin [5] proved that this method is as efficient as the MLE estimators and consistent under more general conditions. We define the uniform spacings of a random sample from the xgamma distribution as:

$$D_i(\theta)=F\left(X_{i:n} \mid \theta\right)-F\left(X_{i-1:n} \mid \theta\right), \qquad i=1,2,\ldots,n,$$

where $F(X_{0:n} \mid \theta)=0$ and $F(X_{n+1:n} \mid \theta)=1$. Clearly $\sum_{i=1}^{n+1} D_i(\theta)=1$. The maximum product of spacings estimator $\hat{\theta}_{MPSE}$, of the parameter θ is obtained by maximizing, with respect to θ, the geometric mean of the spacings:

$$G(\theta)=\left[\prod_{i=1}^{n+1}D_i(\theta)\right]^{\frac{1}{n+1}} \tag{7}$$

or, equivalently, by maximizing the function

$$H(\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\theta) \tag{8}$$

The estimator $\hat{\theta}_{MPSE}$ of the parameter θ can be obtained by solving the non-linear equation

$$\frac{d}{d\theta}H(\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\theta)}\left[\eta_1(X_{i:n}|,\theta) - \eta_1(X_{i-1:n}|,\theta)\right] \quad = \quad 0$$

where, $\eta_1(\cdot \mid \theta)$ is given in Equation (6).

3.5. Anderson-Darling and Right-tail Anderson-Darling estimators

The Anderson-Darling (AD) test [see Anderson and Darling [2]] is an alternative to other statistical tests for detecting sample distribution's departure from normality. Specifically, the Anderson-Darling test converges very quickly towards the asymptote [see Anderson and Darling [3], Pettitt [11] and Stephens [15]]. The **Anderson-Darling estimator** $\hat{\theta}_{ADE}$ of the parameter θ are obtained by minimizing, with respect to θ, the function:

$$A(\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F(X_{i:n} \mid \theta) + \log \overline{F}(X_{n+1-i:n} \mid \theta)\right\}. \tag{9}$$

This estimator can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\eta_1(X_{i:n} \mid \theta)}{F(X_{i:n} \mid \theta)} - \frac{\eta_1(X_{n+1-i:n} \mid \theta)}{\overline{F}(X_{n+1-i:n} \mid \theta)}\right] \quad = \quad 0$$

where, $\eta_1(\cdot \mid \theta)$ is defined in Equation (6).

The **right-tail Anderson-Darling estimator** $\hat{\theta}_{RADE}$ of the parameter θ is obtained by minimizing, with respect to θ, the function:

$$R(\theta) = \frac{n}{2} - 2\sum_{i=1}^{n}F(X_{i:n} \mid \theta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \overline{F}(X_{n+1-i:n} \mid \theta). \tag{10}$$

These estimators can also be obtained by solving the non-linear equations:

$$-2\sum_{i=1}^{n}\eta_1(X_{i:n} \mid \theta) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\eta_1(X_{n+1-i:n} \mid \theta)}{\overline{F}(X_{n+1-i:n} \mid \theta)} \quad = \quad 0$$

where, $\eta_1(\cdot \mid \theta)$ is defined in Equations (6).

4. Simulation Study

In this section, we have carried out a Monte Carlo simulation study to assess the performance of the proposed estimators (MLE, LSE, WLSE, CvME, MPSE, ADE and RADE) of the parameter θ for the xgamma distribution. First, we generate random data from the xgamma distribution where we can use the fact that the xgamma distribution is a special mixture of the exponential(θ) and gamma($3, \theta$) distributions. To generate random data X_i, i = 1, 2, 3, ... n, from the xgamma distribution with parameter θ, we can use the following algorithm:

1. Generate $U_i \sim$ uniform(0, 1), i = 1, 2, 3, ... n
2. Generate $V_i \sim$ exponential(θ), i = 1, 2, 3, ... n
3. Generate $W_i \sim$ gamma(3, θ), i = 1, 2, 3, ... n
4. If $U_i \leq \theta/(1+\theta)$, then set Z_i = V_i. Otherwise, set Z_i = W_i

Table 1: *True value of θ and the average bias of the different estimation procedures for xgamma distribution*

θ	n	MLE	LSE	WLSE	CvME	MPSE	ADE	RADE
0.1	10	0.003704	0.002111	0.001874	0.002768	0.001992	0.001884	0.001008
	20	0.001981	0.001151	0.001088	0.001505	0.001511	0.001123	0.000638
	40	0.000934	0.000556	0.000529	0.000735	0.001200	0.000531	0.000310
	70	0.000544	0.000318	0.000318	0.000421	0.000845	0.000302	0.000156
	100	0.000392	0.000223	0.000233	0.000296	0.000657	0.000219	0.000120
0.5	10	0.025356	0.016170	0.014216	0.021056	0.009505	0.014093	0.008516
	20	0.011770	0.007034	0.006230	0.009586	0.009145	0.006256	0.003748
	40	0.005585	0.003072	0.002902	0.004403	0.007156	0.002951	0.001367
	70	0.003144	0.002050	0.002021	0.002805	0.005178	0.001866	0.000982
	100	0.001619	0.000868	0.000826	0.001398	0.003546	0.000768	0.000143
1	10	0.066489	0.051949	0.046934	0.063382	0.021548	0.041516	0.027305
	20	0.027416	0.017687	0.015768	0.023747	0.016970	0.014983	0.008669
	40	0.017020	0.012473	0.012031	0.015611	0.013533	0.011465	0.007898
	70	0.008023	0.004010	0.004044	0.005801	0.011038	0.003737	0.002373
	100	0.005219	0.002871	0.003082	0.004134	0.008104	0.002752	0.001560
1.5	10	0.107576	0.079362	0.071569	0.097780	0.039168	0.062468	0.040920
	20	0.051824	0.037330	0.033998	0.047283	0.029162	0.032012	0.021111
	40	0.024598	0.015411	0.014735	0.020502	0.023889	0.013955	0.009203
	70	0.014383	0.009096	0.009045	0.012023	0.017182	0.008117	0.005517
	100	0.011339	0.008878	0.008546	0.010944	0.011182	0.008025	0.005675

A Monte Carlo simulation study was carried out considering N = 5000 times for selected values of n, θ. For the first simulation, samples of sizes 10, 20, 40, 70 and 100 were considered, and values of θ were taken as 0.1, 0.5, 1.0, 1.5. The required numerical evaluations are carried out using R 3.1.1 software. The following two measures were computed:

1. Average bias of the simulated estimate $\hat{\theta}$, for i=1, 2, 3, ...,N is $\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)$
2. Average Mean Square Error (MSE) of the simulated estimate $\hat{\theta}$, for i=1, 2, 3, ..., N is $\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2$

In Table 1, we have calculated the average bias of the parameter θ using MLE, LSE, WLSE, CvME, MPSE, ADE and RADE.
In Table 2, we have calculated the average MSEs of the parameter θ using MLE, LSE, WLSE, CvME, MPSE, ADE and RADE.
Table 1 shows that
(i) Bias decreases as n increases.
(ii) Bias decreases as the values of θ increases.
Table 2 shows that
(i) MSE decreases as n increases.
(ii) MSE decreases as the values of θ increases.
Comparing the Tables $1-2$ and Figures $1-2$, even though the MLE is comparatively easy to calculate, the ADE or RADE is preferable from the bias and MSE point of view.

5. Data Analysis

The data set is given by Murthy et al. [9] and represents the failure time of 20 components. The data are 0.072, 4.763, 8.663, 12.089, 0.477, 5.284, 9.511, 13.036, 1.592, 7.709, 10.636, 13.949, 2.475, 7.867, 10.729, 16.169, 3.597, 8.661, 11.501 and 19.809. A summary of these data is: n = 20, $\bar{x} = 8.42945$, s = 5.322056, skewness = 0.1769692, kurtosis = 2.430915. The box plot and the Total Time on Test (TTT) plot of these observations are displayed in Figure 3. The box plot indicates

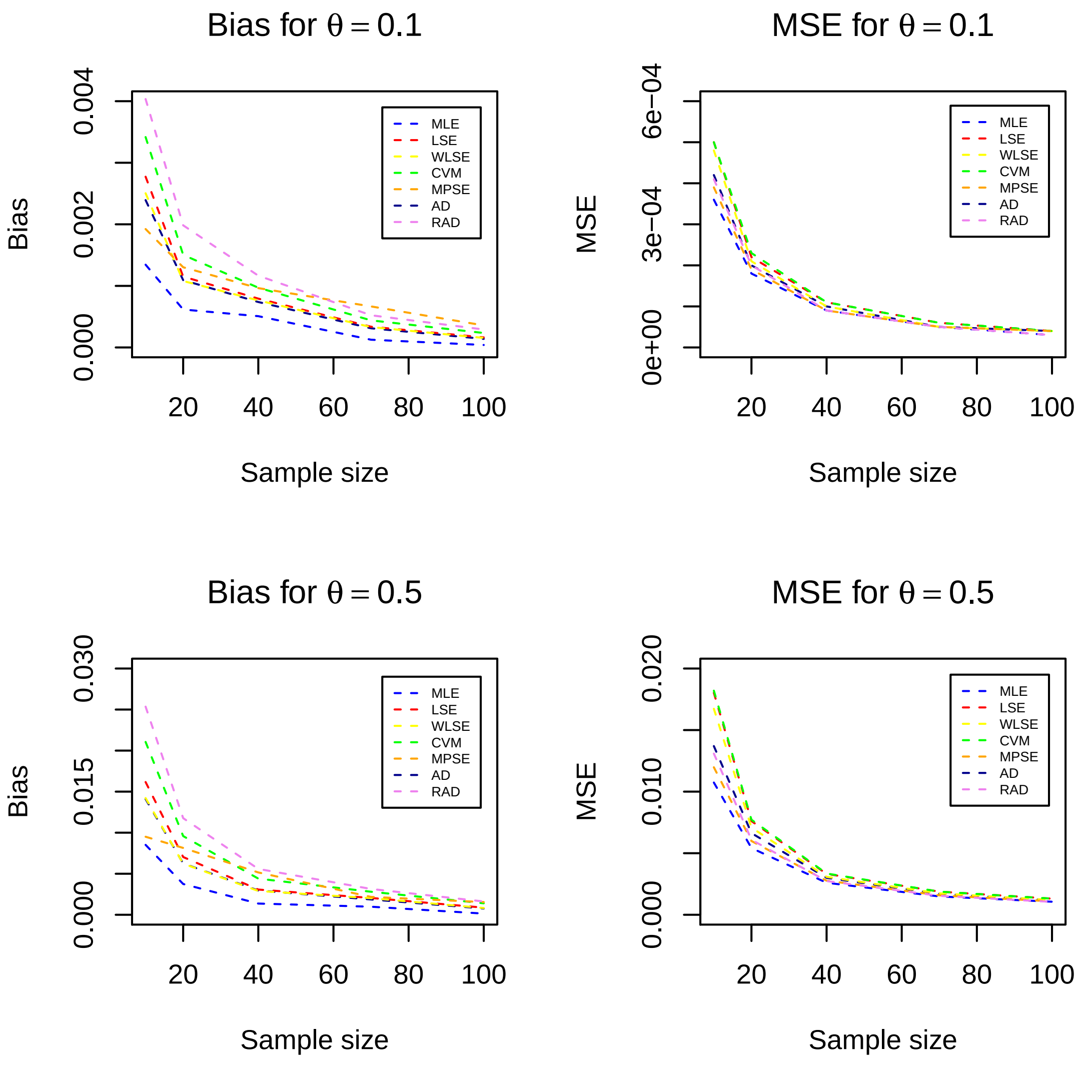


Figure 1: *Average bias and MSE of the estimator of xgamma distribution for different estimation procedures*

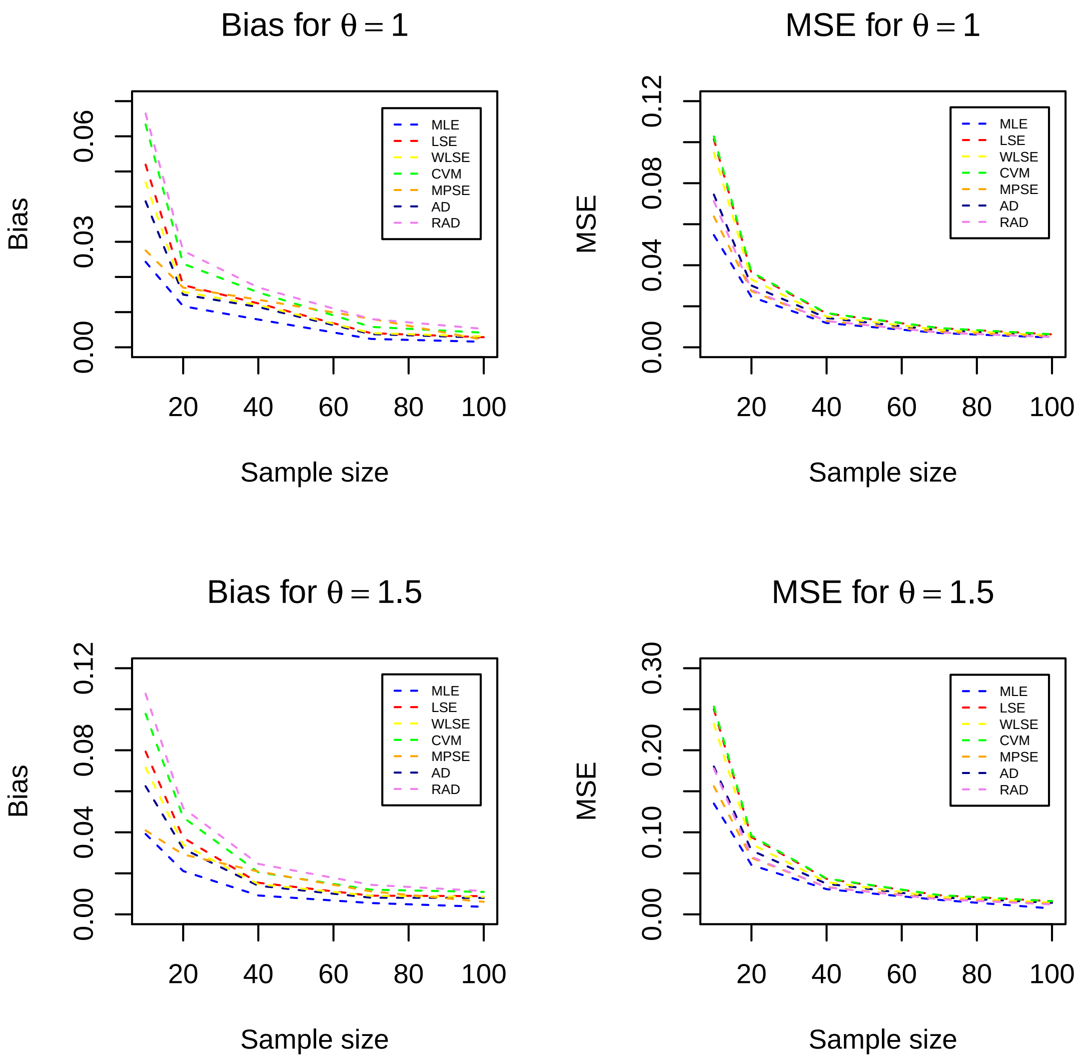
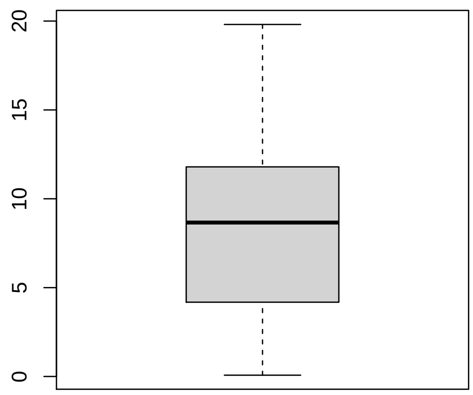


Figure 2: *Average bias and MSE of the estimator of xgamma distribution for different estimation procedures*

Table 2: *True value of θ and the average MSEs of the different estimation procedures for xgamma distribution*

θ	n	MLE	LSE	WLSE	CvME	MPSE	ADE	RADE
0.1	10	0.000439	0.000502	0.000483	0.000506	0.000381	0.000443	0.000414
	20	0.000199	0.000229	0.000217	0.000230	0.000184	0.000209	0.000197
	40	0.000091	0.000106	0.000100	0.000106	0.000088	0.000097	0.000093
	70	0.000052	0.000061	0.000057	0.000062	0.000050	0.000056	0.000053
	100	0.000036	0.000042	0.000039	0.000042	0.000036	0.000039	0.000037
0.5	10	0.013628	0.016936	0.015879	0.017242	0.011157	0.013783	0.012303
	20	0.005962	0.007405	0.006890	0.007501	0.005361	0.006442	0.005854
	40	0.002634	0.003267	0.003026	0.003294	0.002502	0.002930	0.002677
	70	0.001539	0.001940	0.001776	0.001949	0.001497	0.001739	0.001597
	100	0.001069	0.001308	0.001205	0.001312	0.001057	0.001189	0.001106
1	10	0.069448	0.100324	0.093373	0.101749	0.053900	0.072068	0.062812
	20	0.028016	0.034550	0.031993	0.035101	0.024756	0.029757	0.027279
	40	0.012726	0.015990	0.014680	0.016163	0.011771	0.014040	0.012747
	70	0.007001	0.008904	0.008123	0.008952	0.006748	0.007925	0.007184
	100	0.004781	0.006223	0.005606	0.006245	0.004748	0.005514	0.004999
1.5	10	0.178790	0.256766	0.239198	0.260282	0.136610	0.181424	0.156670
	20	0.073687	0.097266	0.089498	0.098933	0.063242	0.080475	0.071668
	40	0.032243	0.041780	0.037965	0.042180	0.029828	0.036235	0.032962
	70	0.017389	0.022625	0.020359	0.022757	0.016632	0.019824	0.018033
	100	0.012620	0.016351	0.014757	0.016432	0.007166	0.014528	0.013152



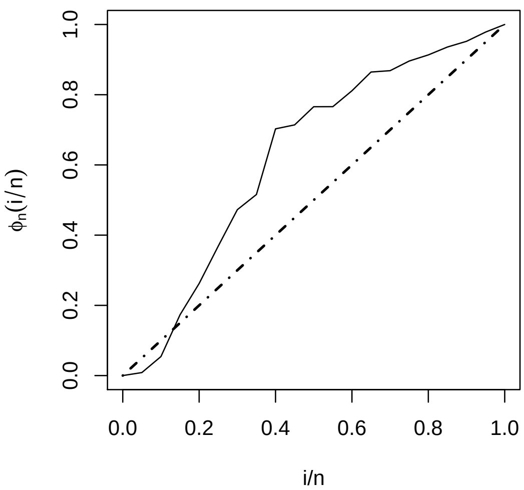


Figure 3: *Box plot and TTT plot for the failure time data*

Table 3: *Summarized results of fitting different distributions for failure time data set*

	xgamma	Akash	Exponential	Lindley	Shankar	Sujatha
ℓ	-60.0189	-61.6744	-62.6346	-61.3792	-62.2797	-61.8345
AIC	122.0378	125.3488	127.2693	124.7583	126.5595	125.6689
BIC	123.0336	126.3445	128.2650	125.7541	127.5552	126.6647
$CAIC$	122.5357	125.5710	127.4915	124.9805	126.7817	125.8912
$HQIC$	122.2322	125.5431	127.4636	124.9527	126.7538	125.8633

Table 4: *Estimated values with SEs for fitting different distributions for failure time data set*

	xgamma	Akash	Exponential	Lindley	Shankar	Sujatha
$\hat{\theta}_{MLE}$	0.3005	0.3427	0.1186	0.2162	0.2352	0.3293
(SE)	(0.0105)	(0.0254)	(0.0315)	(0.0202)	(0.0237)	(0.0247)
$\hat{\theta}_{LSE}$	0.2503	0.3150	0.0954	0.1930	0.2067	0.3026
(SE)	(0.0112)	(0.0235)	(0.0235)	(0.0164)	(0.0200)	(0.0225)
$\hat{\theta}_{WLSE}$	0.2610	0.3192	0.1041	0.2002	0.2133	0.3073
(SE)	(0.0124)	(0.0237)	(0.0265)	(0.0176)	(0.0208)	(0.0229)
$\hat{\theta}_{CvME}$	0.2512	0.3309	0.1003	0.2019	0.2179	0.3174
(SE)	(0.0113)	(0.0245)	(0.0252)	(0.0178)	(0.0214)	(0.0237)
$\hat{\theta}_{MPSE}$	0.3024	0.3171	0.0964	0.1945	0.2083	0.3047
(SE)	(0.0178)	(0.0236)	(0.0239)	(0.0167)	(0.0201)	(0.0227)
$\hat{\theta}_{ADE}$	0.2690	0.3206	0.1071	0.2134	0.2166	0.3208
(SE)	(0.0133)	(0.0238)	(0.0275)	(0.0197)	(0.0212)	(0.0240)
$\hat{\theta}_{RADE}$	0.2381	0.3268	0.1034	0.1969	0.2284	0.3285
(SE)	(0.0107)	(0.0243)	(0.0263)	(0.0170)	(0.0227)	(0.0246)

that the distribution is right-skewed. The TTT plot suggests an increasing failure rate; thus, the xgamma distribution could be appropriate for modeling the current data. Figure 4 shows the fitted probability distribution and empirical distribution function of the xgamma distribution based on different estimates of the parameter to the data set. Table 3 summarises the results of fitting different distributions. Based on the results listed in the table, we conclude that the xgamma distribution provides the best fit with the lowest values of model selection criteria. The xgamma model provides the closest fit to the data. In Table 4, we have presented different estimates of θ under various distributional assumptions to the data and their corresponding standard error (SE). It is also noticed that the SE is the least for the assumption of xgamma distribution, and the MLE and RADE are efficient estimates.

6. Concluding remarks

In this paper, different estimation procedures of the parameter of the xgamma distribution have been studied. Simulation studies are carried out for seven different initial values. As the sample size increases, the MSEs and biases of all estimators decrease and become close to each other. In a small sample situation, the MSEs of the ADE and RADE are smaller than the others. A real data application is conducted to show the appropriateness of xgamma distribution in practical data modelling. The xgamma distribution was compared with some known distributions and presented the estimates according to various parameter estimators. The xgamma distribution

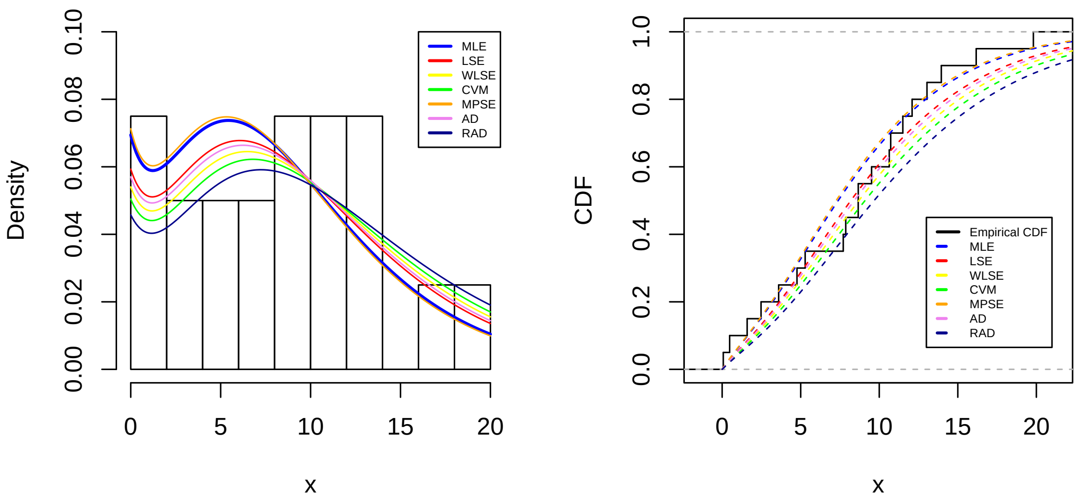


Figure 4: *The fitted pdfs and cdfs of xgamma distribution for different methods of estimation based on failure time data*

is the best-fitting model for some failure time data, and the ADE and RADE are preferable for estimating the parameter even though the MLE has computational ease.

Conflicts of interest

The authors declare that there is no conflict of interest.

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