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DENSITY BY MODULI AND LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

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Abstract

In this paper, we introduced and studied the concept of lacunary statistical convergence of double sequence with respect to modulus function where the modulus function is an unbounded double sequence. We also introduced the concept of lacunary strong convergence of double sequence via modulus function. We further characterized those lacunary convergence of double sequence for which the lacunary statistically convergent of double sequence with respect to modulus function equals statistically convergent of double sequence with respect to modulus function. Finally, we established some inclusion relations between these two lacunary methods and proved some essential analogue for double sequence.

Keywords: modulus function, statistical convergence, lacunary strong convergence, lacunary statistical convergence, double sequence.

1. Introduction

The concept of statistical convergence was formally introduced by [1] and [2] independently. Although statistical convergence was introduced over fifty years ago, it has become an active area of research in recent years. It has been applied in various areas such as summability theory [3] and [4], topological groups [5] and [6], topological spaces [7], locally convex spaces [8], measure theory [9], [10 and [11], Fuzzy Mathematics [12] and [13]. In recent years generalization of statistical convergence has appeared in the study of strong summability and the structure of ideals of bounded functions, [14]. Extension of the notion of statistical convergence of single sequence to double sequences by proposed by [15]. The concept of lacunary statistical convergence of single sequence was introduced by [16]. The extension of the concept of lacunary statistical of single sequence to double sequences was proposed by [17]. The notion of modulus function was introduced by [18]. Following [19] and [20], we recall that a function $f\colon [0,\infty) \to [0,\infty)$ is said to be a modulus function if it satisfies the following properties

(1) $f(x) = 0$ if and only if $x = 0$

(2) $f(x+y) \le f(x) + f(y)$ for $x \ge 0, y \ge 0$,

(3) f is increasing,

(4) f is continuous from the right at 0.

It follows that f is continuous on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x)=\frac{x}{x+1}$, then $f(x)$ is bounded. But, $0<p<1$, $f(x)=x^p$ is not bounded.

The definition of a new concept of density with help of an unbounded modulus function was proposed by [21], as a consequence, they obtained a new concept of non-matrix convergence, namely, f-statistical convergence, which is intermediate between the ordinary convergence and statistical and agrees with the statistical convergence when the modulus function is the identity mapping.

Quite recently, [22] and [23] have introduced and studied the concepts of f-statistical convergence of order α and f-statistical boundedness, respectively, by using approach of [21]. Quite recently, [24] introduced and studied the concept of f-lacunary statistical convergence and the concept of strong lacunary statistical convergence with respect to modulus function. We further extended and introduced some analogues results of double in line with that of [24].

Definition 1.1: ([15]): A real double sequence $x=(x_{jk})$ is statistically convergent to a number l if for each $\varepsilon>0$, the set

$$\{(j,k), j\leq n \text{ and } k\leq m: |x_{jk}-l|\geq\varepsilon\} \tag{1}$$

has double natural density zero. In this case we write $st_2-\lim_{jk} x_{jk}=l$ and we denote the set of all statistically convergent double sequences by st_2.

Definition 1.2 ([17]). The double sequence $\theta_{r,s}=(j_r,k_s)$ is called double lacunary if there exist two increasing sequences of integers such that $j_0=0, h_r=j_r-j_{r-1}\to\infty$ as $r\to\infty$ and $k_0=0, h_s=k_s-k_{s-1}\to\infty$ as $s\to\infty$. Let $j_{r,s}=j_r k_s, h_{r,s}=h_r\overline{h_s}$ and $\theta_{r,s}$ is determined by $I_{r,s}=\{(j,k): j_{r-1}<j\leq j_r \text{ and } k_{s-1}<k\leq k_s\}, q_r=\frac{j_r}{j_{r-1}}, \overline{q_s}=\frac{k_s}{k_{s-1}}$ and $q_{r,s}=q_r\overline{q_s}$.

Definition 1.3 ([17]): Let $\theta_{r,s}$ be a double lacunary sequence, the double number sequence x is double lacunary statistical convergent to L provided that for every $\varepsilon>0$,

$$\lim_{r,s}\frac{1}{h_{r,s}}\left|\{(j,k)\in I_{r,s}: |x_{j,k}-L|\geq\varepsilon\}\right|=0. \tag{2}$$

Throughout this paper s, L_2^∞ and c will denote the spaces of all, bounded and convergent double sequences of real numbers, respectively.

Now in this paper we introduce the concept of $f_{j,k}$-lacunary statistical convergence of double sequence, where $f_{j,k}$ is an unbounded modulus functions of double sequence.

Definition 1.4: Let $f_{j,k}$ be an unbounded modulus functions of double sequence. Let $\theta_{r,s}=(j_r,k_s)$ be double lacunary sequence. A double sequence $x=(x_{jk})$ is said to be $f_{j,k}$-lacunary statistically convergent of double sequence to L or $S_{\theta_{r,s}}^{f_{j,k}}$- convergent to L, if, for each $\varepsilon>0$,

$$\lim_{r,s\to\infty}\frac{1}{f_{j,k}(h_{r,s})}f_{j,k}\left(\left|\{(j,k)\in I_{r,s}: |x_{jk}-L|\geq\varepsilon\}\right|\right)=0. \tag{3}$$

In this case we write

$$S_{\theta_{r,s}}^{f_{j,k}}-\lim x_{jk}=L \text{ or } x_{jk}\to L\left(S_{\theta_{r,s}}^{f_{j,k}}\right).$$

For a given double lacunary sequence $\theta_{r,s}=(j_r,k_s)$ and unbounded modulus function $f_{j,k}$, by $S_{\theta_{r,s}}^{f_{j,k}}$ we denote the set of all $f_{j,k}$-lacunary statistically convergent of double sequences.

2. Methods

2.1 $f_{j,k}$-Lacunary Statistical Convergence of Double Sequence

We begin by establishing elementary connections between convergence of double sequence, $f_{j,k}$-lacunary statistical convergence of double sequence and double lacunary statistical convergence.

Theorem 2.1: Every convergent double sequence is $f_{j,k}$-lacunary statistically convergent of sequence, that is $c \subset S_{\theta_{r,s}}^{f_{j,k}}$ for any unbounded modulus functions f of double sequence and double lacunary statistical convergence sequence $\theta_{r,s}$.

Proof: Let $x = (x_{jk})$ be any convergent double sequence. Then, for each $\varepsilon > 0$, the set

$$\{(j,k) \in \mathbb{N} \times \mathbb{N}: |x_{jk} - L| \geq \varepsilon\} \text{ is finite. Suppose } \{(j,k) \in \mathbb{N} \times \mathbb{N}: |x_{jk} - L| \geq \varepsilon\} = g_0.$$

Now, since $\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\} \subset \{(j,k) \in \mathbb{N} \times \mathbb{N}: |x_{jk} - L| \geq \varepsilon\}$ and $f_{j,k}$ is modulus increasing, therefore

$$\frac{f_{j,k}(|\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\}|)}{f_{j,k}(h_{r,s})} \leq \frac{f(g_0)}{f_{j,k}(h_{r,s})}.$$

Taking limit as $r,s \to \infty$, on both sides, we get

$$\lim_{r,s \to \infty} \frac{f_{j,k}\left((j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\right)}{f_{j,k}(h_{r,s})} = 0,$$

as $f_{j,k}(h_{r,s}) \to \infty$ as $r,s \to \infty$.

Theorem 2.2: Every $f_{j,k}$-lacunary statistical convergent double sequence is double lacunary statistical convergent.

Proof: Suppose $x = (x_{jk})$ is $f_{j,k}$-lacunary statistically convergent double sequence to L. Then by the definition of limit and the fact that $f_{j,k}$ being modulus is subadditive, for every $p \in \mathbb{N}$, there exist $r_0, s_0 \in \mathbb{N}$ such that, for $r,s \geq r_0, s_0$, we have

$$f_{j,k}(|\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\}|) \leq \frac{1}{p} f_{j,k}(h_{r,s}) \leq \frac{1}{p} f_{j,k}\left(\frac{h_{r,s}}{p}\right) = f_{j,k}\left(\frac{h_{r,s}}{p}\right)$$

Since $f_{j,k}$ is increasing, we have

$$\frac{1}{h_{r,s}} |\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\}| \leq \frac{1}{p}.$$

Hence, $x= (x_{jk})$ is a double lacunary statistically convergent to L.

Remark 2.1: It seems that the inclusion $S_{\theta_{r,s}}^{f_{j,k}} \subset S_{\theta_{r,s}}$ is strict. But right now we are not in a position to give an example of a double sequence which is $S_{\theta_{r,s}}$-convergent but not $S_{\theta_{r,s}}^{f_{j,k}}$-convergent. So it is left as an open problem.

Remark 2.2: From theorem 2.1 and 2.2, we can say that the concept of $f_{j,k}$-lacunary statistical convergence is intermediate between the usual notion of convergence of double sequence and the double lacunary statistical convergence of double sequences.

We now establish a relationship between $f_{j,k}$-lacunary statistical convergence of double sequences and double lacunary strong convergence with respect to modulus functions $f_{j,k}$ of double sequence.

Theorem 2.3 Let $\theta_{r,s} = (j_r, k_s)$ be a double lacunary sequence, then consider the following:

(a) For any unbounded modulus functions f for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and there is a positive constant c such that $f(xy) \geq cf(x)f(y)$, for all $x \geq 0, y > 0$,

(i) $x_{jk} \to L\left(N_{\theta_{r,s}}^{f_{j,k}}\right)$ implies $x_{jk} \to L\left(S_{\theta_{r,s}}^{f_{j,k}}\right)$,

(ii) $N_{\theta_{r,s}}^{f_{j,k}}$ is a proper subset of $S_{\theta_{r,s}}^{f_{j,k}}$.

(b) $x \in L_\infty^2$ and $x_{jk} \to L\left(N_{\theta_{r,s}}^{f_{j,k}}\right)$ imply $x_{jk} \to L\left(S_{\theta_{r,s}}^{f_{j,k}}\right)$, for any unbounded modulus functions $f_{j,k}$ of double sequence.

(c) $N_{\theta_{r,s}}^{f_{j,k}} \cap L_\infty^2 = S_{\theta_{r,s}}^{f_{j,k}} \cap L_\infty^2$ for any unbounded modulus function $f_{j,k}$ of double sequence for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and there is a positive constant c such that $f(xy) \geq cf(x)f(y)$, for all $x \geq 0, y \geq 0$.

Proof: (a) (i) For any double sequence $x = (x_{jk})$ and $\varepsilon > 0$, by the definition of a modulus function (1) and (3) we have

$$\frac{1}{h_{r,s}} \sum_{j,k\in I_{r,s}} \sum f_{j,k}\left(|x_{jk} - L|\right) \geq \frac{1}{h_{r,s}} f_{j,k}\left(\sum_{j,k\in I_{r,s}} \sum |x_{jk} - L|\right) \geq \frac{1}{h_{r,s}} f_{j,k}\left(\sum_{\substack{j,k\in I_{r,s} \\ |x_{jk}-L|\geq\varepsilon}} \sum |x_{jk} - L|\right)$$

$$\geq \frac{1}{h_{r,s}} f_{j,k}\left(\left|\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\}\right|\varepsilon\right) \geq \frac{c}{h_{r,s}} f_{j,k}\left(\left|\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\}\right|\right) f(\varepsilon)$$

$$= \frac{c}{h_{r,s}} \frac{f_{j,k}\left(\left|\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq \varepsilon\}\right|\right)}{f_{j,k}(h_{r,s})} f_{j,k}(h_{r,s}) f(\varepsilon)$$

From where it follows that $x \in S_{\theta_{r,s}}^{f_{j,k}}$ as $x \in N_{\theta_{r,s}}^{f_{j,k}}$ and $\lim_{r,s\to\infty}\left(f_{j,k}(h_{r,s}) \big/ h_{r,s}\right) > 0$.

(ii) To show the strictness of inclusion, consider the double sequence $x = (x_{jk})$ such that x_{jk} is to be 1,2, ...,$\left[\sqrt{h_{r,s}}\right]$ at the first $\left[\sqrt{h_{r,s}}\right]$ integers in $I_{r,s}$, and x_{jk}=0 otherwise. Note that (x_{jk}) is not bounded. Also, for every $\varepsilon > 0$,

$$\frac{1}{f_{j,k}(h_{r,s})} f_{j,k}\left(\left|\{(j,k) \in I_{r,s}: |x_{jk} - L| \geq 0\}\right|\right) = \frac{f_{j,k}(\sqrt{h_{r,s}})}{f_{j,k}(h_{r,s})} = \frac{f_{j,k}(\sqrt{h_{r,s}})}{\sqrt{(h_{r,s})}} \times \frac{h_{r,s}}{f_{j,k}(h_{r,s})} \times \frac{\left[\sqrt{h_{r,s}}\right]}{h_{r,s}} \to \infty \text{ as } r \to \infty,$$

Because $\lim_{r,s\to\infty}\left(f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]\right) \big/ \left(\left[\sqrt{h_{r,s}}\right]\right)\right)$, $\lim_{r,s\to\infty}\left(f_{j,k}\,(h_{r,s}) \big/ h_{r,s}\right)$ are positive and $\lim_{r,s\to\infty}\left(f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]\right) \big/ \left(\left[\sqrt{h_{r,s}}\right]\right)\right) = 0$.

Thus, $x_{jk} \to 0\left(S_{\theta_{r,s}}^{f_{j,k}}\right)$. On other hand,

$$\frac{1}{h_{r,s}} \sum_{j,k\in I_{r,s}} \sum f_{j,k}\left(|x_{jk} - L|\right) = \frac{f_{j,k}(1) + f_{j,k}(2) + \cdots + f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]\right)}{h_{r,s}} \geq \frac{f_{j,k}\left(1 + 2 + \cdots + \left[\sqrt{h_{r,s}}\right]\right.}{h_{r,s}}$$

$$= \frac{f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]\left(\left(\left[\sqrt{h_{r,s}}\right] + 1\right) \big/ 2\right)\right)}{h_{r,s}} \geq c \frac{f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]\right) f_{j,k}\left(\left(\left[\sqrt{h_{r,s}}\right] + 1\right) \big/ 2\right)}{h_{r,s}}$$

$$= c \times \frac{f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]\right)}{\left[\sqrt{h_{r,s}}\right]} \times \frac{f_{j,k}\left(\left(\left[\sqrt{h_{r,s}}\right] + 1\right) \big/ 2\right)}{\left(\left[\sqrt{h_{r,s}}\right] + 1\right) \big/ 2} \times \frac{\left[\sqrt{h_{r,s}}\right]\left(\left(\left[\sqrt{h_{r,s}}\right] + 1\right) \big/ 2\right)}{h_{r,s}} > 0$$

As c, $\lim_{r,s\to\infty}\left(f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]\right)/\left[\sqrt{h_{r,s}}\right]\right)$, $\lim_{r,s\to\infty}\left(f_{j,k}\left(\left[\sqrt{h_{r,s}}\right]+1\right)/2\right)/\left(\left(\left[\sqrt{h_{r,s}}\right]+1\right)/2\right)$, and

$\lim_{r,s\to\infty}\left(\left[\sqrt{h_{r,s}}\right]\right)\left(\left(\left[\sqrt{h_{r,s}}\right]+1\right)/2\right)/h_{r,s}\right)$ are positive. Hence $x_{jk} \nrightarrow 0\left(N_{\theta_{r,s}}^{f_{j,k}}\right)$.

(b) Suppose that $x_{jk} \to L\left(S_{\theta_{r,s}}^{f_{j,k}}\right)$ and $x \in L_\infty^2$, say $|x_{jk}-L| \le H$ for all $j,k \in \mathbb{N}$. Given $\varepsilon > 0$, we have

$$\frac{1}{h_{r,s}}\sum_{j,k\in I_{r,s}}\sum f_{j,k}(|x_{jk}-L|) = \frac{1}{h_{r,s}}\sum_{\substack{j,k\in I_{r,s}\\ |x_{jk}-L|\ge\varepsilon}}\sum f_{j,k}(|x_{jk}-L|) + \sum_{\substack{j,k\in I_{r,s}\\ |x_{jk}-L|<\varepsilon}}\sum f_{j,k}(|x_{jk}-L|) \le \frac{1}{h_{r,s}}\left|\{(j,k) \in I_{r,s}: |x_{jk}-L| \ge \varepsilon\}\right| f_{j,k}(H) + \frac{1}{h_{r,s}} h_{r,s} f_{j,k}(\varepsilon).$$

Taking limit on both sides as $r,s\to\infty$, we get $\lim_{r,s\to\infty}\left(\frac{1}{h_{r,s}}\right)\sum_{j,k\in I_{r,s}}\sum f_{j,k}(|x_{jk}-L|) = 0$, in view of theorem 2.2 and the fact that $f_{j,k}$ is increasing.

(c) This is an immediate consequence of (a) and (b)

Remark 2.3 The example given in part (a) of the above theorem shows that the boundedness condition cannot be omitted from the hypothesis of part (b).

3. Results

3.1 $f_{j,k}$-Lacunary Statistical Convergence of Double Sequence Versus $f_{j,k}$-Statistical Convergence of Double Sequence

In this section we study the inclusion $S_{\theta_{r,s}}^{f_{j,k}} \subset S^{f_{j,k}}$ and $S^{f_{j,k}} \subset S_{\theta_{r,s}}^{f_{j,k}}$ under certain restrictions on $\theta_{r,s}$ and $f_{j,k}$.

Lemma 3.1.1: For any double lacunary sequence $\theta_{r,s}$ and unbounded modulus function $f_{j,k}$ for which $\lim_{t\to\infty}(f(t)/t) > 0$ and there is a positive constant c such that $f(xy) \ge cf(x)f(y)$, for all $x \ge 0$, $y \ge 0$, one has $S^{f_{j,k}} \subset S_{\theta_{r,s}}^{f_{j,k}}$ if and only if $\lim\inf_{r,s} q_{r,s} > 1$.

Proof: Sufficiency: If $\lim\inf_{r,s} q_{r,s} > 1$ then there exists $\delta > 0$ such that $q_{r,s} \ge 1+\delta$ for sufficiently large r,s. Since $h_{r,s} = k_{r,s} - k_{r-1,s-1}$, we have

$$\frac{h_{r,s}}{k_{r,s}} \ge \left(\frac{\delta}{1+\delta}\right)^2$$

For sufficiently large r,s. If $x_{jk} \to L(S^{f_{j,k}})$, then, for given $\varepsilon > 0$ and sufficiently large r,s we have

$$\frac{1}{f_{j,k}(j_r k_s)} f_{j,k}\left(\left|\{j \le j_r \textit{ and } k \le k_s: |x_{jk}-L| \ge \varepsilon\}\right|\right) \ge \frac{f_{j,k}\left(\left|\{(j,k)\in I_{r,s}: |x_{jk}-L|\ge\varepsilon\}\right|\right)}{f_{j,k}(j_r k_s)} = \frac{f_{j,k}(h_{r,s})}{f_{j,k}(j_r k_s)} \times \frac{f_{j,k}\left(\left|\{(j,k)\in I_{r,s}: |x_{jk}-L|\ge\varepsilon\}\right|\right)}{f_{j,k}(h_{r,s})} = \left(\frac{f_{j,k}(h_{r,s})}{h_{r,s}}\right)\cdot\left(\frac{j_r k_s}{f_{j,k}(j_r k_s)}\right)\left(\frac{h_{r,s}}{j_r k_s}\right)\frac{f_{j,k}\left(\left|\{(j,k)\in I_{r,s}: |x_{jk}-L|\ge\varepsilon\}\right|\right)}{f_{j,k}(j_r k_s)} \ge \left(\frac{f_{j,k}(h_{r,s})}{h_{r,s}}\right)\left(\frac{j_r k_s}{f_{j,k}(j_r k_s)}\right)\left(\frac{\delta}{1+\delta}\right)^2 \cdot \frac{f_{j,k}\left(\left|\{(j,k)\in I_{r,s}: |x_{jk}-L|\ge\varepsilon\}\right|\right)}{f_{j,k}(h_{r,s})}.$$

This proves the sufficiency.

Necessity: Assume that $\lim\inf_{r,s} q_{r,s} = 1$. We can select a subsequence $(j_{r(i)}k_{s(j)})$ of $\theta_{r,s}$ satisfying

$$\frac{j_{r(i)}k_{s(j)}}{j_{r(i)-1}k_{s(j)-1}} < 1 + \frac{1}{ij}, \frac{j_{r(i)}k_{s(j)}}{j_{r(i)-1}k_{s(j)-1}} > ij,$$

Where $r(i) \geq r(i-1)+2$ and $s(j) \geq s(j-1)+2$.
Define a bounded double sequence by

$$x_{jk} = \begin{cases} 1 \; if \; j,k \in I_{r,s(i,j)}, for \; some \; i,j = 1,2,3,\dots \\ 0 \qquad\qquad\qquad\qquad otherwise. \end{cases}$$

It is shown that $x \notin N_{\theta_{r,s}}$ but $x \in \omega$. Thus, we have $x \notin S_{\theta_{r,s}}^{f_{j,k}}$. Hence $S^{f_{j,k}} \not\subset S_{\theta_{r,s}}^{f_{j,k}}$. But this is a contradiction to the assumption that $S^{f_{j,k}} \subset S_{\theta_{r,s}}^{f_{j,k}}$. This contradiction shows that our assumption is wrong. Hence $\lim \inf_{r,s} q_{r,s} > 1$.

Remark 3.1.1: The double sequence $x = (x_{jk})$, constructed in the necessity part of the above lemma, is an example of $f_{j,k}$-statistically convergent double sequence which is not $f_{j,k}$-lacunary statistically convergent of double sequence.

Lemma 3.1.2: For any double lacunary sequence $\theta_{r,s}$ and unbounded modulus functions $f_{j,k}$ for which $\lim_{t\to\infty}(f(t)/t) > 0$ and there is a positive constant c such that $f(xy) > cf(x)f(y)$, for all $x \geq 0, y \geq 0$ one has $S_{\theta_{r,s}}^{f_{j,k}} \subset S^{f_{j,k}}$ if and only if $\lim \sup_{r,s} q_{r,s} > 1$.

Proof: Sufficiency: If $lim \sup_{r,s} q_{r,s}$, then there is $H > 0$ such that $q_{r,s} < H$ for all r, s. Now, suppose that $x_{jk} \to L\left(S_{\theta_{r,s}}^{f_{j,k}}\right)$ and $\lim_{r,s\to\infty}\left(f(h_{r,s})/h_{r,s}\right) = L'$. Therefore, for given $\varepsilon > 0$, there exist $r_0, s_0 \in \mathbb{N}$ such that for all $r, s > r_0, s_0$

$$\frac{f_{j,k}(h_{r,s})}{h_{r,s}} < L' + \varepsilon,$$

$$\frac{1}{f_{j,k}(h_{r,s})} f_{j,k}\left(\left|\left\{(j,k) \in I_{r,s} : \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right) < \varepsilon.$$

Let $N_{r,s} = \left|\left\{(j,k) \in I_{r,s} : \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|$. Using this notion, we have

$$\frac{f_{j,k}(N_{r,s})}{f_{j,k}(h_{r,s})} < \varepsilon \; \forall \; r,s > r_0, s_0.$$

Now, let $M = \max\{f_{j,k}(N_{1,1}), f_{j,k}(N_{2,2}), \dots, f_{j,k}(N_{r_0,s_0})\}$ and let m, n be integers such that $j_{r-1} < m \leq j_r$ and $k_{s-1} < n < k_s$, then we can write

$$\frac{1}{f_{j,k}(mn)} f_{j,k}\left(\left|\left\{j \leq m, k \leq n : \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right) \leq \frac{1}{f_{j,k}(j_{r-1}k_{s-1})} \cdot f_{j,k}\left(\left|\left\{j \leq m, k \leq n : \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right)$$

$$= \frac{1}{f_{j,k}(j_{r-1}k_{s-1})} f_{j,k}\left(N_{1,1}, N_{2,2} + \dots + N_{r_0,s_0} + N_{r_0+1,s_0+1} + \dots + N_{r,s}\right)$$

$$\leq \frac{1}{f_{j,k}(j_{r-1}k_{s-1})}\left(f_{j,k}(N_{1,1}) + f_{j,k}(N_{2,2}) + \dots + f_{j,k}(N_{r_0,s_0}) + f_{j,k}(N_{r_0+1,s_0+1}) + \dots + f_{j,k}(N_{r,s})\right)$$

$$\leq \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})} + \left[f_{j,k}(N_{r_0+1,s_0+1}) + \dots + f_{j,k}(N_{r,s})\right]$$

$$= \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})}\left[\frac{f_{j,k}\left(h_{r_0+1,s_0+1}\right)}{h_{r_0+1,s_0+1}} \frac{f_{j,k}\left(N_{r_0+1,s_0+1}\right)}{f_{j,k}\left(h_{r_0+1,s_0+1}\right)} h_{r_0+1,s_0+1} + \cdots\right.$$

$$\left. + \frac{f_{j,k}(h_{r,s})}{h_{r,s}} \frac{f_{j,k}(N_{r,s})}{f_{j,k}(h_{r,s})} h_{r,s}\right] + \frac{1}{f_{j,k}(j_{r-1}k_{s-1})}\left[(L' + \varepsilon)\varepsilon h_{r_0+1,s_0+1} + \dots + (L' + \varepsilon)\varepsilon h_{r,s}\right]$$

$$= \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})} + \frac{1}{f_{j,k}(j_{r-1}k_{s-1})}\varepsilon(L'+\varepsilon)\left[h_{r_0+1,s_0+1}+\cdots+h_{r,s}\right]$$
$$= \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})} + \frac{1}{f_{j,k}(j_{r-1}k_{s-1})}\varepsilon(L'+\varepsilon)\left[j_r k_s - j_{r_0}k_{s_0}\right]$$
$$< \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})} + \varepsilon(L'+\varepsilon)\left(\frac{j_r k_s}{f_{j,k}(j_{r-1}k_{s-1})}\right)$$
$$= \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})} + \varepsilon(L'+\varepsilon)\frac{1}{f_{j,k}(j_{r-1}k_{s-1})/j_{r-1}k_{s-1}}\frac{j_r k_s}{j_{r-1}k_{s-1}}$$
$$= \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})} + \varepsilon(L'+\varepsilon)q_{r,s}\frac{1}{f_{j,k}(j_{r-1}k_{s-1})/j_{r-1}k_{s-1}}$$
$$< \frac{r_0 s_0 M}{f_{j,k}(j_{r-1}k_{s-1})} + \varepsilon(L'+\varepsilon)H\cdot\frac{1}{f_{j,k}(j_{r-1}k_{s-1})/j_{r-1}k_{s-1}},$$

From where the sufficiency follows immediately, in view of the above fact that

$$\lim_{r,s\to\infty}\left(f_{j,k}(j_{r-1}k_{s-1})/j_{r-1}k_{s-1}\right) > 0.$$

Necessity: Suppose that $\lim\sup_{r,s} q_{r,s} = \infty$. We can select a subsequence $\left(j_{r(i)}k_{s(j)}\right)$ of double lacunary sequence $\theta_{r,s}$ such that $q_{r(i),s(j)} > ij$. Define a bounded double sequence $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} 1 \; if \; j_{r(i)}k_{s(j)} < jk \le 2j_{r(i)-1}k_{s(j)-1}, for\ some\ i,j = 1,2,3,\dots, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad otherwise. \end{cases}$$

It is shown that $x \in N_\theta$ but $x \notin \omega$. We conclude that $x \in S_{\theta_{r,s}}^{f_{j,k}}$, but $x \notin S^{f_{j,k}}$, for every $f_{j,k}$-statistically convergent of double sequence is statistically convergent double sequence. $S_{\theta_{r,s}}^{f_{j,k}} \not\subset S^{f_{j,k}}$. But this is a contradiction to the assumption that $S_{\theta_{r,s}}^{f_{j,k}} \subset S^{f_{j,k}}$. This contradiction shows that $\lim\sup_{r,s} q_{r,s} < \infty$.

Remark 3.2.1: The double sequence $x = (x_{jk})$, constructed in the necessity part of the above lemma, is an example of $f_{j,k}$-lacunary statistically convergent double sequence which is not $f_{j,k}$-statistically convergent double sequence.

Combining lemma 3.1.1 and 3.2.1 we have the following.

Theorem 3.1.1: For any double lacunary sequence $\theta_{r,s}$ and unbounded modulus functions $f_{j,k}$ for which $\lim_{t\to\infty}(f(t)/t) > 0$ and there is positive constant c such that $f(xy) \ge cf(x)f(y)$, for all $x \ge 0, y \ge 0$, one has $S_{\theta_{r,s}}^{f_{j,k}} = S^{f_{j,k}}$ if and only if $1 < \lim\inf_{r,s} q_{r,s} < \lim\sup_{r,s} q_{r,s} < \infty$.

Theorem 3.2.1: For any double lacunary sequence $\theta_{r,s}$ and unbounded modulus functions $f_{j,k}$ for which $\lim_{t\to\infty}(f(t)/t) > 0$ and there is positive constant c such that $f(xy) \ge cf(x)f(y)$, for all $x \ge 0, y \ge 0$, one has

$$S^{f_{j,k}} = \bigcap_{\lim\inf_{r,s} q_{r,s}>1} S_{\theta_{r,s}}^{f_{j,k}} = \bigcup_{\lim\sup_{r,s} q_{r,s}<\infty} S_{\theta_{r,s}}^{f_{j,k}}. \tag{4}$$

Proof: In view lemma 3.1, we have $S^{f_{j,k}} \subset \bigcap_{\lim\inf_{r,s} q_{r,s}>1} S_{\theta_{r,s}}^{f_{j,k}}$. Suppose if possible $x = (x_{jk}) \in \bigcap_{\lim\inf_{r,s} q_{r,s}>1} S_{\theta_{r,s}}^{f_{j,k}}$ but $x \notin S^{f_{j,k}}$. We have $(x_{jk}) \in S_{\theta_{r,s}}^{f_{j,k}}$ for all $\theta_{r,s} = (j_r, k_s)$ for which $\lim\inf_{r,s} q_{r,s} > 1$. If we take $\theta_{r,s} = (2^{r+s})$, then, in view theorem 3.1, we have $S_{\theta_{r,s}}^{f_{j,k}} = S^{f_{j,k}}$ and so $x \in S^{f_{j,k}}$, contrary to our assumption. Hence $S^{f_{j,k}} = \bigcap_{\lim\inf_{r,s} q_{r,s}>1} S_{\theta_{r,s}}^{f_{j,k}}$. The remaining part can be proved similarly and hence is omitted.

Remark 3.3.1: The double sequence $x = (x_{jk})$ constructed in part (a) of theorem 2.1 belongs to $S_{\theta_{r,s}}^{f_{j,k}}$

for every double lacunary sequence $\theta_{r,s}$, as well unbounded modulus functions $f_{j,k}$ for which $\lim_{t\to\infty}(f(t)/t) > 0$ and there is a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$. Hence $\cap \lim_{r,s} inf\ q_{r,s}\, S_{\theta_{r,s}}^{f_{j,k}} \neq \phi$.

3.2 Inclusion Between two Lacunary Methods of $f_{j,k}$-Statistical Convergence.

Our first results shows that, for certain modulus function $f_{j,k}$, if $\theta'_{r,s}$ is a lacunary refinement of the double lacunary sequence $\theta_{r,s}$ $S_{\theta'_{r,s}}^{f_{j,k}} \subset S_{\theta_{r,s}}^{f_{j,k}}$. To establish this result, we first recall the definition of double lacunary refinement of double sequence.

Definition 3.2.1: The double lacunary sequence $\theta'_{r,s} = (j'_r, k'_s)$ is called a double lacunary refinement of double lacunary sequence $\theta_{r,s} = (j_r, k_s)$ if $(j_r, k_s) \subset (j'_r, k'_s)$.

Theorem 3.2.1: If $\theta'_{r,s} = (j'_r, k'_s)$ is a double lacunary refinement of $\theta_{r,s} = (j_r, k_s)$ and $f_{j,k}$ is an unbounded modulus functions of double sequence such that

$$\left|f_{j,k}(x) - f_{j,k}(y)\right| = f_{j,k}(|x-y|), \forall\, x > 0, y > 0, \tag{5}$$

Then $x \in S_{\theta'_{r,s}}^{f_{j,k}}$ implies $x \in S_{\theta_{r,s}}^{f_{j,k}}$.

Proof: Suppose each $I_{r,s}$ of $\theta_{r,s}$ contains the points $\left(j'_{r(i)}k'_{s(j)}\right)_{i,j=1}^{v(r,s)}$ of $\theta'_{r,s}$ so that

$$j_{r-1}, k_{s-1} < j'_{r,1}, k'_{s,1} < j'_{r,2}, k'_{s,2} < \cdots < j'_{r,v(r)}, k'_{s,v(s)} = j_r, k_s$$

where $I'_{r,s} = \{(j,k): j'_{r-1} < j' \leq j'_{r-1}\ and\ k'_{s-1} < k' \leq k'_s\}$.

Note that, for all $(r,s), v(r,s) \geq 1$ because $(j_r, k_s) \subset (j'_{r-1}, k'_{s-1})$. Let $x_{jk} \to L\left(S_{\theta'_{r,s}}^{f_{j,k}}\right)$. Therefore, for each $\varepsilon > 0$, we have

$$\lim_{\substack{r,s\to\infty\\ 1\leq i,j\leq v(r,s)}} \frac{1}{f_{j,k}\left(h'_{r(i),s(j)}\right)} f_{j,k}\left(\left|\left\{(j,k) \in I'_{r(i),s(j)}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right) = 0,$$

where $h'_{r(i),s(j)} = k'_{r(i),s(j)} - k'_{r(i-1),s(j-1)}$ and $h'_{(r,1),(s,1)} = k'_{(r,1),(s,1)} - k'_{r-1,s-1,}$ whence

$$\lim_{r,s\to\infty} \sum_{\substack{I'_{r(i),s(m)} \subset I_{r(i),s(m)}\\ 1\leq i,j\leq v(r,s)}} \sum \frac{1}{f_{j,k}\left(h'_{r(i),s(j)}\right)} f_{j,k}\left(\left|\left\{(j,k) \in I'_{r(i),s(j)}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right) = 0.$$

For each $\varepsilon > 0$, we have

$$\frac{1}{f_{j,k}\left(h_{r,s}\right)} f_{j,k}\left(\left|\left\{(j,k) \in I_{r,s}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right)$$

$$= \frac{1}{f_{j,k}\left(h_{r,s}\right)} \cdot f_{j,k}\left(\left|\left\{(j,k) \in \bigcup_{\substack{I'_{r(i),s(j)} \subset I_{r,s}\\ 1\leq i,j\leq v(r,s)}} I'_{r(i),s(j)}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right)$$

$$= \frac{1}{f_{j,k}\left(h_{r,s}\right)} \cdot f_{j,k}\left(\sum_{\substack{I'_{r(i),s(j)} \subset I_{r,s}\\ 1\leq i,j\leq v(r,s)}} \sum \left|\left\{(j,k) \in I'_{r(i),s(j)}: \left|x_{jk} - L\right| \geq \varepsilon\right\}\right|\right)$$

$$\leq \frac{1}{f_{j,k}(h_{r,s})} \sum_{\substack{I'_{r(i),s(j)} \subset I_{r,s} \\ 1 \leq i,j \leq v(r,s)}} \sum f_{j,k}\left(\{(j,k) \in I'_{r(i),s(j)} : |x_{jk} - L| \geq \varepsilon\}\right)$$

$$= \frac{1}{f_{j,k}\left(h'_{r(i),s(j)}\right)} \sum_{\substack{I'_{r(i),s(j)} \subset I_{r,s} \\ 1 \leq i,j \leq v(r,s)}} \sum f_{j,k}\left(h'_{r(i),s(j)}\right)$$

$$\cdot \frac{1}{f_{j,k}\left(h'_{r(i),s(j)}\right)} f_{j,k}\left(\{(j,k) \in I'_{r(i),s(j)} : |x_{jk} - L| \geq \varepsilon\}\right).$$

Also, in view of the choice of unbounded modulus functions f and using the fact that $\theta'_{r,s} = (j'_r, k'_s)$ is increasing, we have

$$\sum_{\substack{I'_{r(i),s(j)} \subset I_{r,s} \\ 1 \leq i,j \leq v(r,s)}} \sum f_{j,k}\left(h'_{(r,i),(s,j)}\right) = f_{j,k}\left(h'_{(r,1),(s,1)}\right) + f_{j,k}\left(h'_{(r,2),(s,2)}\right) + \cdots + f_{j,k}\left(h'_{(r,s),v(r,s)}\right)$$

$$\begin{aligned}
&= f_{j,k}\left(j'_{(r,1)}, k'_{(s,1)}\right) + f_{j,k}\left(j'_{(r,2)}, k'_{(s,2)}\right) + \cdots \\
&+ f_{j,k}\left(\left(j'_{(r,v(r))}, k'_{(s,v(s))}\right) - \left(j'_{(r,v(r)-1)}, k'_{(s,v(s)-1)}\right)\right) \\
&= f_{j,k}\left(\left|\left(j'_{r,1}, k'_{s,1}\right) - \left(j'_{r-1}, k'_{s-1}\right)\right|\right) + f_{j,k}\left(\left|\left(j'_{r,2}, k'_{s,2}\right) - \left(j'_{r,1}, k'_{s,1}\right)\right|\right) + \cdots \\
&+ f_{j,k}\left(\left|\left(j'_{r,v(r)}, k'_{s,v(s)}\right) - \left(j'_{(r,v(r)-1}, k'_{(s,v(s)-1)}\right)\right|\right) \\
&= \left|f_{j,k}\left(j'_{r,1}, k'_{s,1}\right) - f_{j,k}\left(j'_{r-1}, k'_{s-1}\right)\right| + \left|f_{j,k}\left(j'_{r,2}, k'_{s,2}\right) - f_{j,k}\left(j'_{r,1}, k'_{s,1}\right)\right| + \cdots \\
&+ \left|f_{j,k}\left(j'_{r,v(r)}, k'_{s,v(s)}\right) - f_{j,k}\left(j'_{(r,v(r)-1}, k'_{(s,v(s)-1)}\right)\right| \\
&= f_{j,k}\left(j'_{r,1}, k'_{s,1}\right) - f_{j,k}\left(j'_{r-1}, k'_{s-1}\right) + f_{j,k}\left(j'_{r,2}, k'_{s,2}\right) - f_{j,k}\left(j'_{r,1}, k'_{s,1}\right) + \cdots \\
&+ f_{j,k}\left(j'_{r,v(r)}, k'_{s,v(s)}\right) = f_{j,k}\left(j'_{(r,v(r)-1}, k'_{(s,v(s)-1}\right) = \left|f_{j,k}\left(j'_{(r,v(r)-1}, k'_{(s,v(s)-1}\right)\right| \\
&= f_{j,k}\left(\left|j'_{(r,v(r)-1}, k'_{(s,v(s)-1}\right|\right) = f_{j,k}\left(|h_{r,s}|\right) = f_{j,k}(h_{r,s}).
\end{aligned}$$

Thus, we have

$$\frac{1}{f_{j,k}(h_{r,s})} f_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right) \leq \frac{1}{\sum_{\substack{I'_{r(i),s(j)} \subset I_{r,s} \\ 1 \leq i,j \leq v(r,s)}} \sum f_{j,k}(h'_{r(i),s(j)})} \sum_{\substack{I'_{r(i),s(j)} \subset I_{r,s} \\ 1 \leq i,j \leq v(r,s)}} \sum f_{j,k}\left(h'_{r(i),s(j)}\right) t_{r(i),s(j)},$$

(6)
where

$$t_{r(i),s(j)} = \left(f_{j,k}(h'_{r(i),s(j)})\right)^{-1} f_{j,k}\left(\left|\{(j,k) \in I'_{r(i),s(j)} : |x_{jk} - L| \geq \varepsilon\}\right|\right).$$

Since the term on the right hand of (6) is regular weighted mean transformation of the double sequence $t_{r(i),s(j)}$, which tend to zero as $r, s \to \infty$, therefore the term on the right hand side of (6) also tends to zero as $r, s \to \infty$. Thus,

$\frac{1}{f_{j,k}(h_{r,s})} f_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right) \to 0$ as $r, s \to \infty$. Hence $x \in S_{\theta_{r,s}}^{f_{j,k}}$.

Theorem 3.2.2: Let $f_{j,k}$ be an unbounded modulus functions and $\theta'_{r,s} = (j'_m, k'_n)$ is a double lacunary refinement of double lacunary sequence

$\theta_{r,s} = (j_r, k_s)$. Let $I_{r,s} = \{(j,k) : j_{r-1} < j \leq j_r \ and \ k_{s-1} < k \leq k_s\}$, $h_r = j_r - j_{r-1}$ and $h_s = k_s - k_{s-1}$,

where

$h_{r,s} = h_r \overline{h_s}$, $r, s = 1,2,3, \ldots$, and $I'_{m,n} = \{(j,k) : j'_{m-1} < j' \leq j'_m \ and \ k'_{n-1} < k' \leq k'_n\}$, $h'_m = j'_m - j'_{m-1}$ and $h'_n = k'_n - k'_n$, where $h'_{m,n} = h'_m h'_n$, $m, n = 1,2,3, \ldots$, if there exists $\delta > 0$, such that

$\frac{f_{j,k}(h'_{m,n})}{f_{j,k}(h_{r,s})} \geq \delta$ for every $I'_{m,n} \subset I_{r,s}$,

Then $x \in S_{\theta_{r,s}}^{f_{j,k}}$ and $x \in S_{\theta'_{r,s}}^{f_{j,k}}$.

Proof: For any $\varepsilon > 0$, and for $I'_{m,n} \subset I_{r,s}$, we can find $I_{r,s}$ such that $I'_{m,n} \subset I_{r,s}$, then we have

$$\begin{aligned}
\frac{1}{f_{j,k}(h'_{m,n})} f_{j,k}\left(\left|\{(j,k) \in I'_{m,n} : |x_{jk} - L| \geq \varepsilon\}\right|\right) &\leq \frac{1}{f_{j,k}(h'_{m,n})} f_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right) \\
&= \frac{f_{j,k}(h_{r,s})}{f_{j,k}(h'_{m,n})} \frac{1}{f_{j,k}(h_{r,s})} f_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right) \\
&\leq \frac{1}{\delta} \frac{1}{f_{j,k}(h_{r,s})} f_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right)
\end{aligned}$$

From where it follows that $S_{\theta_{r,s}}^{f_{j,k}} \subset S_{\theta'_{r,s}}^{f_{j,k}}$.

In the next theorem we deal with a more general situation.

Theorem 3.2.3: Let f and g be any two modulus functions of double sequence such that $f(x) \leq g(x)$, for all $x \in [0, \infty)$, and $\theta'_{r,s} = (j'_m, k'_n)$ is a double lacunary refinement of the double sequence $\theta_{r,s} = (j_r, k_s)$. Let $I_{r,s} = \{(j,k) : j_{r-1} < j \leq j_r \text{ and } k_{s-1} < k \leq k_s\}$, $h_r = j_r - j_{r-1}$ and $h_s = k_s - k_{s-1}$, where $h_{r,s} = h_r \overline{h_s}$, $r, s = 1,2,3, \ldots$, and $I'_{m,n} = \{(j,k) : j'_{m-1} < j' \leq j'_m \text{ and } k'_{n-1} < k' \leq k'_n\}$, $h'_m = j'_m - j'_{m-1}$ and $h'_n = k'_n - k'_n$, where $h'_{m,n} = h'_m h'_n$, $m, n = 1,2,3, \ldots$, if there exists $0 < \delta \leq 1$, such that $\frac{f_{j,k}(h'_{m,n})}{g_{j,k}(h_{r,s})} \geq \delta$ for every $I'_{m,n} \subset I_{r,s}$,
Then $x \in S_{\theta_{r,s}}^{g_{j,k}}$ and $x \in S_{\theta'_{r,s}}^{f_{j,k}}$.

Proof: For any $\varepsilon > 0$, and every $I'_{m,n}$, we can find $I_{r,s}$ such that $I'_{m,n} \subset I_{r,s}$, then we have

$$\begin{aligned}
\frac{1}{f_{j,k}(h'_{m,n})} f_{j,k}\left(\left|\{(j,k) \in I'_{m,n} : |x_{jk} - L| \geq \varepsilon\}\right|\right) &\leq \frac{1}{f_{j,k}(h'_{m,n})} g_{j,k}\left(\left|\{(j,k) \in I'_{m,n} : |x_{jk} - L| \geq \varepsilon\}\right|\right) \\
&\leq \frac{1}{f_{j,k}(h'_{m,n})} g_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right) \\
&= \frac{g_{j,k}(h_{r,s})}{f_{j,k}(h'_{m,n})} \frac{1}{g_{j,k}(h_{r,s})} g_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right) \\
&\leq \frac{1}{\delta} \frac{1}{g_{j,k}(h_{r,s})} g_{j,k}\left(\left|\{(j,k) \in I_{r,s} : |x_{jk} - L| \geq \varepsilon\}\right|\right)
\end{aligned}$$

From where it follows that $S_{\theta_{r,s}}^{g_{j,k}} \subset S_{\theta'_{r,s}}^{f_{j,k}}$.

In the next theorem we show that the inclusion $S_{\theta_{r,s}}^{f_{j,k}} \subset S_{\theta'_{r,s}}^{f_{j,k}}$ is possible if even if none of $\theta_{r,s}$ and $\theta'_{r,s}$ is refinement of the other.

Theorem 3.2.4: Let f be an unbounded modulus functions such that

$$|f(x) - f(y)| = f(|x - y|), \forall\, x \geq 0, y \geq 0. \tag{7}$$

Suppose $\theta'_{r,s} = (j'_m, k'_n)$ and $\theta_{r,s} = (j_r, k_s)$. Let $I_{r,s} = \{(j,k) : j_{r-1} < j \leq j_r \text{ and } k_{s-1} < k \leq k_s\}$, $h_r = j_r - j_{r-1}$ and $h_s = k_s - k_{s-1}$, where $h_{r,s} = h_r \overline{h_s}$, $r, s = 1,2,3, \ldots$, and $I'_{m,n} = \{(j,k) : j'_{m-1} < j' \leq j'_m \text{ and } k'_{n-1} < k' \leq k'_n\}$, $h'_m = j'_m - j'_{m-1}$ and $h'_n = k'_n - k'_n$, where $h'_{m,n} = h'_m h'_n$, $m, n = 1,2,3, \ldots$, and $I_{p,q,m,n} = I_{p,q} \cap I_{m,n}$, $p, q, m, n = 1,2,3, \ldots$, if there exists $\delta > 0$ such that
$\frac{f_{j,k}(\sigma_{p,q,m,n})}{g_{j,k}(h_{r,s})} \geq \delta$ for every $p, q, m, n = 1,2,3, \ldots$,
provided $\sigma_{p,q,m,n} > 0$, where $\sigma_{p,q,m,n}$ denotes the length of the interval $I_{p,q,m,n}$ then $x \in S_{\theta_{r,s}}^{f_{j,k}}$ implies

$S_{\theta'_{r,s}}^{f\,j,k}$.

Remark 3.2.1: If the condition in theorem 4.4 is replaced by $f(\sigma_{p,q,m,n})/f(h'_{m,n}) \geq \delta$ for every $r, s, m, n = 1,2,3, \ldots$, provided $\sigma_{p,q,m,n} > 0$, where $\sigma_{p,q,m,n}$ denotes the length of the interval $I_{p,q,m,n} = I_{p,q} \cap I_{m,n}$, $p, q, m, n = 1,2,3, \ldots$, it can be seen that $x \in S_{\theta'_{r,s}}^{f\,j,k}$ implies $x \in S_{\theta_{r,s}}^{f\,j,k}$.

Combining remark 4.1 and theorem 4.4, we get the following.

Theorem 3.2.5: Let f be an unbounded modulus functions such that

$$|f(x) - f(y)| = f(|x - y|), \forall\, x \geq 0, y \geq 0. \tag{8}$$

Suppose $\theta'_{r,s} = (j'_m, k'_n)$ and $\theta_{r,s} = (j_r, k_s)$ are two double lacunary sequences. Let $I_{r,s} = \{(j,k): j_{r-1} < j \leq j_r \text{ and } k_{s-1} < k \leq k_s\}$, $h_r = j_r - j_{r-1}$ and $h_s = k_s - k_{s-1}$, where $h_{r,s} = h_r \overline{h_s}$, $r, s = 1,2,3, \ldots$, $I'_{m,n} = \{(j,k): j'_{m-1} < j' \leq j'_m \text{ and } k'_{n-1} < k' \leq k'_n\}$, $h'_m = j'_m - j'_{m-1}$ and $h'_n = k'_n - k'_n$, where $h'_{m,n} = h'_m h'_n$, $m, n = 1,2,3, \ldots$, and $I_{p,q,m,n} = I_{p,q} \cap I_{m,n}$, $p, q, m, n = 1,2,3, \ldots$, if there exists $\delta > 0$ such that

$$\frac{f_{j,k}(\sigma_{p,q,m,n})}{g_{j,k}(h_{r,s} + h'_{m,n})} \geq \delta \text{ for every } p, q, m, n = 1,2,3, \ldots,$$

provided $\sigma_{p,q,m,n} > 0$, where $\sigma_{p,q,m,n}$ denotes the length of the interval $I_{p,q,m,n}$ then $S_{\theta_{r,s}}^{f\,j,k} = S_{\theta'_{r,s}}^{f\,j,k}$.

4. Discussion

The concept of modulus lacunary statistical convergence of double sequence was introduced via modulus functions where the modulus function is bounded or unbounded. We have also introduced the concept of lacunary strong convergence of double sequence with respect modulus function. We established some inclusion relations between these two lacunary methods and proved some essential analogues results for double sequence. This concept can be further extended in the direction of fuzzy numbers of double sequence.

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