# PARAMETER ESTIMATION OF SCALE MUTH DISTRIBUTION(SMD) UNDER TYPE-1 CENSORING USING CLASSICAL AND BAYESIAN APPROACHES

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**Abstract**

*The lifetime distributions are used to understand and explain the real life circumstances in various fields (medical, engineering, etc.). Many times it is very tough task to complete an experiment with complete data due to lack of time, money or some other factors and get the data in incomplete form. to draw the information from such type of data (incomplete data), we use some censoring techniques. In the field of statistics, there are several censoring techniques available where type-I censoring is most commonly used due to its simplicity. In this article, the scale Muth distribution (SMD) is considered as a lifetime distribution under type-I censoring scheme. The parameter estimation has been done by classical as well as Bayesian approach. Under the classical paradigm, two most popular methods were used maximum likelihood estimation (MLE) and the maximum product of spacing estimation (MPSE). And under the Bayesian paradigm, we used the informative priors for each parameter and obtained the estimates by considering the squared error loss function using an approximation method, Metropolis Hasting (MH) algorithm. The performance of each estimator is evaluated by their mean squared error or simulated risk. Also, a real data set is used to illustrate the real phenomena and to estimate the parameter using above-mentioned techniques under type-I censoring scheme.*

**Keywords:** Scale Muth distribution, type-I censoring, classical approach, Bayesian approach, real data analysis.

## 1. Introduction

A new popularized lifetime distribution, Muth distribution was introduced by J. E. Muth [1] in the field of reliability theory. Muth distribution has less probability mass on right tail in comparison of some other well known distribution like Weibull, Gamma and log-normal. The cumulative distribution function (cdf) and probability density function (pdf) of this distribution are defined respectively:

$$F(y;\alpha) = 1 - e^{\left\{\alpha y - \frac{1}{\alpha}(e^{\alpha y}-1)\right\}} \qquad y > 0, \quad \alpha \in (0,1] \tag{1}$$

$$f(y;\alpha) = \begin{cases} (e^{\alpha y} - \alpha)e^{\left\{\alpha y - \frac{1}{\alpha}(e^{\alpha y}-1)\right\}} & y > 0, \quad \alpha \in (0,1] \\ 0 & otherwise \end{cases} \tag{2}$$

A very few properties of this distribution was given in [1] and mainly focuses on its strict positive memory and mean residual lifetime. After this Rinne [2] used this distribution as lifetime model with German data set of car prices for fitting. In 2008, Leemis and McQueston [3] given a well defined figure and showed the relationship with exponential distribution. When the shape parameter $\alpha \to 0$ it converges to standered exponential distribution having parameter 1, and given its name as Muth distribution. In 2015, Jodra et.al. [4] again highlighted this distribution and derived some other statistical properties such as, moment generating function (mgf), Quantile function, $r^{th}$ moments, moments of order statistics , mode, median and given a scale transformation form of this distribution named as SMD with scale parameter $\beta > 0$ and shape parameter $\alpha \in (0,1]$.

Estimation of parameters of SMD with classical methods has been done in [4]. After that many of authors used this distribution with different transformation and derived their properties and applications such as Jodra et. al. [5] define power Muth distribution, Chesneau and Agiwal [6] given inverse power Muth distribution and observed that there is no restriction to select the values of $\alpha$ in eq.[1] as $\alpha < 0$ and $\alpha \in (0,1]$ but $\alpha \neq 0$. In 2022, Saroj et.al. [7] given a new lifetime distribution with up-side down bathtub (UBT) shape hazard, named as inverse Muth distribution(IMD) by taking inverse transformation of Muth distribution, and derived some statistical properties of IMD where it was found that the moment of any order for IMD does not exist for $\alpha \in (0,1]$. Also, the scale transformation was considered from the IMD named as scale inverse Muth distribution with real data applicability [7]. Recently in 2022 Sonker et. al.[8] estimated the parameter under simple stress-strength reliability and multi-component stress strength. In 2018, Almarashi and Elgarhy [9] given a Muth generated (M-G) class of distribution with the help of Muth distribution as a generator and T-X generated family [10]. Then Al-Babtain et. al. introduced a Transmuted M-G class of distribution [11] and Almarashi et. al. introduced a new truncated M-G family of distribution[12]. Some censoring schemes are also used by some authors for different form of Muth distribution such as power Muth distribution under progressive censoring scheme [13] and SMD under type-II censoring scheme [14].

Generally lifetime distributions are used under the life testing experiments, where we get the data in ordered form by failure (or death) of testing units. Many of the times, it is very tough task to record failure time of each event due to lack of time cost effectiveness and many more. In all these cases we need a technique that helps in drawing inferences about population after observing a part of testing units. For this, there are several censoring techniques are available for different cases and situations. There are many censoring methods available in statistics one of them is type-I censoring scheme which is a simplest censoring technique and the form of right censoring. Under the type-I censoring scheme, we select a time point $T_o$ to terminate the experiment and record the failure time of the event which fails before $T_o$, and after $T_o$ it assumed that all the remaining items are censored. A very impressive experimental example for real life situation under type-I censoring was given in [15]. For the more details about type-I censoring see [16] & [17].

In this article, classical as well as Bayesian method of estimation are used to estimate the parameter of SMD under type-I censoring scheme. In classical method two most popular method MLE and MPSE are used and Bayesian estimation is performed under squared error loss function (SELF) and an approximation method is used to find the estimates because our posterior is not in simplest form.

## 2. Classical methods of estimation

### 2.1. Maximum Likelihood Estimation

In the classical inference, one of the most popular estimation method was proposed by R. A. Fisher in 1920 based on likelihood function called MLE. The main principle of this method is to search for the value of parameter which maximizes the likelihood function. In this article we have to estimate the parameter of SMD under type-I censoring scheme. In this case the likelihood function was defined by Cohen(1965)[18] as:

$$L(\theta|x) = C\prod_{i=1}^{n}[f(x_i)]^{\delta_i}[1-F(T_o)]^{(1-\delta_i)} \tag{3}$$

where $\delta_i$ is an indicator function which indicates that an experimental unit is observed or censored. If $T_o$ is the pre-assigned time point to terminate the experiment then $\delta_i$ is defined for the $i^{th}; i = 1, 2, 3, ..., n$ unit as:

$$\delta_i = \begin{cases} 1 & if \quad x_i < T_o \\ 0 & if \quad x_i > T_o \end{cases}$$

Let us suppose that the failure time of m units are observed from n before time point $T_o$, then the constant C will be defined as $\frac{n!}{(n-m)!}$, and the Likelihood function is:

$$L(\theta|x) = \prod_{i=1}^{m}[f(x_i)][1-F(T_o)]^{(n-m)} \tag{4}$$

Let $x \sim SMD(\alpha, \beta)$ having pdf and cdf as:

$$F(x;\alpha,\beta) = 1 - e^{\left\{\frac{\alpha}{\beta}x - \frac{1}{\alpha}\left(e^{\frac{\alpha}{\beta}x}-1\right)\right\}} \qquad x > 0, \quad \alpha \in (0,1] \tag{5}$$

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta}(e^{\frac{\alpha}{\beta}x} - \alpha)e^{\left\{\frac{\alpha}{\beta}x - \frac{1}{\alpha}(e^{\frac{\alpha}{\beta}x}-1)\right\}} & x > 0, \quad \alpha \in (0,1] \\ 0 & otherwise \end{cases} \tag{6}$$

then the Likelihood function is expressed as:

$$L(\alpha,\beta|x) = \frac{n!}{(n-m)!}\prod_{i=1}^{m}\left[\frac{1}{\beta}(e^{(\frac{\alpha}{\beta}x_i)}-\alpha)e^{\left\{\frac{\alpha}{\beta}x_i-\frac{1}{\alpha}(e^{(\frac{\alpha}{\beta}x_i)}-1)\right\}}\right]\left[1-\left\{1-e^{\left\{\frac{\alpha}{\beta}x_i-\frac{1}{\alpha}(e^{(\frac{\alpha}{\beta}x_i)}-1)\right\}}\right\}\right]^{n-m}$$

$$L(\alpha,\beta/x) = C\frac{1}{\beta^m}\prod_{i=1}^{m}\left[(e^{(\frac{\alpha}{\beta}x_i)}-\alpha)e^{\left\{\frac{\alpha}{\beta}x_i-\frac{1}{\alpha}(e^{(\frac{\alpha}{\beta}x_i)}-1)\right\}}\right]\left[e^{\left\{\frac{\alpha}{\beta}x_i-\frac{1}{\alpha}(e^{(\frac{\alpha}{\beta}x_i)}-1)\right\}}\right]^{n-m} \tag{7}$$

where $C=\frac{n!}{(n-m)!}$. after taking the Log of both side of eq.[7] the log-likelihood defined as

$$log(L(\alpha,\beta/x)) = log(C) - mlog(\beta) + \sum_{i=1}^{m} log\left(e^{(\frac{\alpha}{\beta}x_i)}-\alpha\right) + \sum_{i=1}^{m}\left(\frac{\alpha}{\beta}x_i - \frac{1}{\alpha}(e^{(\frac{\alpha}{\beta}x_i)}-1)\right) + (n-m)\left(\frac{\alpha}{\beta}T_o - \frac{1}{\alpha}(e^{(\frac{\alpha}{\beta}T_o)}-1)\right) \tag{8}$$

To obtain the MLE for $\alpha$ and $\beta$ we have to search the value of $\alpha$ and $\beta$ which maximizes the eq.[8]. now, differentiate the eq.[8] w.r.t. $\alpha$ and $\beta$ and equating to 0 we get,

$$\frac{\partial}{\partial\alpha}log(L(\alpha,\beta/x)) = 0$$

$$\sum_{i=1}^{m}\frac{x_i e^{\frac{\alpha}{\beta}x_i}-\beta}{\beta(e^{\frac{\alpha}{\beta}x_i}-\alpha)} + \sum_{i=1}^{m}\left\{\frac{x_i}{\beta}+\frac{e^{\frac{\alpha}{\beta}x_i}-1)}{\alpha^2}-\frac{x_i\, e^{\frac{\alpha}{\beta}x_i}}{\alpha\beta}\right\} + (n-m)\sum_{i=1}^{m}\left\{\frac{T_o}{\beta}+\frac{e^{\frac{\alpha}{\beta}T_o}-1)}{\alpha^2}-\frac{x_i\, e^{\frac{\alpha}{\beta}T_o}}{\alpha\beta}\right\} = 0 \tag{9}$$

and,

$$\frac{\partial}{\partial\beta}log(L(\alpha,\beta/x)) = 0$$

$$-\frac{m}{\beta} - \sum_{i=1}^{m}\frac{\left(\alpha x_i e^{\frac{\alpha}{\beta}x_i}\right)}{\left(e^{\frac{\alpha}{\beta}x_i}-\alpha\right)} + \frac{1}{\beta^2}\sum_{i=1}^{m}x_i\left(e^{\frac{\alpha}{\beta}x_i}-\alpha\right) + \frac{T_o(n-m)}{\beta^2}\sum_{i=1}^{m}\left(e^{\frac{\alpha}{\beta}T_o}-\alpha\right) = 0 \tag{10}$$

now solve these two non-linear eq.[9] and eq.[10] to get the value of $\alpha$and $\beta$ with satisfying the condition

$$\left(\frac{\partial\ log(L(\alpha,\beta|x))}{\partial\alpha\partial\beta}\right)_{\alpha_{ml},\ \beta_{ml}} < 0 \tag{11}$$

As we can see that eq.[9] and eq.[10] are the complex function for $\alpha$ and $\beta$ i.e. we can not solve both the equations analytically. To obtain the value of $\alpha$ and $\beta$ we used Newton Raphson iteration method and get the the value of $\alpha$ and $\beta$ as $\hat{\alpha}_{ml}$ and $\hat{\beta}_{ml}$ which satisfying the condition eq.[11] and maximize the eq.[8].

## 2.2. Maximum Product of Spacing Estimation

This (MPSE) method is an another method of estimation alternative to MLE which is originally proposed by Cheng and Amin in 1983 [19] and Ranneby in 1984 [20]. It provides the consistent and asymptotically unbiased estimate. This method cover the drawback of MLE where MLE fails to estimate three or more parameter distribution like gamma, weibull and log-normal [21] and also the major limitation of MLE is that it does not work satisfactorily for Heavy tailed continuous distribution with unknown location and scale parameter [22]. The estimators obtained by MPSE method possess most of the large sample properties like sufficiency, consistency and asymptotic efficiency, possessed by MLE under more general condition [[23] & [21]] and MPSE holds the invariance property same as the MLE, shown by Coolen and Newby [24]. The consistency of MPS estimator have been discussed in detailed in [25]. MPS method is mostly similar as MLE where in case of MLE we use the likelihood function and search the value of parameter which maximises the likelihood function but in the case of MPSE method we tried to find the estimated value of parameter which maximises the product of spacing function based on cdf of the target density function.

Let $x_1, x_2, ..., x_n$, be an ordered random sample of size n from the univariate distribution. For the MPSE, the general condition is that the density function is considered to be strictly positive in an interval (a,b) and zero elsewhere ($a = -\infty \quad and \quad b = \infty$). In this article we have the cdf F(x) and pdf f(x) defined in eq.[5] and [6] respectively with supporting value $x > 0$; $0 < \alpha \leq 1$ and $\beta > 0$. By this $F(x) = 0$ and $f(x) = 0$ for $x < 0$, and $F(x) = 1$ and $f(x) = 0$ for $x > \infty$. The $i^{th}$ spacing function is defined as $D_i = F(x_i) - F(x_{i-1})$ and the product of spacing function is $S = \prod_{i=1}^{n+1} D_i$. The average of this is measured by geometric mean of spacing denoted by 'G' and defined as $G = S^{\frac{1}{n+1}}$

If the sample is very likely or most probable, it is assumed that they gives the magnitude of spacing more or less equal. Hence, MPSE will be that value of parameter which maximizes the S or G. This study considered under type-I censoring so, the sample values belong in (0,∞) can be divided in no. of parts of interval like $[0, x_1]$, $(x_1, x_2]$, $(x_2, x_3]...(x_{k-1}, \ x_k]$, $(x_k, T_o]$, $(T_o, \infty)$. MPSE method described in detail for the type-I censoring case in [14]. According to [14] we may write the product of spacing function for chosen distribution as :

$$S(\alpha, \beta|x) \propto \begin{cases} f(T_o) \cdot \prod_{i=1}^{m} D_i \cdot \left\{\prod_{j=1}^{n+1} D_{m+j}\right\}; & if \quad |T_o - x_m| < \epsilon \\ D_{\xi} \cdot \{\prod_{i=1}^{m} D_i\} \cdot \left\{\prod_{j=1}^{n+1} D_{m+j}\right\}; & if \quad otherwise \end{cases} \tag{12}$$

where, $D_i = F(x_i) - F(x_{i-1})$, $D_{m+j} = (1 - F(T_o))/(n-m)$, $D_{\xi} = F(T_o) - F(x_m)$ and $F(x)$, $f(x)$ given in eq.(5), (6) respectively. Thus the MPSE can be obtained by maximizing $S(\alpha, \beta|x) = log(S(\alpha, \beta|x))$. Now differentiating log of eq.(12) w.r.t. $\alpha$ and $\beta$ we get,

$$\frac{\partial}{\partial \alpha} S(\alpha, \beta|x) \propto \begin{cases} \frac{f'(T_o)}{f(T_o)} + \sum_{i=1}^{m} \frac{F'(x_i) - F'(x_{i-1})}{F(x_i) - F(x_{i-1})} - \frac{(n-m)F'(T_o)}{1 - F(T_o)} & if \quad |T_o - x_m| < \epsilon \\ \\ \frac{F'(T_o) - F'(x_m)}{F(T_o) - F(x_m)} + \sum_{i=1}^{m} \frac{F'(x_i) - F'(x_{i-1})}{F(x_i) - F(x_{i-1})} - \frac{(n-m)F'(T_o)}{1 - F(T_o)} & otherwise \end{cases} \tag{13}$$

where,

$$\frac{\partial}{\partial \alpha} f'(x) = \frac{A_1}{\alpha^2 \beta} \left[(\alpha - e^{\frac{\alpha}{\beta}x}) \left\{\alpha x(e^{\frac{\alpha}{\beta}x} - \alpha) - \beta e^{\frac{\alpha}{\beta}x} + \beta\right\} + \alpha^2 (x e^{\frac{\alpha}{\beta}x} - \beta)\right]$$

$$\frac{\partial}{\partial \alpha} F'(x) = \frac{A_1}{\alpha^2 \beta} \left[\left\{\alpha x(e^{\frac{\alpha}{\beta}x} - \alpha) - \beta e^{\frac{\alpha}{\beta}x} + \beta\right\}\right]$$

and
$$A_1 = e^{(\frac{\alpha}{\beta}x - \frac{1}{\alpha}(e^{\frac{\alpha}{\beta}x} - 1))}$$

And the partial derivative of log of eq.[12] w.r.t. $\beta$ will be similar as eq.[13]

$$\frac{\partial}{\partial \beta} S(\alpha, \beta|x) \propto \begin{cases} \frac{f'(T_o)}{f(T_o)} + \sum_{i=1}^{m} \frac{F'(x_i) - F'(x_{i-1})}{F(x_i) - F(x_{i-1})} - \frac{(n-m)F'(T_o)}{1 - F(T_o)} & if \quad |T_o - x_m| < \epsilon \\ \\ \frac{F'(T_o) - F'(x_m)}{F(T_o) - F(x_m)} + \sum_{i=1}^{m} \frac{F'(x_i) - F'(x_{i-1})}{F(x_i) - F(x_{i-1})} - \frac{(n-m)F'(T_o)}{1 - F(T_o)} & o.w. \end{cases} \tag{14}$$

where,

$$\frac{\partial}{\partial\beta}f'(x) = -\frac{A_1}{\beta^2}\left[\beta(e^{\frac{\alpha}{\beta}x}-\alpha)+\alpha x e^{\frac{\alpha}{\beta}x}+x(e^{\frac{\alpha}{\beta}x}-\alpha)^2\right]$$

$$\frac{\partial}{\partial\beta}F'(x) = -\frac{A_1}{\beta^2}\left(e^{\frac{\alpha}{\beta}x}-\alpha\right)$$

Now equating eq.[13], [14] to zero and solve it for $\alpha$ and $\beta$, we get the estimated value of $\alpha$ and $\beta$ which maximizes the eq.[12]. But the analytical solution is not possible because those equations are not in closed form. Hence we use Newton Raphson method to solve it numerically and get estimated value as $\hat{\alpha}_{mp}$ and $\hat{\beta}_{mp}$.

## 3. Asymptotic Confidence Interval

In this section, we define the interval estimation for $\alpha$and $\beta$ . Since the exact distribution of MLE and MPSE is not easy to find because of the maximum likelihood and product of spacing function is not in the explicit form. Several authors show that the MPSE method is asymptotically equivalent to the MLE see [[19], [26], [27], [28]]. So, by using the large sample theory we may propose the Asymptotic Confidence Interval based on MLE and MPSE function and we may write the asymptotic distribution of both kind of estimator is:

$$(\hat{\theta}-\theta) \equiv N\big(0, I^{-1}(\hat{\theta})\big); \tag{15}$$

where,
$\hat{\theta}$ is the estimate of parameter
$\theta$ is the true value of parameter
$I^{-1}(\hat{\theta})$ is the inverse of Fisher information matrix

The $100(1-\alpha^*)$%confidence interval is defined as:

$$\hat{\theta} \ \pm \ Z_{\alpha^*/2}\sqrt{Var(\hat{\theta})}; \tag{16}$$

where $Var(\hat{\theta}) = -E\left(\frac{\partial^2(L)}{\partial x^2}\right)^{-1}_{\theta=\hat{\theta}}$ is the digonal element of the inverse of Information matrix $I^{-1}(\hat{\theta})$
In our case the information matrix is defined as:

$$I(\hat{\alpha};\hat{\beta}) = \begin{bmatrix} I_{1,1} & I_{1,2} \\ I_{2,1} & I_{2,2} \end{bmatrix}$$

Here $I_{1,1} = \frac{\partial^2}{\partial\alpha^2}$, $I_{2,2} = \frac{\partial^2}{\partial\beta^2}$ *and* $I_{1,2} = I_{2,1} = \frac{\partial^2}{\partial\alpha\partial\beta}$ are the second order derivative of log-likelihood function eq.[8] and the log of the product spacing function eq.[12] . So, for $\alpha$ the $100(1-\alpha^*)$% confidence interval is defined as:

$$\hat{\alpha} \ \pm \ Z_{\alpha^*/2}\sqrt{Var(\hat{\alpha})}; \qquad where \ \hat{\alpha} \ is \ \hat{\alpha}_{ml} \ and \ \hat{\alpha}_{mp} \tag{17}$$

and for $\beta$

$$\hat{\beta} \ \pm \ Z_{\alpha^*/2}\sqrt{Var(\hat{\beta})}; \qquad where \ \hat{\beta} \ is \ \hat{\beta}_{ml} \ and \ \hat{\beta}_{mp} \tag{18}$$

## 4. Bayes estimates of $\alpha$ and $\beta$ under squared error loss function

In this section we considered Bayesian approach to estimate the parameter $\alpha$ and $\beta$ under type-I censoring scheme. In the Bayesian method it is assumed that our parameter are random in nature instead of a fixed quantity. As the random nature of the parameter, we can associate it with distribution function such as called prior distribution. Let us consider that gamma prior for scale parameter $\beta$ and beta prior for shape parameter $\alpha$, as informative prior, therefore

$$\pi(\alpha) \propto \alpha^{a-1} \cdot (1-\alpha)^{b-1} \tag{19}$$

and

$$\pi(\beta) \propto \beta^{c-1} \cdot e^{-d\beta} \tag{20}$$

where, a,b and c,d are hyper parameters for beta prior and gamma prior respectively. Thus the joint prior of $\alpha$ and $\beta$ is

$$\pi(\alpha,\beta) \propto \alpha^{a-1} \cdot (1-\alpha)^{b-1} \cdot \beta^{c-1} \cdot e^{-d\beta} \tag{21}$$

Now, according to the Bayes rule the formula for joint posterior of $\alpha$ and $\beta$ is

$$\Pi(\alpha,\beta|x) = \frac{\pi(\alpha,\beta)L(\alpha,\beta|x)}{\int \pi(\alpha,\beta)L(\alpha,\beta|x)d\alpha\, d\beta} \tag{22}$$

where, $L(\alpha,\beta|x)$ is likelihood given in eq.[7]. So, combining the eq.[7] and [21] the posterior for $\alpha$ and $\beta$ can be written as:

$$\Pi(\alpha,\beta|x) \propto C_1^{-1} \cdot \beta^{(c-m-1)} \cdot \alpha^{(a-1)} \cdot (1-\alpha)^{(b-1)} \cdot e^{(n-m)(\frac{\alpha}{\beta}T_o - \frac{1}{\alpha}(e^{\frac{\alpha}{\beta}T_o}-1)) - d\beta} \cdot e^{\sum(\frac{\alpha}{\beta}x_i - \frac{1}{\alpha}(e^{\frac{\alpha}{\beta}x_i}-1))} \tag{23}$$

where, $C_1 = \int \beta^{(c-m-1)} \cdot \alpha^{(a-1)} \cdot (1-\alpha)^{(b-1)} \cdot e^{(n-m)(\frac{\alpha}{\beta}T_o - \frac{1}{\alpha}(e^{\frac{\alpha}{\beta}T_o}-1)) - d\beta} \cdot e^{\sum(\frac{\alpha}{\beta}x_i - \frac{1}{\alpha}(e^{\frac{\alpha}{\beta}x_i}-1))}\, d\alpha\, d\beta$
In the case of Bayesian techniques, loss function play very important role. In this article, to evaluate the Bayes estimator of $\alpha$ and $\beta$ we use a symmetric loss function, squared error loss function (SELF) which gives the equal weight for under and over estimation problem. The general form of SELF can be expressed as:

$$L(\hat{\theta},\theta) = \lambda(\hat{\theta}-\theta)^2; \qquad \lambda > 0 \tag{24}$$

where $\hat{\theta}$ is the estimate of parameter $\theta$. the Bayes estimator of parameter under SELF is posterior mean i.e. posterior expectation $E_\theta[\theta|x]$. In our case it is found that the posterior mean for both parameter is the ratio of two mathematically non-tractable integrals; so we use the Markov Chain Monte Carlo (MCMC) with Metropolis-Hasting (MH) algorithm to obtain Bayes estimate. MH algorithm is generalization form of basic algorithm originated by Metropolis and Ulman(1949) [29], then Metropolis et. al. (1953) used first time in statistics study the equation of a state of a two-dimensional rigid sphere system. In 1970, W. K. Hasting gives the original method by using arbitrary proposal distribution and it popularized by name MH algorithm. For more detail see[[30], [31] & [32]]. By using this technique we can generate the sample from posterior eq.[23] and evaluate the Bayes estimate. The following iterative steps are involved in MH algorithm:
1. Set an initial guess value $\theta^0$.

2. By using previous point Generate the next value of $\theta$ as $\theta^j$ from the proposal density $p^*(\theta^j|\theta^{j-1})$. Here we take normal distribution as the proposal density for parameter.
3. Then we generate $u$ $U(0,1)$and calculate the ratio:

$$r = \frac{p(\theta^j|\underset{\sim}{x}) \cdot p^*(\theta^{j-1}|\theta^j)}{p(\theta^{j-1}|\underset{\sim}{x}) \cdot p^*(\theta^j|\theta^{j-1})}$$

4. Set

$$\theta^j = \begin{cases} \theta^j & if \;\; u \leq \frac{p(\theta^j|\underset{\sim}{x}) \cdot p^*(\theta^{j-1}|\theta^j)}{p(\theta^{j-1}|\underset{\sim}{x}) \cdot p^*(\theta^j|\theta^{j-1})} \\ \theta^{j-1} & otherwise \end{cases}$$

5. Repeat step (2-4) for j=1,2,3,...,M and obtain $\theta_1$,$\theta_2$,...,$\theta_M$
6. To find Bayes estimate under SELF, we take

$$E_p\left(\theta|\underset{\sim}{x}\right) = \left(\frac{1}{M-M_0}\sum_{M_0+1}^{M}\theta\right)$$

where $M_0$ is burn-in period of Markove chain.

## 5. Highest Posterior density interval

Nowadays, in the field of Bayesian inference is considered that the parameter of interest behaves like a random variable, then what is probability that the parameter (say $\theta$) lies within a specified interval. Edwards et al. (1963) [33] suggested a method to calculate this interval and named it credible interval. By using this many of the statistician summarizes the marginal posterior density by $100(1-\alpha^*)\%$ posterior credible intervals for the parameters. These intervals can be easily calculated by analytically or MCMC simulation technique. The shortest interval among all of the Bayesian credible intervals is called highest posterior density (HPD) interval. The HPD intervals for parameter can be obtained by using method explained by Chen and Shao (1999)[34] on the basis of MCMC sample from posterior density in Eq.[23]. There are also mentioned that when a marginal distribution is not symmetric, a HPD interval is more desirable. Box and Tiao (1973)[35] was already discused that the HPD interval has two main properties as: (i) the density for the points which are lie within the interval is more than that for the points outside the interval, and (ii) for a given probability say, (1 -$\alpha^*$), the interval is of the shortest length. If we have the posterior as $p(\theta|x)$ then the credible interval for $\theta$ is defined as:

$$(\theta_{[(\alpha/2)M]}, \theta_{[1-(\alpha/2)M]})$$

and the HPD interval is defined as:

$$(\theta_{j^*}, \theta_{j^*+[1-(\alpha/2)]})$$

where, $j^*$ is chosen by which length $l_{j^*} = min(\theta_{j+[1-(\alpha/2)]} - \theta_j)$ for $1 \leq j \leq M - [(1-\alpha)M]$

## 6. Simulation Study

Under this section, we study about the behavior of the estimated value of parameters using the simulation study and compare the performances of MLE, MPSE and Bayes estimator on the basis of their mean squared error (MSE) values. To compute the value of MSE, we use the formula as; $MSE(\theta)=\frac{1}{N}\sum_1^N(\theta-\hat{\theta}_i)^2$, where N is the number of simulated random sample (we choose N=10000), $\theta$ is the true value of parameter and $\hat{\theta}_i$ is the estimated value of the parameter for $i^{th}$ simulated random sample. all the computation was done by R-software like sample generation from the considered distribution and finding the estimated value. To generate the random sample, we considered the following steps:

Step 1. Provide some selected values of the parameters, $\alpha$ and $\beta$, and sample size n.

Step 2. Then generate a random sample from a standard uniform distribution U(0, 1), which also be size n denoted as $u_i; \quad i = 1, 2, 3..., n$

Step 3. Now by using the quantile function of SMD given in [4], obtain $u^{th}$ quantile (random sample) from the SMD $x_i$.

Step 4. Repeat steps 2 and 3, for each i = 1, 2, ... , n we get a random sample of size n from the SMD.

In step 3 the Lambert-W function $W_{-1}(.)$ can be computed using R-command lambertWm1() from 'lamW ' package, see Adler [2]. In the simulation study, we choose the different combination of parameter values as $(\alpha = 0.3, \ \ \beta = 0.5)$, $(\alpha = 0.5, \ \ \beta = 0.5)$, $(\alpha = 0.7, \ \ \beta = 0.5)$, $(\alpha = 0.3, \ \ \beta = 2.5)$, $(\alpha = 0.5, \ \ \beta = 2.5)$, $(\alpha = 0.7, \ \ \beta = 2.5)$, and observed the behavior of MSE by varying sample size. For this we choose the different sample sizes as $n = (30, \ 50, \ 70)$. In case of Bayesian estimate we need the value of hyper parameters for which we take prior means equal to true value of parameter and the variances can be chosen as, for small value of parameter ($\alpha$=(0.3,0.5) & $\beta$=0.5) we take small variance for gamma and beta prior and for the large value of parameter ($\alpha$=0.7 & $\beta$=2.5) we take some large variance for the both prior. An another issue in this simulation study is the choice of $T_o$. We take $T_o = (0.5, 0.9)$ for $\beta$=0.5 with different value of $\alpha$, and $T_o = (3, 5)$ for $\beta$=2.5 with different value of $\alpha$. On observing the simulation table [1-6] we found that:

In general, as sample size increases the MSE of all estimators decreases. This shows that the estimators are consistent. And this decreasing nature of MSE is also found with increasing $T_o$.

The average length of asymptotic confidence interval as well as HPD interval decreases as sample size increases.

MSE of Bayes estimates is least in comparison of MLE and MPSE both, and the MSE of MPSE is less that MLE.

Width of the HPD interval is smaller in comparison of width of asymptotic confidence interval and the width of asymptotic confidence interval for MPSE is less than the width for asymptotic confidence interval MLE.

**Table 1:** *Estimates of the parameters, MSE, CI and HPD interval for α=0.3, β=0.5*

| | n | To | MLE Est. | MLE MSE | MLE CI Length | MPS Est. | MPS MSE | MPS CI Length | Bayes Est. | Bayes MSE | Bayes HPDI Length |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha = 0.3$ | 30 | 0.5 | 0.34704 | 0.07774 | 0.72461, 0.06792<br>0.65669 | 0.33273 | 0.06535 | 0.72461, 0.06792<br>0.65669 | 0.49504 | 0.04156 | 0.73352, 0.25224<br>0.48128 |
| | | 0.9 | 0.32891 | 0.04485 | 0.76512, 0.09032<br>0.67480 | 0.31103 | 0.03862 | 0.75007, 0.08840<br>0.66167 | 0.46616 | 0.03210 | 0.68793, 0.24349<br>0.44444 |
| | 50 | 0.5 | 0.32582 | 0.05735 | 0.72646, 0.08661<br>0.63985 | 0.31781 | 0.05174 | 0.71290, 0.09112<br>0.62179 | 0.47571 | 0.03562 | 0.69921, 0.24710<br>0.45211 |
| | | 0.9 | 0.31201 | 0.03038 | 0.72890, 0.10531<br>0.62359 | 0.30214 | 0.02796 | 0.71555, 0.10427<br>0.61129 | 0.43824 | 0.02370 | 0.64006, 0.23556<br>0.40450 |
| | 70 | 0.5 | 0.31557 | 0.04712 | 0.71985, 0.09472<br>0.62513 | 0.31017 | 0.04380 | 0.70609, 0.09875<br>0.60734 | 0.46208 | 0.03090 | 0.67481, 0.24393<br>0.43088 |
| | | 0.9 | 0.29997 | 0.02274 | 0.68941, 0.11159<br>0.57782 | 0.29338 | 0.02154 | 0.67894, 0.11067<br>0.56827 | 0.41580 | 0.01768 | 0.60281, 0.22852<br>0.37429 |
| $\beta = 0.5$ | 30 | 0.5 | 0.51621 | 0.01738 | 0.80992, 0.36543<br>0.44448 | 0.50410 | 0.01401 | 0.76774, 0.36775<br>0.39998 | 0.50820 | 0.01338 | 0.69941, 0.35175<br>0.34765 |
| | | 0.9 | 0.50952 | 0.00725 | 0.67859, 0.38679<br>0.29180 | 0.50958 | 0.00640 | 0.67060, 0.39217<br>0.27843 | 0.52538 | 0.00699 | 0.67103, 0.39446<br>0.27657 |
| | 50 | 0.5 | 0.51008 | 0.01114 | 0.73543, 0.37591<br>0.35951 | 0.50452 | 0.00943 | 0.93305, 0.38101<br>0.55204 | 0.49198 | 0.00676 | 0.63733, 0.36875<br>0.26858 |
| | | 0.9 | 0.50562 | 0.00415 | 0.63644, 0.40359<br>0.23284 | 0.50584 | 0.00366 | 0.63370, 0.40661<br>0.22708 | 0.51580 | 0.00388 | 0.62735, 0.41243<br>0.21492 |
| | 70 | 0.5 | 0.50932 | 0.00790 | 0.70286, 0.38420<br>0.31866 | 0.50536 | 0.00674 | 0.72334, 0.38939<br>0.33395 | 0.48843 | 0.00462 | 0.61332, 0.38087<br>0.23245 |
| | | 0.9 | 0.50449 | 0.00286 | 0.61525, 0.41471<br>0.20054 | 0.50428 | 0.00271 | 0.61537, 0.41601<br>0.19936 | 0.51197 | 0.00259 | 0.60662, 0.42325<br>0.18337 |

**Table 2:** *Estimates of the parameters, MSE, CI and HPD interval for α=0.5, β=0.5*

| | n | To | MLE Est. | MLE MSE | MLE CI Length | MPS Est. | MPS MSE | MPS CI Length | Bayes Est. | Bayes MSE | Bayes HPDI Length |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha = 0.5$ | 30 | 0.5 | 0.48074 | 0.07860 | 0.83626, 0.10341<br>0.73285 | 0.45601 | 0.06665 | 0.82580, 0.11771<br>0.70808 | 0.52302 | 0.00395 | 0.76440, 0.27504<br>0.48936 |
| | | 0.9 | 0.50786 | 0.04226 | 0.85294, 0.17571<br>0.67723 | 0.47783 | 0.03675 | 0.83123, 0.17776<br>0.65346 | 0.52332 | 0.00517 | 0.74687, 0.29615<br>0.45072 |
| | 50 | 0.5 | 0.49154 | 0.05688 | 0.84875, 0.15769<br>0.69106 | 0.47462 | 0.05084 | 0.83208, 0.16627<br>0.66582 | 0.52207 | 0.00485 | 0.75113, 0.28546<br>0.46566 |
| | | 0.9 | 0.50121 | 0.02726 | 0.79694, 0.23084<br>0.56610 | 0.48342 | 0.02516 | 0.77794, 0.22849<br>0.54944 | 0.51618 | 0.00582 | 0.71989, 0.30895<br>0.41094 |
| | 70 | 0.9 | 0.48544 | 0.04791 | 0.83548, 0.18795<br>0.64754 | 0.47320 | 0.04426 | 0.81647, 0.19507<br>0.62140 | 0.51835 | 0.00532 | 0.73660, 0.29214<br>0.44445 |
| | | 0.9 | 0.50248 | 0.01965 | 0.75472, 0.26570<br>0.48902 | 0.48975 | 0.01847 | 0.73761, 0.26445<br>0.47316 | 0.51345 | 0.00587 | 0.70190, 0.32155<br>0.38035 |
| $\beta = 0.5$ | 30 | 0.5 | 0.53008 | 0.01885 | 0.87626, 0.36571<br>0.51055 | 0.51941 | 0.01454 | 0.88980, 0.36510<br>0.52470 | 0.53692 | 0.01431 | 0.74564, 0.36762<br>0.37801 |
| | | 0.9 | 0.50709 | 0.00447 | 0.65383, 0.39495<br>0.25888 | 0.50638 | 0.00415 | 0.65251, 0.39464<br>0.25787 | 0.51898 | 0.00463 | 0.65208, 0.39763<br>0.25445 |
| | 50 | 0.5 | 0.52053 | 0.01047 | 0.75021, 0.38034<br>0.36987 | 0.51528 | 0.00892 | 0.73297, 0.38130<br>0.35167 | 0.52322 | 0.00794 | 0.67951, 0.39025<br>0.28926 |
| | | 0.9 | 0.50307 | 0.00248 | 0.61212, 0.41396<br>0.19816 | 0.50265 | 0.00238 | 0.71388, 0.41340<br>0.30048 | 0.51130 | 0.00250 | 0.61151, 0.41741<br>0.19410 |
| | 70 | 0.5 | 0.51661 | 0.00723 | 0.70762, 0.39148<br>0.31614 | 0.51257 | 0.00643 | 1.18937, 0.39147<br>0.79790 | 0.51562 | 0.00486 | 0.64536, 0.40269<br>0.24267 |
| | | 0.9 | 0.50260 | 0.00179 | 0.59260, 0.42653<br>0.16608 | 0.50227 | 0.00173 | 0.59251, 0.42607<br>0.16643 | 0.50890 | 0.00179 | 0.59275, 0.42927<br>0.16348 |

**Table 3:** *Estimate of the parameters, MSE, CI and HPD interval for α=0.7, β=0.5*

| | | | MLE | | | MPS | | | Bayes | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | n | To | Est. | MSE | CI<br>Length | Est. | MSE | CI<br>Length | Est. | MSE | CI<br>Length |
| $\alpha = 0.7$ | 30 | 0.5 | 0.63980 | 0.06248 | 0.92610, 0.14204<br>0.78406 | 0.59994 | 0.05700 | 0.91185, 0.17602<br>0.73583 | 0.55456 | 0.02370 | 0.79820, 0.30191<br>0.49629 |
| | | 0.9 | 0.50786 | 0.04226 | 0.91664, 0.26414<br>0.65250 | 0.64827 | 0.02526 | 0.89101, 0.28691<br>0.60410 | 0.58975 | 0.01594 | 0.80713, 0.36497<br>0.44216 |
| | 50 | 0.5 | 0.68032 | 0.03730 | 0.92746, 0.24556<br>0.68190 | 0.65050 | 0.03410 | 0.90945, 0.27204<br>0.63741 | 0.57610 | 0.01872 | 0.80553, 0.33467<br>0.47086 |
| | | 0.9 | 0.70544 | 0.01720 | 0.89096, 0.37177<br>0.51919 | 0.67673 | 0.01564 | 0.87044, 0.38060<br>0.48984 | 0.61656 | 0.01144 | 0.80940, 0.41673<br>0.39268 |
| | 70 | 0.9 | 0.69356 | 0.02819 | 0.91520, 0.31706<br>0.59813 | 0.67052 | 0.02584 | 0.89762, 0.33618<br>0.56143 | 0.59264 | 0.01592 | 0.80882, 0.36402<br>0.44481 |
| | | 0.9 | 0.70790 | 0.01276 | 0.87322, 0.43324<br>0.43999 | 0.68693 | 0.01162 | 0.85563, 0.43526<br>0.42037 | 0.63296 | 0.00900 | 0.80789, 0.45184<br>0.35605 |
| $\beta = 0.5$ | 30 | 0.5 | 0.53196 | 0.01545 | 0.93145, 0.37106<br>0.56039 | 0.52256 | 0.01226 | 0.80758, 0.36741<br>0.44017 | 0.56326 | 0.01757 | 0.78578, 0.38201<br>0.40377 |
| | | 0.9 | 0.50076 | 0.00278 | 0.61889, 0.40570<br>0.21319 | 0.49952 | 0.00257 | 0.61928, 0.40339<br>0.21588 | 0.50111 | 0.00277 | 0.61721, 0.39407<br>0.22314 |
| | 50 | 0.5 | 0.51477 | 0.00624 | 0.69256, 0.39453<br>0.29802 | 0.51045 | 0.00563 | 0.79455, 0.39222<br>0.40234 | 0.54068 | 0.00797 | 0.70099, 0.40394<br>0.29705 |
| | | 0.9 | 0.50108 | 0.00166 | 0.58866, 0.42670<br>0.16196 | 0.49997 | 0.00158 | 0.60593, 0.42482<br>0.18110 | 0.49808 | 0.00166 | 0.58369, 0.41675<br>0.16695 |
| | 70 | 0.5 | 0.51100 | 0.00391 | 0.64484, 0.41075<br>0.23409 | 0.50826 | 0.00365 | 0.64154, 0.40867<br>0.23287 | 0.53239 | 0.00491 | 0.66268, 0.41861<br>0.24407 |
| | | 0.9 | 0.50143 | 0.00115 | 0.57417, 0.43798<br>0.13619 | 0.50052 | 0.00110 | 0.57392, 0.43666<br>0.13726 | 0.49782 | 0.00109 | 0.56874, 0.42956<br>0.13918 |

**Table 4:** *Estimates of the parameters, MSE, CI and HPD interval for α=0.3, β=2.5*

| | | | MLE | | | MPS | | | Bayes | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | n | To | Est. | MSE | CI<br>Length | Est. | MSE | CI<br>Length | Est. | MSE | HPDI<br>Length |
| $\alpha = 0.3$ | 30 | 3 | 0.33908 | 0.06482 | 0.74655, 0.07368<br>0.67287 | 0.32510 | 0.05521 | 0.73790, 0.07829<br>0.65961 | 0.48950 | 0.04125 | 0.79820, 0.30191<br>0.49629 |
| | | 5 | 0.33122 | 0.04056 | 0.76794, 0.09554<br>0.67240 | 0.31172 | 0.03484 | 0.75338, 0.09133<br>0.66206 | 0.46244 | 0.03133 | 0.67978, 0.24478<br>0.43500 |
| | 50 | 3 | 0.32277 | 0.04611 | 0.74687, 0.09224<br>0.65463 | 0.31534 | 0.04189 | 0.73531, 0.09535<br>0.63996 | 0.46766 | 0.03350 | 0.68422, 0.24752<br>0.43670 |
| | | 5 | 0.31184 | 0.02651 | 0.71418, 0.10936<br>0.60482 | 0.30053 | 0.02438 | 0.70142, 0.10662<br>0.59480 | 0.43199 | 0.02197 | 0.62896, 0.23485<br>0.39411 |
| | 70 | 3 | 0.31052 | 0.03601 | 0.73439, 0.09710<br>0.63729 | 0.30563 | 0.03362 | 0.72240, 0.10037<br>0.62202 | 0.45009 | 0.02728 | 0.65550, 0.24180<br>0.41370 |
| | | 5 | 0.30539 | 0.01948 | 0.66915, 0.11946<br>0.54969 | 0.29735 | 0.01840 | 0.65943, 0.11694<br>0.54248 | 0.41117 | 0.01666 | 0.59270, 0.22909<br>0.36360 |
| $\beta = 2.5$ | 30 | 3 | 2.56213 | 0.33418 | 3.79115, 1.82942<br>1.96174 | 2.53662 | 0.25619 | 3.64655, 1.85827<br>1.78828 | 2.48647 | 0.13565 | 3.18897, 1.86317<br>1.32580 |
| | | 5 | 2.52823 | 0.18019 | 3.34280, 1.92556<br>1.41723 | 2.53977 | 0.14785 | 3.33731, 1.94855<br>1.38876 | 2.58703 | 0.12178 | 3.19483, 2.02135<br>1.17349 |
| | 50 | 3 | 2.53769 | 0.21201 | 3.47349, 1.90637<br>1.56711 | 2.52595 | 0.16525 | 4.80067, 1.93116<br>2.86951 | 2.47033 | 0.10858 | 3.04908, 1.94862<br>1.10045 |
| | | 5 | 2.52399 | 0.09420 | 3.15481, 2.02575<br>1.12907 | 2.52977 | 0.08328 | 3.15171, 2.03905<br>1.11266 | 2.57688 | 0.07407 | 3.07648, 2.10567<br>0.97081 |
| | 70 | 3 | 2.54392 | 0.14758 | 3.34860, 1.96835<br>1.38025 | 2.53228 | 0.12293 | 3.41558, 1.98083<br>1.43475 | 2.47440 | 0.07359 | 2.98561, 2.01011<br>0.97550 |
| | | 5 | 2.51170 | 0.07450 | 3.04064, 2.07754<br>0.96310 | 2.51796 | 0.06407 | 3.04172, 2.08817<br>0.95355 | 2.55707 | 0.06053 | 2.99013, 2.14544<br>0.84470 |

**Table 5:** *Estimates of the parameters, MSE, CI and HPD interval for α=0.5, β=2.5*

| | | | MLE | | | MPS | | | Bayes | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | n | To | Est. | MSE | CI<br>Length | Est. | MSE | CI<br>Length | Est. | MSE | HPDI<br>Length |
| $\alpha = 0.5$ | 30 | 3 | 0.49063 | 0.06249 | 0.86146, 0.12443<br>0.73703 | 0.46567 | 0.05350 | 0.84526, 0.13707<br>0.70819 | 0.52890 | 0.00457 | 0.76369, 0.28798<br>0.47572 |
| | | 5 | 0.51364 | 0.03845 | 0.84534, 0.18898<br>0.65635 | 0.48145 | 0.03331 | 0.82155, 0.18757<br>0.63398 | 0.52551 | 0.00554 | 0.74427, 0.30314<br>0.44113 |
| | 50 | 3 | 0.49569 | 0.04508 | 0.84606 , 0.18054<br>0.66552 | 0.47960 | 0.04058 | 0.82739, 0.18762<br>0.63978 | 0.52462 | 0.00526 | 0.74473, 0.29795<br>0.44678 |
| | | 5 | 0.50622 | 0.02396 | 0.78083, 0.24494<br>0.53589 | 0.48643 | 0.02201 | 0.75979, 0.24123<br>0.51856 | 0.51811 | 0.00584 | 0.71663, 0.31660<br>0.40003 |
| | 70 | 3 | 0.49200 | 0.03487 | 0.82882, 0.21163<br>0.61719 | 0.48054 | 0.03235 | 0.81042, 0.21597<br>0.59444 | 0.51908 | 0.00542 | 0.72744, 0.30437<br>0.42307 |
| | | 5 | 0.50179 | 0.01767 | 0.73912, 0.27530<br>0.46382 | 0.48773 | 0.01671 | 0.72061, 0.27285<br>0.44775 | 0.51249 | 0.00587 | 0.69522, 0.32677<br>0.36845 |
| $\beta = 2.5$ | 30 | 3 | 2.60200 | 0.27134 | 3.90702, 1.86206<br>2.04496 | 2.56793 | 0.22632 | 4.41924, 1.86555<br>2.55369 | 2.56169 | 0.11156 | 3.26219, 1.93257<br>1.32962 |
| | | 5 | 2.52288 | 0.10553 | 3.22612, 1.97815<br>1.24797 | 2.52547 | 0.09680 | 3.23023, 1.97960<br>1.25063 | 2.55712 | 0.07207 | 3.11970, 2.02915<br>1.09055 |
| | 50 | 3 | 2.56577 | 0.14969 | 3.42124, 1.96766<br>1.45359 | 2.54752 | 0.13165 | 3.35205, 1.97103<br>1.38102 | 2.54999 | 0.07483 | 3.12237, 2.03016<br>1.09221 |
| | | 5 | 2.51464 | 0.05957 | 3.03958, 2.08200<br>0.95758 | 2.51666 | 0.05730 | 3.04707, 2.08138<br>0.96569 | 2.54589 | 0.04696 | 2.99657, 2.11663<br>0.87995 |
| | 70 | 3 | 2.55300 | 0.09713 | 3.25062, 2.02987<br>1.22075 | 2.53941 | 0.08819 | 5.83190, 2.02799<br>3.80391 | 2.54610 | 0.05616 | 3.04397, 2.08901<br>0.95496 |
| | | 5 | 2.51102 | 0.04241 | 2.94791, 2.13973<br>0.80818 | 2.51263 | 0.04122 | 2.95148, 2.14001<br>0.81147 | 2.53840 | 0.03553 | 2.92507, 2.16716<br>0.75791 |

**Table 6:** *Estimates of the parameters, MSE, CI and HPD interval for α=0.7, β=2.5*

| | | | MLE | | | MPS | | | Bayes | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | n | To | Est. | MSE | CI<br>Length | Est. | MSE | CI<br>Length | Est. | MSE | HPDI<br>Length |
| $\alpha = 0.7$ | 30 | 3 | 0.66784 | 0.04386 | 0.93602, 0.18502<br>0.75101 | 0.62763 | 0.04023 | 0.91603, 0.21852<br>0.69750 | 0.57242 | 0.01912 | 0.80563, 0.32935<br>0.47628 |
| | | 5 | 0.69938 | 0.02430 | 0.91249, 0.27819<br>0.63430 | 0.65117 | 0.02269 | 0.88475, 0.29866<br>0.58609 | 0.59734 | 0.01431 | 0.80928, 0.37808<br>0.43120 |
| | 50 | 3 | 0.68838 | 0.02839 | 0.91716, 0.29125<br>0.62591 | 0.66076 | 0.02607 | 0.89869, 0.31178<br>0.58691 | 0.59256 | 0.01550 | 0.80859, 0.36665<br>0.44193 |
| | | 5 | 0.70956 | 0.01542 | 0.88856, 0.38549<br>0.50307 | 0.67790 | 0.01394 | 0.86624, 0.39026<br>0.47598 | 0.62341 | 0.01003 | 0.81088, 0.42952<br>0.38137 |
| | 70 | 3 | 0.69760 | 0.01970 | 0.89736, 0.36228<br>0.53508 | 0.67665 | 0.01817 | 0.88071, 0.37265<br>0.50806 | 0.60782 | 0.01306 | 0.80874, 0.39669<br>0.41204 |
| | | 5 | 0.71161 | 0.01148 | 0.87161, 0.44849<br>0.42313 | 0.68855 | 0.01042 | 0.85369, 0.44483<br>0.40886 | 0.63988 | 0.00781 | 0.80919, 0.46494<br>0.34425 |
| $\beta = 2.5$ | 30 | 3 | 2.55373 | 0.14866 | 3.46516, 1.92668<br>1.53848 | 2.52432 | 0.13224 | 3.39144, 1.90963<br>1.48181 | 2.57792 | 0.08864 | 3.25633, 1.96686<br>1.28947 |
| | | 5 | 2.49284 | 0.06864 | 3.06494, 2.02924<br>1.03570 | 2.49091 | 0.06396 | 3.07687, 2.01978<br>1.05709 | 2.47366 | 0.04880 | 2.97784, 1.99879<br>0.97905 |
| | 50 | 3 | 2.53912 | 0.07923 | 3.14974, 2.06092<br>1.08882 | 2.52252 | 0.07473 | 3.13833, 2.04526<br>1.09306 | 2.57224 | 0.06356 | 3.11481, 2.07595<br>1.03886 |
| | | 5 | 2.50142 | 0.04142 | 2.92997, 2.13616<br>0.79381 | 2.49971 | 0.03929 | 2.93508, 2.12985<br>0.80523 | 2.47471 | 0.03182 | 2.86599, 2.09898<br>0.76701 |
| | 70 | 3 | 2.52649 | 0.05159 | 3.00072, 2.13151<br>0.86921 | 2.51528 | 0.04996 | 2.99912, 2.11911<br>0.88001 | 2.56087 | 0.04968 | 3.02008, 2.13653<br>0.88355 |
| | | 5 | 2.50640 | 0.02923 | 2.86355, 2.19413<br>0.66942 | 2.50501 | 0.02801 | 2.86692, 2.18957<br>0.67735 | 2.48004 | 0.02321 | 2.81090, 2.15919<br>0.65172 |

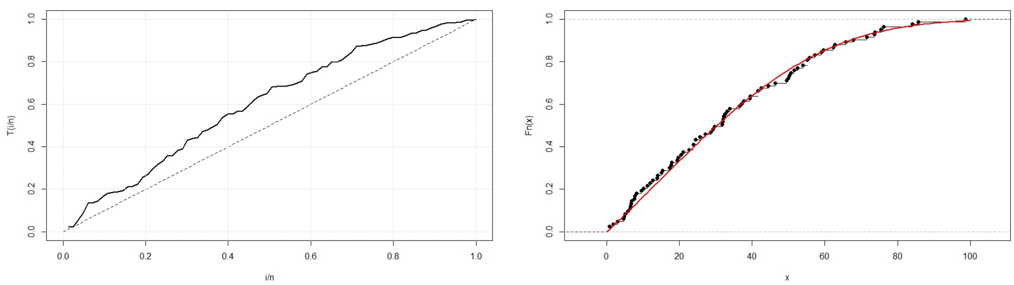
## 7. Real Data Analysis

In this section, we use a real data set to illustrate the applicability of the proposed work in real life situation. The data-set recently considered in [4], which is taken from the website of the Bureau of Meteorology of the Australian Government (www.bom.gove.au). It represent the monthely rainfall (in mm) between the period of January 2000 to February 2007 in the rain guege station of Carrol, located in the State of New South Wales on the east coast of Australia. The data are given as:

**Table 7:** *Carrol Data-set (n=83)*

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 22.7 | 75.5 | 28.6 | 65.8 | 39.4 | 33.1 | 84 | 41.6 | 62.3 | 52.5 | 13.9 | 15.4 | 31.9 |
| 32.5 | 37.7 | 9.5 | 49.9 | 31.8 | 32.2 | 50.2 | 55.8 | 20.4 | 5.9 | 10.1 | 44.5 | 19.7 | 6.4 |
| 29.2 | 42.5 | 19.4 | 23.8 | 55.2 | 7.7 | 0.8 | 6.7 | 4.8 | 73.8 | 5.1 | 7.6 | 25.7 | 50.7 |
| 59.7 | 57.2 | 29.7 | 32 | 24.5 | 71.6 | 15 | 17.7 | 8.2 | 23.8 | 46.3 | 36.5 | 55.2 | 37.2 |
| 33.9 | 53.9 | 51.6 | 17.3 | 85.7 | 6.6 | 4.7 | 1.8 | 98.7 | 62.8 | 59 | 76.1 | 67.9 | 73.7 |
| 27.2 | 39.5 | 6.9 | 14 | 3 | 41.6 | 49.5 | 11.2 | 17.9 | 12.7 | 0.8 | 21.1 | 24.5 | |

To ensure that this data-set is appropriate for the illustration of proposed work, first we draw the Total Time on Test (TTT) plot Fig.[1(a)], which shows that the Hazard rate of considered data-set same as the considered distribution (increasing Hazard rate). The Kolmogorov-Smirnov (K-S) test has been used to verify that this data fitted or not on the considered distribution, and found that the the value of K-S statistic is 0.057005 and p-value is 0.9502 which were shows that the considered data fitted on SMD. Graphical representation of K-S test is shown in Fig[1(b)].




**(a)** *TTT Plot Carrol Data-set*

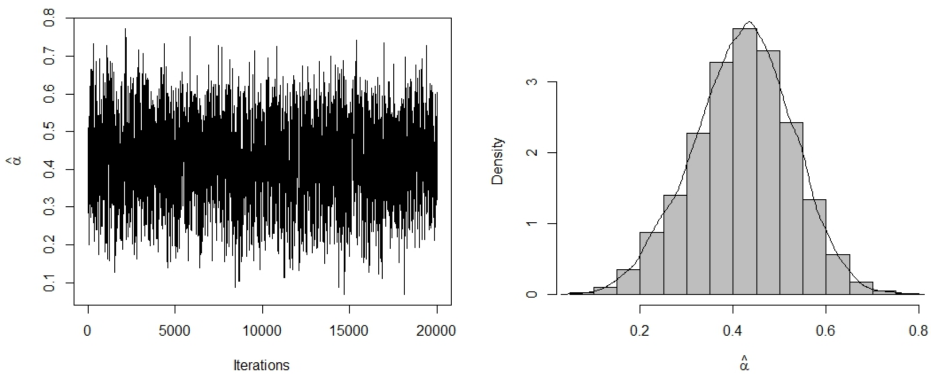
**(b)** *empirical cdf and fitted cdf plot*

**Figure 1:** *(a) plot shows the increasing hazard rate and (b) show fitted cdf of real data on SMD*

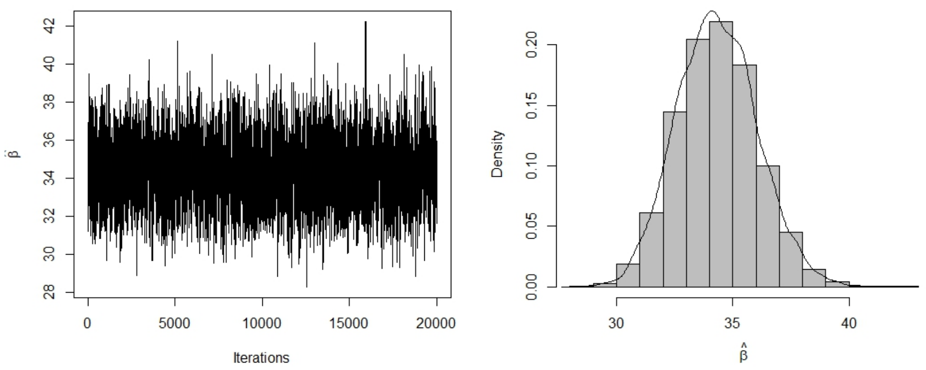
In [4] it is already shows that SMD gives best fit for this data-set in comparison of some other lifetime distribution (Weibull, Gamma, log-Normal and exponential distribution), on basis of their AIC and BIC values shown in table[8]. The MLE, MPSE and Bayes estimate of the parameter for this data-set in case of type-1 censoring given in table[9].

**Table 8:** *MLE, AIC and BIC values for the Carrol Data-set*

| Model | MLE | AIC | BIC |
|---|---|---|---|
| SMD $(\alpha, \beta)$ | $\hat{\alpha} = 0.4608, \hat{\beta} = 33.9049$ | 740.3600 | 741.5220 |
| Weibull $(a, b)$ | $\hat{a} = 1.3665, \hat{b} = 36.9120$ | 744.4891 | 745.6511 |
| Gamma$(a, b)$ | $\hat{a} = 1.5160, \hat{b} = 22.3838$ | 747.3087 | 748.4706 |
| Exponential $(\lambda)$ | $\hat{\lambda} = 0.0295$ | 753.0520 | 753.6330 |
| Log-Normal $(\mu, \sigma)$ | $\hat{\mu} = 3.1597, \hat{\sigma} = 1.0253$ | 767.1983 | 768.3602 |




**(a)** *traceplot and marginal density plot for α*




**(b)** *traceplot and marginal density plot for β*

**Figure 2:** *Shows that MCMC traceplot and marginal density plot with histogram*

**Table 9:** *Estimate of parameter for Carrol Data-set*

| | | | MLE | | MPSE | | Bayes | |
|---|---|---|---|---|---|---|---|---|
| To | m | | est | CI | est | CI | est | HPD |
| 25 | 36 | $\alpha$ | 0.4079 | 0.8835, 0.0589<br>0.8246 | 0.3912 | 0.8807, 0.0529<br>0.8278 | 0.4328 | 0.7031, 0.1414<br>0.5617 |
| | | $\beta$ | 33.4852 | 51.7903, 21.6500<br>30.1403 | 33.2816 | 51.4880, 21.5130<br>29.9749 | 33.4929 | 37.3285, 29.2367<br>8.0918 |
| 50 | 60 | $\alpha$ | 0.3060 | 0.7129, 0.0726<br>0.6403 | 0.2985 | 0.7081, 0.0694<br>0.6387 | 0.3565 | 0.6010, 0.1126<br>0.4884 |
| | | $\beta$ | 36.3427 | 45.1023, 29.2843<br>15.8180 | 36.2114 | 44.9145, 29.1947<br>15.7197 | 36.4217 | 40.0513, 32.6227<br>7.4286 |
| 75 | 78 | $\alpha$ | 0.4266 | 0.6486 , 0.2307<br>0.4179 | 0.4152 | 0.6382 , 0.2223<br>0.4160 | 0.4254 | 0.6131, 0.2169<br>0.3962 |
| | | $\beta$ | 34.1870 | 39.8435, 29.3336<br>10.5099 | 34.2052 | 39.8796, 29.3382<br>10.5415 | 34.3796 | 37.6232, 31.0938<br>6.5295 |

## 8. Conclusion

In this article, the considered model SMD is another form of Muth distribution by adding a scale parameter. To estimate the parameters of this distribution, we used classical and Bayesian approach under type-1 censoring scheme. In classical approach MLE and MPSE are obtain for time censored data. For the Bayes estimate of the parameters, beta and gamma prior are considered for shape parameter $\alpha$ and scale parameter $\beta$ respectively. The simulation study was also done by using Monte Carlo method of simulation. From the simulation study it is found that the Bayes estimate perform better than classical method (MLE and MPSE) on the basis of their MSE and length of interval estimation.

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